MA477 - Financial Risk Management and Modelling
Credit Rating Migration

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1 Introduction

In this project, we study the phenomenon of Credit Rating Migration. It has been observed by rating agencies that companies are quite likely to face changes in their credit quality across years. This shift of credit quality (and thus, rating) is called a *credit migration*, the probability of which is collected in a *migration matrix*. Therefore, we would need to use stochastic methods such as Markov Chains to reliably model this phenomenon. This constitutes the bulk of this report. We begin with the setup from [2], Chapter 6 and discuss the Mixture of Markov Chains Model from [1].

2 Markov Chain Model

We study credit migration within the framework of a time-homogeneous Markov Chain. Of course, such a model comes along with the assumptions underlying a Markov Chain summarized as follows:

1. (*Markov Property*) The current year's rating for a company is independent of the full history of its ratings and only depends on last year's rating.

2. (*Time-homogeneity*) The migration probabilities are time-invariant.

These assumptions may be an over-simplification of reality. Nonetheless, we proceed with such a model as it helps understanding credit migration better.

So, we assume that the migration process is controlled via a Markov Chain where the states are the possible credit ratings and the transition probabilities reflect the probability of migrating (for a given time horizon, typically 1 year) from one state to another.

Concretely, we have the finite state space,

$$\begin{aligned}\Omega &= \{AAA, AA, A, BBB, BB, B, CCC, D\} \\ &:= \{1, \ldots, 8\}\end{aligned} \tag{1}$$

where the state D represents default. Now we represent the rating of a particular company for each year with a sequence of random variables $\{R_n\}$ taking values in Ω. The Markov property implies that,

$$\mathbb{P}\left(R_n = j | R_{n-1} = i, R_{n-2} = i_{n-2}, \ldots, R_0 = i_0\right) = \mathbb{P}\left(R_n = j | R_{n-1} = i\right) \tag{2}$$

The migration matrix $M \in \mathbb{R}^{8\times 8}$ such that, $M = [[m_{ij}]]_{i,j=1,\ldots,8}$ is defined as the transition matrix of the chain, that is,

$$m_{ij} = \mathbb{P}\left(R_n = j | R_{n-1} = i\right) := \mathbb{P}\left(i \to j\right) \quad i, j = 1, \ldots, 8 \tag{3}$$

where $i \to j$ represents a transition from state i to j, over the course of an year. As M is simply a transition matrix, it comes with all the generic properties of transition matrices. That is, $m_{ij} \geq 0$ for all i, j and all row sums equal 1 (i.e., it is a stochastic matrix). Additionally, the default state is absorbing. Thus, $m_{8j} = 0$ for all $j = 1, \ldots, 7$ and $m_{88} = 1$.

Let us denote the n-year transition matrix by M_n whose entries are of the form $\mathbb{P}\left(R_n = j | R_0 = i\right)$. As a simple consequence of the Markov property we get the following relation,

$$M_n = M_1^n = M^n \tag{4}$$

We would also like to impose the so-called "plausibility constraints" as in [2] on M to reflect certain key properties of migration. These are:

1. Low-risk states do not show higher default probability than high risk states. That is,
$$m_{i8} \leq m_{j8} \tag{5}$$
whenever $i < j$.

2. It should be more likely to migrate to closer ratings than distant ones. That is, as we go further from the diagonal element (on either side) of any row, the probabilities must decrease. So,
$$m_{i(i+1)} \geq m_{i(i+2)} \cdots \geq m_{i8} \tag{6}$$
$$m_{i(i-1)} \geq m_{i(i-2)} \cdots \geq m_{i1} \tag{7}$$

3. It should be more likely to migrate into a certain rating from closer ratings than distant ones. That is, as we go further from the diagonal element (on either side) of any column, the probabilities must decrease. So,
$$m_{(i+1)i} \geq m_{(i+2)i} \cdots \geq m_{8i} \tag{8}$$
$$m_{(i-1)i} \geq m_{(i-2)i} \cdots \geq m_{1i} \tag{9}$$
4. We may present the first constraint in a generalized manner by imposing that the probability of migrating to a class at least as worse as rating k must be a non-decreasing function of the initial class i. That is, $\sum_{j \geq k} m_{ij}$ is non-decreasing in i for any fixed k.

