```
% HW 8
% Q1

clc
close all

% a
% Define the matrices A and B
A = [1 2 1; 0 4 3; 0 0 2];
B = [1; 1; 0];

% Number of states (rows of A)
n = size(A, 1);

% Initialize the controllability matrix C
C = B;

% Construct the controllability matrix
for i = 1:n-1
    C = [C, A^i * B];
end

% Check if the system is controllable by verifying the rank of C
isControllable = rank(C) == n;

% Display result
if isControllable
    disp('The system is controllable.');
else
    disp('The system is not controllable.');
end

% b
% Extract the first two columns of C, which span the controllable subspace
V = C(:, 1:2);

% Initialize the orthonormal basis vectors
orthonormalBasis = zeros(size(V));

% First orthonormal vector (normalized first vector of V)
orthonormalBasis(:, 1) = V(:, 1) / norm(V(:, 1));

% Second vector orthogonalized against the first
v2_orthogonal = V(:, 2) - proj(V(:, 2), orthonormalBasis(:, 1));

% Normalizing the second orthogonal vector
orthonormalBasis(:, 2) = v2_orthogonal / norm(v2_orthogonal);

% Display the orthonormal basis
disp('Orthonormal basis for the controllable subspace:');
disp(orthonormalBasis);
```

```
% c and d
% Compute e^(At) and integral_part (common for both parts c and d)
t = 1; % Time of transition
eAt = expm(A*t);
integral_part = integral(@(tau) expm(A*(t-tau)) * B, 0, t, 'ArrayValued',
 true);

% c
% Define the initial and final states for part c
x0_c = [1; 1; 1];
xf_c = [0; 0; 0];

% Solve for u(t) for part c
u_t_c = integral_part \ (xf_c - eAt*x0_c);

% Display the input for part c
disp('Input u(t) that drives the state from x(0) to x(1) for part c:');
disp(u_t_c);

% d
% Define the initial and final states for part d
x0_d = [0; 0; 0];
xf_d = [1; 1; 0];

% Solve for u(t) for part d
u_t_d = integral_part \ (xf_d - eAt*x0_d);

% Display the input for part d
disp('Input u(t) that drives the state from x(0) to x(1) for part d:');
disp(u_t_d);

% Function to calculate the projection of v onto u
function p = proj(v, u)
    p = (dot(v, u) / dot(u, u)) * u;
end
```

```
The system is not controllable.
Orthonormal basis for the controllable subspace:
    0.7071   -0.7071
    0.7071    0.7071
         0         0

Input u(t) that drives the state from x(0) to x(1) for part c:
   -9.0380

Input u(t) that drives the state from x(0) to x(1) for part d:
    0.0849
```

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```
% HW 8
% Q2

clc
close all

% a
% Define the matrices A and B
A = [-1 0 -1; 0 -3 1; 0 0 -2];
B = [1; 0; 2];

% Number of states (rows of A)
n = size(A, 1);

% Initialize the controllability matrix C
C = B;

% Construct the controllability matrix
for i = 1:n-1
    C = [C, A^i * B];
end

% Check if the system is controllable by verifying the rank of C
isControllable = rank(C) == n;

% Display result
if isControllable
    disp('The system is controllable.');
else
    disp('The system is not controllable.');
end

% b
% Extract the columns of C that span the controllable subspace
V = C(:, 1:rank(C));

% Initialize the orthonormal basis vectors
orthonormalBasis = zeros(size(V));

% First orthonormal vector (normalized first vector of V)
orthonormalBasis(:, 1) = V(:, 1) / norm(V(:, 1));

% Second vector orthogonalized against the first and normalized
v2_orthogonal = V(:, 2) - proj(V(:, 2), orthonormalBasis(:, 1));
orthonormalBasis(:, 2) = v2_orthogonal / norm(v2_orthogonal);

% Third vector orthogonalized against the first two and normalized
v3_orthogonal = V(:, 3) - proj(V(:, 3), orthonormalBasis(:, 1)) - proj(V(:,
 3), orthonormalBasis(:, 2));
orthonormalBasis(:, 3) = v3_orthogonal / norm(v3_orthogonal);

% Display the orthonormal basis
```

```
disp('Orthonormal basis for the controllable subspace:');
disp(orthonormalBasis);

% c
% Define the time interval
t0 = 0;
t1 = 1;

% Calculate the controllability Gramian
Wc = integral(@(tau) expm(A*(t1-tau))*B*B'*expm(A'*(t1-tau)), t0,
 t1, 'ArrayValued', true);

% Display the controllability Gramian
disp('Controllability Gramian from t = 0 to t = 1:');
disp(Wc);

% d
% Define the initial and final states
x0 = [1; 1; 1];
xf = [0; 0; 0];

% Time of transition
t = 1;

% Compute e^(At)
eAt = expm(A*t);

% Solving for the integral part (as in previous part)
integral_part = integral(@(tau) expm(A*(t-tau)) * B, 0, t, 'ArrayValued',
 true);

