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Chapter 1

Introduction

In quantum theory, the state of system can be depicted with a wave function. A wave function can be described in Hilbert space as a superposition of all possible states with probability amplitude. It contains all the information to think about a given quantum system and physical quantities are determined from the given wave function. Quantum states fulfill superposition behavior. The superposition must be destroyed by the measurements on quantum system. Real quantum systems are influenced by their surrounding environments. These unavoidable mutual interaction results in dissipation and quantum system loses coherence which finally leads to state laps or loss of useful information. This is an irreversible loss and severely damage or lose the overall quantum state. Decohernce causes both phenomenon like, local dynamics, in which single particle dissipate, decay, and diffuse and global dynamics, that causes the possible disappearance of entanglement. The decohernce occurs due to spontaneous emission in which atoms come to their ground level due to vacuum fluctuations [1]. second well known cause of decohernce is presence of thermal noise in environment. Due to increase in temperature the entanglement may finish in a finite amount of time is known as sudden death of entanglement (SDE) [2]. Similarly, Quantum systems such as,

thermal and vacuum environments strongly effect the entanglement dynamics of two two-level atoms which can be solved by using density matrix approach. These quantum systems are represented in pure state as well as in mixed state. The system is in pure state when $Tr\left(\rho^2\right) = 1$ and is mixed state when $Tr\left(\rho^2\right) < 1$. The pure state can be defined by a wavefunction

$$|\psi\rangle = \sum_i \alpha_i |\psi_i\rangle. \tag{1.0.1}$$

Such that $\sum_i |\alpha_i|^2 = 1$. $\alpha_i's$ are the probability amplitudes. The density matrix formalism represents the mixed quantum system, defined as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{1.0.2}$$

where $\sum_i \rho_i = 1$. If many body systems represent in state $|\psi_i\rangle$ then the correlation between two or more subsystems can be represented by mixed state of density matrix formalism. Quantum entanglement is another property of quantum theory that cannot be described classically. Quantum entanglement systems cannot be portrayed as individual system and contain correlation in the composite state of atleast two sub-systems. Among this correlation between the composite states of two sub-systems, the quantum entanglement states cannot be written in product form [3]. The best example to exhibit bipartite entangled system are the maximally entangled bell states [4] , given by

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}\left[|a_1, b_2\rangle \pm |b_1, a_2\rangle\right], \tag{1.0.3}$$

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}\left[|a_1, a_2\rangle \pm |b_1, b_2\rangle\right], \tag{1.0.4}$$

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where a_i and b_i are the excited and ground states of $i'th$ atom ($i = 1, 2$). Once entanglement is generated it can further be used to setup many mechanical task such as quantum teleportation, super dense coding, quantum cryptography [3, 5] and many others. These developments may lead to achieve the ambition of a quantum computers [6] which is a physical device and matchless as compared to the classical computers in every aspect. Once entanglement is generated we always need to quantify the amount of entanglement. For such purpose, concurrence as a quantitative measure of entanglement is used. Concurrence ranges from $c(t) = 0$ for a separable state and $c(t) = 1$ for an entangled state.

In this thesis, the first part depends on a cavity QED based atom field interaction Hamiltonian in which quantized electromagnetic field is trapped inside a cavity of specific length. At that point the interaction between single two-level atom with quantized electromagnetic field would be happened inside the cavity [1]. This mathematical model is extended for a two two-level atoms interacting with quantized electromagnetic field. After this phase, in the second part of thesis we explain the reduced density matrix equation for an atom by using density matrix formalism and observe the entanglement dynamics of two qubit system in dissipative environments. In the last part we determined the concurrence graphically and numerically as a quantitative measure of entanglement. Concurrence is basically used to quantify the amount of entanglement.

Chapter 2

Density Matrix: General Formalism

This chapter covers the preliminaries of atom field interaction and the general formalism of density matrix and its properties. First, we will built the basic Hamiltonian [1] for single two-level atom interacting with quantized electromagnetic field in the controlled environment of a high Q-cavity. This model is extended for a two two-level atoms interacting with quantized electromagnetic field . This model is further utilized in our further work in coming chapters. Second part of this chapter we will elaborate the general mathematical foundation about density matrix formalism and its basic properties.

2.1 Atom field interaction Hamiltonian

Consider a single two level atom trapped inside the controlled environment of a cavity and interacting with a single mode electromagnetic field. The total energy for this system consist of three parts, i.e. the energy of atom represented by H_A, the energy of quantized electromagnetic field, represented by H_F and the energy

which takes part in the interaction of atom and field. Therefore, this system can be defined by total Hamiltonian in a simple form as [1] ,

$$H = H_A + H_F - H_{AF}. \tag{2.1.1}$$

As field interacts with an atom the interaction between field and atom takes place so that last term represents the atom field interaction Hamiltonian H_{AF}. The electromagnetic field trapped inside the cavity of length L, can be expressed by the Maxwell's equations [7]. Here, we consider the source free Maxwell's equations. The solution of wave equation in this source free region can be built and given in standard text [7]. The solution either for electric or magnetic field, can be expanded in terms of the normal modes of the cavity [1]. Only those frequency $v_j = j\pi c/L$ which are resonant with cavity length. For such system the classical Hamiltonian can be written as [7]

$$H = \frac{1}{2} \int\limits_V d\tau \left(\epsilon_o E_x^2 + \mu_o H_y^2 \right). \tag{2.1.2}$$

By placing the Electric and magnetic field values and further simplification leads us to represent the above equation in term of dynamical variables. such as

$$H = \frac{1}{2} \sum_j \left(m_j \nu_j^2 q_j^2 + m_j \dot{q_j}^2 \right), \tag{2.1.3}$$

Where m_j is a constant of dimensions like mass and introduced here to create a similarity between the single-mode of electromagnetic field and simple Harmonic oscillator. The above Eq. (2.1.3) expresses that the electromagnetic field trapped inside a cavity looks like a radiation field which is summation of independent oscillator energies. The dynamical variables q_j and p_j will obey the uncertainty relation $[q_j, p_{j'}] = i\hbar\delta_{jj'}$, and $[q_j, q_{j'}] = [p_j, p_{j'}] = 0$. From here we can replace

these dynamical variables with operators a_j and $a_j^\dagger$

$$a_j e^{-iv_j t} = \frac{1}{\sqrt{2m_j \hbar v_j}} \left(m_j v_j q_j + i p_j \right), \tag{2.1.4}$$

$$a_j^\dagger e^{iv_j t} = \frac{1}{\sqrt{2m_j \hbar v_j}} \left(m_j v_j q_j - i p_j \right). \tag{2.1.5}$$

After multiplying above Eq's. (2.1.4) and (2.1.5) this formalism leads us to quantize the electromagnetic field trapped inside a cavity thus the energy free field part H_F in Eq. (2.1.1) can be written as

$$H_F = \sum_k \hbar v_k \left(a^\dagger a + \frac{1}{2} \right).$$

Here, the zero point energy $\left(\frac{1}{2} \hbar v_k \right)$ is similar as like the Harmonic oscillator and this past of energy remains with the system in any case, resulting it as a constant energy. This part of energy is responsible for the vacuum fluctuations, spontaneous decay and for the lamb shift. Here, in this system this constant term is ignored due to simplicity, so our Hamiltonian reduces to,

$$H_F = \sum_k \hbar v_k a^\dagger a. \tag{2.1.6}$$

Now, we can express the total energy of an atom H_A in any state $|i\rangle$ gives the eigen energy value of corresponding state such as, $H_A |i\rangle = E_i |i\rangle$. The completeness relation for an eigen states can be written as, $\sum_i |i\rangle \langle i| = 1$ By using these relations total energy of an atom can be represented in terms of transition operator that [1] is

$$H_A = \sum_i E_i |i\rangle \langle i| = \sum_i E_i \sigma_{ii}. \tag{2.1.7}$$

For a two-level atom having the energy E_a for above level and E_b for below level the above expression can be written in this form

$$H_A = (E_a \sigma_{aa} + E_b \sigma_{bb}).\qquad(2.1.8)$$

The difference of energy level can be written as, $(E_a - E_b) = \hbar\omega$. Thus, the above equation can be simplified as

$$
\begin{aligned}
(E_a \sigma_{aa} + E_b \sigma_{bb}) &= (\hbar\omega + E_b)\sigma_{aa} + (E_a - \hbar\omega)\sigma_{bb} \\
&= \hbar\omega(\sigma_{aa} - \sigma_{bb}) + E_b \sigma_{aa} + E_a \sigma_{bb}. \qquad(2.1.9)
\end{aligned}
$$

By, simplification and using the outer product form of Pauli matrix

$$\sigma_z = \sigma_{aa} - \sigma_{bb} = |a\rangle\langle a| - |b\rangle\langle b|.\qquad(2.1.10)$$

The above Eq. (2.1.9) can be rewritten as

$$(E_a \sigma_{aa} + E_b \sigma_{bb}) = \frac{1}{2}(E_a + E_b + \hbar\omega\sigma_z).\qquad(2.1.11)$$

Here, the total energy $(E_a + E_b/2)$ is an constant term so it can be ignored. This gives us a very simplified form of Hamiltonian for a two level atoms

$$H_A = \frac{1}{2}\hbar\omega\sigma_z.\qquad(2.1.12)$$

Now solving interaction energy term H_{AF} represented in Eq. (2.1.1) of a two level atom coupled with radiation field and field is thought to be uniform over the entire atom. This assumption is known as Electric dipole approximation [1, 8]. Thus, the interaction Hamiltonian can be described by using dipole approximation as

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$$H_{AF} = \hbar \sum_{i,j} \sum_{k} \frac{\wp_{ij} \sigma_{ij} \hat{\epsilon_k} \xi_k \left(a_k + a_k^\dagger \right)}{\hbar}, \qquad (2.1.13)$$

$\wp_{ij} = e < i \mid r \mid j >$ is electric dipole transition matrix. "ϵ_k" is a unit polarization vector and "ξ_k" is an intensity of an electric field, has dimensions of electric field. Where,

$$\xi_k = \left(\frac{\hbar v_k}{2\varepsilon_o V} \right)^{\frac{1}{2}}. \qquad (2.1.14)$$

Now putting all the values of H_A, H_F and H_{AF} from Eq's. (2.1.6), (2.1.12), (2.1.13), into an Eq. (2.1.1) , we get

$$H = \sum_k \hbar_k a^\dagger a + \frac{1}{2} \hbar \omega \sigma_z + \hbar \sum_k \sum_{i,j} g_k^{i,j} \sigma_{ij} \left(a_k + a_k^\dagger \right), \qquad (2.1.15)$$

Where $a_k^\dagger, a_k$ are creation and annihilation operator for kth normal mode with frequency v_k of electromagnetic field. Here the coupling constant is

$$g_k^{i,j} = -\frac{\wp_{ij} . \hat{\epsilon_k} \xi_k}{\hbar}. \qquad (2.1.16)$$

For two level atoms $g_k = g_k^{ab} = g_k^{ba}$, Thus following form of Hamiltonian can be written as

$$H = \sum_k \hbar \nu_k a^\dagger a + \frac{1}{2} \hbar \omega \sigma_z + \hbar \sum_k g_k \left(\sigma_{ab} + \sigma_{ba} \right) \left(a_k + a_k^\dagger \right), \qquad (2.1.17)$$

using these expressions

$$\sigma_{ab} = \sigma_+ = |a\rangle\langle b|, \tag{2.1.18}$$

$$\sigma_{ba} = \sigma_- = |b\rangle\langle a|, \tag{2.1.19}$$

The Hamiltonian takes the form as

$$
\begin{aligned}
H &= \sum_k \hbar\nu_k a^\dagger a + \frac{1}{2}\hbar\omega\sigma_z + \hbar\sum_k g_k\left(\sigma_+ + \sigma_-\right)\left(a_k + a_k^\dagger\right), \\
&= \sum_k \hbar\nu_k a^\dagger a + \frac{1}{2}\hbar\omega\sigma_z + \hbar\sum_k g_k\left(\sigma_+ a_k + \sigma_+ a_k^\dagger + \sigma_- a_k + \sigma_- a_k^\dagger\right) \tag{2.1.20}
\end{aligned}
$$

In above Eq. the last term of atom field interaction can be described by using rotating wave approximation [9, 1, 2]. The atomic operators (σ_+, σ_-) are used to make transition between atomic levels to move the atom from upper state to lower state and vice versa. The σ_+ operator is used to make transition from lower level to upper level whereas, the σ_- operator is used to make transition from upper level to lower level. $\sigma_+ a_k$ defines the process in which atom make transition from lower level to upper level while a_k field operator make transition from upper level to lower level. At the end of this process energy of conservation takes place and a photon of k mode is present. similarly, $\sigma_- a_k^\dagger$ defines the process in which atom make transition from upper level to lower level while a_k field operator make transition from lower level to upper level. At the end of this process energy of conservation takes place and a photon of k mode is present. The term $\sigma_+ a_k^\dagger$ describe the process in which the atomic operator σ_+ make transition from lower state to upper state and a photon is created which results in gain of energy $2\hbar\omega$ so, energy is not remain conserved. this term can be neglected. Similarly $a_k\sigma_-$ results in loss of energy $2\hbar\omega$ thus, this term can also be neglected.

After applying rotating wave approximation, simplified Hamiltonian can be written as

$$H = \sum_k \hbar\nu_k a^\dagger a + \frac{1}{2}\hbar\omega\sigma_z + \hbar\sum_k g_k \left(\sigma_+ a_k + a_k^\dagger \sigma_-\right),\qquad (2.1.21)$$

The third term represents the interaction of two level atom with the electromagnetic field of kth modes. Many examples of atom field interaction can be determined by using this Hamiltonian. In above equation the first terms described the unpurtubed part of Hamiltonian as

$$H_o = \sum_k \hbar\nu_k a^\dagger a + \frac{1}{2}\hbar\omega\sigma_z,\qquad (2.1.22)$$

and atom-field interaction Hamiltonian can be express as

$$H_1 = \hbar\sum_k g_k \left(\sigma_+ a_k + a_k^\dagger \sigma_-\right),\qquad (2.1.23)$$

Now total Hamiltonian $H = H_o + H_1$ can be solved in interaction picture by using $\nu = e^{iH_o t} H_1 e^{-iH_o t}$, by solving this interaction term we obtained the simplified resulting Hamiltonian as

$$H = \sum_k \hbar\nu_k a^\dagger a + \frac{1}{2}\hbar\omega\sigma_z + \hbar\sum_k g_k \left[a_k^\dagger \sigma^- e^{-\imath(\omega-\nu_k)t} + \overset{+}{\sigma} a_k e^{\imath(\omega-\nu_k)t}\right].\qquad (2.1.24)$$

2.2   Bipartite system interact with Environment

We consider a bipartite system of two two-level atoms interacting with their local environments. The local environment is modeled as the field annihilation operator $a_k\,(b_k)$ and field creation operator $a_k^\dagger\,\left(b_k^\dagger\right)$ for the photons of the reservoir. The excited level and ground level of atom are represented by $|a_i\rangle$ and $|b_i\rangle$ respectively whereas, $i = 1, 2$ is the atom number. These atoms are placed in separate cavities.

Here, we assume that there is no direct coupling with each other but initially these atoms constitute in an entangled state [2]. These atoms are independently interact with their local environments as shown in Figure. (2.2.1)The total Hamiltonian
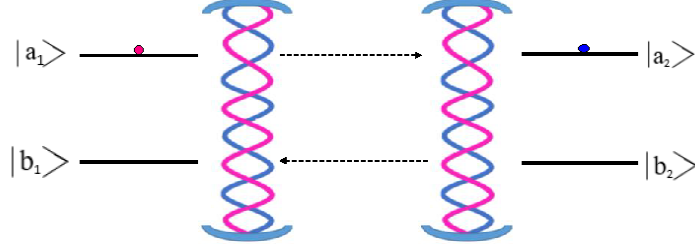
Figure 2.2.1: Two two-level atom independently interact with their local reservoir.

for a single two level atom interacts with kth mode of the field is discussed and given in Eq. (2.1.24). Similarly the Hamiltonian for a bipartite system interacting with their local environments can be written in interaction picture Hamiltonian using rotating wave approximation is

$$
\begin{aligned}
H &= \hbar \sum_k \left[g_k^{(1)} a_k^\dagger \sigma^- e^{-\imath(\omega - \nu_k)t} + \sigma^+ a_k e^{\imath(\omega - \nu_k)t} \right] \\
&+ \sum_k \left[g_k^{(2)} b_k^\dagger \sigma^- e^{-\imath(\omega - \nu_k)t} + \sigma^+ b_k e^{\imath(\omega - \nu_k)t} \right],
\end{aligned}
\qquad (2.2.1)
$$

For our simplicity we assume $\hbar = 1$. An atomic operators σ^+ represents the transitions from lower state to excited state whereas, the σ^- represents the transition from upper state to lower state. These two atomic operators can be defined as, $\sigma_i^+ = |a_i\rangle\langle b_i|$ and $\sigma_i^- = |b_i\rangle\langle a_i|$.

The difference between atomic states can be act as the frequency ω and ν_k is the frequency of kth modes of field. $g_k^{(1)}$ is the coupling constant for one atom and $g_k^{(2)}$ is the coupling constant for second atom. The generalized form of Eq. (2.2.1

) can be written in compact notation as

$$\upsilon\left(t\right) = \sum_{k}\sum_{i=1}^{2} g_k^{(i)} \left[b_k^\dagger \sigma_i^- e^{-\imath(\omega-\nu_k)t} + \sigma_i^+ b_k e^{\imath(\omega-\nu_k)t} \right] + H.C. \tag{2.2.2}$$

This is total interaction picture Hamiltonian of two two-level atoms which interact with reservoir. As the interaction takes place between two atoms and two modes of fields spontaneous decay occur. This spontaneous emission provide us to study the dynamic of entanglement of this system under different environments These types of system can be delt effectively using Heisenberg operator method as well as density matrix approach. It is convenient to use density matrix operator method to dealt with such statistical ensemble and it also tackle the damping effect which arise in system due to environment. In the coming section we will revisit the general density matrix formalism and its properties.

2.3   Density Matrix

Density matrix is a approach to describe the statistical mixture of system. The density matrix is helpful in dealing mixed states as well as pure state [10]. The density matrix is a general formalism to describe a quantum state by its sate vector (ket) or by statistical mixture of state vectors (kets). When we have imperfect information about the system, density matrix can be used as a tool to perform statistical averages in order to describe the quantum observable. The density operator can be defined in terms of outer product [3]

$$\rho = |\psi\rangle \langle\psi| . \tag{2.3.1}$$

The above Eq. (2.3.1) implies that if you find an operator A in state $|\psi\rangle$, then the $\langle\psi| A |\psi\rangle$ gives the expectation value of finding an operator A in state $|\psi\rangle$.

$$\langle A \rangle = \langle \psi | A | \psi \rangle. \qquad (2.3.2)$$

If we subdivide a system $|\psi\rangle$ into subsystems forming an ensemble $|\psi_i\rangle$, then the expectation or average value for an observable "A" in statistical mixture occupying the state $|\psi_i\rangle$ with probabilities p_i can be represented in terms of density matrix "ρ". such as

$$
\begin{aligned}
\langle A \rangle &= \sum_{k,l=1}^{n} \langle \psi_l | A | \psi_k \rangle \langle \psi_k | A | \psi_l \rangle, \\
&= \sum_{k,l=1}^{n} \langle \psi_l | \rho \left(\sum_{k=1}^{n} | \psi_k \rangle \langle \psi_k | \right) A | \psi_l \rangle, \\
&= \sum_{l=1}^{n} \langle \psi_l | \rho A | \psi_l \rangle, \\
\langle A \rangle &= Tr\left(\rho A\right).
\end{aligned}
$$

The density matrix is a practical tool to calculate the expectation values of pure state as well as mixed state. Now, here we discuss about pure state and mixed state [3] as

(a) Pure State

Pure state is the pure quantum state which is described mathematically by a vector in Hilbert space. A pure state represents an ensemble of system for which,when you do a measurement of a well chosen variable you always get the same outcome. When the system is in pure state, there is always a complete set of observable in which measurements on observable are completely determined. In this way if you measure all these observable several times the state remains same and you don't get more information each time that you repeat the measurement.

