ME 564 - HW1

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Q1. Linear i-o sys:

$$\Sigma: \quad \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b u(t)$$

where $y(0)=0$, $\dot{y}(0)=0$, $a_1, a_0, b \in \mathbb{R}$. For $i=1,2$, let $\phi_i(t)$ be the sol. of Σ corresponding to $u(t)=u_i(t)$, for $t \in [0,T]$

a) Show that for every pair of $\alpha_1, \alpha_2 \in \mathbb{R}$, $\phi_3(t) = \alpha_1 \phi_1(t) + \alpha_2 \phi_2(t)$ is a sln. corr. to $u_3(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$

for $\phi_1(t)$ and $u_1(t) \rightarrow$ $\phi_1(t)$ is the sol. cor. to $u(t) = u_1(t)$, $\phi_1(0)=0$, $\dot{\phi}_1(0)=0$

(1) so $\ddot{\phi}_1(t) + a_1 \dot{\phi}_1(t) + a_0 \phi_1(t) = b \alpha_1 u_1(t)$

for $\phi_2(t)$ and $u_2(t) \rightarrow u(t) = u_2(t)$, $\phi_2(0) = 0$, $\dot{\phi}_2(0) = 0$,

(2) $\ddot{\phi}_2(t) + a_1 \dot{\phi}_2(t) + a_0 \phi_2(t) = b \alpha_2 u_2(t)$

→ now we need to solve that $\phi_3(t) = \alpha_1 \phi_1(t) + \alpha_2 \phi_2(t)$ is a sol. cor. to $u_3(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$

plug $\phi_3(t)$ & $u_3(t)$ into Σ:

$$\ddot{\phi}_3(t) + a_1 \dot{\phi}_3(t) + a_0 \phi_3(t) = b u_3(t)$$

substitute ϕ_3 & u_3

(3) $(\alpha_1 \ddot{\phi}_1(t) + \alpha_2 \ddot{\phi}_2(t)) + a_1(\alpha_1 \dot{\phi}_1(t) + \alpha_2 \dot{\phi}_2(t)) + a_0(\alpha_1 \phi_1(t) + \alpha_2 \phi_2(t)) = b(\alpha_1 u_1(t) + \alpha_2 u_2(t))$

Now, comparing LHS and RHS expressions, we see they're equal.
Therefore given $\phi_3(t)$ is a sol. corr. to $u_3(t)$.

b) Show that the set of sol.s to Σ over the time interval $[0,T]$ forms a vector space over the reals.

It should satisfy vector space axioms.

- part a showed that linear comb. of sol.s to Σ are also sol.s to Σ. (closure under addition)
- part a also showed that if $\phi_1(t)$ is a sol. to Σ, then $\alpha \phi_1(t)$ is also a sol. to Σ for any $\alpha \in \mathbb{R}$ (closure under scalar multiplication)
- zero vector in this context is the solution where $y(t)=0$ for all t. This is a sol. to Σ as it satisfies the dif. eqn. and the initial conditions. (contains zero vector)

Since the set of solutions is closed under addition, scalar multiplication, and contains the zero vector it forms a vector space over reals.

Q2. Consider the set of polynomials in s of degree k or less, w/ real coefficients,

a) Show that this is a vector space over the field of real numbers. → I will use it as polynomials in short

b) what's the dimension of this vector space?

a) We need to demonstrate that it satisfies the vector space axioms.

✓ closure under addition : If we take two polynomials and add them, the result'll still be a polynomial of degree k or less.

✓ closure under scalar multip: If you multiply a poly. by a real #, result'll be still a polynom.

✓ commutative property of addition & associative : Addition of poly.s is commutative and associative (a property inherited from real #s)

✓ existence of a zero vector : the zero vector in this vector space is the zero polynomial. (all coef. equal to zero)

✓ existence of additive inverses : For any poly. p(s) its add. inverse is −p(s) which is also a poly. of degree k or less w/ real coef.

✓ scalar multip. distributive over vector addition & scalar addition : yes, polynomial operations satisfy these.