However, one cannot expect these properties to be satisfied by transition matrix sampled from historic data. There are two typical approaches to estimate these transition matrices. One follows from using a discrete-time approach and the other with a continuous-time approach. We begin with the discrete-time approach.

2.1 Cohort or Discrete Multinomial approach

In this method, we estimate M over a 1 year horizon through a very simple idea. Let $N_i(t)$ denote the number of firms in rating class i at the start of year t and let $N_{ij}(t+1)$ denote the number of firms that migrated to class j from i at the end of year t, before the beginning of $t+1$.

Under the assumption of a time-homogeneous Markov chain, we have the maximum likelihood estimate of the migration probability,

$$\hat{m}_{ij} = N_{ij}(t+1)/N_i(t) \tag{10}$$

Invoking time-homogeneity, we have the n-year estimates of the migration matrices as,
$$\hat{M}_n = \hat{M}^n \tag{11}$$
where $\hat{M} = [[\hat{m}_{ij}]]$. These are typically published migration estimates by the rating agencies. However, this approach has one major flaw. It ignores both within year migrations as well as the duration of the rating.

For example, if we have a firm rated as AA at the start of 2021, followed by a downgrade to A in the second quarter of 2021 and finally upgraded back to AA at the end of 2021, this firm would not be considered in the count of $N_{ij}(t+1)$. Also, if we have another firm rated as BBB only very briefly in the first quarter of 2021 and is upgraded to A and remains there for the rest of the year, it would be considered in the $N_i(t)$ calculation of BBB, which might not lead to entirely accurate results.

To overcome these problems, we may use the continuous time approach. To this end, we study continous-time Markov chains.

2.2 Continuous time Markov Chains and Generators

Typically, the shortest time intervals from which a transition matrix is estimated is one year. However we may be interested in knowing the transition probabilities and chance of default for loans with terms less than a year (for example, for the estimation problem mentioned in the previous section). For this, we try to embed a discrete-time Markov process with a transition matrix M into a continuous time Markov chain. We define a continuous time Markov Chain with the help of pure jump processes as follows:

Definition 2.1. Let $S_0 = 0$ and define a sequence of random variables $\{S_n\}$ such that $S_n > S_{n-1}$ for all $n \in \mathbb{N}$ and $S_n \to \infty$ as $n \to \infty$. Let $\{X_n\}$ be a sequence of random variables taking values in a countable state space (such as Ω). The process $\{Y_t, t \geq 0\}$ such that $Y_t = X_n$ for all $S_n \leq t < S_{n+1}$ is called a pure jump process. Further, the holding times are defined as $T_n = S_{n+1} - S_n$.

Now, we define a continuous time Markov Chain as a pure jump process where X_n is a discrete-time Markov chain (with transition matrix, say $P = [[p_{ij}]]$) and $T_n \sim \text{Exp}(\lambda_{X_n})$. That is, we remain in a state for a random amount of time whose distribution is only determined by that state. By the property of the exponential distribution, we remain in a given state i for an average amount of $1/\lambda_i$ time units.

(Note that, if we have $p_{ii} > 0$, we can simply adjust λ_i such that it accounts for self-loops in the discrete-time chain. Thus, WLOG we can assume that $p_{ii} = 0 \iff \lambda_i > 0$ and $p_{ii} = 1 \iff \lambda_i = 0$.)

Now, we define $M(t)$ as the time-dependent transition matrix of the chain Y_t,

$$M(t)_{ij} = \mathbb{P}(Y_t = j | Y_0 = i) \tag{12}$$

By time-homogeneity of the underlying discrete-time chain, it is easy to see that $M(t+s) = M(t)M(s)$, $M(0) = I$, where I is the identity matrix.

Further, for small Δt and $i \neq j$, we have

$$\begin{aligned} M(\Delta t)_{ij} &= \mathbb{P}(Y_{\Delta t} = j | Y_0 = i) \\ &= \mathbb{P}(T_0 < \Delta t)\, \mathbb{P}(X_1 = j | X_0 = i) \\ &= (1 - \exp(-\lambda_i \Delta t))\, p_{ij} \\ &\approx \lambda_i \Delta t p_{ij} \end{aligned} \tag{13}$$

Similarly, we have for $i = j$, $M_{ii}(\Delta t) = \exp(-\lambda_i \Delta t) \approx 1 - \lambda_i \Delta t$.