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The quantum Subsystem is depicted in Hilbert space by a state vector $|\psi\rangle$ of a basis function $\{\phi_1, \phi_2, \dots \phi_{\bar{n}}\}$ which express the collection of particles from an ensemble according to,

$$|\psi\rangle = \sum_i c_i |\phi_i\rangle.$$

The coefficient "c_i" represents the time propagation of the quantum subsystem, In the linear combination

$$c_i = \langle \phi_i | \psi_i \rangle.$$

The density operator for a state $|\psi\rangle$ can be defined in the form of projection operator. such as

$$\rho(t) = |\psi(t)\rangle \langle \psi(t)|.$$

To find the probability of the system in a state $|\phi_i\rangle$ is depicted by diagonal elements "ρ_{ii}" and the coherent superposition is defined by the polarization between $|\phi_i\rangle$ and $|\phi_j\rangle$ that is included in off diagonal element ρ_{ij}.

$$Tr(\rho) = \sum_i \rho_{ii} = 1.$$

The expectation value for an operator can be written as

$$\langle \psi| \hat{A} |\psi\rangle = \sum_i \sum_j c_i^* c_j \langle \phi_i| \hat{A} |\phi_j\rangle = c_i^* c_j \hat{A}_{ij},$$

$$= \sum_i \sum_j \rho_{ji}^* \hat{A}_{ij} = \sum_i \sum_j \rho_{ij} \hat{A}_{ij},$$

$$\langle \psi| \hat{A} |\psi\rangle = Tr\left[\rho \hat{A}\right]. \tag{2.3.3}$$

(b) Mixed State

The mixed state is a quantum thermodynamical ensemble that mixed quantum and classical thermodynamical or statistical informations, that described mathematically in Hilbert space by the density matrix. In other words,it can be says that mixed state is the combination of classical probabilities of the information about quantum state of the quantum system. Mixed state does not describe the state of system completely. If we measure the outcome of a complete set of observable,then at the end of measurements the state is completely changed and contain more information rather than at the start. of course,after do all the measurements,if the system is considered isolated,you get a pure state.

A mixed state is incoherent ensemble of a pure state $|\psi_j\rangle$ with sum of all classical probabilities ρ_j such as

$$\sum_{j=1}^{n} \rho_j = 1.$$

The expectation value for an operator "A" in mixed state is given as

$$\left\langle \hat{A} \right\rangle = \sum_{j=1}^{n} \rho_j \left\langle \psi_j \right| \hat{A} \left| \psi_j \right\rangle.$$

The density matrix for a mixed sate involved in pure state and their statistical weights ρ_j we can write as

$$\rho = \sum_{j=1}^{n} \rho_j \left| \psi_j \right\rangle \left\langle \psi_j \right|.$$

The density matrix defined above has the following properties [11].

1) ρ is Hermitian: $\rho_{nm}^{*} = \rho_{nm}$

2)   Normalization $Tr\left(\rho\right) = 1$

3)   Cyclic invariance $Tr\left(ABC\right) = Tr\left(CAB\right) = Tr\left(BCA\right)$

4) $\quad Tr\left(\rho^2\right)\begin{cases} =1 & \text{For pure state} \\ \\ <1 & \text{For mixed state} \end{cases}$

The last property reflect the fact that, for a pure state the diagonal matrix elements can be 0 and 1 but for a mixed state the diagonal matrix elements lies between 0 and 1. For a statistical ensemble or mixed states the time derivatives of density operator gives rise to the time evolution of the density operator which is discussed in next section.

2.4   Time Evolution of the Density Operator

We want to find out time evolution of density operator by using application of time dependent Schrodinger wave equation [11] and its Hermitian conjugate.

$$i\hbar\frac{d}{dt}\left|\psi\right\rangle = H\left|\psi\right\rangle, \tag{2.4.1}$$

Since $H = H^\dagger$,

$$-i\hbar\frac{d}{dt}\left\langle\psi\right| = \left\langle\psi\right|H, \tag{2.4.2}$$

The density operator can be defined in terms of outer product form, such as.

$$\rho = \left|\psi\right\rangle\left\langle\psi\right|,$$

By taking derivative of above Eq. (2.4.3) we get,

$$\frac{d}{dt}\rho = \left(\frac{d}{dt}\left|\psi\right\rangle\right)\left\langle\psi\right| + \left|\psi\right\rangle\left(\frac{d}{dt}\left\langle\psi\right|\right), \tag{2.4.3}$$

Using Eq(2.4.1)and Eq(2.4.2) the result becomes

$$\dot{\rho} = -\frac{i}{\hbar}\left[v,\rho\right]. \tag{2.4.4}$$

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This equation is known as Von -Neumann equation [12] and represents how the density operator evolves with time.

2.5   General reservoir theory

Firstly we consider in general the system S interacts with reservoir R with interaction Hamiltonian. The combined system can be denoted by ρ_{SR}. The equation of motion for reduced density matrix of the system then follows by taking the trace over reservoir

$$\rho_s = Tr_R\left(\rho_{SR}\right).$$

For quantum statistical mechanics the Schrodinger equation offers ascend to Von-Neumann equation for density operator as discuss above in Eq. (2.4.4)

$$i\hbar\dot{\rho}_{SR} = \left[\upsilon\left(t\right),\rho_{SR}\left(t\right)\right], \tag{2.5.1}$$

By taking Integral of above equation on both sides we get

$$\rho_{SR}\left(t\right) = \rho_{SR}\left(t_i\right) - \frac{i}{\hbar}\int_{t_i}^{t}\left[\upsilon\left(t'\right),\rho_{SR}\left(t'\right)\right]dt', \tag{2.5.2}$$

It is expected that interaction begins at time "t_i" and two systems are uncorrelated before this time.

putting this value in R.H.S of Eq. (2.5.1) the equation of motion can be expressed as

$$\dot{\rho}_{SR} = -\frac{i}{\hbar}\left[\upsilon\left(t\right),\rho_{SR}\left(t_i\right) - \frac{i}{\hbar}\int_{t_i}^{t}\left[\upsilon\left(t'\right),\rho_{SR}\left(t'\right)\right]\right]dt',$$

To expand this equation the above expression can be written as

$$\dot{\rho}_{SR} = -\frac{i}{\hbar} [\upsilon(t), \rho_{SR}(t_i)] - \frac{1}{\hbar^2} \left[\upsilon(t), \int_{t_i}^{t} \left[\upsilon(t'), \rho_{SR}(t') \right] \right] dt', \qquad (2.5.3)$$

Suppose that interaction between system and reservoir is independent to each other and density operator ρ_{SR} can be written as a direct product form $\rho_{SR}(t) = \rho_S(t) \otimes \rho_R(t_i)$.

Now the reduced density matrix for a system (ρ_{atom}) can be obtained by taking trace over all reservoir variables. We have

$$\dot{\rho}_S = -\frac{i}{\hbar} Tr_R [\upsilon(t), \rho_s(t_i) \otimes \rho_R(t_i)] - \frac{1}{\hbar^2} Tr_R \int_{t_i}^{t} \left[\upsilon(t), \left[\upsilon(t'), \rho_s(t') \otimes \rho_R(t_i) \right] \right] dt',$$

$$(2.5.4)$$

As indicated by this expression the behavior of system relies upon the past events in time interval $t = t_i$ to t' and integral contain $\rho_S(t')$. Now, withstanding, the system "S" coupled with reservoir "R" which cause damping that demolishes the "information" of past event. These contemplations lead to the suppositions that the system loses all memory of past. Thus the substitution

$$\rho_S(t') = \rho_S(t),$$

is necessary which is known as Markovian approximation. Markovian approximation [1, 13, 14] can be defined as

Markovian Approximation

Markovian approximation is generally excellent guess. The key thought is that there might be a perfect division between the correlation time of fluctuations and

the time scale of evolution that we need to pursue. Future behavior cannot be predicted from its past behavior except the current and present behavior. In general,the reduced density matrix is used to find out the statistical properties of system and depends on past behavior $t = t_i - t'$. since the reservoir retains a memory of information .As damping occur between atomic levels, memory is lost.

The exact solution of reduced density matrix for an atom using Markovian approximation [1] is

$$\dot{\rho_S} = -\frac{i}{\hbar} Tr_R \left[\upsilon(t), \rho_s(t_i) \otimes \rho_R(t_i) \right] - \frac{1}{\hbar^2} Tr_R \int\limits_{t_i}^{t} \left[\upsilon(t), \left[\upsilon\left(t'\right), \rho_s(t) \otimes \rho_R(t_i) \right] \right] dt'.$$

(2.5.5)

This is the valid equation for reduced density matrix for system ρ_S which is coupled with the reservoir ρ_R. Example of system-reservoir interaction will be considered in next section.

2.6   Density Matrix Equation For Atomic Decay In Dissipative Environments

Two two-level atoms interacting with quantum field (Electromagnetic field) inside the cavity. The decay of an atoms from excited state to lower state can be easily understood by simple model of Harmonic oscillator. In the same way the decay of electromagnetic field inside cavity can be understood by a model in which modes of field is coupled to the whole set of reservoir modes.

We initially consider that two two-level atoms decayed inside the reservoir is portrayed by annihilation operator (b_k) and creation operator $\left(b_k^\dagger\right)$ with density distribution frequencies $\nu_k = ck$. In interaction picture under certain condition of

rotating wave approximation the Hamiltonian [2] is simply,$(\hbar = 1)$

$$\upsilon(t) = \sum_{i=1}^{2}\sum_{k} g_k^i \left[b_k^\dagger \sigma_i^- e^{-\imath(\omega-\nu_k)t} + \sigma_i^+ b_k e^{\imath(\omega-\nu_k)t} \right] + H.C \qquad (2.6.1)$$

where $\sigma^+ = |a\rangle\langle b|$ and $\sigma^- = |b\rangle\langle a|$ here $|a\rangle$ is the excited state of an atom and $|b\rangle$ is the lower state of an atom. Putting Eq. (2.6.1) into an Eq. (2.5.5). Using expression $Tr_R[b_k \rho_R(t_i)] = \langle b_k \rangle$, $Tr_R\left[b_k b_{k'}^\dagger \rho_R(t)\right] = \left\langle b_k b_{k'}^\dagger \right\rangle$, $Tr_R[b_k b_{k'} \rho_R(t)] = \langle b_k b_{k'} \rangle$, $Tr_R\left[b_k^\dagger b_{k'}^\dagger \rho_R(t)\right] = \left\langle b_k^\dagger b_{k'}^\dagger \right\rangle$, $Tr_R\left[b_k^\dagger b_{k'} \rho_R(t)\right] = \left\langle b_k^\dagger b_{k'} \right\rangle$, and solving commutation relations. we get

$$
\begin{aligned}
\dot{\rho}_S &= -i\sum_{i=1}^{2}\sum_{k} g_k \left\langle b_k^\dagger \right\rangle \left[\sigma_i^-, \rho_s(t_i)\right] e^{-\imath(\omega-\nu_k)t} \\
&\quad -\int_{t_i}^{t} dt' \sum_{i=1}^{2}\sum_{k,k'} g_k g_{k'} \left\{ \left[\sigma_i^- \sigma_i^- \rho_{atom} - 2\sigma_i^- \rho_{atom}\sigma_i^- + \rho_{atom}\sigma_i^- \sigma_i^- \right] e^{-\imath(\omega-\nu_k)t - \imath(\omega-\nu_k)t'} \left\langle b_k^\dagger b_{k'}^\dagger \right\rangle \right. \\
&\quad + \left[\sigma_i^+ \sigma_i^- \rho_{atom} - \sigma_i^- \rho_{atom}\sigma_i^+ + \rho_{atom}\sigma_i^+ \sigma_i^- \right] e^{-\imath(\omega-\nu_k)t + \imath(\omega-\nu_k)t'} \left\langle b_k b_{k'}^\dagger \right\rangle \\
&\quad \left. + \left[\sigma_i^- \sigma_i^+ \rho_{atom} - \sigma_i^+ \rho_{atom}\sigma_i^- + \rho_{atom}\sigma_i^- \sigma_i^+ \right] e^{-\imath(\omega-\nu_k)t + \imath(\omega-\nu_k)t'} \left\langle b_k b_{k'}^\dagger \right\rangle \right\} + H.C \qquad (2.6.2)
\end{aligned}
$$

This is a general equation for reduced density matrix of a system to incorporate different environments in general reservoir theory. Now we can go proceed to calculate the $\dot{\rho}_S$ for the two environments that is thermal reservoir and vacuum reservoir [2].

Chapter 3

Thermal Reservoir

As is well-known, quantum mechanics originated with Planck's discovery of radiation law. we allude,obviously to the law depicting the radiation emitted and absorbed by a perfect body,is known as black body. A black body can be considered as a cavity, containing radiation at thermal equilibrium with its walls. The radiation is interacted to a heat bath but assuming that coupling between heat bath and radiation is weak. We think about at that point, a single mode field in thermal equilibrium with the walls of a cavity at temperature T.

3.1 Mathematical solution of average thermal photon number

In statistical mechanics,the density matrix at thermal equilibrium is characterized by thermally distributed population in the nth level of cavity [11, 12] is,

$$P_n = \frac{e^{-\beta E_n}}{Z} \tag{3.1.1}$$

"Z" is the partition function,"β" is the thermodynamic beta,defined as $\frac{1}{K_B T}$ and exponential factor $e^{-\beta E_{\overline{n}}}$ is known as Boltzman's factor. This pursues normally from the general definition of density matrix [11] such as,

$$\rho = \frac{e^{-\beta H_n}}{Z}. \tag{3.1.2}$$

We know that partition function

$$Z = Tr\left(e^{-\beta H_n}\right). \tag{3.1.3}$$

the above expression can be written as

$$Z = Tr\left(e^{-\frac{H_n}{K_B T}}\right),$$

where,

$$Tr\left[\exp\left(-\frac{H_n}{K_B T}\right)\right] = \sum_{\overline{n}=0}^{\infty} \langle n|\exp\left(-\frac{H_n}{K_B T}\right)|n\rangle,$$
$$Z = \sum_{n=0}^{\infty} \exp\left(-\frac{E_n}{K_B T}\right).$$

In the case of Harmonic oscillator, $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$ by subsituate in above expression we get,

$$Z = \exp\left(-\frac{\hbar\nu}{2K_B T}\right)\sum_{n=0}^{\infty}\exp\left(-\frac{\hbar\nu n}{K_B T}\right). \tag{3.1.4}$$

Since $\exp\left(-\frac{\hbar\omega}{K_B T}\right) < 1$, so it can be written as sum of geometric series

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$$\sum_{n=0}^{\infty} \exp\left(-\frac{\hbar\nu n}{K_B T}\right) = \frac{1}{1 - \exp\left(-\frac{\hbar\nu}{K_B T}\right)}$$

Putting this value in Eq. (3.1.4) we get,

$$Z = \exp\left(-\frac{\hbar\nu}{2K_B T}\right) \cdot \frac{1}{1 - \exp\left(-\frac{\hbar\nu}{K_B T}\right)}. \tag{3.1.5}$$

Putting the value of Z in Eq. (3.1.1) we have

$$P_n = \langle n | \rho | n \rangle = \frac{1}{Z} \exp\left(-\frac{E_n}{K_B T}\right). \tag{3.1.6}$$

The density operator itself can be written as

$$\rho = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \left| n' \right\rangle \left\langle n' \right| \rho | n \rangle \langle n |, \tag{3.1.7}$$

$$= \frac{1}{Z} \sum_{n=0}^{\infty} \exp\left(-\frac{E_n}{K_B T}\right) | n \rangle \langle n |, \tag{3.1.8}$$

$$= \sum_{n=0}^{\infty} P_n | n \rangle \langle n |. \tag{3.1.9}$$

The average thermal photon number can be calculated as

$$\overline{n} = \langle \widehat{n} \rangle = Tr\left(\overline{n}\rho\right) = \sum_{n=0}^{\infty} \langle n | n\rho | n \rangle, \tag{3.1.10}$$

Putting the value of "ρ" from Eq. (3.1.9) into an Eq. (3.1.10) we get,

$$= \sum_{n=0}^{\infty} n P_{\overline{n}} = \exp\left(-\frac{\hbar\nu}{2K_B T}\right) \frac{1}{Z} \sum_{n=0}^{\infty} n \exp\left(-\frac{\hbar\nu n}{K_B T}\right). \tag{3.1.11}$$

Let us assume that $x = \frac{\hbar\omega}{K_B T}$ we have

$$\sum_{n=0}^{\infty} n e^{-nx} = -\frac{d}{dx} \sum_{\overline{n}=0}^{\infty} e^{-nx},$$

$$= -\frac{d}{dx} \left(\frac{1}{1 - e^{-x}} \right),$$

After taking derivative of above expression we get this result,

$$= \frac{e^{-x}}{\left(1 - e^{-x}\right)^2}, \tag{3.1.12}$$

Now replacing x by $\frac{\hbar\omega}{K_B T}$ we have

$$= \frac{\exp\left(-\frac{\hbar\nu}{K_B T}\right)}{1 - \exp\left(-\frac{\hbar\nu}{K_B T}\right)}, \tag{3.1.13}$$

Multiplying and dividing above expression by $\exp\left(-\frac{\hbar\omega}{K_B T}\right)$, after simplification the average thermal photon number is

$$\overline{n} = \frac{1}{\exp\left(\frac{\hbar\nu}{K_B T}\right) - 1}. \tag{3.1.14}$$

For multi mode fields the reduced density matrix can be written in terms of Gibb's function as

$$\rho(x) = Z^{-1} e^{-\frac{H(x)}{K_B T}}, \tag{3.1.15}$$

using Eq. (3.1.13) in above expression we can write as

$$= \frac{1 - \exp\left(-\frac{\hbar\nu}{K_B T}\right)}{\exp\left(-\frac{\hbar\nu}{2K_B T}\right)} . e^{-\left(\frac{\hbar\nu b^\dagger b}{K_B T}\right)} . e^{-\left(\frac{\hbar\nu}{2K_B T}\right)}, \tag{3.1.16}$$

After simplifying this equation we obtained reduced density matrix for reservoir [1] of multi mode field is,

$$\rho_R = \prod_k \left[1 - \exp\left(-\frac{\hbar\nu_k}{K_B T} \right) \right] \exp\left(-\frac{\hbar\nu_k b_k^\dagger b_k}{K_B T} \right).$$ (3.1.17)

This equation shows the distribution of reservoir variable in the mixture of states which is in uncorrelated thermal equilibrium. K_B is the Boltzmann constant and T is the temperature.

3.2   Entanglement Dynamics in a thermal reservoir

As we have already found the reduced density matrix equation for an atom. Now, the expectation values for a thermal reservoir in Eq. (2.6.2) can be deduced as

$$\left\langle b_k^\dagger \right\rangle = \langle b_k \rangle = 0,$$ (3.2.1)

$$\left\langle b_k^\dagger b_{k'}^\dagger \right\rangle = \langle b_k b_{k'} \rangle = 0,$$ (3.2.2)

$$\left\langle b_k^\dagger b_{k'} \right\rangle = \overline{n}_k \delta_{kk'},$$ (3.2.3)

$$\left\langle b_k b_{k'}^\dagger \right\rangle = \left(\overline{n}_k + 1 \right) \delta_{kk'}.$$ (3.2.4)

$\overline{n}_k$ is average thermal photon number and can be defined as

$$\overline{n}_k = \frac{1}{\exp\left(\frac{\hbar\nu_k}{K_B T} \right) - 1}.$$

Putting these expectation values in Eq. (2.6.2) we get following equation for reduced density matrix of an atom $\dot{\rho}_S$ for a thermal reservoir . An atoms interacting

with thermal reservoir of mean thermal photon numbers $\overline{m}$ and $\overline{n}$ such as

$$
\begin{aligned}
\dot{\rho}_S &= -\int_{t_i}^{t} dt' \sum_k g_k^2 \\
&\quad \left\{ e^{i(\omega-\nu_k)(t-t')} (\overline{n}+1) [\sigma_+^1 \sigma_-^1 \rho_{atom} - 2\sigma_-^1 \rho_{atom} \sigma_+^1 + \rho_{atom} \sigma_+^1 \sigma_-^1] \right. \\
&\quad + e^{-i(\omega-\nu_k)(t-t')} \overline{n} [\sigma_-^1 \sigma_+^1 \rho_{atom} - 2\sigma_+^1 \rho_{atom} \sigma_-^1 + \rho_{atom} \sigma_-^1 \sigma_+^1] \\
&\quad + e^{i(\omega-\nu_k)(t-t')} (\overline{m}+1) [\sigma_+^2 \sigma_-^2 \rho_{atom} - 2\sigma_-^2 \rho_{atom} \sigma_+^2 + \rho_{atom} \sigma_+^2 \sigma_-^2] \\
&\quad \left. + e^{-i(\omega-\nu_k)(t-t')} \overline{m} [\sigma_-^2 \sigma_+^2 \rho_{atom} - 2\sigma_+^2 \rho_{atom} \sigma_-^2 + \rho_{atom} \sigma_-^2 \sigma_+^2] \right\}, \quad (3.2.5)
\end{aligned}
$$

Now the sum over k can be solved by using weisskopf wigner theory of spontaneous emission [1] by replacing

$$
\sum_k \longrightarrow 2\frac{V}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\infty} dk k^2 \qquad (3.2.6)
$$

Here, coupling constant is defined as,

$$
g_k = -\frac{\wp_{ab} \cdot \hat{\epsilon}_k \xi_k}{\hbar}, \qquad (3.2.7)
$$

using $\xi_k = \left(\frac{\hbar\nu_k}{2\epsilon_o V}\right)^{\frac{1}{2}}$ and taking square of above equation we get the result,

$$
g_k^2 = \frac{\nu_k \wp_{ab}^2}{2\hbar\epsilon_o V} \cos^2\theta. \qquad (3.2.8)
$$