↳ So, the set of polynomials in s of degree k or less w/ real coefficients is a vector space over the field of real numbers.

b) for dimension → max # of linearly independent polynomials that can be formed within this space.

a poly. of degree k or less → $p(x) = \underbrace{a_0 + a_1 x + \dots + a_k x^k}$

there are k+1 coefficients $(a_0, a_1, \dots, a_k)$ and we can consider k+1 linearly independent polynomials.

$(1, x, x^2, \dots, x^k)$; these k+1 polynomials are lin. indep. because no linear comb. of them results in zero polynomial, (each has different degree, and coef. are distinct)

So the dimension = $\underline{k+1}$

Q3. Consider the set of poly.s in s of degree less than or equal to four, w/ $\mathbb{R}$ coefficients, w/the conditions :

$$\Sigma: \quad p'(0) + p(0) = 0 \quad (1)$$
$$p(1) = 0 \quad (2)$$

a) Show that this is a vector space.

It needs to satisfy the vector space axioms.

1) $P(s)$, $Q(s)$: assume two poly.s of degree ≤ 4 satisfies given cond.s

$P(s) + Q(s)$ ∵ degree ≤ 4

put in sys Σ :

(1) $(P(s) + Q(s))'(0) + (P(s) + Q(s))(0) = 0$

(2) $(P(s) + Q(s))(1) = 0$

$$P'(0) + Q'(0) + P(0) + Q(0) = \underbrace{(P'(0) + P(0))}_{0} + \underbrace{(Q'(0) + Q(0))}_{0} = 0$$

(2) $(P(s) + Q(s))(1) = 0$

$$\underbrace{P(1)}_{0} + \underbrace{Q(1)}_{0} = 0$$

So,
* set is closed under addition.

2) $P(s)$, α : satisfies the conditions.

(1) $(\alpha P(s))'(0) + (\alpha P(s))(0) = 0$

$$\alpha P'(0) + \alpha P(0) = 0$$
$$\alpha \underbrace{(P'(0) + P(0))}_{0} = 0$$

— satisfies

(2) $(\alpha P(s))(1) = 0$

$$\alpha \underbrace{P(1)}_{0} = 0$$

→ satisfies

* So, set is closed under scalar multip.

3) Identity element : zero polynomial

4) $P(s) \longrightarrow -P(s)$ inverse exists

additive inverse

5) all other axioms are hold for poly.s

* so, it's a vector space

b) Find a basis for this vector space.

→ we need to determine a set of linearly indep. pol.s to span the space. They should satisfy (1) & (2).

(2) $p(1)=0 \rightarrow (s-1)$ as a factor

(1) $p'(0)+p(0)=0 \rightarrow s$ as a factor

$\searrow (s-1)^2$

$s(s-1)^2(s-1)$
$s(s-1)^2$
$s(s-1)$
s

lin. indep. & span.
dim = 4

c) what's the dim. of this VS?

4

Q4. Consider the following two bases for $\mathbb{R}^2$:

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$B_2 = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

a) let $v = \begin{bmatrix} 20 \\ 12 \end{bmatrix}$ in the standard basis. Find the coordinates for v in bases B_1 & B_2.

we need to express v as a lin. comb. of the basis vectors in each of these bases. The coordinates in B_1 & B_2 → coefficients.

for B_1: $a \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 20 \\ 12 \end{bmatrix} \rightarrow$ $a+b=20$, $a-b=12$

$a = 16$, $b = 4$) coordinates

for B_2: $c \begin{bmatrix} 2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 20 \\ 12 \end{bmatrix} \rightarrow$ $2c+d=20$, $c-2d=12$

$c = \frac{52}{5}$

$d = -\frac{4}{5}$

b) Construct a matrix that would be used to convert a vector space expressed in basis B_2 to a vector expressed in B_1.