From this, we see that the "rate of moving from state i to j" is given by $q_{ij} = \lambda_i p_{ij}$. Thus, we collect these values in a so-called *rate matrix* or *generator matrix* Q given by,

$$Q = [[q_{ij}]] \tag{14}$$

where,

$$q_{ij} = \begin{cases} -\lambda_i & i = j \\ \lambda_i p_{ij} & i \neq j \end{cases} \tag{15}$$

From this definition, it is clear that $-q_{ii} > 0$ for all i, $q_{ij} \geq 0$ for all $i \neq j$ and $\sum_j q_{ij} = 0$ (as there can only be two cases, $p_{ii} = 0$ or $p_{ii} = 1$, both of which result in the row sums of Q being 0).

It can be shown that given a Q matrix, we can find the underlying transition matrix P and the rates $\{\lambda_i\}$, uniquely. That is, a Q matrix describes the continuous time Markov chain equivalently to the information contained in P and $\{\lambda_i\}$.

Further, a critical property of continuous time Markov Chains is that they follow the forward and backward Kolmogorov equations given by,

$$dM(t) = M(t)Qdt \tag{16}$$

$$dM(t) = QM(t)dt \tag{17}$$

respectively.

Under the boundary condition $M(0) = I$, the solution to the above differential equation is the matrix exponential,

$$M(t) = e^{tQ} = \sum_{k=0}^{\infty} \frac{(tQ)^k}{k!} \tag{18}$$

If instead, we take equation (17) as the starting point, then we have the following theorem from [2],

Theorem 2.1. *$M(t)$ defined by (17) is a stochastic matrix for all $t \geq 0$ iff $Q = [[q_{ij}]]$ satisfies the following properties:*

1. $0 \leq -q_{ii} < \infty$ for all $i = 1, \ldots, 8$
2. $q_{ij} \geq 0$ for all $i \neq j$
3. $\sum_{j=1}^{8} q_{ij} = 0$ for all $i = 1, \ldots, 8$

Given that we have single-period (one year) transition matrix from empirical data (that is, $M = e^Q$), such as those from a cohort estimator, we use results seen in [2] to estimate Q as follows,

Theorem 2.2. Let $M = [[m_{ij}]]$ be an $n \times n$ strictly diagonally dominant Markov transition matrix, i.e $m_{ii} > 0.5$ for all i. Then the series

$$\tilde{Q} = \sum_{k=1}^{l} (-1)^{k+1} \frac{(M-I)^k}{k} \tag{19}$$

converges geometrically quickly for $l \to \infty$ giving rise to an $n \times n$ matrix $\tilde{Q}$ having row-sums zero. That is, we get an estimate for the matrix logarithm $\log(M)$.

Note that diagonal dominance is an exogenous condition that need not be satisfied by empirically obtained migration matrices. Further, the $\tilde{Q}$ obtained from (19) may end up having negative off diagonal entries. This problem is solved by either simply setting these entries to 0 or by redistributing them to all the other entries in a given row. These schemes are discussed thoroughly in [2].

2.3 Hazard-rate or Duration approach

(The authors of [1] use the notation Λ in place of Q and $Q(t)$ in place of $M(t)$ in contrast to our discussion.) [1] shows a direct Q estimator as,

$$\hat{q}_{ij} = \frac{N_{ij}(0,T)}{\int_0^T Z_i(s)ds} \quad i \neq j \tag{20}$$

where $N_{ij}(0,T)$ is the total number of migrations from i to j within time T and $Z_i(t)$ is the number of firms with rating i at time t.

Thus, the integral in the denominator gives the total time spent by all considered firms in a given state i within $[0,T]$, called the *rating duration*. Also, note that the diagonal elements can be obtained by using the fact that the row sum must be 0. With this, we have our estimate of Q as simply, $\hat{Q} = [[\hat{q}_{ij}]]$.

This estimator of Q, followed by finding its exponential to get $M(t)$ overcomes the problems of the cohort estimator as N_{ij} considers all transitions and not just the initial and final states by the end of the year.

In conclusion, given the migration matrix we may estimate Q via (19) which can be used for estimating migration probabilities at arbitrary points in time. This serves the user side of things. Alternatively, from the raw data of migrations, we can also estimate Q via (20) and then estimate the migration matrix by using the fact that $M(t) = e^{tQ}$. This serves the rating agency side of things. The MMC model that follows is towards this end.