θ is the angle between electric field polarization vector $\hat{\epsilon}_k$ and atomic dipole moment $\wp_{ab}$. using $k = \frac{\nu_k}{c}$ and putting above these relation in Eq. (3.2.5) we have,

$$
\begin{aligned}
\dot{\rho_S} = &-\int_0^t dt' \frac{\nu_k \wp_{ab}^2}{2\hbar\epsilon_o c^3 V} \frac{V}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta \cos^2\theta \, d\theta \int_0^\infty d\nu_k \nu_k^2 \\
&\left\{ e^{i(\omega-\nu_k)(t-t')} (\overline{n}+1) \left[\sigma_+^1\sigma_-^1\rho_{atom} - 2\sigma_-^1\rho_{atom}\sigma_+^1 + \rho_{atom}\sigma_+^1\sigma_-^1\right] \right. \\
&+ e^{-i(\omega-\nu_k)(t-t')} \overline{n} [\sigma_-^1\sigma_+^1\rho_{atom} - 2\sigma_+^1\rho_{atom}\sigma_-^1 + \rho_{atom}\sigma_-^1\sigma_+^1] \\
&+ e^{i(\omega-\nu_k)(t-t')} (\overline{m}+1) [\sigma_+^2\sigma_-^2\rho_{atom} - 2\sigma_-^2\rho_{atom}\sigma_+^2 + \rho_{atom}\sigma_+^2\sigma_-^2] \\
&\left. + e^{-i(\omega-\nu_k)(t-t')} \overline{m}[\sigma_-^2\sigma_+^2\rho_{atom} - 2\sigma_+^2\rho_{atom}\sigma_-^2 + \rho_{atom}\sigma_-^2\sigma_+^2] \right\}, \quad (3.2.9)
\end{aligned}
$$

Solving integrals we get following results

$$
\int_0^{2\pi} d\phi = 2\pi, \tag{3.2.10}
$$

$$
\int_0^\pi \sin\theta \cos^2\theta \, d\theta = \frac{2}{3}. \tag{3.2.11}
$$

According to emission spectrum the intensity of light is related with the emitted radiation field ν is going to be resonant the atomic transition frequency ω such that $\nu_k = \omega$. we can replace the quantity ν_k varies little around ω. and the lower limit in the ν_k integration by $-\infty$. The integral becomes

$$
\int_{-\infty}^\infty d\nu_k e^{i(\omega-\nu_k)(t-t')} = 2\pi\delta\left(t-t'\right). \tag{3.2.12}
$$

Putting the values of integral from Eq. (3.2.11) and 3.2.12 into an Eq. (3.2.9) and after simplification we can write as

$$= -\frac{1}{2}\frac{1}{4\pi\epsilon_o}\frac{4\omega^3\wp_{ab}^2}{3\hbar c^3}\int\limits_0^t dt'\,\delta\left(t-t'\right)$$

$$\left\{(\overline{n}+1)\left[\sigma_+^1\sigma_-^1\rho_{atom} - 2\sigma_-^1\rho_{atom}\sigma_+^1 + \rho_{atom}\sigma_+^1\sigma_-^1\right]\right.$$

$$+\overline{n}[\sigma_-^1\sigma_+^1\rho_{atom} - 2\sigma_+^1\rho_{atom}\sigma_-^1 + \rho_{atom}\sigma_-^1\sigma_+^1]$$

$$+(\overline{m}+1)\left[\sigma_+^2\sigma_-^2\rho_{atom} - 2\sigma_-^2\rho_{atom}\sigma_+^2 + \rho_{atom}\sigma_+^2\sigma_-^2\right]$$

$$\left.+\overline{m}[\sigma_-^2\sigma_+^2\rho_{atom} - 2\sigma_+^2\rho_{atom}\sigma_-^2 + \rho_{atom}\sigma_-^2\sigma_+^2]\right\}, \qquad (3.2.13)$$

Here, we define Dirac delta function that is $\int\limits_0^t dt'\,\delta\left(t-t'\right) = 1$, where the decay constant

$$\gamma = \frac{1}{4\pi\epsilon_o}\frac{4\omega^3\wp_{ab}^2}{3\hbar c^3}. \qquad (3.2.14)$$

Using these expression in above equation we have,

$$\dot{\rho_S} = -\frac{1}{2}\gamma_1(\overline{n}+1)\left[\sigma_+^1\sigma_-^1\rho_{atom} - 2\sigma_-^1\rho_{atom}\sigma_+^1 + \rho_{atom}\sigma_+^1\sigma_-^1\right]$$

$$-\frac{1}{2}\gamma_1\overline{n}[\sigma_-^1\sigma_+^1\rho_{atom} - 2\sigma_+^1\rho_{atom}\sigma_-^1 + \rho_{atom}\sigma_-^1\sigma_+^1]$$

$$-\frac{1}{2}\gamma_2(\overline{m}+1)\left[\sigma_+^2\sigma_-^2\rho_{atom} - 2\sigma_-^2\rho_{atom}\sigma_+^2 + \rho_{atom}\sigma_+^2\sigma_-^2\right]$$

$$-\frac{1}{2}\gamma_2\overline{m}[\sigma_-^2\sigma_+^2\rho_{atom} - 2\sigma_+^2\rho_{atom}\sigma_-^2 + \rho_{atom}\sigma_-^2\sigma_+^2], \qquad (3.2.15)$$

This is an equation of motion for reduced density matrix of an atom in thermal reservoir. Now we find Equation of motion for Two two-level atomic density matrix elements by using above equation for a thermal reservoir. Here levels are illustrate according to Fig. (2.0.1). as $|1\rangle = |a_1, a_2\rangle, |2\rangle = |a_1, b_2\rangle, |3\rangle = |b_1, a_2\rangle, |4\rangle = |b_1, b_2\rangle$. So

$$\dot{\rho}_{11} = \langle 1| \dot{\rho}_s |1 \rangle, \tag{3.2.16}$$

$$
\begin{aligned}
= \quad \langle a_1, a_2| \Big\{ &-\frac{1}{2}\gamma_1 \left(\overline{n}+1\right) [\sigma^1_+\sigma^1_-\rho_{atom} - 2\sigma^1_-\rho_{atom}\sigma^1_+ + \rho_{atom}\sigma^1_+\sigma^1_-] \\
&-\frac{1}{2}\gamma_1\overline{n}[\sigma^1_-\sigma^1_+\rho_{atom} - 2\sigma^1_+\rho_{atom}\sigma^1_- + \rho_{atom}\sigma^1_-\sigma^1_+] \\
&-\frac{1}{2}\gamma_2 \left(\overline{m}+1\right) [\sigma^2_+\sigma^2_-\rho_{atom} - 2\sigma^2_-\rho_{atom}\sigma^2_+ + \rho_{atom}\sigma^2_+\sigma^2_-] \\
&-\frac{1}{2}\gamma_2\overline{m}[\sigma^2_-\sigma^2_+\rho_{atom} - 2\sigma^2_+\rho_{atom}\sigma^2_- + \rho_{atom}\sigma^2_-\sigma^2_+] \Big\} |a_1, a_2\rangle, \tag{3.2.17}
\end{aligned}
$$

Using some epression such as

$$
\begin{aligned}
\sigma^+_i &= |a_i\rangle \langle b_i|, \\
\sigma^-_i &= |b_i\rangle \langle a_i|, \tag{3.2.18}
\end{aligned}
$$

σ^+_i and σ^-_i are rising and lowering operator of an atom where $(i = 1, 2)$. After simplification the result of $\dot{\rho}_{11}$ will be,

$$\dot{\rho}_{11} = - \left[\left(\overline{n}+1\right)\gamma_1 + \left(\overline{m}+1\right)\gamma_2\right]\rho_{11} + \overline{m}\gamma_2\rho_{22} + \overline{n}\gamma_1\rho_{33}, \tag{3.2.19}$$

Similarly the results for atomic desity matrix elements are given as

$$\dot{\rho}_{22} = -\left[(\overline{n} + 1)\gamma_1 + \overline{m}\gamma_2\right]\rho_{22} + (\overline{m} + 1)\gamma_2\rho_{11} + \overline{n}\gamma_1\rho_{44}, \qquad (3.2.20)$$

$$\dot{\rho}_{33} = -\left[\overline{n}\gamma_1 + (\overline{m} + 1)\gamma_2\right]\rho_{33} + (\overline{n} + 1)\gamma_1\rho_{11} + \overline{m}\gamma_2\rho_{44}, \qquad (3.2.21)$$

$$\dot{\rho}_{44} = -\left[\overline{n}\gamma_1 + \overline{m}\gamma_2\right]\rho_{44} + (\overline{n} + 1)\gamma_1\rho_{22} + (\overline{m} + 1)\gamma_2\rho_{33}, \qquad (3.2.22)$$

$$\dot{\rho}_{12} = -\left[(\overline{n} + 1)\gamma_1 + (\overline{m} + 1/2)\gamma_2\right]\rho_{12} + \overline{n}\gamma_1\rho_{34}, \qquad (3.2.23)$$

$$\dot{\rho}_{34} = -\left[\overline{n}\gamma_1 + (\overline{m} + 1/2)\gamma_2\right]\rho_{34} + (\overline{n} + 1)\gamma_1\rho_{12}, \qquad (3.2.24)$$

$$\dot{\rho}_{13} = -\left[(\overline{n} + 1/2)\gamma_1 + (\overline{m} + 1)\gamma_2\right]\rho_{13} + \overline{m}\gamma_2\rho_{24}, \qquad (3.2.25)$$

$$\dot{\rho}_{24} = -\left[(\overline{n} + 1/2)\gamma_1 + \overline{m}\gamma_2\right]\rho_{24} + (\overline{m} + 1)\gamma_1\rho_{13}, \qquad (3.2.26)$$

$$\dot{\rho}_{21} = -\left[(\overline{n} + 1)\gamma_1 + (\overline{m} + 1/2)\gamma_2\right]\rho_{21} + \overline{n}\gamma_1\rho_{43}, \qquad (3.2.27)$$

$$\dot{\rho}_{43} = -\left[\overline{n}\gamma_1 + (\overline{m} + 1/2)\gamma_2\right]\rho_{43} + (\overline{n} + 1)\gamma_1\rho_{21}, \qquad (3.2.28)$$

$$\dot{\rho}_{31} = -\left[(\overline{n} + 1/2)\gamma_1 + (\overline{m} + 1)\gamma_2\right]\rho_{31} + \overline{m}\gamma_2\rho_{42}, \qquad (3.2.29)$$

$$\dot{\rho}_{42} = -\left[(\overline{n} + 1/2)\gamma_1 + \overline{m}\gamma_2\right]\rho_{42} + (\overline{m} + 1)\gamma_2\rho_{31}, \qquad (3.2.30)$$

$$\dot{\rho}_{23} = -\left[(\overline{n} + 1/2)\gamma_1 + (\overline{m} + 1/2)\gamma_2\right]\rho_{23}, \qquad (3.2.31)$$

$$\dot{\rho}_{32} = -\left[(\overline{n} + 1/2)\gamma_1 + (\overline{m} + 1/2)\gamma_2\right]\rho_{32}, \qquad (3.2.32)$$

$$\dot{\rho}_{14} = -\left[(\overline{n} + 1/2)\gamma_1 + (\overline{m} + 1/2)\gamma_2\right]\rho_{14}, \qquad (3.2.33)$$

$$\dot{\rho}_{41} = -\left[(\overline{n} + 1/2)\gamma_1 + (\overline{m} + 1/2)\gamma_2\right]\rho_{41}. \qquad (3.2.34)$$

The detailed solution of these atomic density matrix elements are given in Appendix A. These are coupled first order differential equations. To solve these coupled differential equations, we firstly consider the initial pure state [2] for a two-qubit atomic system that is

$$|\psi(0)\rangle = \alpha_1|1\rangle + \alpha_2 e^{i\phi_1}|2\rangle + \alpha_3 e^{i\phi_2}|3\rangle + \alpha_4 e^{i\phi_3}|4\rangle, \qquad (3.2.35)$$

where $|1\rangle = |a_1, a_2\rangle$, $|2\rangle = |a_1, b_2\rangle$, $|3\rangle = |a_2, b_1\rangle$ and $|4\rangle = |b_1, b_2\rangle$ in which α_i's are probability amplitude where $\sum_i |\alpha_i|^2 = 1$ $(i = 1, 2, 3, 4)$ and ϕ_i are the relative phases. By using this initial state, density matrix can be find out as

$$\rho = |\psi(0)\rangle \langle \psi(0)| . \tag{3.2.36}$$

For this initial state, the density matrix elements ρ_{ij} are determined from coupled first order linear differential equations and their detailed solution are given in

Appendix B. The final result of density matrix elements are given below

$$\rho_{11} = \frac{\overline{m}\gamma_2 e^{-at}\left(\overline{n}\gamma_1 + \alpha_1^2 + \alpha_2^2\right)}{b} + \frac{\overline{m}\gamma_1 e^{-bt}\left(\overline{m}\gamma_2 + \alpha_1^2 + \alpha_3^2\right)}{a} \quad (3.2.37)$$
$$+ \frac{c\gamma_1\gamma_2 e^{-(a+b)t}}{ab},$$

$$\rho_{22} = \frac{\left(\overline{m}+1\right)\gamma_2 e^{-at}\left(\overline{n}\gamma_1 + \alpha_1^2 + \alpha_2^2\right)}{b} - \frac{\overline{n}\gamma_1 e^{-bt}\left(\overline{m}\gamma_2 + \alpha_1^2 + \alpha_3^2\right)}{a} (3.2.38)$$
$$- \frac{c\gamma_1\gamma_2 e^{-(a+b)t}}{ab},$$

$$\rho_{33} = \frac{\overline{m}\gamma_2 e^{-at}\left(\overline{n}\gamma_1 + \alpha_1^2 + \alpha_2^2\right)}{b} + \frac{\left(\overline{n}+1\right)\gamma_1 e^{-bt}\left(\overline{m}\gamma_2 + \alpha_1^2 + \alpha_3^2\right)}{a} (3.2.39)$$
$$- \frac{c\gamma_1\gamma_2 e^{-(a+b)t}}{ab},$$

$$\rho_{44} = 1 - \left(\rho_{11} + \rho_{22} + \rho_{33}\right), \quad (3.2.40)$$

$$\rho_{12} = \frac{\gamma_1 e^{-i\phi_2}}{a}\left[\overline{n}e^{-bt}\left(\alpha_1\alpha_2 e^{i(\phi_1+\phi_2)} + \alpha_3\alpha_4 e^{i\phi_3}\right)\right.$$
$$\left.+ e^{-(2a+b)t/2}\left(\alpha_1\alpha_3 e^{i(\phi_1+\phi_2)}\left(\overline{n}+1\right) - \overline{n}\alpha_3\alpha_4 e^{i\phi_3}\right)\right], \quad (3.2.41)$$

$$\rho_{13} = \frac{\gamma_2 e^{-i\phi_1}}{a}\left[\overline{m}e^{-at}\left(\alpha_1\alpha_3 e^{i(\phi_1+\phi_2)} + \alpha_2\alpha_4 e^{i\phi_3}\right)\right.$$
$$\left.+ e^{-(2a+b)t/2}\left(\alpha_1\alpha_3 e^{i(\phi_1+\phi_2)}\left(\overline{m}+1\right) - \overline{m}\alpha_2\alpha_4 e^{i\phi_3}\right)\right], \quad (3.2.42)$$

$$\rho_{14} = e^{-(a+b)t/2}\alpha_1\alpha_4 e^{i\phi_3}, \quad (3.2.43)$$

$$\rho_{23} = e^{-(a+b)t/2}\alpha_2\alpha_3 e^{i(-\phi_1+\phi_2)}, \quad (3.2.44)$$

$$\rho_{24} = \frac{\gamma_2 e^{-i\phi_1}}{b}\left[\left(\overline{m}+1\right)e^{-at/2}\left(\alpha_1\alpha_3 e^{i(\phi_1+\phi_2)} + \alpha_2\alpha_4 e^{i\phi_3}\right)\right.$$
$$\left.- e^{-(a+2b)t/2}\left(\alpha_1\alpha_3 e^{i(\phi_1+\phi_2)}\left(\overline{m}+1\right) - \overline{m}\alpha_2\alpha_4 e^{i\phi_3}\right)\right], \quad (3.2.45)$$

$$\rho_{34} = \frac{\gamma_1 e^{-i\phi_2}}{b}\left[\left(\overline{n}+1\right)e^{-bt/2}\left(\alpha_1\alpha_2 e^{i(\phi_1+\phi_2)} + \alpha_3\alpha_4 e^{i\phi_3}\right)\right.$$
$$\left.- e^{-(2a+b)t/2}\left(\alpha_1\alpha_2 e^{i(\phi_1+\phi_2)}\left(\overline{n}+1\right) - \overline{n}\alpha_3\alpha_4 e^{i\phi_3}\right)\right]. \quad (3.2.46)$$

and

$$\rho_{21} = \rho_{12}^\star, \ \rho_{31} = \rho_{13}^\star, \ \rho_{41} = \rho_{14}^\star, \ \rho_{42} = \rho_{24}^\star, \ \rho_{43} = \rho_{34}^\star, \ \rho_{32} = \rho_{23}^\star,$$

In the above equations we assume that

$$a = (2\overline{n} + 1)\gamma_1, \tag{3.2.47}$$

$$b = (2\overline{m} + 1)\gamma_2, \tag{3.2.48}$$

$$c = \alpha_1^2 (1 + \overline{m} + \overline{n}) - \overline{m}(1 + 2\overline{n})\alpha_2^2$$
$$-\overline{n}(1 + 2\overline{m})\alpha_3^2 - \overline{mn}(a + b). \tag{3.2.49}$$

These density matrix elements ρ_{ij} can be represented in the form of 4×4 matrix. such as

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}, \tag{3.2.50}$$

The complete solution of this 4×4 matrix is not possible by applying the initial condition (3.2.35) because the result is more complicated and we cannot find the general solution with all these variables. The exact solution of time dependence entanglement can be explained by considering the initial entangled 2-qubit states in thermal reservoir. .

3.3   Entangled states

In simple words, Entangled state [15] is a state of composite system that cannot be written in the form of product state. Entangled states are extremely interesting states since they show relationship that have no classical analogy. They are of specific significance in quantum information and quantum computation. Let us consider the entangled state of a pure bipartite system is,

$$\left|\psi^{+}\right\rangle = \frac{1}{\sqrt{2}} \left(\left|01\right\rangle_{AB} + \left|10\right\rangle_{AB}\right), \tag{3.3.1}$$

This state cannot be written in the form of product state. so it is maximally entangled. This implies that if we measure 1 in System A in any basis the outcome will be totally superposition (0 and 1 with equal probability $\frac{1}{\sqrt{2}}$). Be that as it may, there is a perfect correlation. If we measure 1 in system A, then with equal certainty of 0 in system B and vice versa. The composite states of bipartite system named as A and B can be written in form of bell states such as

$$\left|\Psi^{(+)}\right\rangle_{AB} = \frac{1}{\sqrt{2}} \left(\left|0_A, 1_B\right\rangle + \left|1_A, 0_B\right\rangle\right), \tag{3.3.2}$$

$$\left|\Psi^{(-)}\right\rangle_{AB} = \frac{1}{\sqrt{2}} \left(\left|0_A, 1_B\right\rangle - \left|1_A, 0_B\right\rangle\right), \tag{3.3.3}$$

$$\left|\Phi^{(+)}\right\rangle_{AB} = \frac{1}{\sqrt{2}} \left(\left|0_A, 0_B\right\rangle + \left|1_A, 1_B\right\rangle\right), \tag{3.3.4}$$

$$\left|\Phi^{(-)}\right\rangle_{AB} = \frac{1}{\sqrt{2}} \left(\left|0_A, 0_B\right\rangle - \left|1_A, 1_B\right\rangle\right), \tag{3.3.5}$$

These are maximally Entangled four bell states or EPR states [4]. On the other hand, if the states of any bipartite system such as A and B are written in the form of tensor product then it is called separable state. These states are not entangled or must be separable if it can be written as

$$\left|\Psi\right\rangle_{AB} = \frac{1}{\sqrt{2}} \left(\left|0_A, 1_B\right\rangle + \left|0_A, 1_B\right\rangle\right) = \frac{1}{\sqrt{2}} \left|0_A\right\rangle \otimes \left(\left|1_B\right\rangle + \left|0_B\right\rangle\right), \tag{3.3.6}$$

This state of the system A and B does not contain any correlation because if we measured the state of the system A then the state of system B remains in superposition. For a mixed system, if composite state ρ_{AB} can be written in the

form of tensor product for two subsystems such as ρ_A and ρ_B then the state is separable or unentangled. we can write as

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i. \qquad (3.3.7)$$

Mixed system is entangled if the subsystem ρ_A and ρ_B cannot be written in above form. Mixed state cannot be written with state vector whereas pure state can be defined with state vector. Thus mixed entangled state is a state of system that cannot be produced by local convex operations and classical communications. The entanglement characterization approach indicate that entanglement is generated in the system but we always need to quantify the amount of it. To quantify the amount of entanglement we use different entanglement measurement techniques which are discuss in next section.