We need to find the change of matrix from $B_2 \rightarrow B_1$. ($P_{B_2 \rightarrow B_1}$)

$P_{B_2 \rightarrow B_1} = \Big[$ coordinates of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in B_2 coord. of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in $B_2 \Big]$

basis vectors of B_2: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ → find their coord. in B_1. let's say x & y. a & b

for $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$: $x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow x = \frac{3}{2}, y = \frac{1}{2}$

$x+y=2$
$x-y=1$

for $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$: $a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightarrow a = -\frac{1}{2}, b = \frac{3}{2}$

$a+b=1$
$a-b=-2$

$$P_{B_2 \rightarrow B_1} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

c) Verify that the change of basis matrix you constructed above can be used to convert the representation of v between ~~bases~~.

$$\underbrace{v = \begin{bmatrix} 20 \\ 12 \end{bmatrix}}_{\text{in the standard basis}} \quad , \quad P_{B_2 \to B_1} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$$v_{B_2} = P_{B_2 \to B_1} \cdot v$$

$$v_{B_2} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 20 \\ 12 \end{bmatrix} = \begin{bmatrix} 24 \\ 28 \end{bmatrix} \rightarrow v \text{ in basis } B_2$$

$$v_{B_1} = P_{B_2 \to B_1} \cdot v_{B_2}$$

$$= \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 24 \\ 28 \end{bmatrix} = \begin{bmatrix} 22 \\ 54 \end{bmatrix} \rightarrow v \text{ in basis } B_1$$

* So $P_{B_2 \to B_1}$ can be used to convert vector v from basis $B_2 \to$ basis B_1.

Q5 Linearly dependent or independent?

a) $\left\{ \begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^3$ over $\mathbb{R}$

assume $c_1, c_2, c_3 \in \mathbb{R}$, not all zero.

$$c_1 \begin{bmatrix} -1 \\ -9 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -c_1 + c_2 + 2\cancel{c_3}^{0} = 0 &\rightarrow c_1 = c_2 \\ -9c_1 + 3c_2 - 2\cancel{c_3}^{0} = 0 &\rightarrow c_1 = 0 \\ \underline{c_3 = 0} \quad & \quad \underline{c_2 = 0} \end{aligned}\right\} \text{ vectors are } \underline{\text{linearly independent.}}$$

b) $\left\{ \begin{bmatrix} 4 \\ -9 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 13 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^3$ over $\mathbb{R}$

assume $c_1, c_2, c_3 \in \mathbb{R}$, not all zero.

$$c_1 \begin{bmatrix} 4 \\ -9 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 13 \\ 10 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2/ \quad & 4c_1 + 2c_2 + 2c_3 = 0 \quad (1) \\ & -9c_1 + 13c_2 - 4c_3 = 0 \quad (2) \\ -4/ \quad & c_1 + 10c_2 + c_3 = 0 \quad (3) \end{aligned}$$

$$\left.\begin{aligned} (1) - 4\times(3) &: \ -38c_2 - 2c_3 = 0 \rightarrow c_3 = -19c_2 \\ (2) + 2\times(1) &: \ -c_1 + 17c_2 = 0 \rightarrow c_1 = 17c_2 \end{aligned}\right)$$

c_1, c_2, c_3 are linearly related. So, the set of vectors $\underline{\text{linearly dependent.}}$

c) $\left\{ \begin{bmatrix} 2-j \\ -j \end{bmatrix}, \begin{bmatrix} 1+2j \\ -j \end{bmatrix}, \begin{bmatrix} -j \\ 3+4j \end{bmatrix} \right\}$ in $\mathbb{C}^2$ over $\mathbb{R}$.

$$c_1 \begin{bmatrix} 2-j \\ -j \end{bmatrix} + c_2 \begin{bmatrix} 1+2j \\ -j \end{bmatrix} + c_3 \begin{bmatrix} -j \\ 3+4j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(Re + Im)

$\hookrightarrow (2c_1 + c_2) + (-c_1 + 2c_2 - c_3)j = 0$

$\underbrace{(3c_3)}_{c_3 = 0} + (-c_1 - c_2 + 4c_3)j = 0$

$\hookrightarrow$ Re: $2c_1 + c_2 = 0$

Im: $-c_1 + 2c_2 = 0 \rightarrow c_1 = 2c_2$

$-c_1 - c_2 = 0 \rightarrow c_1 = -c_2$

$c_3 = 0$

These eqn.s are inconsistent unless $c_1 = 0$ and $c_2 = 0$.
There's no non-trivial sln for c_1, c_2, c_3