3 Mixture of Markov Chains Model

In this section, we present a model that takes into account the time-evolution of the business cycle as well. As a result, this model mixes two time-homogeneous Markov Chains, one for the evolution of the business cycle and another for the evolution of ratings. Here, we consider a model with two possible states for the business cycle, expansion (denoted by E) and contraction (denoted by C).

Let us denote the one-period (typically a quarter) transition matrix for this chain (denoted $\{B_n\}$) by $S = S(1)$ given by,

$$S = \begin{bmatrix} \theta & 1-\theta \\ 1-\phi & \phi \end{bmatrix} \tag{21}$$

where, $\theta = \mathbb{P}(B_1 = E | B_0 = E)$ and $\phi = \mathbb{P}(B_1 = C | B_0 = C)$.

We can estimate S using the Hazard-rate approach described in the previous section. Here,

$$\hat{q}_{EC} = \frac{N_{EC}(0,T)}{T_E} \tag{22}$$

where $N_{EC}(0,T)$ is the total number of $E \to C$ transitions in the sample time (T) and T_E is the total duration of expansion within T, measured in, say, months. Similarly, we may estimate $\hat{q}_{CE}$ as well. Assuming the one-period length is one-quarter and that the durations are measured in months, we would have the estimate for $S(1)$ as simply, $\hat{S} = e^{3\hat{Q}}$ as 1 month $= \frac{1}{3}$ quarter.

Now, conditional on the economic phase, the ratings follow another time homogeneous Markov Chain with transition matrices M_E and M_C. Thus, given that we are in expansion at time t, the probability that in the next period (i.e, time $t+1$) we are in an expansion and we make a rating migration from i to j is given by $\theta(M_E)_{ij}$ and similarly for other transitions.

In short, we may describe the full migration dynamics with the following migration matrix $M \in \mathbb{R}^{2k \times 2k}$, $k =$ number of rating classes,

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \tag{23}$$

where $M_1 = \theta M_E$, $M_2 = (1-\theta)M_C$, $M_3 = (1-\phi)M_E$, $M_4 = \phi M_C$. That is, each sub-matrix corresponds to making a rating migration and an economic phase transition.

Let us denote by $M^E(1)$ and $M^C(1)$, the one period transition matrices given current expansion and contraction respectively. Clearly, these are given by,

$$\begin{aligned} M^E(1) &= \theta M_E + (1-\theta)M_C \\ M^C(1) &= (1-\phi)M_E + \phi M_C \end{aligned} \tag{24}$$

In general, we have the following theorem.

Theorem 3.1. For $n \geq 2$, we have the following relations,

$$\begin{aligned} M^E(n) &= F_1 M^{n-2} L' \\ M^C(n) &= F_2 M^{n-2} L' \end{aligned} \tag{25}$$

where M is as in (23) and $F_1 = (M_1, M_2)$, $F_2 = (M_3, M_4)$, $L = (M^E(1), M^C(1))$.

Proof. We prove by induction on n.

Base case ($n = 2$): Clearly, $M^E(2)$ is given by,

$$\begin{aligned} M^E(2) &= \theta M_E\left(\theta M_E + (1-\theta)M_C\right) + (1-\theta)M_C\left((1-\phi)M_E + \phi M_C\right) \\ &= \theta M_E M^E(1) + (1-\theta)M_C M^C(1) \\ &= (M_1, M_2)(M^E(1), M^C(1))' \\ &= F_1 L' \end{aligned} \tag{26}$$

as required. Entirely similar arguments can be used for $M^C(2)$.

Induction step: Assume (25) is true for $n = k-1$. By the Markov property of the phase transitions, we can say that,

$$\begin{aligned} M^E(k) &= \theta M_E(M^E(k-1)) + (1-\theta)M_C(M^C(k-1)) \\ &= \theta M_E\left(F_1 M^{k-3} L'\right) + (1-\theta)M_C\left(F_2 M^{k-3} L'\right) \\ &= (M_1, M_2)\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} M^{k-3} L' \\ &= F_1 M^{k-2} L' \end{aligned} \tag{27}$$

as required. Here, the second equality follows from the induction hypothesis and the last equality follows simply from the fact that $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = M$. Entirely similar arguments hold for $M^C(k)$ as well. $\square$

This approach improves the simple Markovian approach in many ways, some of which are listed below:

1. Unconditionally, rating migration probabilities are time-heterogeneous. This is because of the explicit dependence on the economic phase variable.