3.4 Entanglement Measures

Entanglement measures is used to quantify the amount of entanglement in a bipartite system or quantum state. There are many ways to characterize the amount of entanglement like Schmidt decomposition [16], Negativity [12, 1], Von-Neumann entropy [1] and Wooters concurrence .

But here we only explain Wooters concurrence [17, 18].

Wooters concurrence

The concurrence formula is characterized for two qubit system and also depends on bit flip operator σ_y. consider the concurrence in which state $|\psi\rangle$ overlap with bit flip operator σ_y thus,we can write concurrence as

$$c(\psi) = \left| \langle \psi | \tilde{\psi} \rangle \right|, \qquad (3.4.1)$$

where $\left| \breve{\psi} \right\rangle = \sigma_y \left| \psi \right\rangle$ and $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. The concurrence is defined in terms of density operator for a two qubit system by considering the matrix M given as,

$$M = \rho \left(\sigma_y \otimes \sigma_y \right) \rho^\star \left(\sigma_y \otimes \sigma_y \right), \tag{3.4.2}$$

Eigenvalues have been calculated by using Eq. (3.4.2) of 4×4 matrix M and these eigenvalues are used to defines the concurrence as,

$$C \left(t \right) = \max \left[0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right], \tag{3.4.3}$$

These are eigenvalues of 4×4 matrix are written in descending order. concurrence ranges from $c \left(t \right) = 0$ for a separable state and $c \left(t \right) = 1$ for an entangled state.

The entanglement dynamics of an entangled state in thermal reservoir [19] can be calculate as,

Case 1

Consider the initial entangled 2-qubit state in which both atoms are in upper level or lower level with arbitrary amplitude α_1 and α_4. The state is written as

$$\left| \psi_1 \left(0 \right) \right\rangle = \alpha_1 \left| a_1, a_2 \right\rangle + \alpha_4 e^{i\phi_3} \left| b_1, b_2 \right\rangle, \tag{3.4.4}$$

Now the density matrix elements are determined by using $\rho = \left| \psi \left(t \right) \right\rangle \langle \psi \left(t \right) |$ and detailed solution are given in Appendix B for a thermal reservoir. where only the terms are survived with α_1 and α_4 whereas α_2 and α_3 is equal to zero. After simplification we get 4×4 matrix. By putting this 4×4 matrix in Eq. (3.4.2) and taking tensor product of Pauli matrix σ_y we get the general form of 4×4 matrix M. Now calculate the eigenvalues of matrix M to determined the entanglement dynamics. The detailed solution to calculate the eigenvalues are given in Appendix

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C. The results of the calculated eigenvalues are as follows and here it is repeated for consistency.

$$
\begin{aligned}
\lambda_1 \;=\; \lambda_2 &= \frac{e^{-4(2\bar{n}+1)\gamma t}\left(-1+e^{(2\bar{n}+1)\gamma t}\right)^2}{(2\bar{n}+1)^4} \\
&\quad \times \left[\bar{n}\left(\bar{n}+1\right)e^{(2\bar{n}+1)\gamma t}+\bar{n}^2+(2\bar{n}+1)\alpha_1^2\right]^2, \\
\lambda_3 &= \frac{e^{-4(2\bar{n}+1)\gamma t}}{(2\bar{n}+1)^4}\Big[\big\{(\bar{n}+1)^2\left(2\bar{n}e^{3(2\bar{n}+1)\gamma t}+\bar{n}^2e^{4(2\bar{n}+1)\gamma t}-2\left(\bar{n}+1\right)e^{(2\bar{n}+1)\gamma t}\right. \\
&\quad +(\bar{n}+1)^2+e^{2(2\bar{n}+1)\gamma t}\times\left(1-2\bar{n}\left(\bar{n}+1\right)\right)\big)\alpha_1^4+e^{2(2\bar{n}+1)\gamma t} \\
&\quad \times\left(4\bar{n}\left(\bar{n}+1\right)\cosh\left(\gamma t\right)+4\bar{n}^2\left(\bar{n}+1\right)^2\cosh\left(2\gamma t\right)+1\right. \\
&\quad +4\bar{n}\left(\bar{n}+1\right)\times\left(1+3\bar{n}\left(\bar{n}+1\right)\right)\big)\alpha_1^2\alpha_4^2 \\
&\quad +\bar{n}^2\left(2\bar{n}e^{(2\bar{n}+1)\gamma t}-2\left(1+\bar{n}\right)e^{3(2\bar{n}+1)\gamma t}\right. \\
&\quad +\bar{n}^2+(1+\bar{n})^2\,e^{4(2\bar{n}+1)\gamma t} \\
&\quad +\left(1-2\bar{n}\left(1+\bar{n}\right)\right)e^{2(2\bar{n}+1)\gamma t}\big)\alpha_4^2\big\}^{\frac{1}{2}} \\
&\quad -\alpha_1\alpha_4\left(2\bar{n}+1\right)^2e^{(2\bar{n}+1)\gamma t}\Big]^2, \\
\lambda_4 &= \frac{e^{-4(2\bar{n}+1)\gamma t}}{(2\bar{n}+1)^4}\Big[\big\{(\bar{n}+1)^2\left(2\bar{n}e^{3(2\bar{n}+1)\gamma t}+\bar{n}^2e^{4(2\bar{n}+1)\gamma t}-2\left(\bar{n}+1\right)e^{(2\bar{n}+1)\gamma t}\right. \\
&\quad +(\bar{n}+1)^2+e^{2(2\bar{n}+1)\gamma t}\times\left(1-2\bar{n}\left(\bar{n}+1\right)\right)\big)\alpha_1^4+e^{2(2\bar{n}+1)\gamma t} \\
&\quad \times\left(4\bar{n}\left(\bar{n}+1\right)\cosh\left(\gamma t\right)+4\bar{n}^2\left(\bar{n}+1\right)^2\cosh\left(2\gamma t\right)+1\right. \\
&\quad +4\bar{n}\left(\bar{n}+1\right)\times\left(1+3\bar{n}\left(\bar{n}+1\right)\right)\big)\alpha_1^2\alpha_4^2 \\
&\quad +\bar{n}^2\left(2\bar{n}e^{(2\bar{n}+1)\gamma t}-2\left(1+\bar{n}\right)e^{3(2\bar{n}+1)\gamma t}\right. \\
&\quad +\bar{n}^2+(1+\bar{n})^2\,e^{4(2\bar{n}+1)\gamma t} \\
&\quad +\left(1-2\bar{n}\left(1+\bar{n}\right)\right)e^{2(2\bar{n}+1)\gamma t}\big)\alpha_4^2\big\}^{\frac{1}{2}} \\
&\quad +\alpha_1\alpha_4\left(2\bar{n}+1\right)^2e^{(2\bar{n}+1)\gamma t}\Big]^2. \tag{3.4.5}
\end{aligned}
$$

This is the four eigenvalues for initial entangled state in which both atoms are in upper state or lower state. To quantify the amount of entanglement it is necessary

to calculate the concurrence. The amount of concurrence for these four eigenvalues can be calculated by using Eq. (3.4.3). After simplification we get the concurrence in this form such as

$$C_1(t) = \frac{2e^{-2(2\overline{n}+1)\gamma t}}{(2\overline{n}+1)^2} \left[(2\overline{n}+1)^2 \alpha_1 \alpha_4 e^{(2\overline{n}+1)\gamma t} - \left(-1 + e^{(2\overline{n}+1)\gamma t} \right) \right.$$
$$\left. \left[\overline{n}(\overline{n}+1) e^{(2\overline{n}+1)\gamma t} + \overline{n}^2 + (2\overline{n}+1) \alpha_1^2 \right] \right], \qquad (3.4.6)$$

This is the final analytical result of concurrence for initial entangled state Eq. (3.4.4) which demonstrate the dependence of concurrence on various parameters. Now, the result of concurrence verses decay terms are shown graphically in Figure (3.4.1).For an entangled qubit, the dynamics of decay must be finite in time which
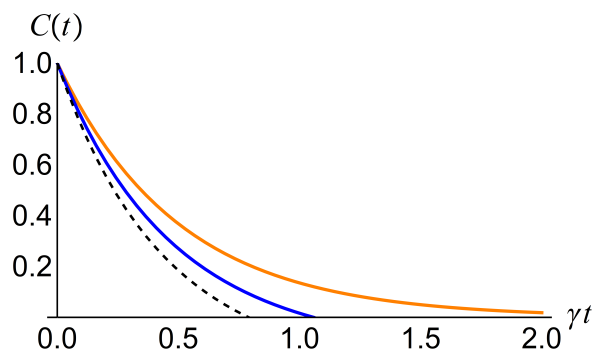
Figure 3.4.1: Entanglement dynamics of EPR states $|\psi_1(0)\rangle = \frac{1}{\sqrt{2}} [|a_1, a_2\rangle + |b_1, b_2\rangle]$ at different temperature in thermal reservoir. Orange for $\overline{n} = 0$, Blue for $\overline{n} = 0.1$, Dashed Black for $\overline{n} = 0.2$.

is well known as sudden death of entanglement. This plot shows the sudden death of entanglement (SDE) [20, 21] due to the presence of thermal photon numbers in the cavities. SDE occur due to thermal environments. This graph shows that decay of entanglement is faster due to presence of doubly excited atoms $|a_1, a_2\rangle$ in the cavities because no more atoms remain in excited state for a long time due to

spontaneous emission.

In order to see the effects of dynamical decay with rise of temperature we draw three plots with change in temperature, shown in figure (3.4.2) which shows the initial entangled state is taken as $|\psi_1(0)\rangle = \alpha_1|a_1, a_2\rangle + \sqrt{1 - \alpha_1^2}|b_1, b_2\rangle$ and plot the concurrence for different values of α_1. Here, the dynamical decay of entangled qubit shows the different behavior of concurrence due to the presence of probability amplitude α_1 of the state $|a_1, a_2\rangle$, where both atoms are in excited state. These plots represent that sudden death time decrease as we increase the temperature or average thermal photon numbers in the cavities.

Case 2

Now consider the initial entangled state in which both atoms are in superposition. The superposition state of two atoms can be written with arbitrary amplitude α_2 and α_3 such as

$$|\psi_2(0)\rangle = \alpha_2|a_1, b_2\rangle + \alpha_3 e^{i\phi_2}|b_1, a_2\rangle, \qquad (3.4.7)$$

The solution of density matrix elements ρ_{ij} are given in Appendix B. With this initial state only terms are survived with arbitrary amplitude α_2 and α_3 rest of terms are going to be zero. After simplification we get the density matrix ρ. Put this 4×4 matrix in Eq. (3.4.2) and taking tensor product of bit flip operator σ_y After solving all these terms we determined the compact form of 4×4 matrix M. To find out the entanglement dynamics we have to calculate the eigenvalues of matrix M by using Eq. (3.4.2). The detail solution to calculate the eigenvalues are given in Appendix D. The results are
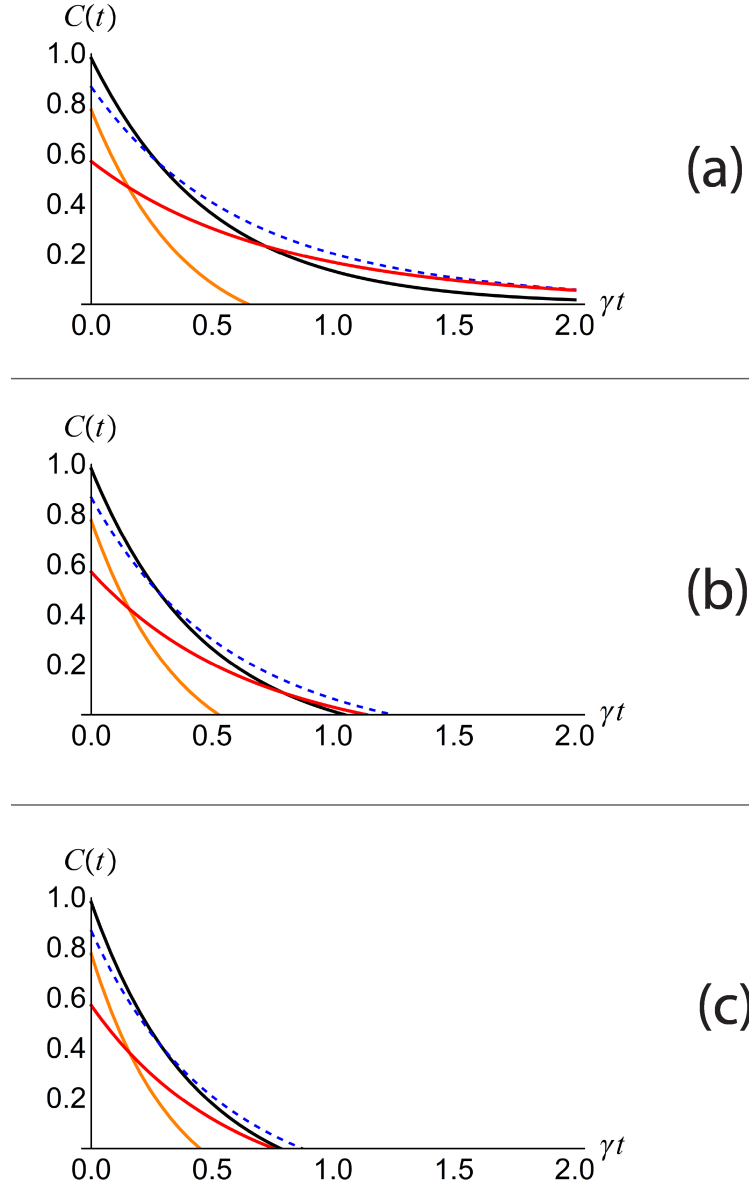
Figure 3.4.2: Entanglement dynamics of EPR states $|\psi_1(0)\rangle = \alpha_1 \left[|a_1, a_2\rangle + \sqrt{1-\alpha_1^2}|b_1, b_2\rangle \right]$ at different initial temperatures. (a) $\bar{n} = 0$ (b) $\bar{n} = 0.1$ (c) $\bar{n} = 0.2$. Black for $\alpha_1 = 0.7$, Blue Dashed for $\alpha_1 = 0.5$, Orange for $\alpha_1 = 0.9$, Red for $\alpha_1 = 0.3$.

$$\lambda_1 = \lambda_2 = \frac{e^{-4(2\overline{n}+1)\gamma t}\left(-1 + e^{(2\overline{n}+1)\gamma t}\right)^2}{(2\overline{n}+1)^4}$$

$$\times \overline{n}\left(\overline{n}+1\right)\left(1 + \overline{n}\left(1 + e^{(2\overline{n}+1)\gamma t}\right)\right)\left(\overline{n} + \left(\overline{n}+1\right)\right)e^{(2\overline{n}+1)\gamma t},$$

$$\lambda_3 = \frac{e^{-4(2\overline{n}+1)\gamma t}}{(2\overline{n}+1)^4}\left[\left\{2\overline{n}\left(\overline{n}+1\right)e^{2(2\overline{n}+1)\gamma t}\left(\cosh\left(\gamma t\right) + \overline{n}\left(\overline{n}+1\right)\right.\right.\right.$$

$$\times \cosh\left(2\gamma t\right) - \left(1 + \overline{n}\left(\overline{n}+1\right)\right)\right)\left(\alpha_2^4 + \alpha_3^4\right) + e^{2(2\overline{n}+1)\gamma t}$$

$$\times \left(4\overline{n}\left(\overline{n}+1\right)\cosh\left(\gamma t\right) + 4\overline{n}^2\left(\overline{n}+1\right)^2\cosh\left(2\gamma t\right) + 1\right.$$

$$\left.+4\overline{n}\left(\overline{n}+1\right)\times\left(1 + 3\overline{n}\left(\overline{n}+1\right)\right)\right)\alpha_2^2\alpha_3^2\Big\}^{\frac{1}{2}}$$

$$\left.-\alpha_2\alpha_3\left(2\overline{n}+1\right)^2 e^{(2\overline{n}+1)\gamma t}\right]^2,$$

$$\lambda_4 = \frac{e^{-4(2\overline{n}+1)\gamma t}}{(2\overline{n}+1)^4}\left[\left\{2\overline{n}\left(\overline{n}+1\right)e^{2(2\overline{n}+1)\gamma t}\left(\cosh\left(\gamma t\right) + \overline{n}\left(\overline{n}+1\right)\right.\right.\right.$$

$$\times \cosh\left(2\gamma t\right) - \left(1 + \overline{n}\left(\overline{n}+1\right)\right)\right)\left(\alpha_2^4 + \alpha_3^4\right) + e^{2(2\overline{n}+1)\gamma t}$$

$$\times \left(4\overline{n}\left(\overline{n}+1\right)\cosh\left(\gamma t\right) + 4\overline{n}^2\left(\overline{n}+1\right)^2\cosh\left(2\gamma t\right) + 1\right.$$

$$\left.+4\overline{n}\left(\overline{n}+1\right)\times\left(1 + 3\overline{n}\left(\overline{n}+1\right)\right)\right)\alpha_2^2\alpha_3^2\Big\}^{\frac{1}{2}}$$

$$\left.+\alpha_2\alpha_3\left(2\overline{n}+1\right)^2 e^{(2\overline{n}+1)\gamma t}\right]^2. \tag{3.4.8}$$

This is the four eigenvalues of initial entangled superposition state. The concurrence for these eigenvalues can be calculated by using Eq. (3.4.3) Basically, the concurrence can be used to quantify the amount of entanglement. The concurrence for these eigenvalues are,

$$C_2\left(t\right) = \frac{2e^{-2(2\overline{n}+1)\gamma t}}{(2\overline{n}+1)^2}\left[\left(2\overline{n}+1\right)^2\alpha_2\alpha_3 e^{(2\overline{n}+1)\gamma t} - \left(-1 + e^{(2\overline{n}+1)\gamma t}\right)\right.$$

$$\left.\times \sqrt{\overline{n}\left(\overline{n}+1\right)\left[\overline{n}\left(\overline{n}+1\right)\left(1 + e^{(2\overline{n}+1)\gamma t}\right)^2 + e^{(2\overline{n}+1)\gamma t}\right]}\right], \quad (3.4.9)$$

The analytical outcomes are displayed to demonstrate the reliance of concurrence on various parameters. Now the results of concurrence verses decay terms can be explain graphically as shown in figure (3.4.3). This figure shows the behavior of concurrence for different values of temperature. we see the finite time disentanglement in the case of non-zero photon numbers or non-zero temperature. In, sudden death entanglement, entangled qubit dynamical decay must be finite in time. It always occurs due to interaction of thermal environments.
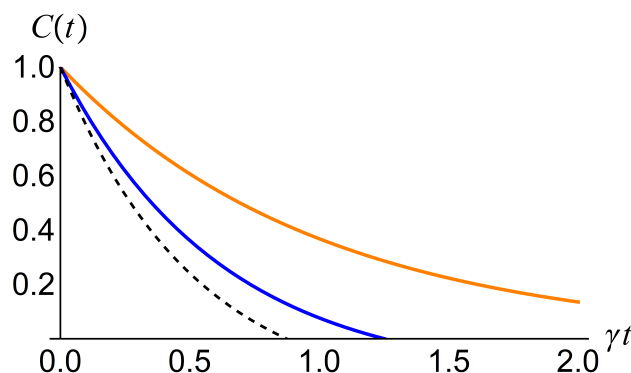
Figure 3.4.3: Entanglement dynamics of EPR states $|\psi_2(0)\rangle = \frac{1}{\sqrt{2}}[|a_1, b_2\rangle + |b_1, a_2\rangle]$ at different temperature in thermal reservoir. Orange for $\overline{n} = 0$, Blue for $\overline{n} = 0.1$, Dashed Black for $\overline{n} = 0.2$.