$\hookrightarrow$ sys is linearly independent.

d) $\left\{ \begin{bmatrix} 1+j \\ 2+3j \end{bmatrix}, \begin{bmatrix} 10+2j \\ 4-j \end{bmatrix}, \begin{bmatrix} -j \\ 3 \end{bmatrix} \right\}$ in $(\mathbb{C}^2, \mathbb{R})$

$$c_1 \begin{bmatrix} 1+j \\ 2+3j \end{bmatrix} + c_2 \begin{bmatrix} 10+2j \\ 4-j \end{bmatrix} + c_3 \begin{bmatrix} -j \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(Re + Im)

$(c_1 + 10c_2) + (c_1 + 2c_2 - c_3)j = 0$

$(2c_1 + 4c_2 + 3c_3) + (3c_1 - c_2)j = 0$

$\hookrightarrow c_2 = 3c_1$

$\hookrightarrow c_2 = -\frac{c_1}{10}$

eqn.s are inconsistent unless $c_1 = c_2 = c_3 = 0$

$\hookrightarrow$ sys is linearly independent

e) $\{ 2s^2+2s-1, -2s^2+2s+1, s^2-s-5 \}$ in the space of poly. over $\mathbb{R}$.

$$c_1(2s^2+2s-1) + c_2(-2s^2+2s+1) + c_3(s^2-s-5) = 0$$

$$(2c_1-2c_2+c_3)s^2 + (2c_1+2c_2-c_3)s + (-c_1+c_2-5c_3) = 0$$

(1) $2c_1 - 2c_2 + c_3 = 0$

(2) $2c_1 + 2c_2 - c_3 = 0$

(3) $-c_1 + c_2 - 5c_3 = 0$

(1)+(2) $\rightarrow 4c_1 = 0 \rightarrow c_1 = 0$

(1) $\rightarrow c_3 = 2c_2$

(3) $\rightarrow -9c_2 = 0 \rightarrow c_2 = 0,\ c_3 = 0$

all constants are zero.

<u>sys. is linearly independent</u>

f) $\left\{ \frac{1}{s+1}, \frac{1}{s+3}, \frac{s+2}{2} \right\}$ in $(\mathbb{R}(s), \mathbb{R})$

$$c_1\left(\frac{1}{s+1}\right) + c_2\left(\frac{1}{s+3}\right) + c_3\left(\frac{s+2}{2}\right) = 0$$

$(2)(s+3)$ $(2)(s+1)$ $(s+1)(s+3)$

$$2c_1(s+3) + 2c_2(s+1) + c_3\underbrace{(s+2)(s+1)(s+3)}_{3^{rd}\text{ order}} = 0$$

$\rightarrow c_3 = 0$

$$\underbrace{(2c_1+2c_2)}_{\rightarrow c_1 = -c_2} s + \underbrace{(6c_1+2c_2)}_{\rightarrow -4c_2 = 0} = 0$$

$c_2 = 0$

$c_1 = 0$

linearly independent.

9) $\{e^{-t}, te^{-t}, e^{-2t}\}$ in $\mathcal{F}(U, \mathbb{R})$ where U is the set of real-valued cont. functs defined on $[0, \infty)$

(1) $c_1 e^{-t} + c_2 te^{-t} + c_3 e^{-2t} = 0$ t on the interval $[0, \infty)$

if $t \to \infty$: $e^{-t} \to 0$

$e^{-2t} \to 0$

$te^{-t} \to 0$ (t is dominated by e^{-t})

So equation (1) $\to 0$: therefore constants c_1, c_2, c_3 must equal to 0 for all t in $[0, \infty)$

$c_1 = c_2 = c_3 = 0$

So, linearly independent.

if $t = 0$: $c_1 e^0 + c_2 \cdot 0 \cdot e^0 + c_3 \cdot e^0 = 0$

$c_1 + c_3 = 0$

$c_1 = -c_3$

cannot hold for all values of t in $[0, \infty)$ → will be violated @ $t=1$ for instance