[illegible]

cont.

Q1. Let $x(t)$ be a solution to $\dot{z} = A(t) \cdot z$ w/ $x(0) = x_0$
Let $y(t)$ be another " " " . w/ $y(0) = y_0$.

Is the following statement true? why?

$x_0 \neq y_0 \rightarrow x(t) \neq y(t)$, $\forall t$

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Is the solution unique?

For linear sys.s of ODEs, the existence & uniqueness theorem:
if $A(t)$ is cont. on an interval I containing t_0
then there exists a unique sol. $z(t)$ s.t. $z(t_0) = z_0$ for each initial cond. z_0.

Here $x(t)$ & $y(t)$ are both sol.s to the same linear ODE $\dot{z}$ w/ different initial conditions ($x_0 \neq y_0$)
by the uniqueness $\rightarrow$ $x(t)$ and $y(t)$ must be different for each t in the I of existance of the sol.s.
So the statement is true.

disc

Q2. Let $x(k)$ be a sol. to $z(k+1) = A(k) \cdot z(k)$ w/ $x(0) = x_0$
Let $y(k)$ " another " " w/ $y(0) = y_0$

Is it true? why?

$x_0 \neq y_0 \longrightarrow x(k) \neq y(k)$, $\forall k$, where k is a nonnegative integer.

Analogous to cont. sys.s in that if $A(k)$ is a sequence of nonsingular matrices, then the discrete sys. has a unique solution for a given initial condition.

So $x_0 \neq y_0$ given, and assuming $A(k)$ is nonsingular for all k
disc. sys. propagate these initial cond.s forward uniquely. So $y(k) \neq x(k)$ & the statement is true.

3. For every sys. $\dot{x} = Ax(t)$, an adjoint sys. is defined by $\dot{p}(t) = -A^T p(t)$
Show that if $\Phi_A(t,\tau)$ is the state-transition matrix for the original system, then the state-transition matrix for the adjoint sys. is $\Phi_A^T(\tau, t)$.

$\rightarrow$ $\Phi_A(t,\tau)$ The state transition matrix satisfies:

1. $\Phi_A(t,\tau)$ is the solution to $\dot{x} = Ax(t)$ w/ the init. cond, $\Phi_A(\tau,\tau) = I$ (identity)
2. For any time t & τ, $\Phi_A(t,\tau)$ maps the state from τ to t.
3. $\Phi_A(t,\tau)$ satisfies $\frac{d}{dt}\Phi_A(t,\tau) = A\,\Phi_A(t,\tau)$

So the adjoint sys: $\dot{p}(t) = -A' p(t)$ — assoc. w/ the transpose of A.
The state trans. matrix for this sys — lets call it $\Phi_{-A^T}(t,\tau)$ must satisfy

$$\frac{d}{dt}\Phi_{-A^T}(t,\tau) = -A^T \Phi_{-A^T}(t,\tau) \text{ w/ } \Phi_{-A^T}(\tau,\tau) = I$$

We must show $\Phi_{-A^T}(t,\tau) = \Phi_A^T(\tau,t)$

$$\Phi_A(t,\tau)\,.\,\Phi_A^{-1}(t,\tau) = I \quad (\text{by def.})$$

$$(\Phi_A(t,\tau)\,.\,\Phi_A^{-1}(t,\tau))^T = I^T$$

$$\Phi_A^{-1}(t,\tau)^T\,.\,\Phi_A(t,\tau)^T = I$$

satisfies the dif. eqn. we can denote it as: $\Phi_{-A^T}(t,\tau)\,.\,\Phi_A(t,\tau)^T = I$

if we use the property: $\Phi_A(\tau,t) = \Phi_A^{-1}(t,\tau)$

$$\Phi_A^T(t,\tau) = \underbrace{\Phi_A^{-1}(\tau,t)^T}_{\text{this is exactly } \Phi_{-A^T}(t,\tau)}$$

So, $\Phi_A^T(\tau,t) = \Phi_{-A^T}(t,\tau)$

and $\Phi_A^T(\tau,t)$ is the state-transition matrix for the adjoint sys.

Q4. Let $A(t) = \begin{bmatrix} 0 & f(t) \\ -f(t) & 0 \end{bmatrix}$ where $f(t): \mathbb{R} \to \mathbb{R}$ is a continuous func.