In order to see the effect of dynamical decay with change in temperature. we draw three different plots shown in figure (3.4.4) for the initial entangled state taken as $|\psi_2(0)\rangle = \alpha_2|a_1, b_2\rangle + \sqrt{1 - \alpha_2^2}|b_1, a_2\rangle$ and plot the concurrence for different values of α_2. Here, the dynamical decay of entangled qubit shows the different behavior of concurrence due to the different values of α_2. These plots represent that sudden death time decrease as we increase the temperature or average thermal photon numbers in the cavities.
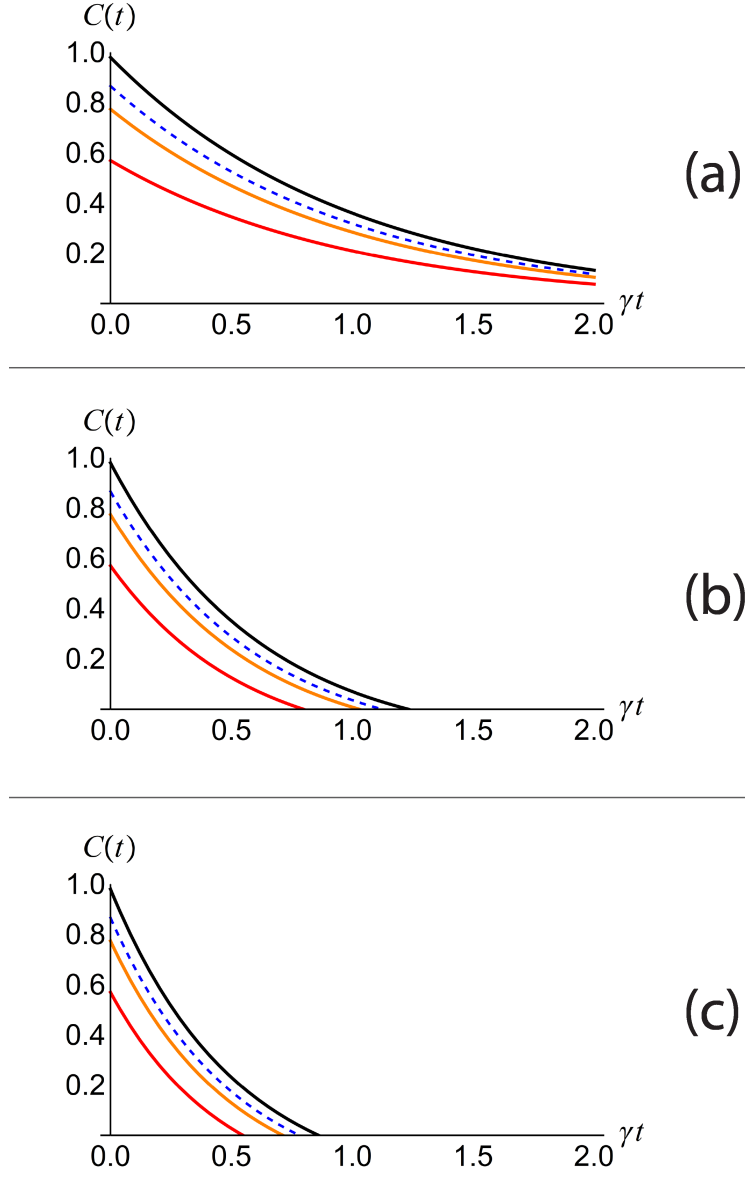
Figure 3.4.4: Entanglement dynamics of EPR states $|\psi_2(0)\rangle = \alpha_2 \left[|a_1, a_2\rangle + \sqrt{1-\alpha_2^2}|b_1, b_2\rangle \right]$ at different initial temperatures. (a) $\overline{n} = 0$ (b) $\overline{n} = 0.1$ (c) $\overline{n} = 0.2$. Black for $\alpha_2 = 0.7$, Blue Dashed for $\alpha_2 = 0.5$, Orange for $\alpha_2 = 0.9$, Red for $\alpha_2 = 0.3$.

In this chapter, we discuss entanglement dynamics in thermal reservoir by graphically and analytically. In the next section, we will discuss entanglement dynamics in vacuum reservoir and deduce the concurrence analytically.

Chapter 4

Vacuum reservoir

There is no mean photon number present in vacuum reservoir. But still noises and decays are present in the cavity due to cavity fluctuations which is due to zero point energy. This part of energy is responsible for the vacuum fluctuations [22], spontaneous decay [23] and for the lamb shift [1]. These noises [24] are incorporated by using equation of motion.

4.1 Entanglement Dynamics in a vacuum reservoir

Now, the equation of motion for reduced density matrix of an atom for a vacuum reservoir can be find out by using Eq. (2.6.2) where expectation values can be calculated as

$$\langle b_k \rangle \quad = \quad \langle 0|b_k|0 \rangle = 0, \tag{4.1.1}$$

$$\langle b_k^\dagger \rangle \quad = \quad \langle 0|b_k^\dagger|0 \rangle = 0, \tag{4.1.2}$$

$$\langle b_k b_{k'}^\dagger \rangle \quad = \quad \langle 0|b_k b_{k'}^\dagger|0 \rangle = \delta_{kk'}., \tag{4.1.3}$$

$$\langle b_k^\dagger b_{k'} \rangle \quad = \quad \langle 0|b_k^\dagger b_{k'}|0 \rangle = 0, \tag{4.1.4}$$

$$\langle b_k^\dagger b_{k'}^\dagger \rangle \quad = \quad \langle 0|b_k^\dagger b_{k'}^\dagger|0 \rangle = 0, \tag{4.1.5}$$

$$\langle b_k b_{k'} \rangle \quad = \quad \langle 0|b_k b_{k'}|0 \rangle = 0. \tag{4.1.6}$$

Putting these expectation values in Eq. (2.6.2) and proceed as in derivation of Eq. (3.2.15) we get equation of motion for vacuum reservoir as

$$
\begin{aligned}
\dot{\rho}_S \quad = \quad & -\frac{1}{2}\gamma_1[\sigma_+^1 \sigma_-^1 \rho_{atom} - 2\sigma_-^1 \rho_{atom}\sigma_+^1 + \rho_{atom}\sigma_+^1 \sigma_-^1] \\
& -\frac{1}{2}\gamma_2[\sigma_+^2 \sigma_-^2 \rho_{atom} - 2\sigma_-^2 \rho_{atom}\sigma_+^2 + \rho_{atom}\sigma_+^2 \sigma_-^2], \tag{4.1.7}
\end{aligned}
$$

Now equation of motion of an atom for different elements can be obtained by putting $\overline{n} = 0$ and $\overline{m} = 0$ in Eq's. (3.2.34) we obtained,

$$\dot{\rho}_{11} = -[\gamma_1 + \gamma_2]\rho_{11}, \tag{4.1.8}$$

$$\dot{\rho}_{22} = -\gamma_1\rho_{22} + \gamma_2\rho_{11}, \tag{4.1.9}$$

$$\dot{\rho}_{33} = -\gamma_2\rho_{33} + \gamma_1\rho_{11}, \tag{4.1.10}$$

$$\dot{\rho}_{44} = \gamma_1\rho_{22} + \gamma_2\rho_{33}, \tag{4.1.11}$$

$$\dot{\rho}_{12} = -[\gamma_1 + +1/2\gamma_2]\rho_{12}, \tag{4.1.12}$$

$$\dot{\rho}_{34} = -1/2\gamma_2\rho_{34} + \gamma_1\rho_{12}, \tag{4.1.13}$$

$$\dot{\rho}_{13} = -[1/2\gamma_1 + \gamma_2]\rho_{13}, \tag{4.1.14}$$

$$\dot{\rho}_{24} = -1/2\gamma_1\rho_{24} + \gamma_2\rho_{13}, \tag{4.1.15}$$

$$\dot{\rho}_{21} = -[\gamma_1 + 1/2\gamma_2]\rho_{21}, \tag{4.1.16}$$

$$\dot{\rho}_{43} = -1/2\gamma_2\rho_{43} + \gamma_1\rho_{21}, \tag{4.1.17}$$

$$\dot{\rho}_{31} = -[1/2\gamma_1 + \gamma_2]\rho_{31}, \tag{4.1.18}$$

$$\dot{\rho}_{42} = -1/2\gamma_1\rho_{42} + \gamma_2\rho_{31}, \tag{4.1.19}$$

$$\dot{\rho}_{23} = -[1/2\gamma_1 + 1/2\gamma_2]\rho_{23}, \tag{4.1.20}$$

$$\dot{\rho}_{32} = -[1/2\gamma_1 + 1/2\gamma_2]\rho_{32}, \tag{4.1.21}$$

$$\dot{\rho}_{14} = -[1/2\gamma_1 + 1/2\gamma_2]\rho_{14}, \tag{4.1.22}$$

$$\dot{\rho}_{41} = -[1/2\gamma_1 + 1/2\gamma_2]\rho_{41}. \tag{4.1.23}$$

The solution of these equations of initial bipartite system can be given by putting $(\overline{m} = \overline{n} = 0)$ in Eq's. (3.2.46) For sake of our simplicity we put $\gamma_1 = \gamma_2 = \gamma$ and $\phi_1 + \phi_2 - \phi_3 = \Phi$. we obtained results as

$$\rho_{11} = \alpha_1^2 e^{-2\gamma t}, \tag{4.1.24}$$

$$\rho_{12} = \alpha_1 \alpha_2 e^{\frac{-3\gamma t}{2}}, \tag{4.1.25}$$

$$\rho_{13} = \alpha_1 \alpha_3 e^{\frac{-3\gamma t}{2}}, \tag{4.1.26}$$

$$\rho_{14} = \alpha_1 \alpha_4 e^{-\gamma t}, \tag{4.1.27}$$

$$\rho_{22} = -e^{-2\gamma t} \alpha_1^2 + e^{-\gamma t} \left(\alpha_1^2 + \alpha_2^2 \right), \tag{4.1.28}$$

$$\rho_{23} = \alpha_2 \alpha_3 e^{-\gamma t}, \tag{4.1.29}$$

$$\rho_{24} = e^{\frac{-3\gamma t}{2}} \left(\left(-1 + e^{\gamma t} \right) \alpha_1 \alpha_3 + e^{\gamma t} \alpha_2 \alpha_4 \right), \tag{4.1.30}$$

$$\rho_{33} = -e^{-2\gamma t} \alpha_1^2 + e^{-\gamma t} \left(\alpha_1^2 + \alpha_3^2 \right), \tag{4.1.31}$$

$$\rho_{34} = e^{\frac{-3\gamma t}{2}} \left(\left(-1 + e^{\gamma t} \right) \alpha_1 \alpha_2 + e^{\gamma t} \alpha_3 \alpha_4 \right), \tag{4.1.32}$$

$$\rho_{44} = 1 + e^{-2\gamma t} \alpha_1^2 - e^{-\gamma t} \left(\alpha_1^2 + \alpha_2^2 \right) - e^{-\gamma t} \left(\alpha_1^2 + \alpha_3^2 \right). \tag{4.1.33}$$

These density matrix elements ρ_{ij} corresponds to 4×4 matrix which can be written as,

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}, \tag{4.1.34}$$

The Eigenvalues of matrix M can be calculated by using this 4×4 matrix in Eq. (3.4.2) where σ_y is Pauli matrix can be defined as

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{4.1.35}$$

The solution to find eigenvalues of matrix M can be given in Appendix E. For entanglement dynamics, we have the eigenvalues of matrix M as

$$\lambda_1 = e^{-4t\gamma} \left\{ \left[\alpha_1^4 - 2\alpha_1^4 e^{\gamma t} + e^{2\gamma t} \left(\alpha_1^2 - \alpha_1^2 \alpha_2^2 - \alpha_1^2 \alpha_3^2 + \alpha_2^2 \alpha_3^2 - 2e^{2\gamma t} \alpha_1 \alpha_2 \alpha_3 \alpha_4 \cos(\Phi) \right) \right]^{1/2} \right.$$
$$\left. + e^{t\gamma} \left[\alpha_1^2 \alpha_4^2 + \alpha_2^2 \alpha_3^2 - 2\alpha_1 \alpha_2 \alpha_3 \alpha_4 \cos(\Phi) \right]^{1/2} \right\}^2, \qquad (4.1.36)$$

$$\lambda_2 = e^{-4t\gamma} \left\{ \left[\alpha_1^4 - 2\alpha_1^4 e^{\gamma t} + e^{2\gamma t} \left(\alpha_1^2 - \alpha_1^2 \alpha_2^2 - \alpha_1^2 \alpha_3^2 + \alpha_2^2 \alpha_3^2 - 2e^{2\gamma t} \alpha_1 \alpha_2 \alpha_3 \alpha_4 \cos(\Phi) \right) \right]^{1/2} \right.$$
$$\left. - e^{t\gamma} \left[\alpha_1^2 \alpha_4^2 + \alpha_2^2 \alpha_3^2 - 2\alpha_1 \alpha_2 \alpha_3 \alpha_4 \cos(\Phi) \right]^{1/2} \right\}^2, \qquad (4.1.37)$$

$$\lambda_3 = \lambda_4 = e^{-4\gamma t} \alpha_1^4 \left(-1 + e^{t\gamma} \right)^2. \qquad (4.1.38)$$

The concurrence [17, 18] by using Eq. (3.4.3) for these eigenvalues can be given as

$$C(t) = \max \left\{ 0, e^{-\gamma t} C(0) - 2\alpha_1^2 e^{-2\gamma t} \left(e^{\gamma t} - 1 \right) \right\}, \qquad (4.1.39)$$

where $C(0)$ is initial concurrence can be defined as,

$$C(0) = 2\sqrt{\alpha_1^2 \alpha_4^2 + \alpha_2^2 \alpha_3^2 - 2\alpha_1 \alpha_2 \alpha_3 \alpha_4 \cos(\Phi)}. \qquad (4.1.40)$$

The EPR assumes a central role in quantum communication and is absolutely of significance to study its entanglement decay by the surrounding. Here we assume two cases.

Case 1

Let us consider the combination of state in which both atoms are in excited state or in lower state with arbitrary amplitudes α_1 and α_4. given as

$$\left| \psi_1(0) \right\rangle = \alpha_1 \left| a_1, a_2 \right\rangle + \alpha_4 e^{i\phi_3} \left| b_1, b_2 \right\rangle, \qquad (4.1.41)$$

For this state, In 4×4 density matrix only α_1 and α_4 are considered whereas α_2 and α_3 is equal to zero.

we determined the eigenvalues of this state to define entanglement dynamics which depends upon the relation between α_1 and α_4. The detailed solution of matrix M are given in Appendix F. we get the eigenvalues of matrix M as

$$\lambda_1 = \alpha_1^2 e^{-4t\gamma} \left[\sqrt{\alpha_1^2 \left(1 - 2e^{\gamma t}\right) + e^{2\gamma t}} + \alpha_4 e^{\gamma t} \right]^2, \qquad (4.1.42)$$

$$\lambda_2 = \alpha_1^2 e^{-4t\gamma} \left[\sqrt{\alpha_1^2 \left(1 - 2e^{\gamma t}\right) + e^{2\gamma t}} - \alpha_4 e^{\gamma t} \right]^2, \qquad (4.1.43)$$

$$\lambda_3 = \lambda_4 = e^{-4t\gamma} \alpha_1^4 \left(-1 + e^{t\gamma}\right)^2. \qquad (4.1.44)$$

and the concurrence is

$$C\left(t\right) = max\left\{ 0, 2\left(\alpha_1 \alpha_4 e^{-\gamma t} - \alpha_1^2 e^{-2\gamma t}\left(e^{\gamma t} - 1\right)\right)\right\}.$$

For this kind of entangled state, sudden death of entanglement takes place faster due to the presence of $|a_1, a_2\rangle$ in the cavities because no more atoms remain in excited state for a long time due to spontaneous emission.

Case 2

Now consider the superposition state like Bell state with arbitrary amplitudes α_2 and α_3 the state can be written as,

$$|\psi_2\left(0\right)\rangle = \alpha_2 |a_1, b_2\rangle + \alpha_3 e^{i\phi_2} |b_1, a_2\rangle, \qquad (4.1.45)$$

For 4×4 matrix, only the arbitrary amplitudes α_2 and α_3 are included whereas the arbitrary amplitudes with α_1 and α_4 terms are excluded. we have to calculate

the eigenvalues to defined the entanglement dynamics. The detailed solution are given in Appendix G. we have the eigenvalue of matrix M as

$$\lambda_1 = 4\alpha_2^2\alpha_3^3 e^{-2\gamma t}, \tag{4.1.46}$$

$$\lambda_2 = \lambda_3 = \lambda_4 = 0. \tag{4.1.47}$$

and the concurrence can be calculated by using formula (3.4.3). we have,

$$C(t) = max\left\{0, 2\alpha_2\alpha_3 e^{-\gamma t}\right\}. \tag{4.1.48}$$

This is a maximum concurrence for equal values of α_2 and α_3. For this kind of entangled state, sudden death of entanglement is never seen.

Chapter 5

Conclusion

The present thesis is concerned with the reduced density matrix equation for a system by using density matrix formalism and observe the entanglement dynamics in dissipative environments. The decays in dissipative environments takes place due to atom field interaction in the reservoir. In first chapter we have given a historical review about atom field interaction Hamiltonian, density matrix formalism, decays in dissipative environments and calculated concurrence analytically and then graphically. In the second chapter a brief review of field quantization in cavity QED is explained. we start with electromagnetic field theory both classically and quantum mechanically and see how to quantize the electromagnetic field in a cavity. Later on, we explained the single two-level atom field interaction Hamiltonian by using Electric dipole and rotating wave approximation. Further this model is extended for a two, two-level atoms interacting with quantized electromagnetic field. In second part of this chapter we calculated the general mathematical formalism of density matrix. This chapter contains detailed explanation of mathematical tool that were later on used for further engineering.

In the 3rd chapter we explained that atomic decays happens due to interaction with thermal environments. we calculated atomic decays by using Von-Neumann

equation. In the next section of this chapter we calculate the entanglement dynamics of our system by using analytical and graphical technique of concurrence. By these techniques we elaborated that due to increase in temperature or presence of mean thermal photon numbers sudden death time decrease.

In chapter fourth, we discussed the phenomenon in which there is no mean thermal photon numbers present in the cavity but decays takes place due to vacuum fluctuations. we calculated the concurrence mathematically and then graphically which shows that EPR states has the finite time disentanglement. The results also shows that doubly excited atoms must be decay faster than superposition state of atoms.

Appendix A

Solution of density matrix element for a thermal reservoir

Here using following terms to illustrate that according to figure

$$|1\rangle = |a_1, a_2\rangle, |2\rangle = |a_1, b_2\rangle, |3\rangle = |b_1, a_2\rangle, |4\rangle = |b_1, b_2\rangle \qquad (A.0.1)$$

Next we find

$$\dot{\rho}_{11} = \langle 1| \dot{\rho} |1\rangle$$

$$
\begin{aligned}
= \quad & \langle a_1, a_2| \left\{ -\frac{1}{2}\gamma_1 \left(\overline{n}+1\right) [\sigma_+^1 \sigma_-^1 \rho_{atom} - 2\sigma_-^1 \rho_{atom}\sigma_+^1 + \rho_{atom}\sigma_+^1 \sigma_-^1] \right. \\
& -\frac{1}{2}\gamma_1 \overline{n}[\sigma_-^1 \sigma_+^1 \rho_{atom} - 2\sigma_+^1 \rho_{atom}\sigma_-^1 + \rho_{atom}\sigma_-^1 \sigma_+^1] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m}+1\right) [\sigma_+^2 \sigma_-^2 \rho_{atom} - 2\sigma_-^2 \rho_{atom}\sigma_+^2 + \rho_{atom}\sigma_+^2 \sigma_-^2] \\
& \left. -\frac{1}{2}\gamma_2 \overline{m}[\sigma_-^2 \sigma_+^2 \rho_{atom} - 2\sigma_+^2 \rho_{atom}\sigma_-^2 + \rho_{atom}\sigma_-^2 \sigma_+^2] \right\} |a_1, a_2\rangle \qquad (A.0.2)
\end{aligned}
$$

$$\sigma_i^+ = |a_i\rangle \langle b_i|$$

$$\sigma_i^- = |b_i\rangle \langle a_i|$$

Here

$$\sigma_+^1 \sigma_-^1 = |a_1\rangle \langle b_1|b_1\rangle \langle a_1| \tag{A.0.3}$$

$$\sigma_-^1 \sigma_+^1 = |b_1\rangle \langle a_1|a_1\rangle \langle b_1| \tag{A.0.4}$$

Inner product of this term $\langle b_1|b_1\rangle = 1$ and $\langle a_1|a_1\rangle = 1$ so above equation can be written as,

$$\sigma_+^1 \sigma_-^1 = |a_1\rangle \langle a_1|$$

$$\sigma_-^1 \sigma_+^1 = |b_1\rangle \langle b_1|$$

After using these relations in Eq A.0.2 we can write as

$$\langle a_1, a_2| \left\{ -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[|a_1\rangle \langle a_1| \rho_{atom} - 2|b_1\rangle \langle a_1| \rho_{atom} |a_1\rangle \langle b_1| + \rho_{atom} |a_1\rangle \langle a_1|\right] \right.$$

$$-\frac{1}{2}\gamma_1\overline{n} \left[|b_1\rangle \langle b_1| \rho_{atom} - 2|a_1\rangle \langle b_1| \rho_{atom}\sigma_-^1 + \rho_{atom} |b_1\rangle \langle b_1|\right]$$

$$-\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[|a_2\rangle \langle a_2| \rho_{atom} - 2|b_2\rangle \langle a_2| \rho_{atom} |a_2\rangle \langle b_2| + \rho_{atom} |a_2\rangle \langle a_2|\right]$$

$$\left. -\frac{1}{2}\gamma_2\overline{m}[|b_2\rangle \langle b_2| \rho_{atom} - 2|a_2\rangle \langle b_2| \rho_{atom} |b_2\rangle \langle a_2| + \rho_{atom} |b_2\rangle \langle b_2|] \right\} |a_1, a_2\rangle \tag{A.0.5}$$

