Pricing European, American and Barrier Options using Spectral Collocation Methods

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Table of Contents

Table of Contents

Black Scholes Framework

In this project, we focus on solving the PDEs that arise from pricing options under the Black Scholes framework. We assume that the price of the underlying asset (S) evolves according to the SDE:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{1}$$

where μ and σ are the (constant) drift and volatility and $W(t)$ is the standard Wiener process. For the sake of simplicity, we assume that the underlying asset does not pay dividends.

Using this setup along with the no-arbitrage principle, we may obtain pricing formulas for several options as solutions to the Black-Scholes PDE with different boundary and initial conditions.

For any option, let T be its time of maturity and $V(S, \tau)$ be its value at time $0 \leq t \leq T$, where $\tau = T - t$. We state the IBVPs for $V(S, \tau)$ below: (In all the equations below, E is the strike price of the option and r is the risk-free rate.)

The Black Scholes PDEs

We obtain the following PDEs under the Black Scholes framework for the different options under consideration:
European call: For $0 < \tau \leq T$ and $S \geq 0$, V satisfies,

$$\frac{\partial V(S,\tau)}{\partial \tau} = rS\frac{\partial V(S,\tau)}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V(S,\tau)}{\partial S^2} - rV(S,\tau) \qquad (2)$$

with initial condition,

$$V(S,0) = \max(S-E,0)$$

and boundary conditions,

$$V(0,\tau) = 0$$
$$\lim_{S\to\infty} V(S,\tau) = S - E\exp\left(-r\tau\right)$$

Up and Out Barrier Call: For $0 < \tau \leq T$ and $0 \leq S < B$, where B is the barrier, V satisfies,

$$\frac{\partial V(S,\tau)}{\partial \tau} = rS\frac{\partial V(S,\tau)}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V(S,\tau)}{\partial S^2} - rV(S,\tau) \qquad (3)$$

with initial condition,

$$V(S,0) = \max(S - E, 0)$$

and boundary conditions,

$$V(0,\tau) = 0$$
$$V(B,\tau) = 0$$

Both the European call and Up and Out barrier call options have well-known closed form solutions.

American Put: For $0 < \tau \leq T$, V satisfies the free-boundary PDE,

$$\frac{\partial V(S,\tau)}{\partial \tau} = rS\frac{\partial V(S,\tau)}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V(S,\tau)}{\partial S^2} - rV(S,\tau) \quad S > B(\tau) \tag{4}$$

$$V(S,\tau) = E - S \quad 0 \leq S \leq B(\tau)$$

with the smooth pasting condition,

$$\frac{\partial V(B(\tau),\tau)}{\partial S} = -1$$

initial condition,

$$V(S,0) = \max(E - S, 0)$$

and finally boundary condition,

$$\lim_{S\to\infty} V(S,\tau) = 0$$

Where $B(\tau)$ is the free-boundary or the early-exercise curve. Also, to avoid arbitrage, V must also satisfy,

$$V(S,\tau) \geq \max(E - S, 0) \quad S \geq 0, 0 \leq \tau \leq T$$

Penalty Formulation

To resolve the complications arising from the free-boundary in (4), we replace the original PDE with a non-linear PDE by adding a penalty term. The reformulated PDE is as follows:

$$\frac{\partial V(S,\tau)}{\partial \tau} = rS\frac{\partial V(S,\tau)}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V(S,\tau)}{\partial S^2} - rV(S,\tau) + g(V(S,\tau)) \tag{5}$$

with initial condition,

$$V(S,0) = \max(E - S, 0)$$

and boundary conditions,

$$V(0,\tau) = E$$
$$\lim_{S\to\infty} V(S,\tau) = 0$$

where $g(V(S,\tau))$ is the penalty term.

The function g is such that for the American put,

$$g(V(S,\tau)) = \frac{\epsilon C}{V(S,\tau) + \epsilon - E + S} \tag{6}$$

and for the other two options $g(V(S,\tau)) = 0$, so that we may write the PDE for all three options together as,

$$\frac{\partial V(S,\tau)}{\partial \tau} = rS\frac{\partial V(S,\tau)}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 V(S,\tau)}{\partial S^2} - rV(S,\tau) + g(V(S,\tau)) \tag{7}$$

Thus, for every option under consideration, we solve equation (7) with initial conditions $\psi(S)$ and boundary conditions $\phi_1(\tau)$ at $S = 0$ and $\phi_2(\tau)$ at $S = \infty$, where each function takes the form corresponding to the option as presented above.