Let $F(t) = \int_0^t f(\tau)\,d\tau$. Show that the state-trans. matrix ass. w/ $A(t)$ is

$$\Phi(t,0) = \begin{bmatrix} \cos(F(t)) & \sin(F(t)) \\ -\sin(F(t)) & \cos(F(t)) \end{bmatrix}$$

→ The sol. to the sys. $\dot{x} = A(t).x(t)$ can be expressed in terms of the state trans. matrix

$$\frac{d}{dt}\Phi(t,\tau) = A(t)\,.\,\Phi(t,\tau)$$ ← & $\Phi(t,0)$ can be found by solving the dif. eqn.

w/ $\Phi(0,0) = I$

Let's start w/ $\dot{x} = A(t).x(t)$ & solve it using $A(t)$ given

let's assume $\Phi(t,0)$ has the form of a rotation matrix (because $A(t)$ is skew-symmetric)

A general rot. matrix $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, θ: rotation angle

$$\frac{d\theta}{dt} = f(t)$$

$$\theta(t) = F(t)$$

→ Substituting $\theta(t)$ w/ $F(t)$ in the rot. matrix we get the proposed $\Phi(t,0)$

To conform → we must show that $\frac{d}{dt}\Phi(t,0)$ satisfies the matrix dif. equation w/ the initial condition $\Phi(0,0) = I$

So → $$\frac{d}{dt}\Phi(t,0) = \frac{d}{dt}\begin{bmatrix} \cos F(t) & \sin(F(t)) \\ -\sin(F(t)) & \cos(F(t)) \end{bmatrix}$$

$$= \begin{bmatrix} -\sin(F(t)) & \cos(F(t)) \\ -\cos(F(t)) & -\sin(F(t)) \end{bmatrix} \underbrace{\frac{dF(t)}{dt}}_{f(t)}$$

$$= \begin{bmatrix} 0 & f(t) \\ -f(t) & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos(F(t)) & \sin(F(t)) \\ -\sin(F(t)) & \cos(F(t)) \end{bmatrix}$$

$$= A(t) \cdot \Phi(t,0)$$

→ So our proposed $\Phi(t,0)$ satisfies the eqn.
Also @ $t=0$, $F(0) = 0$ & $\Phi(0,0) = I$

* The state-transition matrix is correct.

Q5. Given sys: $$\dot{x} = \underbrace{\begin{bmatrix} -\frac{4}{t} & -\frac{2}{t^2} \\ 1 & 0 \end{bmatrix}}_{A(t)} x$$

a) Is $\phi_1(t) = \begin{bmatrix} \frac{1}{t^2} \\ -\frac{1}{t} \end{bmatrix}$ a solution to this system?

check if it satisfy $\dot{x} = A(t).x$

LHS / So → $$\dot{\phi}_1(t) = \frac{d}{dt}\left(\begin{bmatrix} \frac{1}{t^2} \\ -1/t \end{bmatrix}\right) = \begin{bmatrix} -2/t^3 \\ 1/t^2 \end{bmatrix}$$

RHS / $$A(t) \cdot \phi_1(t) = \begin{bmatrix} -4/t & -2/t^2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/t^2 \\ -1/t \end{bmatrix} = \begin{bmatrix} -2/t^3 \\ 1/t^2 \end{bmatrix}$$

LHS = RHS &
$\dot{\phi}_1(t) = A(t)\,\phi_1(t)$
So $\phi_1(t)$ is a sol. to the sys.

b) Is $\phi_2(t) = \begin{bmatrix} 2/t^3 \\ -1/t^2 \end{bmatrix}$ a sol.?

LHS / $$\dot{\phi}_2(t) = \frac{d}{dt}\left(\begin{bmatrix} 2/t^3 \\ -1/t^2 \end{bmatrix}\right) = \begin{bmatrix} -6/t^4 \\ 2/t^3 \end{bmatrix}$$

RHS / $$A(t)\,\phi_2(t) = \begin{bmatrix} -4/t & -2/t^2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2/t^3 \\ -1/t^2 \end{bmatrix} = \begin{bmatrix} -6/t^4 \\ 2/t^3 \end{bmatrix}$$

LHS = RHS
$\dot{\phi}_2(t) = A(t) \cdot \phi_2(t)$
So $\phi_2(t)$ is a sol. to the sys.

c) Find the state-transition matrix $\Phi(t,\tau)$ for this sys.

$\Phi(t,\tau) \rightarrow$ fund. sol. to $\dot{x} = A(t)x$ & satisfies $\Phi(\tau,\tau) = I$

$$A(t) = \begin{bmatrix} -4/t & -2/t^2 \\ 1 & 0 \end{bmatrix}$$

$$\frac{d\,\Phi(t)}{dt} = A(t) \cdot \Phi(t) \quad \text{w/ the cond. } \Phi(\tau,\tau) = I$$

From a & b we know that

$\phi_1(t) = \begin{bmatrix} 1/t^2 \\ -1/t \end{bmatrix}$ & $\phi_2(t) = \begin{bmatrix} 2/t^3 \\ -1/t^2 \end{bmatrix}$ are two linearly indep. sol.s to the sys.

We can construct $\Phi(t)$ as:

$$* \quad \Phi(t) = \begin{bmatrix} 1/t^2 & 2/t^3 \\ -1/t & -1/t^2 \end{bmatrix} \quad \text{— satisfies } \frac{d}{dt}\Phi(t) = A(t) \cdot \phi(t)$$

(However it doesn't necessarey satisfy $\Phi(\tau,\tau) = I$ for $\tau \neq t$.
To ensure this condition $\Phi(t)$ should be normalized s.t. $\Phi(\tau) = I$.
& for $\Phi(t,\tau)$ we need to solve the matrix IVP from τ to t w/ $\Phi(\tau,\tau) = I$
Requires integrating $A(t)$ from τ to t (may or not be feasible))

d) Find $\phi(t)$ w/ $x(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\phi(t) = \Phi(t,1)\, x(1)$ * I've used MATLAB for approximate calculatio

Q6 Consider the sys : $\dot{x}(t) = t \cdot x(t)$
Is it stable in the sense of Lyapunov?