After solving these terms we get result

$$= -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[\langle a_2,a_1|\rho|a_1,a_2\rangle + \langle a_2,a_1|\rho|a_1,a_2\rangle\right]$$
$$-\frac{1}{2}\gamma_1\overline{n}\left[-2\langle a_2,b_1|\rho|b_1,a_2\rangle\right]$$
$$-\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[\langle a_1,a_2|\rho|a_1,a_2\rangle + \langle a_1,a_2|\rho|a_2,a_1\rangle\right]$$
$$-\frac{1}{2}\gamma_2\overline{m}\left[-2\langle a_1,b_2|\rho|b_2,a_1\rangle\right] \tag{A.0.6}$$

In simplifying form,this equation becomes.

$$= -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[\rho_{11}+\rho_{11}\right] - \frac{1}{2}\gamma_1\overline{n}\left[-2\rho_{33}\right]$$
$$-\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[\rho_{11}+\rho_{11}\right] - \frac{1}{2}\gamma_2\overline{m}\left[-2\rho_{22}\right] \tag{A.0.7}$$

Exact solution of ρ_{11} will be,

$$\dot{\rho}_{11} = -\left[\left(\overline{n}+1\right)\gamma_1 + \left(\overline{m}+1\right)\gamma_2\right]\rho_{11} + \overline{m}\gamma_2\rho_{22} + \overline{n}\gamma_1\rho_{33} \tag{A.0.8}$$

Similarly for

$$\dot{\rho}_{22} = \langle 2|\dot{\rho}|2\rangle \tag{A.0.9}$$

using Equ A.0.5.where $|2\rangle = |a_1,b_2\rangle$ then

$$= \langle a_1,b_2|\left\{-\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[|a_1\rangle\langle a_1|\rho_{atom} - 2|b_1\rangle\langle a_1|\rho_{atom}|a_1\rangle\langle b_1| + \rho_{atom}|a_1\rangle\langle a_1|\right]\right.$$
$$-\frac{1}{2}\gamma_1\overline{n}\left[|b_1\rangle\langle b_1|\rho_{atom} - 2|a_1\rangle\langle b_1|\rho_{atom}\sigma_-^1 + \rho_{atom}|b_1\rangle\langle b_1|\right]$$
$$-\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[|a_2\rangle\langle a_2|\rho_{atom} - 2|b_2\rangle\langle a_2|\rho_{atom}|a_2\rangle\langle b_2| + \rho_{atom}|a_2\rangle\langle a_2|\right]$$
$$\left.-\frac{1}{2}\gamma_2\overline{m}[|b_2\rangle\langle b_2|\rho_{atom} - 2|a_2\rangle\langle b_2|\rho_{atom}|b_2\rangle\langle a_2| + \rho_{atom}|b_2\rangle\langle b_2|]\right\}|a_1,b_2\rangle \tag{A.0.10}$$

After solving

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[\langle b_2, a_1|\rho|a_1, b_2\rangle + \langle a_1, b_2|\rho|a_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_1\overline{n}\left[-2\langle b_2, b_1|\rho|b_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[-2\langle a_1, a_2|\rho|a_2, a_1\rangle\right] \\
& -\frac{1}{2}\gamma_1\overline{m}\left[\langle a_2, b_2|\rho|a_1, b_2\rangle + \langle a_1, b_2|\rho|b_2, a_1\rangle\right]
\end{aligned}
$$

After simplifying

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[\rho_{22} + \rho_{22}\right] - \frac{1}{2}\gamma_1\overline{n}\left[-2\rho_{44}\right] \\
& -\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[\rho_{11} + \rho_{11}\right] - \frac{1}{2}\gamma_2\overline{m}\left[-2\rho_{22}\right]
\end{aligned}
$$

Final solution of ρ_{22} will be

$$
\dot{\rho}_{22} = -\left[\left(\overline{n}+1\right)\gamma_1 + \overline{m}\gamma_2\right]\rho_{22} + \left(\overline{m}+1\right)\gamma_2\rho_{11} + \overline{n}\gamma_1\rho_{44} \qquad (A.0.11)
$$

Now find ρ_{33} such as

$$
\dot{\rho}_{33} = \langle 3|\,\dot{\rho}\,|3\rangle
$$

using equ A.0.5 for ρ_{33} where $|3\rangle = |b_1, a_2\rangle$

$$
\begin{aligned}
= \;& \langle b_1, a_2| \left\{ -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[|a_1\rangle\langle a_1|\,\rho_{atom} - 2\,|b_1\rangle\langle a_1|\,\rho_{atom}\,|a_1\rangle\langle b_1| + \rho_{atom}\,|a_1\rangle\langle a_1|\right] \right. \\
& -\frac{1}{2}\gamma_1\overline{n}\left[|b_1\rangle\langle b_1|\,\rho_{atom} - 2\,|a_1\rangle\langle b_1|\,\rho_{atom}\sigma_-^1 + \rho_{atom}\,|b_1\rangle\langle b_1|\right] \\
& -\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[|a_2\rangle\langle a_2|\,\rho_{atom} - 2\,|b_2\rangle\langle a_2|\,\rho_{atom}\,|a_2\rangle\langle b_2| + \rho_{atom}\,|a_2\rangle\langle a_2|\right] \\
& \left. -\frac{1}{2}\gamma_2\overline{m}[|b_2\rangle\langle b_2|\,\rho_{atom} - 2\,|a_2\rangle\langle b_2|\,\rho_{atom}\,|b_2\rangle\langle a_2| + \rho_{atom}\,|b_2\rangle\langle b_2|]\right\} |b_1, a_2\rangle
\end{aligned}
\tag{A.0.12}
$$

After solving

$$
\begin{aligned}
= \;& -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[-2\langle a_2, a_1|\rho|a_1, a_2\rangle\right] \\
& -\frac{1}{2}\gamma_1\overline{n}\left[\langle a_2, b_1|\rho|b_1, a_2\rangle + \langle b_1, a_2|\rho|b_1, a_2\rangle\right] \\
& -\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[\langle b_1, a_2|\rho|b_1, a_2\rangle + \langle b_1, a_2|\rho|b_1, a_2\rangle\right] \\
& -\frac{1}{2}\gamma_2\overline{m}\left[-2\langle b_1, b_2|\rho|b_2, b_1\rangle\right]
\end{aligned}
$$

Simplifying above equation

$$
\begin{aligned}
= \;& -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[-2\rho_{11}\right] - \frac{1}{2}\gamma_1\overline{n}\left[2\rho_{33}\right] \\
& -\frac{1}{2}\gamma_1\left(\overline{m}+1\right)\left[2\rho_{33}\right] - \frac{1}{2}\gamma_2\overline{m}\left[-2\rho_{44}\right]
\end{aligned}
\tag{A.0.13}
$$

The final solution of ρ_{33} is

$$
\dot{\rho}_{33} = -\left[\overline{n}\gamma_1 + \left(\overline{m}+1\right)\gamma_2\right]\rho_{33} + \left(\overline{n}+1\right)\gamma_1\rho_{11} + \overline{m}\gamma_2\rho_{44}
\tag{A.0.14}
$$

Similarly find the solution of ρ_{44} we have

$$\dot{\rho}_{44} = \langle 4 | \dot{\rho} | 4 \rangle$$

using Eq. A.0.5 where $|4\rangle = |b_1, b_2\rangle$

$$
\begin{aligned}
= \quad & \langle b_1, b_2 | \left\{ -\frac{1}{2}\gamma_1 (\overline{n} + 1) \left[|a_1\rangle \langle a_1| \rho_{atom} - 2 |b_1\rangle \langle a_1| \rho_{atom} |a_1\rangle \langle b_1| + \rho_{atom} |a_1\rangle \langle a_1| \right] \right. \\
& -\frac{1}{2}\gamma_1 \overline{n} \left[|b_1\rangle \langle b_1| \rho_{atom} - 2 |a_1\rangle \langle b_1| \rho_{atom} \sigma_-^1 + \rho_{atom} |b_1\rangle \langle b_1| \right] \\
& -\frac{1}{2}\gamma_2 (\overline{m} + 1) \left[|a_2\rangle \langle a_2| \rho_{atom} - 2 |b_2\rangle \langle a_2| \rho_{atom} |a_2\rangle \langle b_2| + \rho_{atom} |a_2\rangle \langle a_2| \right] \\
& \left. -\frac{1}{2}\gamma_2 \overline{m} [|b_2\rangle \langle b_2| \rho_{atom} - 2 |a_2\rangle \langle b_2| \rho_{atom} |b_2\rangle \langle a_2| + \rho_{atom} |b_2\rangle \langle b_2|] \right\} |b_1, b_2\rangle
\end{aligned}
$$

solving above equation we write as

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1 (\overline{n} + 1) \left[-2\langle b_2, a_1 | \rho | a_1, b_2 \rangle \right] \\
& -\frac{1}{2}\gamma_1 \overline{n} \left[\langle b_2, b_1 | \rho | b_1, b_2 \rangle + \langle b_1, b_2 | \rho | b_1, b_2 \rangle \right] \\
& -\frac{1}{2}\gamma_2 (\overline{m} + 1) \left[-2\langle b_1, a_2 | \rho | a_2, b_1 \rangle \right] \\
& -\frac{1}{2}\gamma_2 \overline{m} \left[\langle b_1, b_2 | \rho | b_1, b_2 \rangle + \langle b_1, b_2 | \rho | b_2, b_1 \rangle \right] \quad \text{(A.0.15)}
\end{aligned}
$$

Simplified above equation we get.

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1 (\overline{n} + 1) \left[-2\rho_{22} \right] - \frac{1}{2}\gamma_1 \overline{n} \left[\rho_{44} + \rho_{44} \right] \\
& -\frac{1}{2}\gamma_2 (\overline{m} + 1) \left[-2\rho_{33} \right] - \frac{1}{2}\gamma_2 \overline{m} \left[2\rho_{44} \right] \quad \text{(A.0.16)}
\end{aligned}
$$

after simplification, final result will be

$$\dot{\rho}_{44} = -\left[\overline{n}\gamma_1 + \overline{m}\gamma_2\right]\rho_{44} + (\overline{n}+1)\gamma_1\rho_{22} + (\overline{m}+1)\gamma_2\rho_{33} \qquad \text{(A.0.17)}$$

Now find the solution for $\dot{\rho}_{12}$ such as

$$\dot{\rho}_{12} = \langle 1|\,\dot{\rho}\,|2\rangle$$

where $\langle 1| = \langle a_1, a_2|$ and $|2\rangle = |a_1, b_2\rangle$ using Eq. A.0.5 we can write as

$$
\begin{aligned}
= \quad & \langle a_1, a_2|\left\{ -\frac{1}{2}\gamma_1\,(\overline{n}+1)\left[|a_1\rangle\langle a_1|\,\rho_{atom} - 2\,|b_1\rangle\langle a_1|\,\rho_{atom}\,|a_1\rangle\langle b_1| + \rho_{atom}\,|a_1\rangle\langle a_1|\right] \right. \\
& -\frac{1}{2}\gamma_1\overline{n}\left[|b_1\rangle\langle b_1|\,\rho_{atom} - 2\,|a_1\rangle\langle b_1|\,\rho_{atom}\,|b_1\rangle\langle a_1| + \rho_{atom}\,|b_1\rangle\langle b_1|\right] \\
& -\frac{1}{2}\gamma_2\,(\overline{m}+1)\left[|a_2\rangle\langle a_2|\,\rho_{atom} - 2\,|b_2\rangle\langle a_2|\,\rho_{atom}\,|a_2\rangle\langle b_2| + \rho_{atom}\,|a_2\rangle\langle a_2|\right] \\
& \left. -\frac{1}{2}\gamma_2\overline{m}[|b_2\rangle\langle b_2|\,\rho_{atom} - 2\,|a_2\rangle\langle b_2|\,\rho_{atom}\,|b_2\rangle\langle a_2| + \rho_{atom}\,|b_2\rangle\langle b_2|]\right\}|a_1, b_2\rangle \text{(A.0.18)}
\end{aligned}
$$

After solving above equation we have,

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1\,(\overline{n}+1)\left[\langle a_2, a_1|\rho|a_1, b_2\rangle + \langle a_1, a_2|\rho|a_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_1\overline{n}\left[-2\langle a_2, b_1|\rho|b_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_2\,(\overline{m}+1)\left[\langle a_1, a_2|\rho|a_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_2\overline{m}\left[\langle a_1, a_2|\rho|b_2, a_1\rangle\right] \qquad\qquad \text{(A.0.19)}
\end{aligned}
$$

above equation can also be written as

$$= -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[\rho_{12}+\rho_{12}\right] - \frac{1}{2}\gamma_1\overline{n}\left[-2\rho_{34}\right]$$
$$-\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[\rho_{12}\right] - \frac{1}{2}\gamma_2\overline{m}\left[\rho_{12}\right] \tag{A.0.20}$$

Final solution for $\dot{\rho}_{12}$ can be written in this form

$$\dot{\rho}_{12} = -\left[\left(\overline{n}+1\right)\gamma_1 + \left(\overline{m}+1/2\right)\gamma_2\right]\rho_{12} + \overline{n}\gamma_1\rho_{34}$$

Similarly find out the solution for $\dot{\rho}_{13}$ we have

$$\dot{\rho}_{13} = \langle 1|\,\dot{\rho}\,|3\rangle$$

using Eq. A.0.5 where $\langle 1| = \langle a_1, a_2|$ and $|3\rangle = |b_1, a_2\rangle$

$$\begin{aligned}
= \ &\langle a_1, a_2|\left\{-\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[|a_1\rangle\langle a_1|\rho_{atom} - 2|b_1\rangle\langle a_1|\rho_{atom}|a_1\rangle\langle b_1| + \rho_{atom}|a_1\rangle\langle a_1|\right]\right. \\
&-\frac{1}{2}\gamma_1\overline{n}\left[|b_1\rangle\langle b_1|\rho_{atom} - 2|a_1\rangle\langle b_1|\rho_{atom}|b_1\rangle\langle a_1| + \rho_{atom}|b_1\rangle\langle b_1|\right] \\
&-\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[|a_2\rangle\langle a_2|\rho_{atom} - 2|b_2\rangle\langle a_2|\rho_{atom}|a_2\rangle\langle b_2| + \rho_{atom}|a_2\rangle\langle a_2|\right] \\
&\left.-\frac{1}{2}\gamma_2\overline{m}\left[|b_2\rangle\langle b_2|\rho_{atom} - 2|a_2\rangle\langle b_2|\rho_{atom}|b_2\rangle\langle a_2| + \rho_{atom}|b_2\rangle\langle b_2|\right]\right\}|b_1, a_2\rangle \tag{A.0.21}
\end{aligned}$$

Simplified above equation we get,

$$\begin{aligned}
=\ &-\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[\langle a_2,a_1|\rho|b_1,a_2\rangle\right]\\
&-\frac{1}{2}\gamma_1\overline{n}\left[\langle a_1,a_2|\rho|b_1,a_2\rangle\right]\\
&-\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[\langle a_1,a_2|\rho|b_1,a_2\rangle+\langle a_1,a_2|\rho|a_2,b_1\rangle\right]\\
&-\frac{1}{2}\gamma_2\overline{m}\left[-2\langle a_1,b_2|\rho|b_2,b_1\rangle\right]
\end{aligned} \tag{A.0.22}$$

In simplified form,above equation can be written as

$$\begin{aligned}
=\ &-\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[\rho_{13}\right]-\frac{1}{2}\gamma_1\overline{n}\left[\rho_{13}\right]\\
&-\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[2\rho_{13}\right]-\frac{1}{2}\gamma_2\overline{m}\left[-2\rho_{24}\right]
\end{aligned} \tag{A.0.23}$$

Final solution of $\dot{\rho}_{13}$ will becomes

$$\dot{\rho}_{13}=-\left[\left(\overline{n}+1/2\right)\gamma_1+\left(\overline{m}+1\right)\gamma_2\right]\rho_{13}+\overline{m}\gamma_2\rho_{24} \tag{A.0.24}$$

Similarly,$\dot{\rho}_{14}$ can be written as

$$\dot{\rho}_{14}=\langle 1|\dot{\rho}|4\rangle \tag{A.0.25}$$

using Eq A.0.5 where $\langle 1|=\langle a_1,a_2|$ and $|4\rangle=|b_1,b_2\rangle$ so,

$$
\begin{aligned}
= \quad & \langle a_1, a_2 | \left\{ -\frac{1}{2}\gamma_1 \left(\overline{n}+1\right) \left[|a_1\rangle \langle a_1| \rho_{atom} - 2|b_1\rangle \langle a_1| \rho_{atom} |a_1\rangle \langle b_1| + \rho_{atom} |a_1\rangle \langle a_1|\right] \right. \\
& -\frac{1}{2}\gamma_1 \overline{n} \left[|b_1\rangle \langle b_1| \rho_{atom} - 2|a_1\rangle \langle b_1| \rho_{atom} |b_1\rangle \langle a_1| + \rho_{atom} |b_1\rangle \langle b_1|\right] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m}+1\right) \left[|a_2\rangle \langle a_2| \rho_{atom} - 2|b_2\rangle \langle a_2| \rho_{atom} |a_2\rangle \langle b_2| + \rho_{atom} |a_2\rangle \langle a_2|\right] \\
& \left. -\frac{1}{2}\gamma_2 \overline{m}[|b_2\rangle \langle b_2| \rho_{atom} - 2|a_2\rangle \langle b_2| \rho_{atom} |b_2\rangle \langle a_2| + \rho_{atom} |b_2\rangle \langle b_2|] \right\} |b_1, b_2\rangle \quad \text{(A.0.26)}
\end{aligned}
$$

After solving this we get

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1 \left(\overline{n}+1\right) \left[\langle a_2, a_1|\rho|b_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_1 \overline{n} \left[\langle a_1, a_2|\rho|b_1, a_2\rangle\right] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m}+1\right) \left[\langle a_1, a_2|\rho|b_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_2 \overline{m} \left[\langle a_1, a_2|\rho|b_2, b_1\rangle\right]
\end{aligned}
$$

above equation can be written as

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1 \left(\overline{n}+1\right) [\rho_{14}] - \frac{1}{2}\gamma_1 \overline{n} [\rho_{14}] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m}+1\right) [\rho_{14}] - \frac{1}{2}\gamma_2 \overline{m} [\rho_{14}] \quad \text{(A.0.27)}
\end{aligned}
$$

Final solution of $\dot{\rho}_{14}$ will be,

$$
\dot{\rho}_{14} = - \left[\left(\overline{n}+1/2\right)\gamma_1 + \left(\overline{m}+1/2\right)\gamma_2\right]\rho_{14} \quad \text{(A.0.28)}
$$

Now we find out the solution for $\dot{\rho}_{21}$ in such a way that,

$$\dot{\rho}_{21} = \langle 2| \, \dot{\rho} \, |1\rangle$$

By using Eq A.0.5 where we use $\langle 2| = \langle a_1, b_2|$ and $|1\rangle = |a_1, a_2\rangle$ then

$$
\begin{aligned}
= \quad & \langle a_1, b_2| \left\{ -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[|a_1\rangle \langle a_1| \rho_{atom} - 2 |b_1\rangle \langle a_1| \rho_{atom} |a_1\rangle \langle b_1| + \rho_{atom} |a_1\rangle \langle a_1|\right] \right. \\
& -\frac{1}{2}\gamma_1 \overline{n} \left[|b_1\rangle \langle b_1| \rho_{atom} - 2 |a_1\rangle \langle b_1| \rho_{atom} |b_1\rangle \langle a_1| + \rho_{atom} |b_1\rangle \langle b_1|\right] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[|a_2\rangle \langle a_2| \rho_{atom} - 2 |b_2\rangle \langle a_2| \rho_{atom} |a_2\rangle \langle b_2| + \rho_{atom} |a_2\rangle \langle a_2|\right] \\
& \left. -\frac{1}{2}\gamma_2 \overline{m}[|b_2\rangle \langle b_2| \rho_{atom} - 2 |a_2\rangle \langle b_2| \rho_{atom} |b_2\rangle \langle a_2| + \rho_{atom} |b_2\rangle \langle b_2|] \right\} |a_1, a_2\rangle
\end{aligned}
$$

Solved above equation in simplified terms we have,

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[\langle b_2, a_1|\rho|a_1, a_2\rangle + \langle a_1, b_2|\rho|a_1, a_2\rangle\right] \\
& -\frac{1}{2}\gamma_1 \overline{n} \left[-2\langle b_2, b_1|\rho|b_1, a_2\rangle\right] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[\langle a_1, b_2|\rho|a_2, a_1\rangle\right] \\
& -\frac{1}{2}\gamma_2 \overline{m} \left[\langle a_1, b_2|\rho|a_1, a_2\rangle\right] \quad\quad\quad\quad\quad \text{(A.0.29)}
\end{aligned}
$$

The above equations can also be written as

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[\rho_{12} + \rho_{12}\right] - \frac{1}{2}\gamma_1 \overline{n} \left[-2\rho_{43}\right] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[\rho_{21}\right] - \frac{1}{2}\gamma_2 \overline{m} \left[\rho_{21}\right]
\end{aligned}
$$