Table of Contents

Chebyshev Polynomials and Interpolation

Let $\{T_n(x)\}_{n=0}^{\infty}$ be the class of Chebyshev polynomials. Consider the extreme points of these polynomials (Chebyshev-Gauss-Lobatto, or CGL points) given by,

$$\xi_k = \cos\left(\frac{k\pi}{N}\right) \quad k = 0, \ldots, N \tag{8}$$

We define the Lagrange polynomials with CGL points as:

$$L_j(x) = \prod_{i=0, i\neq j}^{N} \frac{x - x_i}{\xi_j - \xi_i} \quad j = 0, \ldots, N \tag{9}$$

Interpolation

With these polynomials, we may interpolate a given function $u \in L^2(\mathbb{R}^2)$ with domain $[-1, 1] \times [0, T]$ as,

$$p(u(x,t)) = \sum_{j=0}^{N} u(\xi_j, t) L_j(x) \tag{10}$$

The approximation for $\frac{\partial u}{\partial x}$ over the CGL points are,

$$p(u_x(\xi_k, t)) = \sum_{j=0}^{N} u(\xi_j, t) D^{(1)}_{j,k} \tag{11}$$

where $D^{(1)}_{j,k} = \frac{\partial L_j(x)}{\partial x}|_{x=\xi_k}$ and the matrix $D^{(1)} = [[D^{(1)}_{j,k}]]$ is called the matrix of the derivative of the first kind Chebyshev polynomials. Similarly we have the approximation for $\frac{\partial^2 u}{\partial x^2}$ as,

$$p(u_{xx}(\xi_k, t)) = \sum_{j=0}^{N} u(\xi_j, t) D^{(2)}_{j,k} \tag{12}$$

where $D^{(2)}$ is defined analogously to $D^{(1)}$.

Table of Contents

Numerical Solution to the PDEs

We first truncate the infinite domain $[0, \infty]$ of S to $\Omega = [0, S_{max}]$. Choice of S_{max} will be described later.
Further, we decompose the domain Ω into m subdomains of the form $\Omega_i = [S_{i-1}, S_i]$ for $i = 1, \ldots, m$ such that $S_0 = 0 < S_1 \cdots < S_m = S_{max}$. Thus, we would have to solve the BS PDE for each domain,

$$\frac{\partial V_i(S,\tau)}{\partial \tau} = rS\frac{\partial V_i(S,\tau)}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 V_i(S,\tau)}{\partial S^2} - rV_i(S,\tau) + g(V_i(S,\tau)) \tag{13}$$

for $S \in [S_{i-1}, S_i]$, $i = 1, \ldots, m$.

Transformation to $[-1, 1]$

We transform each domain Ω_i to $[-1, 1]$ via the transformation,

$$X_i(x) = S = \frac{S_i - S_{i-1}}{2}x + \frac{S_i + S_{i-1}}{2} \tag{14}$$

Applying this transformation and setting $V_i(S, \tau) = u_i(x, \tau)$ reduces (13) to the set of equations,

$$\frac{\partial u_i(x,\tau)}{\partial \tau} = R_i^{(1)}(x)\frac{\partial u_i(x,\tau)}{\partial x} + R_i^{(2)}(x)\frac{\partial^2 u_i(x,\tau)}{\partial x^2} - ru_i(x,\tau) + g(u_i(x,\tau)) \tag{15}$$

for $x \in [-1, 1]$, $i = 1, \ldots, m$.

Here,

$$\begin{aligned} R_i^{(1)}(x) &= rX_i(x)\frac{2}{S_i - S_{i-1}} \\ R_i^{(2)}(x) &= \frac{1}{2}\sigma^2\left(X_i(x)\frac{2}{S_i - S_{i-1}}\right)^2 \end{aligned} \tag{16}$$

with initial and boundary conditions,

$$\begin{aligned} u_i(x,0) &= \psi(X_i(x)) \quad i = 1,\ldots,m \\ u_1(-1,\tau) &= \phi_1(\tau) \\ u_m(1,\tau) &= \phi_2(\tau) \end{aligned} \tag{17}$$

Smoothness Conditions

Additionally, we impose the smoothness conditions,

$$
\begin{aligned}
u_i(1,\tau) &= u_{i+1}(-1,\tau) \\
\frac{\partial u_i(x,\tau)}{\partial x}\Big|_{x=1} &= \frac{\partial u_{i+1}(x,\tau)}{\partial x}\Big|_{x=-1}
\end{aligned}
\tag{18}
$$

for $i = 1, \ldots, m-1$.
Combining (18) with the interpolation in (11) gives us the following form for $du_i(\tau) := u_i(1,\tau)$,

$$
du_i(\tau) = \frac{1}{D^{(1)}_{0,0} - D^{(1)}_{N,N}} \left(\sum_{j=0}^{N-1} u_{i+1}(\xi_j,\tau) D^{(1)}_{j,N} - \sum_{j=1}^{N} u_i(\xi_j,\tau) D^{(1)}_{j,0} \right)
\tag{19}
$$