Lyapunov $\Rightarrow$ for $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\underbrace{|x(0)|}_{\text{(init. cond)}} < \delta$
then for $\forall t > 0$, $|x(t)| < \varepsilon$

$$\dot{x}(t) = t \cdot x(t)$$
$$\frac{d}{dt}x = t \cdot x$$
$$\int \frac{1}{x}\,dx = \int t\,dt$$
$$\ln|x| = \frac{1}{2}t^2 + C \rightarrow x(t) = Ce^{1/2 t^2}$$
, C: integration cons. det. by initial cond. x(0)

as $t \to \infty$
- if $x(0) = 0 \rightarrow C = 0$ & solutions $= 0$ (stable)
- if $x(0) \neq 0 \rightarrow C \neq 0$ & $x(t)$ will grow w/o bound as t increases (bcs $e^{1/2t^2}$ increases rapidly w/ t)

* Therefore, for $\forall \varepsilon > 0$, no matter how small δ, there'll be t s.t. $|x(t)| \geq \varepsilon$ due to the exponential term.

SYS Not stable in terms of Lyapunov

Q7. sys: $\dot{x} = \underbrace{\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u$

$y = \underbrace{[-1 \ \ 2]}_{C} x$

a) Find all eqm. sol.s x_e

x_e is a constant sol. where the sys's state doesn't change over time.

Occurs when the input (u) is s.t.

$$A x_e + B u = 0$$

$$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} x_e + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_{e_1} + u = 0 \longrightarrow x_{e_1} = \frac{u}{2}$

$0\, x_{e_2} + u = 0 \longrightarrow$ this state is free

$\hookrightarrow u = 0$

$w/ \, u=0 \rightarrow A x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2x_{e_1} = 0 \rightarrow x_{e_1} = 0$

$\hookrightarrow x_{e_2}$ (any)

$\rightarrow x_e = \begin{bmatrix} 0 \\ x_{e_2} \end{bmatrix}$, where x_2 can be any real.

b) For x_e, determine whether it's asymptotically stable.

$\hookrightarrow$ x(t) remains close to x_e for all initial conditions close to x_e for $\forall t$ (Lyapunov)

$\hookrightarrow$ x(t) converges to x_e as $t \to \infty$ for any init. cond

Since $u=0$ @ eqm.

eigenvalues of $A = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$

$\lambda_1 = -2$

$\lambda_2 = 0 \longrightarrow$ not asymptotically stable

(for asy. stability: all eigenvalues must have negative real parts)

So $x_e = \begin{bmatrix} 0 \\ x_{e_2} \end{bmatrix}$ are not asymptotically stable.

c) For x_e determine whether it's stable in the sence of Lyapunov.

the eigenvalues are

$\lambda_1 = -2 \rightarrow$ any perturbation in the x_1 dir. will decay back to the eqm

$\lambda_2 = 0 \rightarrow$ perturbations in the x_2 dir will not grow, but they'll also not decay

So $x_e = \begin{bmatrix} 0 \\ x_{e_2} \end{bmatrix}$ are Lyapunov stable.

d) Is the sys. BIBO stable?

transfer func. $H(s) \rightarrow$ $Y(s) = H(s)U(s)$

$$H(s) = C(sI - A)^{-1}B$$

$$= [-1 \ \ 2] \begin{bmatrix} s+2 & 0 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= [-1 \ \ 2] \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$H(s) = -\frac{1}{s+2} + \frac{2}{s}$$

poles @ $s = -2$ — in the left-half plane does not effect BIBO stability

$s = 0$ on the imaginary axis - indicates a marginally stable sys. in the con of internal stability.

For BIBO → a pole @ the origin is permissible if it corresponds to a const mode in the sys's response

Therefore, the sys is BIBO stable

(as long as the pole @ the origin does not correspond to an integrator)

e) Is the sys. Controllable?

controllability matrix: $C = [B, AB, A^2B, \ldots, A^{n-1}B]$ n: # of states, $n = 2$

$A = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$ $AB = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$C = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} \rightarrow$ C has 2 rank, The sys. is controllable.

f) Find $B \neq 0$ s.t. the sys. becomes uncontrollable.

find B s.t. C won't have full rank

One way to guarantee → select B to be in the N(A), bcz any vector v in N(A), $Av = 0$

AB would be a zero vector → lin. dep. w/ B.

$A = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow$ since 2nd row is 0, any vector w/ a zero in the first component will be in N(A)

* $B = \begin{bmatrix} 0 \\ b \end{bmatrix} \rightarrow$ Therefore this'll make the sys. uncontrollable

let's compute $AB = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \rightarrow$ uncontrollable.