Final solution for $\dot{\rho}_{21}$ is,

$$\dot{\rho}_{21} = -\left[\left(\overline{n} + 1\right)\gamma_1 + \left(\overline{m} + 1/2\right)\gamma_2\right]\rho_{21} + \overline{n}\gamma_1\rho_{43} \qquad (A.0.30)$$

Similarly solution for $\dot{\rho}_{23}$ can be find out as

$$\dot{\rho}_{23} = \langle 2|\,\dot{\rho}\,|3\rangle$$

using Eq A.0.5 in which $\langle 2| = \langle a_1, b_2|$ and $|3\rangle = |b_1, a_2\rangle$ so we can write as

$$
\begin{aligned}
= \quad & \langle a_1, b_2| \left\{ -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[|a_1\rangle\langle a_1|\,\rho_{atom} - 2|b_1\rangle\langle a_1|\,\rho_{atom}\,|a_1\rangle\langle b_1| + \rho_{atom}\,|a_1\rangle\langle a_1|\right] \right. \\
& -\frac{1}{2}\gamma_1\overline{n}\left[|b_1\rangle\langle b_1|\,\rho_{atom} - 2|a_1\rangle\langle b_1|\,\rho_{atom}\,|b_1\rangle\langle a_1| + \rho_{atom}\,|b_1\rangle\langle b_1|\right] \\
& -\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[|a_2\rangle\langle a_2|\,\rho_{atom} - 2|b_2\rangle\langle a_2|\,\rho_{atom}\,|a_2\rangle\langle b_2| + \rho_{atom}\,|a_2\rangle\langle a_2|\right] \\
& \left. -\frac{1}{2}\gamma_2\overline{m}[|b_2\rangle\langle b_2|\,\rho_{atom} - 2|a_2\rangle\langle b_2|\,\rho_{atom}\,|b_2\rangle\langle a_2| + \rho_{atom}\,|b_2\rangle\langle b_2|\right] \right\}|b_1, a_2\rangle (A.0.31)
\end{aligned}
$$

After solving above equation we get

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[\langle b_2, a_1|\rho|b_1, a_2\rangle\right] \\
& -\frac{1}{2}\gamma_1\overline{n}\left[\langle a_1, b_2|\rho|b_1, a_2\rangle\right] \\
& -\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[\langle a_1, b_2|\rho|a_2, b_1\rangle\right] \\
& -\frac{1}{2}\gamma_2\overline{m}\left[\langle a_1, b_2|\rho|b_1, a_2\rangle\right] \qquad (A.0.32)
\end{aligned}
$$

Simplified form of above equation can be written as

$$= -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[\rho_{23}\right] - \frac{1}{2}\gamma_1\overline{n}\left[\rho_{23}\right]$$
$$-\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[\rho_{23}\right] - \frac{1}{2}\gamma_2\overline{m}\left[\rho_{23}\right] \qquad (A.0.33)$$

Final result of $\dot{\rho}_{23}$ is,

$$\dot{\rho}_{23} = -\left[\left(\overline{n}+1/2\right)\gamma_1 + \left(\overline{m}+1/2\right)\gamma_2\right]\rho_{41} \qquad (A.0.34)$$

Now find the solution for $\dot{\rho}_{24}$ such as

$$\dot{\rho}_{24} = \langle 2|\,\dot{\rho}\,|4\rangle$$

Using Eq A.0.5 where $\langle 2| = \langle a_1, b_2|$ and $|4\rangle = |b_1, b_2\rangle$ we get

$$\begin{aligned}
= \;\; & \langle a_1, b_2| \left\{ -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[|a_1\rangle\langle a_1|\,\rho_{atom} - 2\,|b_1\rangle\langle a_1|\,\rho_{atom}\,|a_1\rangle\langle b_1| + \rho_{atom}\,|a_1\rangle\langle a_1|\right]\right. \\
& -\frac{1}{2}\gamma_1\overline{n}\left[|b_1\rangle\langle b_1|\,\rho_{atom} - 2\,|a_1\rangle\langle b_1|\,\rho_{atom}\,|b_1\rangle\langle a_1| + \rho_{atom}\,|b_1\rangle\langle b_1|\right] \\
& -\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[|a_2\rangle\langle a_2|\,\rho_{atom} - 2\,|b_2\rangle\langle a_2|\,\rho_{atom}\,|a_2\rangle\langle b_2| + \rho_{atom}\,|a_2\rangle\langle a_2|\right] \\
& \left. -\frac{1}{2}\gamma_2\overline{m}[|b_2\rangle\langle b_2|\,\rho_{atom} - 2\,|a_2\rangle\langle b_2|\,\rho_{atom}\,|b_2\rangle\langle a_2| + \rho_{atom}\,|b_2\rangle\langle b_2|]\right\} |b_1, b_2\rangle
\end{aligned}$$

after simplifying above equation we can write as

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[\langle b_2, a_1|\rho|b_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_1 \overline{n} \left[\langle a_1, b_2|\rho|b_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[-2\langle a_1, a_2|\rho|a_2, b_1\rangle\right] \\
& -\frac{1}{2}\gamma_2 \overline{m} \left[\langle a_1, b_2|\rho|b_1, b_2\rangle + \langle a_1, b_2|\rho|b_2, b_1\rangle\right] \quad\quad \text{(A.0.35)}
\end{aligned}
$$

above equation can be written as

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[\rho_{24}\right] - \frac{1}{2}\gamma_1 \overline{n} \left[\rho_{24}\right] \\
= \quad & -\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[-2\rho_{13}\right] - \frac{1}{2}\gamma_2 \overline{m} \left[\rho_{24} + \rho_{24}\right] \quad\quad \text{(A.0.36)}
\end{aligned}
$$

Final result of above equation for $\dot{\rho}_{24}$ can be written as,

$$
\dot{\rho}_{24} = -\left[\left(\overline{n} + 1/2\right)\gamma_1 + \overline{m}\gamma_2\right]\rho_{24} + \left(\overline{m} + 1\right)\gamma_2\rho_{13} \quad\quad \text{(A.0.37)}
$$

Now find the solution for $\dot{\rho}_{31}$

$$
\dot{\rho}_{31} = \langle 3|\,\dot{\rho}\,|1\rangle
$$

using Eq A.0.5 where $\langle 3| = \langle b_1, a_2|$ and $|1\rangle = |a_1, a_2\rangle$

$$= \langle b_1, a_2 | \left\{ -\frac{1}{2}\gamma_1 \left(\overline{n}+1\right) \left[|a_1\rangle \langle a_1| \rho_{atom} - 2 |b_1\rangle \langle a_1| \rho_{atom} |a_1\rangle \langle b_1| + \rho_{atom} |a_1\rangle \langle a_1| \right] \right.$$

$$-\frac{1}{2}\gamma_1 \overline{n} \left[|b_1\rangle \langle b_1| \rho_{atom} - 2 |a_1\rangle \langle b_1| \rho_{atom} |b_1\rangle \langle a_1| + \rho_{atom} |b_1\rangle \langle b_1| \right]$$

$$-\frac{1}{2}\gamma_2 \left(\overline{m}+1\right) \left[|a_2\rangle \langle a_2| \rho_{atom} - 2 |b_2\rangle \langle a_2| \rho_{atom} |a_2\rangle \langle b_2| + \rho_{atom} |a_2\rangle \langle a_2| \right]$$

$$\left. -\frac{1}{2}\gamma_2 \overline{m} \left[|b_2\rangle \langle b_2| \rho_{atom} - 2 |a_2\rangle \langle b_2| \rho_{atom} |b_2\rangle \langle a_2| + \rho_{atom} |b_2\rangle \langle b_2| \right] \right\} |a_1, a_2\rangle$$

after solving above equation we can write as

$$= -\frac{1}{2}\gamma_1 \left(\overline{n}+1\right) \left[\langle b_1, a_2|\rho|a_1, a_2\rangle \right]$$

$$-\frac{1}{2}\gamma_1 \overline{n} \left[\langle a_2, b_1|\rho|a_1, a_2\rangle \right]$$

$$-\frac{1}{2}\gamma_2 \left(\overline{m}+1\right) \left[\langle b_1, a_2|\rho|a_1, a_2\rangle + \langle b_1, a_2|\rho|a_2, a_1\rangle \right]$$

$$-\frac{1}{2}\gamma_2 \overline{m} \left[-2\langle b_1, b_2|\rho|b_2, a_1\rangle \right] \tag{A.0.38}$$

after simplification we get

$$= -\frac{1}{2}\gamma_1 \left(\overline{n}+1\right) \left[\rho_{31}\right] - \frac{1}{2}\gamma_1 \overline{n} \left[\rho_{31}\right]$$

$$-\frac{1}{2}\gamma_2 \left(\overline{m}+1\right) \left[\rho_{31} + \rho_{31}\right] - \frac{1}{2}\gamma_2 \overline{m} \left[-2\rho_{42}\right] \tag{A.0.39}$$

final result of $\dot{\rho}_{31}$ is,

$$\dot{\rho}_{31} = - \left[\left(\overline{n}+1/2\right)\gamma_1 + \left(\overline{m}+1\right)\gamma_2\right]\rho_{31} + \overline{m}\gamma_2\rho_{42} \tag{A.0.40}$$

Similarly find the solution for $\dot{\rho}_{32}$ such as

$$\dot{\rho}_{32} = \langle 3| \, \dot{\rho} \, |2\rangle$$

By using Eq A.0.5 in which $\langle 3| = \langle b_1, a_2|$ and $|2\rangle = |a_1, b_2\rangle$ we have

$$
\begin{aligned}
= \ & \langle b_1, a_2| \left\{ -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[|a_1\rangle \langle a_1| \, \rho_{atom} - 2 |b_1\rangle \langle a_1| \, \rho_{atom} \, |a_1\rangle \langle b_1| + \rho_{atom} \, |a_1\rangle \langle a_1|\right] \right. \\
& -\frac{1}{2}\gamma_1 \overline{n} \left[|b_1\rangle \langle b_1| \, \rho_{atom} - 2 |a_1\rangle \langle b_1| \, \rho_{atom} \, |b_1\rangle \langle a_1| + \rho_{atom} \, |b_1\rangle \langle b_1|\right] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[|a_2\rangle \langle a_2| \, \rho_{atom} - 2 |b_2\rangle \langle a_2| \, \rho_{atom} \, |a_2\rangle \langle b_2| + \rho_{atom} \, |a_2\rangle \langle a_2|\right] \\
& \left. -\frac{1}{2}\gamma_2 \overline{m} \left[|b_2\rangle \langle b_2| \, \rho_{atom} - 2 |a_2\rangle \langle b_2| \, \rho_{atom} \, |b_2\rangle \langle a_2| + \rho_{atom} \, |b_2\rangle \langle b_2|\right] \right\} |a_1, b_2\rangle
\end{aligned}
$$

After simplified above equation we get

$$
\begin{aligned}
= \ & -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[\langle b_1, a_2|\rho|a_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_1 \overline{n} \left[\langle a_2, b_1|\rho|a_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[\langle b_1, a_2|\rho|a_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_2 \overline{m} \left[\langle b_1, a_2|\rho|b_2, a_1\rangle\right]
\end{aligned}
\tag{A.0.41}
$$

Above equation can be written in this form as

$$
\begin{aligned}
= \ & -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[\rho_{32}\right] - \frac{1}{2}\gamma_1 \overline{n} \left[\rho_{32}\right] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[\rho_{32}\right] - \frac{1}{2}\gamma_2 \overline{m} \left[\rho_{32}\right]
\end{aligned}
\tag{A.0.42}
$$

Final result for $\dot{\rho}_{32}$ can be written as

$$\dot{\rho}_{32} = -\left[\left(\overline{n} + 1/2\right)\gamma_1 + \left(\overline{m} + 1/2\right)\gamma_2\right]\rho_{32} \tag{A.0.43}$$

Now find out the solution for $\dot{\rho}_{34}$ such as

$$\dot{\rho}_{34} = \langle 3| \dot{\rho} |4\rangle$$

using Eq A.0.5 in which $\langle 3| = \langle b_1, a_2|$ and $|4\rangle = |b_1, b_2\rangle$ we get

$$
\begin{aligned}
= \quad & \langle b_1, a_2| \left\{ -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[|a_1\rangle\langle a_1|\rho_{atom} - 2|b_1\rangle\langle a_1|\rho_{atom}|a_1\rangle\langle b_1| + \rho_{atom}|a_1\rangle\langle a_1|\right] \right. \\
& -\frac{1}{2}\gamma_1\overline{n}\left[|b_1\rangle\langle b_1|\rho_{atom} - 2|a_1\rangle\langle b_1|\rho_{atom}|b_1\rangle\langle a_1| + \rho_{atom}|b_1\rangle\langle b_1|\right] \\
& -\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[|a_2\rangle\langle a_2|\rho_{atom} - 2|b_2\rangle\langle a_2|\rho_{atom}|a_2\rangle\langle b_2| + \rho_{atom}|a_2\rangle\langle a_2|\right] \\
& \left. -\frac{1}{2}\gamma_2\overline{m}[|b_2\rangle\langle b_2|\rho_{atom} - 2|a_2\rangle\langle b_2|\rho_{atom}|b_2\rangle\langle a_2| + \rho_{atom}|b_2\rangle\langle b_2|] \right\} |b_1, b_2\rangle
\end{aligned}
$$

Solved above equation and after simplification we can write as

$$
\begin{aligned}
= \quad & -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[-2\langle a_2, a_1|\rho|a_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_1\overline{n}\left[\langle a_2, b_1|\rho|b_1, b_2\rangle + \langle b_1, a_2|\rho|b_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[\langle b_1, a_2|\rho|b_1, b_2\rangle\right] \\
& -\frac{1}{2}\gamma_2\overline{m}\left[\langle b_1, a_2|\rho|b_2, b_1\rangle\right] \tag{A.0.44}
\end{aligned}
$$

Above equation can be written in simple terms as

$$= -\frac{1}{2}\gamma_1 (\overline{n}+1) [-2\rho_{12}] - \frac{1}{2}\gamma_1 \overline{n} [\rho_{34} + \rho_{34}]$$
$$-\frac{1}{2}\gamma_2 (\overline{m}+1) [\rho_{34}] - \frac{1}{2}\gamma_2 \overline{m} [\rho_{34}] \tag{A.0.45}$$

Final result for $\dot{\rho}_{34}$ can be written as

$$\dot{\rho}_{34} = -\left[\overline{n}\gamma_1 + (\overline{m} + 1/2)\,\gamma_2\right]\rho_{34} + (\overline{n}+1)\,\gamma_1 \rho_{12} \tag{A.0.46}$$

Similarly the solution for $\dot{\rho}_{41}$ is given as

$$\dot{\rho}_{41} = \langle 4| \dot{\rho} |1\rangle$$

Using Eq A.0.5 where $\langle 4| = \langle b_1, b_2|$ and $|1\rangle = |a_1, a_2\rangle$ we have

$$= \langle b_1, b_2| \left\{ -\frac{1}{2}\gamma_1 (\overline{n}+1) [|a_1\rangle \langle a_1| \rho_{atom} - 2 |b_1\rangle \langle a_1| \rho_{atom} |a_1\rangle \langle b_1| + \rho_{atom} |a_1\rangle \langle a_1|] \right.$$
$$-\frac{1}{2}\gamma_1 \overline{n} [|b_1\rangle \langle b_1| \rho_{atom} - 2 |a_1\rangle \langle b_1| \rho_{atom} |b_1\rangle \langle a_1| + \rho_{atom} |b_1\rangle \langle b_1|]$$
$$-\frac{1}{2}\gamma_2 (\overline{m}+1) [|a_2\rangle \langle a_2| \rho_{atom} - 2 |b_2\rangle \langle a_2| \rho_{atom} |a_2\rangle \langle b_2| + \rho_{atom} |a_2\rangle \langle a_2|]$$
$$\left. -\frac{1}{2}\gamma_2 \overline{m}[|b_2\rangle \langle b_2| \rho_{atom} - 2 |a_2\rangle \langle b_2| \rho_{atom} |b_2\rangle \langle a_2| + \rho_{atom} |b_2\rangle \langle b_2|] \right\} |a_1, a_2\rangle$$

After simplification,above equation can be written as

$$= -\frac{1}{2}\gamma_1 (\overline{n}+1) [\langle b_1, b_2|\rho|a_1, a_2\rangle]$$
$$-\frac{1}{2}\gamma_1 \overline{n} [\langle b_2, b_1|\rho|a_1, a_2\rangle]$$
$$-\frac{1}{2}\gamma_2 (\overline{m}+1) [\langle b_1, b_2|\rho|a_2, a_1\rangle]$$
$$-\frac{1}{2}\gamma_2 \overline{m} [\langle b_1, b_2|\rho|a_1, a_2\rangle]$$

Above equation can be written in simple terms such as

$$= -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) [\rho_{41}] - \frac{1}{2}\gamma_1 \overline{n} [\rho_{41}]$$
$$-\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) [\rho_{41}] - \frac{1}{2}\gamma_2 \overline{m} [\rho_{41}] \qquad \text{(A.0.47)}$$

Final result of $\dot{\rho}_{41}$ can be written as

$$\dot{\rho}_{41} = - \left[\left(\overline{n} + 1/2\right)\gamma_1 + \left(\overline{m} + 1/2\right)\gamma_2\right]\rho_{41} \qquad \text{(A.0.48)}$$

Now find out the solution for $\dot{\rho}_{42}$ can be determined as

$$\dot{\rho}_{42} = \langle 4| \dot{\rho} |2\rangle$$

By using Eq A.0.5 where $\langle 4| = \langle b_1, b_2|$ and $|2\rangle = |a_1, b_2\rangle$ For this,

$$\begin{aligned}
= \ & \langle b_1, b_2| \Bigg\{ -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) [|a_1\rangle \langle a_1| \rho_{atom} - 2 |b_1\rangle \langle a_1| \rho_{atom} |a_1\rangle \langle b_1| + \rho_{atom} |a_1\rangle \langle a_1|] \\
& -\frac{1}{2}\gamma_1 \overline{n} [|b_1\rangle \langle b_1| \rho_{atom} - 2 |a_1\rangle \langle b_1| \rho_{atom} |b_1\rangle \langle a_1| + \rho_{atom} |b_1\rangle \langle b_1|] \\
& -\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) [|a_2\rangle \langle a_2| \rho_{atom} - 2 |b_2\rangle \langle a_2| \rho_{atom} |a_2\rangle \langle b_2| + \rho_{atom} |a_2\rangle \langle a_2|] \\
& -\frac{1}{2}\gamma_2 \overline{m} [|b_2\rangle \langle b_2| \rho_{atom} - 2 |a_2\rangle \langle b_2| \rho_{atom} |b_2\rangle \langle a_2| + \rho_{atom} |b_2\rangle \langle b_2|] \Bigg\} |a_1, b_2\rangle
\end{aligned}$$

After solved this equation we get

$$= -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[\langle b_1, b_2 | \rho | a_1, b_2 \rangle\right]$$
$$-\frac{1}{2}\gamma_1 \overline{n} \left[\langle b_2, b_1 | \rho | a_1, b_2 \rangle\right]$$
$$-\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[-2\langle b_1, a_2 | \rho | a_2, a_1 \rangle\right]$$
$$-\frac{1}{2}\gamma_2 \overline{m} \left[\langle b_1, b_2 | \rho | a_1, b_2 \rangle + \langle b_1, b_2 | \rho | b_2, a_1 \rangle\right] \tag{A.0.49}$$

In simple terms above equation can be written as

$$= -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[\rho_{42}\right] - \frac{1}{2}\gamma_1 \overline{n} \left[\rho_{42}\right]$$
$$-\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[-2\rho_{31}\right] - \frac{1}{2}\gamma_2 \overline{m} \left[\rho_{42} + \rho_{42}\right] \tag{A.0.50}$$

Final result for $\dot{\rho}_{42}$ is

$$\dot{\rho}_{42} = -\left[\left(\overline{n} + 1/2\right)\gamma_1 + \overline{m}\gamma_2\right]\rho_{42} + \left(\overline{m} + 1\right)\gamma_2\rho_{31} \tag{A.0.51}$$

Now find the solution for $\dot{\rho}_{43}$ such as

$$\dot{\rho}_{43} = \langle 4 | \dot{\rho} | 3 \rangle$$

Using Eq A.0.5 where $\langle 4 | = \langle b_1, b_2 |$ and $|3\rangle = |b_1, a_2\rangle$ we have

$$= \langle b_1, b_2 | \left\{ -\frac{1}{2}\gamma_1 \left(\overline{n} + 1\right) \left[|a_1\rangle \langle a_1| \rho_{atom} - 2 |b_1\rangle \langle a_1| \rho_{atom} |a_1\rangle \langle b_1| + \rho_{atom} |a_1\rangle \langle a_1|\right]\right.$$
$$-\frac{1}{2}\gamma_1 \overline{n} \left[|b_1\rangle \langle b_1| \rho_{atom} - 2 |a_1\rangle \langle b_1| \rho_{atom} |b_1\rangle \langle a_1| + \rho_{atom} |b_1\rangle \langle b_1|\right]$$
$$-\frac{1}{2}\gamma_2 \left(\overline{m} + 1\right) \left[|a_2\rangle \langle a_2| \rho_{atom} - 2 |b_2\rangle \langle a_2| \rho_{atom} |a_2\rangle \langle b_2| + \rho_{atom} |a_2\rangle \langle a_2|\right]$$
$$\left.-\frac{1}{2}\gamma_2 \overline{m}[|b_2\rangle \langle b_2| \rho_{atom} - 2 |a_2\rangle \langle b_2| \rho_{atom} |b_2\rangle \langle a_2| + \rho_{atom} |b_2\rangle \langle b_2|]\right\} |b_1, a_2\rangle$$