Obtaining the ODEs

Now, multiplying (15) by the Dirac delta $\delta(x - \xi_j)$ and integrating for $j = 0, \dots, N$ gives us the system of ODEs in τ,

$$\frac{\partial u_i(\xi_j, \tau)}{\partial \tau} = R_i^{(1)}(\xi_j)\frac{\partial u_i(\xi_j, \tau)}{\partial x} + R_i^{(2)}(\xi_j)\frac{\partial^2 u_i(\xi_j, \tau)}{\partial x^2} - ru_i(\xi_j, \tau) + g(u_i(\xi_j, \tau)) \tag{20}$$

for $i = 1, \dots, m$ and $j = 0, \dots N$.

For the partial derivatives with respect to x, we may use the interpolation formulas from (11) and (12) to further simplify this system of ODEs. The final system takes the form:

$$\begin{aligned} \frac{dU(\tau)}{d\tau} &= F(\tau, U(\tau)) \\ U(0) &= U_0 \end{aligned} \tag{21}$$

where $U(\tau) \in \mathbb{R}^{mN-1}$ (as it doesn't include boundary terms) and F takes the form as in [1], (4.23).

Solving the ODEs

To solve the IVP in (21), we adopt a predictor-corrector approach where the predictor step follows the Euler method and the corrector step follows the Trapezoidal method. Also, we use a variable step size approach for fast convergence. The key idea behind variable step size is to choose the next step optimally so as to reduce the chance of unstable numerical behaviour. Concretely, the scheme is summarized as follows:

$$\begin{aligned} U_{n+1}^{(1)} &= U_n + h_n F(\tau_n, U_n) \\ U_{n+1}^{(\mu)} &= U_n + \frac{h_n}{2}\left(F(\tau_n, U_n) + F(\tau_{n+1}, U_{n+1}^{(1)})\right) \end{aligned} \tag{22}$$

Also, we define,

$$\mathrm{Er}_{n+1} = \left\| U_{n+1}^{(\mu)} - U_{n+1}^{(1)} \right\|_2 \tag{23}$$

where Tol is a predefined tolerance level for the scheme.

Evolution of h_n

The step size evolves as follows:

$$\begin{aligned} h^*_{n+1} &= h_n \left(\frac{\alpha \, \mathrm{Tol}}{\mathrm{Er}_{n+1}} \right)^{1/(p+1)} \\ h_{n+1} &= \min(h^*_{n+1}, 2) \end{aligned} \tag{24}$$

In total, we repeat (22) until $\mathrm{Er}_{n+1} < \mathrm{Tol}$, where in every repetition, we halve h_n (We denote the initial step size chosen by the user as h_0). Each such repetition corresponds to a "rejected step". Once $\mathrm{Er}_{n+1} < \mathrm{Tol}$ is achieved, we "accept" the step and set the next step size as per equations in (24) where we choose $\alpha = 0.5$ and p is the order of the method (here, $p = 2$).

Adjustment for American Put

In the case of the American put, we must also add an additional step before after each computation in (22) to maintain $V(S, \tau) \geq E - S$ as follows:

$$U_{n+1}^{(j)} = \max(U_{n+1}^{(j)}, U_0) \tag{25}$$

where max is the component-wise maximum. We now proceed to examine the results obtained from implementing these schemes.

Table of Contents

Choice of S_{max}

As mentioned in [1], for the European call and American put, we always take,

$$S_{max} = \frac{mE}{\lfloor \frac{m}{2} \rfloor} \tag{26}$$

This is so that we encounter minimal error from the truncation of the original half-infinite strip and also to make sure that E is one of the points $\{S_i\}$. Similarly, for the Barrier call $S_{max} = B$. So, given B and E, we make sure that,

$$E = \frac{k}{m} B \tag{27}$$

for some $k \in \{1, \ldots, m\}$.

Observations

(For tables and plots related to the observations, refer the report)

1. We can observe for each option that for a fixed m, the maximal error decreases as we increase the number of collocation points, N.
2. Similarly for a fixed N, we see that the error reduces as we increase m as well.
3. As we decrease Tol, the error decreases.
4. Also, we can observe that the variable step size method outperforms constant step size. As we can see, it takes much more time and steps with constant step size to achieve the same degree of error as the variable step size method.

1. We also notice that the error typically spikes around E as it causes discontinuity in the payoff.
2. For the American Put, we choose $C = rE$ and $\epsilon = 10^{-4}$. With respect to m, N and Tol we observe identical increase and decrease patterns in the error.
3. We can also notice that the error decreases as we decrease ϵ in the penalty term.
4. It is also clear that the variable step size method outperforms the constant step size alternative.

Table of Contents

Bibliography

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