2. This model would allow the default rates of two obligors to be correlated due to the common systematic and macroeconomic factors arising from the economic phase. This is closer to reality than the simple Markovian evolution model.

3. There is also serial dependence (also called *ratings drift*) as the unconditional rating process is not Markovian anymore.

Of course, we must address the question as to what qualifies a period of time as an expansion or a contraction period. This is answered in [1] as follows: "Recession is conceptualized by the National Bureau of Economic Research (NBER) as a significant decline in US economic activity lasting more than a few months, normally visible in real GDP, real income, employment, industrial production, and wholesale-retail sales". Thus, we classify a period as a contraction (or recession) period based on the aforementioned economic variables. Once we classify all such periods in a given dataset as contraction, we may call the rest of the periods as expansion periods. More details regarding the economic variables can be found in [1].

Further, we may generalize this model to any number of economic phases as well. In particular, Appendix B of [1] shows a three-phase model where the phases are expansion, "normal" and contraction. In this case, $S \in \mathbb{R}^{3\times 3}$ and $M \in \mathbb{R}^{3k\times 3k}$, $k =$ number of rating classes. Clearly, as we increase the number of possible phases, the complexity of the model increases tremendously.

4 Empirical Analysis

In this section, we describe the empirical analysis based on the discussed methods to estimate the transition matrix in (23). However, due the lack of rating migration data, we employ a simulation based approach.

[1] reports estimated values of θ and ϕ as,

$$\begin{aligned} 1 - \hat{\theta} &= 2.8\% \\ 1 - \hat{\phi} &= 24.1\% \end{aligned} \tag{28}$$

Also, Table 5 of [1] (pg. 39) reports the conditional (yearly) transition matrices $\hat{M}_E$ and $\hat{M}_C$ in Panel III and IV respectively. (Note that Panel I and II correspond to conditional transition matrices under the assumption that the economic phase changes in a deterministic fashion. For example, there may be a transition from E to C every three quarters and at the end of the last quarter there is a transition from C to E. In this case, $\theta = \phi = 0$. We do not show this approach here as it is just a special case of the more general approach.) Also, the space of ratings is of size 9 and not 8 in [1] as they include the "NR" (Not Rated) category as well. We follow this in our analysis as well. So,

$$\begin{aligned} \Omega &= \{AAA, AA, A, BBB, BB, B, CCC, D, NR\} \\ &:= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \end{aligned} \tag{29}$$

analogous to (1).

We wish to verify their matrices by doing a reverse calculation of them. The procedure is as follows:

1. Using $\hat{M}_E$ and $\hat{M}_C$, find $\hat{Q}_E$ and $\hat{Q}_C$. That is, $\log(\hat{M}_E)$ and $\log(\hat{M}_C)$ respectively, where log is the matrix logarithm.

2. Now, obtain the quarterly transition matrices as $M_{Eq} = \exp(\frac{1}{4}\hat{Q}_E)$ and $M_{Cq} = \exp(\frac{1}{4}\hat{Q}_C)$ respectively.

3. Further, we assume that we start with n companies in each rating category excluding D and including NR. That is, we have $8n$ companies in total.

4. Now, using (28) simulate N_p paths of the phase transition chain where each path is of length N.

5. For each phase transition path, simulate the rating changes of all the companies by sampling from the distribution specified by the company's rating category in the conditional transition matrix. For example, if we are currently in phase E and need to simulate the new rating of a company with current rating A, we simply sample from the discrete distribution specified by the third row (coresp. to A) of M_{Eq}. Let us call all of the rating migrations together as a *migration path*.

6. We simulate N_r migration paths for each phase transition path. For each migration path, we estimate Q_{Em} and Q_{Cm} using (20). For example, for Q_{Em}, $N_{ij}(0, T)$ is the total number of $i \to j$ transitions that occurred in E periods and $\int_0^T Z_i(s)ds$ would be the total time spent in rating i during E periods. This would just be the total number of occurrences of i in E periods in our case, as we simulate only quarterly transitions.

7. Now, we get our estimate of yearly conditional transition matrices, $M_{Em} = \exp(4Q_{Em})$ and $M_{Cm} = \exp(4Q_{Cm})$ for each migration path. The average of the N_r M_{Em}'s would be our estimate of M_E for this particular phase path (and correspondingly for C as well). Let us denote this average by M_{Ep} and M_{Cp}.