After simplification,above equation can be written as

$$
\begin{aligned}
= \ & -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[-2\langle b_2,a_1|\rho|a_1,a_2\rangle\right] \\
& -\frac{1}{2}\gamma_1\overline{n}\left[\langle b_2,b_1|\rho|b_1,a_2\rangle+\langle b_1,b_2|\rho|b_1,a_2\rangle\right] \\
& -\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[\langle b_1,b_2|\rho|a_2,b_1\rangle\right] \\
& -\frac{1}{2}\gamma_2\overline{m}\left[\langle b_1,b_2|\rho|b_1,a_2\rangle\right]
\end{aligned}
\tag{A.0.52}
$$

The above equation in simple form can be written as

$$
\begin{aligned}
= \ & -\frac{1}{2}\gamma_1\left(\overline{n}+1\right)\left[-2\rho_{21}\right]-\frac{1}{2}\gamma_1\overline{n}\left[\rho_{43}+\rho_{43}\right] \\
& -\frac{1}{2}\gamma_2\left(\overline{m}+1\right)\left[\rho_{43}\right]-\frac{1}{2}\gamma_2\overline{m}\left[\rho_{43}\right]
\end{aligned}
$$

Final result for $\dot{\rho}_{43}$ can be written as

$$
\dot{\rho}_{43}=-\left[\overline{n}\gamma_1+\left(\overline{m}+1/2\right)\gamma_2\right]\rho_{43}+\left(\overline{n}+1\right)\gamma_1\rho_{21}
\tag{A.0.53}
$$

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Appendix B

Solution of density matrix element for an initial bipartite pure state

The mathematica code to find the solution of $\dot{\rho}_{21}$ and $\dot{\rho}_{43}$

$$
\begin{aligned}
equa\overline{t}ion1 \; = \; & \left\{ \rho'_{21}\left[t\right] == -\left(\left(\overline{n}+1\right)*\gamma_1 + \left(\overline{m}+1/2\right)*\gamma_2\right)*\rho_{21}\left[t\right] + \overline{n}*\gamma_1*\rho_{43}\left[t\right] , \right. \\
& \rho'_{43}\left[t\right] == -\left(\overline{n}*\gamma_1 + \left(\overline{m}+1/2\right)*\gamma_2\right)*\rho_{43}\left[t\right] + \left(\overline{n}+1\right)*\gamma_1*\rho_{21}\left[t\right] , \\
& \left. \rho_{21}\left[0\right] == \alpha_2*\alpha_1*e^{i\phi_1}, \rho_{43}\left[0\right] == \alpha_4*\alpha_3*e^{\left(i\phi_3-i\phi_2\right)} \right\} ; \\
& DSolve\left[equa\overline{t}ion1, \left\{\rho_{21}\left[t\right], \rho_{43}\left[t\right]\right\}, t\right] \qquad\qquad \text{(B.0.1)}
\end{aligned}
$$

The mathematica code to find the solution of $\dot{\rho}_{42}$ and $\dot{\rho}_{31}$

$$equatio\overline{n}2 \quad = \quad \left\{ \rho'_{31}[t] == -\left((\overline{n}+1/2)*\gamma_1 + (\overline{m}+1)*\gamma_2\right)*\rho_{31}[t] + \overline{m}*\gamma_2*\rho_{42}[t],\right.$$

$$\rho'_{42}[t] == -\left((\overline{n}+1/2)*\gamma_1 + \overline{m}*\gamma_2\right)*\rho_{42}[t] + (\overline{m}+1)*\gamma_2*\rho_{31}[t],$$

$$\rho_{31}[0] == \alpha_3*\alpha_1*e^{i\phi_2}, \rho_{43}[0] == \alpha_4*\alpha_2*e^{(i\phi_3 - i\phi_1)}\Big\};$$

$$DSolve\left[equatio\overline{n}1, \{\rho_{31}[t], \rho_{42}[t]\}, t\right] \qquad\qquad (B.0.2)$$

The mathematica code to find the solution of $\dot{\rho}_{11}$, $\dot{\rho}_{22}$, $\dot{\rho}_{33}$, and $\dot{\rho}_{44}$

$$Eq1 \quad = \quad \rho'_{11}[t] == -\left((\overline{n}+1)*\gamma_1 + (\overline{m}+1)*\gamma_2\right)*\rho_{11}[t] + \overline{m}*\gamma_2*\rho_{22}[t] + \overline{n}*\gamma_1*\rho_{33}[t]; \quad (B.0.3)$$

$$Eq2 \quad = \quad \rho'_{22}[t] == -\left((\overline{n}+1)*\gamma_1 + \overline{m}*\gamma_2\right)*\rho_{22}[t] + (\overline{m}+1)*\gamma_2*\rho_{11}[t] + \overline{n}*\gamma_1*\rho_{44}[t]; \quad (B.0.4)$$

$$Eq3 \quad = \quad \rho'_{33}[t] == -\left(\overline{n}*\gamma_1 + (\overline{m}+1)*\gamma_2\right)*\rho_{33}[t] + (\overline{n}+1)*\gamma_1*\rho_{11}[t] + \overline{m}*\gamma_2*\rho_{44}[t]; \quad (B.0.5)$$

$$Eq4 \quad = \quad \rho'_{44}[t] == -\left(\overline{n}*\gamma_1 + \overline{m}*\gamma_2\right)*\rho_{44}[t] + (\overline{n}+1)*\gamma_1*\rho_{22}[t] + (\overline{m}+1)*\gamma_2*\rho_{33}[t]; \quad (B.0.6)$$

$$DSolve\left[\{Eq1, Eq2, Eq3, Eq4, \rho_{11}[0] == \alpha_1^2, \rho_{22}[0] == \alpha_2^2, \rho_{33}[0] == \alpha_3^2, \rho_{44}[0] == \alpha_4^2\},\right.$$

$$\{\rho_{11}[t], \rho_{22}[t], \rho_{33}[t], \rho_{44}[t]\}, t] \qquad\qquad (B.0.7)$$

The solution of $\dot{\rho}_{14}$, we get

$$\rho'_{14}[t] \quad = \quad -\left[(\overline{n}+1/2)*\gamma_1 + (\overline{m}+1/2)*\gamma_2\right]\rho_{14} \qquad (B.0.8)$$

$$\frac{dp_{14}}{dt} \quad = \quad -\left[(\overline{n}+1/2)*\gamma_1 + (\overline{m}+1/2)*\gamma_2\right]\rho_{14}$$

$$\frac{dp_{14}}{\rho_{14}} \quad = \quad -\left[(\overline{n}+1/2)*\gamma_1 + (\overline{m}+1/2)*\gamma_2\right]dt$$

By taking integration on both sides we get

$$\rho_{14} = e^{-(a+b)t/2}\rho_{14}(0)$$

After applying initial condition, $\rho_{14}(0) = \alpha_1 \alpha_4 e^{i\phi_3}$ we get

$$\rho_{14} = e^{-(a+b)t/2} \alpha_1 \alpha_4 e^{i\phi_3} \tag{B.0.9}$$

where, $a = (2\overline{n} + 1)\gamma_1$, $b = (2\overline{m} + 1)\gamma_2$

Similarly, solution to find $\dot{\rho}_{23}$ we get

$$
\begin{aligned}
\rho'_{23}[t] &= -\left[(\overline{n} + 1/2) * \gamma_1 + (\overline{m} + 1/2) * \gamma_2\right]\rho_{23} \tag{B.0.10} \\
\frac{dp_{23}}{dt} &= -\left[(\overline{n} + 1/2) * \gamma_1 + (\overline{m} + 1/2) * \gamma_2\right]\rho_{23} \\
\frac{dp_{23}}{\rho_{23}} &= -\left[(\overline{n} + 1/2) * \gamma_1 + (\overline{m} + 1/2) * \gamma_2\right] dt
\end{aligned}
$$

By taking integration on both sides we get,

$$\rho_{23} = e^{-(a+b)t/2} \rho_{23}(0)$$

After applying initial condition, $\rho_{23}(0) = \alpha_2 \alpha_3 e^{i\phi_3}$ we get

$$\rho_{23} = e^{-(a+b)t/2} \alpha_2 \alpha_3 e^{i(-\phi_1 + \phi_2)} \tag{B.0.11}$$

Appendix C

Solution of 4×4 matrix by an initial state for a thermal reservoir

For solution of 4×4 matrix we consider $\overline{m} = \overline{n}$, and $\gamma_1 = \gamma_2 = \gamma$, only taking the terms of α_1 and α_4 from the elements of ρ_{ij}.

Entanglement dynamics of the states $|\psi_1(0)\rangle = \alpha_1 |a_1, a_2\rangle + \alpha_4 e^{i\phi_3} |b_1, b_2\rangle$

$$
\rho = \left(\begin{array}{l}
\dfrac{\overline{n}e^{-at}\left(\overline{n}\gamma+\alpha_1^2\right)}{2\overline{n}+1} + \dfrac{\overline{n}e^{-bt}\left(\overline{n}\gamma+\alpha_1^2\right)}{2\overline{n}+1} - \dfrac{\alpha_1^2(2\overline{n}+1)-\overline{n}^2(2(2\overline{n}+1)\gamma)}{(2\overline{n}+1)^2}, \quad 0, \quad 0, \quad \alpha_1\alpha_4 e^{i\phi_3}e^{-(a+b)t/2} \quad , \\[3ex]
0, \quad \dfrac{(\overline{n}+1)e^{-at}\left(\overline{n}\gamma+\alpha_1^2\right)}{2\overline{n}+1} - \dfrac{\overline{n}e^{-bt}\left(\overline{n}\gamma+\alpha_1^2\right)}{2\overline{n}+1} - \dfrac{\alpha_1^2(2\overline{n}+1)-\overline{n}^2(2(2\overline{n}+1)\gamma)}{(2\overline{n}+1)^2}, \quad 0, \quad 0 \quad , \\[3ex]
0, \quad 0, \quad -\dfrac{\overline{n}e^{-at}\left(\overline{n}\gamma+\alpha_1^2\right)}{2\overline{n}+1} + \dfrac{(\overline{n}+1)e^{-bt}\left(\overline{n}\gamma+\alpha_1^2\right)}{2\overline{n}+1} - \dfrac{\alpha_1^2(2\overline{n}+1)-\overline{n}^2(2(2\overline{n}+1)\gamma)}{(2\overline{n}+1)^2}, \quad 0 \quad , \\[3ex]
\alpha_1\alpha_4 e^{-i\phi_3}e^{-(a+b)t/2}, \quad 0, \quad 0, \quad \dfrac{(2\overline{n}+1)-\alpha_1^2+2e^{-2(2\overline{n}+1)\gamma t}\overline{n}^2\gamma-2e^{-(2\overline{n}+1)\gamma t}(1+\overline{n})\left(\alpha_1^2+\overline{n}\gamma\right)}{2\overline{n}+1}
\end{array} \right)
$$

$$
\begin{aligned}
y &= \{\{0, 0, 0, -1\}, \{0, 0, 1, 0\}, \{0, 0, 1, 0\}, \{-1, 0, 0, 0\}\} \, //matrixForm \\
P &= p.y.p^*.y \\
& \quad \text{Ei} genvalues\,[P]
\end{aligned}
\tag{C.0.1}
$$

Appendix D

Solution of 4×4 matrix by an initial state for a thermal reservoir

For solution of 4×4 matrix we consider $\overline{m} = \overline{n}$, and $\gamma_1 = \gamma_2 = \gamma$, only taking the terms of α_2 and α_3 from the elements of ρ_{ij}.

Entanglement dynamics of the states $|\psi_2(0)\rangle = \alpha_2|a_1, b_2\rangle + \alpha_3 e^{i\phi_2}|b_1, b_2\rangle$

$$
\rho \;=\; \left(\begin{array}{l} \dfrac{\overline{n}e^{-at}\left(\overline{n}\gamma+\alpha_2^2\right)}{2\overline{n}+1} + \dfrac{\overline{n}e^{-bt}\left(\overline{n}\gamma+\alpha_3^2\right)}{2\overline{n}+1} - \dfrac{\overline{n}\alpha_2^2(2\overline{n}+1)-\overline{n}(1+2\overline{n})\alpha_3^2-\overline{n}^2(2(2\overline{n}+1)\gamma)}{(2\overline{n}+1)^2}, \;\; 0, \;\; 0, \;\; 0 \;, \\[2ex] 0, \;\; \dfrac{(\overline{n}+1)e^{-at}\left(\overline{n}\gamma+\alpha_2^2\right)}{2\overline{n}+1} - \dfrac{\overline{n}e^{-bt}\left(\overline{n}\gamma+\alpha_3^2\right)}{2\overline{n}+1} - \dfrac{\overline{n}\alpha_2^2(2\overline{n}+1)-\overline{n}(1+2\overline{n})\alpha_3^2-\overline{n}^2(2(2\overline{n}+1)\gamma)}{(2\overline{n}+1)^2}, \;\; \alpha_2\alpha_3 e^{i(-\phi_1+\phi_2)}e^{-(a+b)t/2} \\[2ex] 0, \;\; \alpha_2\alpha_3 e^{-i(-\phi_1+\phi_2)}e^{-(a+b)t/2}, \;\; -\dfrac{\overline{n}e^{-at}\left(\overline{n}\gamma+\alpha_2^2\right)}{2\overline{n}+1} + \dfrac{(\overline{n}+1)e^{-bt}\left(\overline{n}\gamma+\alpha_3^2\right)}{2\overline{n}+1} - \dfrac{\overline{n}\alpha_2^2(2\overline{n}+1)-\overline{n}(1+2\overline{n})\alpha_3^2-\overline{n}^2(2(2\overline{n}+1)}{(2\overline{n}+1)^2} \\[2ex] 0, \;\; 0, \;\; 0, \;\; 1-\left(\rho_{11}+\rho_{22}+\rho_{33}\right) \end{array} \right)
$$

$$\begin{aligned}
y &= \{\{0, 0, 0, -1\}, \{0, 0, 1, 0\}, \{0, 0, 1, 0\}, \{-1, 0, 0, 0\}\}\, //matrixForm \\
P &= p.y.p^*.y\, //matrixForm \\
&\quad \mathrm{Ei}\,genvalues\,[P]
\end{aligned} \qquad (\mathrm{D.0.1})$$

Appendix E

Solution of 4×4 density matrix by an initial state for vacuum reservoir

Solution of 4×4 density matrix for vacuum reservoir where $\gamma_1 = \gamma_2 = \gamma$ and $\phi_1 + \phi_2 - \phi_3 = \Phi$

Entanglement dynamics of state $|\psi(0)\rangle = \alpha_1|a_1, a_2\rangle + \alpha_2 e^{i\phi_1}|a_1, b_2\rangle + \alpha_3 e^{i\phi_2}|b_1, a_2\rangle + \alpha_4 e^{i\phi3}|b_1, b_2\rangle$

$$\rho \;=\; \left\{\left\{ \alpha_1^2 e^{-2\gamma t}, \quad \alpha_1\alpha_2 e^{-3\gamma t/2}, \quad \alpha_1\alpha_3 e^{-3\gamma t/2}, \quad \alpha_1\alpha_4 e^{-\gamma t} \right\}, \right.$$

$$\left\{ \alpha_2\alpha_1 e^{-2\gamma t/2}, \quad e^{-\gamma t}\left(\alpha_1^2 + \alpha_2^2\right) - \alpha_1^2 e^{-2\gamma t}, \right.$$

$$\left. \alpha_2\alpha_3 e^{-\gamma t}, \quad e^{-3\gamma t/2}\left(\left(-1 + e^{\gamma t}\right)\alpha_1\alpha_3\right) + \alpha_2\alpha_4 e^{\gamma t} \right\}, \tag{E.0.1}$$

$$\left\{ \alpha_3\alpha_1 e^{-3\gamma t/2}, \quad \alpha_2\alpha_3 e^{-\gamma t}, \right.$$

$$\left. e^{-\gamma t}\left(\alpha_1^2 + \alpha_3^2\right) - \alpha_1^2 e^{-2\gamma t}, \quad e^{-3\gamma t/2}\left(\left(-1 + e^{\gamma t}\right)\alpha_1\alpha_2 + e^{\gamma t}\alpha_3\alpha_4\right) \right\}, \tag{E.0.2}$$

$$\left\{ \alpha_4\alpha_1 e^{-\gamma t}, \quad e^{-3\gamma t/2}\left(\left(-1 + e^{\gamma t}\right)\alpha_1\alpha_3\right) + \alpha_2\alpha_4 e^{\gamma t}, \right.$$

$$\left. e^{-3\gamma t/2}\left(\left(-1 + e^{\gamma t}\right)\alpha_1\alpha_2 + e^{\gamma t}\alpha_3\alpha_4\right), \quad 1 + e^{-2\gamma t}\alpha_1^2 - e^{-\gamma t}\left(\alpha_1^2 + \alpha_2^2\right) - e^{-\gamma t}\left(\alpha_1^2 + \alpha_3^2\right) \right\} \tag{E.0.3}$$

$$y \;=\; \{\{0,0,0,-1\}, \{0,0,1,0\}, \{0,0,1,0\}, \{-1,0,0,0\}\} \,//matrixFor\overline{m}$$

$$P \;=\; p.y.p^*.y\,//matrixForm$$

$$\mathrm{Ei}\,genvalues\,[P] \tag{E.0.4}$$

Appendix F

Solution of 4×4 density matrix by an initial state for vacuum reservoir

Entanglement dynamics of EPR state for vacuum reservoir $|\psi_1(0)\rangle = \alpha_1|a_1, a_2\rangle + \alpha_4 e^{i\phi_3}|b_1, b_2\rangle$

$$
\begin{aligned}
\rho = \Big\{ &\Big\{ \ \alpha_1^2 e^{-2\gamma t}, \ \ 0, \ \ 0, \ \ \alpha_1\alpha_4 e^{-\gamma t} \ \Big\}, \\
&\Big\{ \ 0, \ \ \alpha_1^2 e^{-\gamma t} - \alpha_1^2 e^{-2\gamma t}, \ \ 0, \ \ 0 \ \Big\}, \\
&\Big\{ \ 0, \ \ 0, \ \ \alpha_1^2 e^{-\gamma t} - \alpha_1^2 e^{-2\gamma t}, \ \ 0 \ \Big\}, \\
&\Big\{ \ \alpha_1\alpha_4 e^{-\gamma t}, \ \ 0, \ \ 0, \ \ 1 + 2e^{-2\gamma t}\alpha_1^2 - \alpha_1^2 e^{-\gamma t} - \alpha_1^2 e^{-\gamma t} \ \Big\} \Big\} \quad \text{(F.0.1)}
\end{aligned}
$$

$$
\begin{aligned}
y &= \{\{0,0,0,-1\}, \{0,0,1,0\}, \{0,0,1,0\}, \{-1,0,0,0\}\} \, //matrixForm \\
P &= p.y.p^*.y \, //matrixForm \\
&\mathrm{Ei}\,genvalues\,[P] \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{(F.0.2)}
\end{aligned}
$$

Appendix G

Solution of 4×4 density matrix by an initial state for vacuum reservoir

Entanglement dynamics of EPR state for vacuum reservoir $|\psi_2(0)\rangle = \alpha_2|a_1, b_2\rangle + \alpha_3 e^{i\phi_2}|b_1, a_2\rangle$

$$
\rho = \left\{ \left\{ 0, \ 0, \ 0, \ 0 \right\}, \right.
$$
$$
\left\{ 0, \ \alpha_2^2 e^{-\gamma t}, \ \alpha_2\alpha_3 e^{-\gamma t}, \ 0 \right\},
$$
$$
\left\{ 0, \ \alpha_2\alpha_3 e^{-\gamma t}, \ \alpha_3^2 e^{\gamma t}, \ 0 \right\},
$$
$$
\left. \left\{ 0, \ 0, \ 0, \ \alpha_2^2 + \alpha_3^2 - \alpha_2^2 e^{-\gamma t} - \alpha_3^2 e^{\gamma t} \right\} \right\} \tag{G.0.1}
$$

$$
y = \{\{0, 0, 0, -1\}, \{0, 0, 1, 0\}, \{0, 0, 1, 0\}, \{-1, 0, 0, 0\}\} // matrixForm
$$
$$
P = p.y.p^*.y // matrixForm
$$
$$
\mathrm{Ei}\, genvalues\,[P] \tag{G.0.2}
$$

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