8. The average over all N_p M_{Ep}'s (M_{Cp}'s) would be our final estimate for M_E (M_C), denoted $\tilde{M}_E$ ($\tilde{M}_C$).

9. Thus, we have the simulation errors,

$$\begin{aligned} \mathrm{Err}_E &= \left\| \hat{M}_E - \tilde{M}_E \right\|_2 \\ \mathrm{Err}_C &= \left\| \hat{M}_C - \tilde{M}_C \right\|_2 \end{aligned} \tag{30}$$

We choose $N = 100$, i.e, we wish to simulate 25 years of migrations and economic phase transitions. Also, we choose $n = 500$, $N_p = 20$ and $N_r = 50$. These choices have been made as they seem to balance the amount of time needed to get results vs the size of simulation errors.

(All the results can be obtained via the program `sim_res.m` which makes use of two additional programs `sim_rating.m` and `sim_phase.m`. All analysis was done on an instance of MATLAB Online R2021a).

Under these choices, we get the simulation errors as,

$$\begin{aligned} \mathrm{Err}_E &= 0.6593 \\ \mathrm{Err}_C &= 0.7678 \end{aligned} \tag{31}$$

The reasons for the large simulation errors are as follows:

1. First of all, the initial distribution of companies and ratings would most likely not be uniform in [1]. However, as no information regarding this was available in the paper itself, we have used the uniform assumption. This would alter the $N_{ij}(0, T)$ and $\int_0^T Z_i(s)ds$ values.

2. Next, [1]'s estimate comes from one particular phase transition path and migration path. Thus, we may expect the average of the properties of several paths to be different from a particular instance.

The obtained estimates $\tilde{M}_E$ and $\tilde{M}_C$ are shown in Figure 1 and 2. We can observe from the tables that the plausibility constraints from section 1 are approximately followed with notable deviations such as the PD for higher ratings.

An immediate application of these transition matrices is that we may read off the probability of default (PD) values for each rating class, conditional on economic phase, from their 8^{th} column. We may obtain a full term structure (with time steps in years) by simply raising $\tilde{M}_E$ (equivalently $\tilde{M}_C$) to appropriate powers and reading the 8^{th} column of the obtained matrix. In this way, we have shown the term structure of PD for the CCC rating class for both economic phases in Figure 3. An immediate observation that agrees with our understanding is that the PD for contraction periods is visibly higher than for expansion periods.

In case we want the term structure on a continuous scale, we would have to do the following:

1. Find the implied generators $\tilde{Q}_E = \log(\tilde{M}_E)$ and $\tilde{Q}_C = \log(\tilde{M}_C)$.

2. The PD term structure for rating $i \in \Omega$, conditional on expansion (contraction), is given by the i^{th} entry of the 8^{th} column of $\exp\left(t\tilde{Q}_E\right)$ $\left(\exp\left(t\tilde{Q}_C\right)\right)$.

Further, the `sim_rating.m` function outputs a matrix `cmps` as well. This variable records the number of companies in each rating at each point of time in the simulation. One observes through this variable that if we run it for a large number of quarters, say $N = 1000$, nearly all the companies default at the end. This agrees with both the PD term structure in Figure 3 as well as the general idea that in the long run, all entities will default.

5 Conclusion

In this project we studied the phenomenon of credit rating migration and the different models used to understand it. We first started with a simple Markovian model and then considered the Mixture of Markov Chains model which takes the current economic state into account as well. Finally, we did a simulation study of this model.

ME_est =

9x9 table

	AAA	AA	A	BBB	BB	B	CCC	D	NR
AAA	0.98229	0.0090039	0.001211	0.00025153	0.00016252	1.203e-05	1.2073e-06	3.4488e-05	0.0070382
AA	0.00085861	0.97801	0.012198	0.0011836	0.00013592	0.00012138	2.168e-05	0.00010761	0.0073651
A	0.00010081	0.0023386	0.97971	0.0088118	0.00070183	0.0002579	1.189e-05	0.00015336	0.0079146
BBB	2.2016e-05	0.00031854	0.0053942	0.97603	0.0071391	0.00082188	9.6193e-05	0.00028758	0.0098926
BB	3.9713e-05	0.00011899	0.00059033	0.0072891	0.96394	0.011874	0.00081128	0.0012538	0.014078
B	1.6548e-06	7.1922e-05	0.00028956	0.00064287	0.0068248	0.961	0.0088559	0.0055548	0.016762
CCC	5.8792e-07	6.5348e-06	0.00053831	0.00079566	0.0013532	0.013155	0.89596	0.068332	0.019859
D	0	0	0	0	0	0	0	1	0
NR	3.4066e-05	0.00012255	0.00028841	0.00049104	0.00053457	0.00063139	3.1646e-05	0.0012002	0.99667

Figure 1: $\tilde{M}_E$

MC_est =

9x9 table

	AAA	AA	A	BBB	BB	B	CCC	D	NR
AAA	0.95622	0.018355	0.005005	0.0014907	0.00020186	4.3504e-05	4.6028e-06	0.00021116	0.01847
AA	0.001156	0.94904	0.023403	0.0039096	0.00071015	0.00015085	2.0113e-05	0.00084691	0.020762
A	0.00017232	0.0040594	0.95721	0.014907	0.0016918	0.00056723	6.0449e-05	0.0011016	0.020226
BBB	1.7691e-05	0.00099094	0.0086	0.95703	0.010037	0.0013907	0.00010302	0.0015273	0.020301
BB	2.6327e-05	0.00018167	0.0018514	0.0094305	0.93879	0.013226	0.0011991	0.006334	0.028966
B	4.0568e-06	0.00011664	0.00059984	0.0017766	0.0066025	0.93623	0.010291	0.014751	0.02963
CCC	2.2298e-06	1.1417e-05	0.00029483	0.00092641	0.001661	0.010413	0.85602	0.10425	0.026422
D	0	0	0	0	0	0	0	1	0
NR	3.8327e-05	0.00024012	0.00049082	0.00092357	0.0007454	0.00073342	6.1543e-05	0.0031739	0.99359

Figure 2: $\tilde{M}_C$

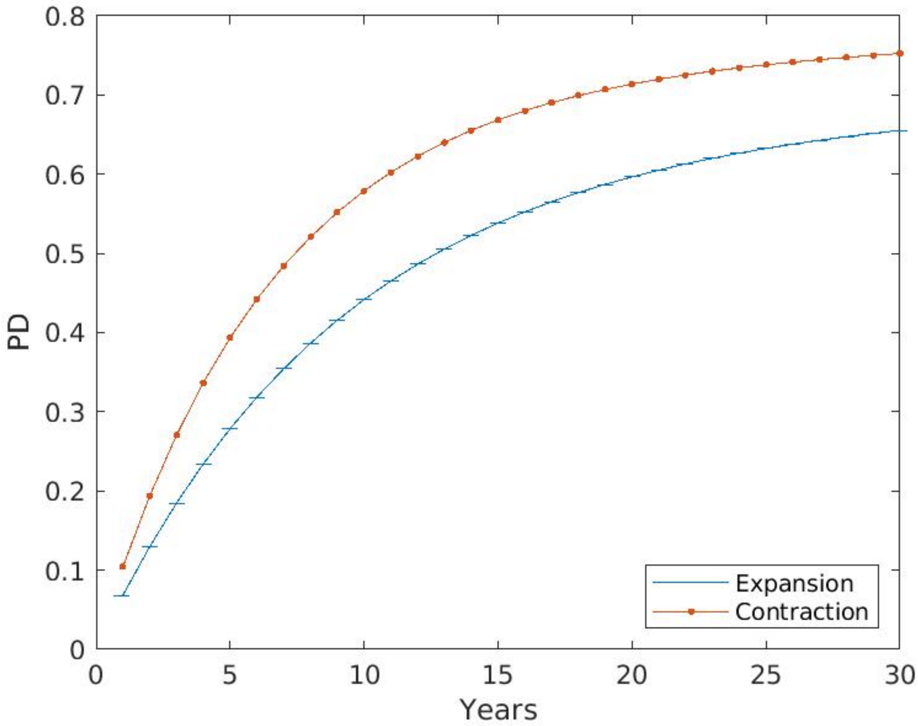

Figure 3: Term structure of PD for CCC rating class

References

[1] F. Fei , A. M. Fuertes , E. Kalotychou. Credit rating migration risk and business cycles. 2012.

[2] C. Bluhm , L. Overbeck , C. Wagner. *An Introduction to Credit Risk Modelling.* CRC Financial Mathematics Series. Chapman Hall, 2003.