% Solve for u(t)
u_t = integral_part \ (xf - eAt*x0);

% Display the input
disp('Input u(t) that drives the state from x(0) to x(1):');
disp(u_t);

% e
% Define Q as B*B'
Q = B * B';

% Solve the Lyapunov equation A'P + PA + Q = 0 for P
P = lyap(A, Q);

% Display the P matrix
disp('Matrix P that solves the Lyapunov equation:');
disp(P);

% f
% Compute the infinite-time controllability Gramian (which should be equal to
 P)
% AW + WA' + BBt = 0
Wc_infinite = lyap(A, B*B');
```

```
% Display the infinite-time controllability Gramian
disp('Controllability Gramian from t = 0 to t = infinity:');
disp(Wc_infinite);


% Function to calculate the projection of v onto u
function p = proj(v, u)
    p = (dot(v, u) / dot(u, u)) * u;
end
```

```
The system is controllable.
Orthonormal basis for the controllable subspace:
    0.4472   -0.3651   -0.8165
         0    0.9129   -0.4082
    0.8944    0.1826    0.4082

Controllability Gramian from t = 0 to t = 1:
    0.1471    0.0444    0.3482
    0.0444    0.0575    0.1871
    0.3482    0.1871    0.9817

Input u(t) that drives the state from x(0) to x(1):
   -0.2102

Matrix P that solves the Lyapunov equation:
    0.1667    0.0333    0.3333
    0.0333    0.0667    0.2000
    0.3333    0.2000    1.0000

Controllability Gramian from t = 0 to t = infinity:
    0.1667    0.0333    0.3333
    0.0333    0.0667    0.2000
    0.3333    0.2000    1.0000
```

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Q3 DT sys w/ $x_0 = x(t_0)$ $n=3$

$$x(k+1) = Ax(k) + Bu(k), \quad A \in \mathbb{R}^{3\times3}, \; B \in \mathbb{R}^3$$

* sys is controllable. Given some desired state x_f, we want to find a sequence of inputs $u(0), u(1), \ldots$ that drives the sys to x_f @ some time $k = k_f$

a) for general A, B, x_0 & x_f — fastest that you can drive the sys. to x_f? What's the smallest possible value of k_f?

$$C = [B \;\; AB \;\; A^2B]^{k_f - 1}$$ since the sys. is controllable $\rho(C) = 3$

So the smallest value of $k_f = 3$.

b) Sequence of inputs for shortest time?

$k_f = 3$ $\quad C\begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_0 \end{bmatrix} = -A^k x_0$

$$[B \;\; AB \;\; A^2B]\begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = x_f - Ax(0)$$

represents state change

$$\begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = [B \;\; AB \;\; A^2B]^{-1}(x_f - Ax(0))$$

general A, B

c) do there exist specific values of x_0 and x_f, fewer steps than (a)

Yes, example: $Ax_0 = x_f$ — no input needed & the sys. transitions to x_f in one step.

d) general x_0, x_f — specific A, fewer steps than (a).

Yes. Certain configurations of A can facilitate faster transitions. Especially ex: $Ax_0 = x_f$ - one step true if A has certain props align sys's dyn. favorably w/ desired

e) Assume $B^{3\times2}$ → the sys's controllability must be re-evaluated.

$C = [B \;\; AB \;\; A^2B \;\; A^{n-1}B]$ → w/ $B \in \mathbb{R}^{3\times2}$ sys has 2 inputs @ each step

Yes, there can be. 2 inputs can increase control capacity.

For instance: A has evals that significantly amplify certain state vars. & B may allow direct influence on these vars.

Q4 The sys: $\dot{x} = Ax + Bu$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$.

If there exists a vector $v \in \mathbb{R}^n$ s.t. $v^TA = \lambda v^T$, $v^TB = 0$, then the sys. is not controllable.

$C = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$; the sys is controllable iff. $\rho(C) = n$

(1) – $v^TB = 0 \rightarrow v^T \perp$ 1st col. of C

(2) – $v^TA = \lambda v^T \rightarrow$ more general: $v^TA^k = \lambda^k v^T$ for $k = 1, \dots, (n-1)$

(1) & (2) : $v^TA^k B = \lambda^k v^T B = 0$ for $k = 1, \dots, (n-1)$

So if we multiply v^T w/ any col. block of C we'll get zero vector.

The existence of v implies C does not have full rank.

So if v exists (lock) then the sys is not controllable.

Q5 $\dot{x} = Ax + Bu$ (1), ~~A~~ Define a new state w s.t. $w = Q^{-1}x$ (2) for some invertible matrix Q

a) rewrite the sys: $\dot{w} = \tilde{A}w + \tilde{B}u$. Find $\tilde{A}, \tilde{B}$ i.t.o. A, B, Q.

$w = Q^{-1}x$

$\dot{w} = \frac{d}{dt}(Q^{-1}x) \overset{TI}{=} Q^{-1}\dot{x}$

$\overset{(1)}{=} Q^{-1}(Ax + Bu)$

(2) $\rightarrow$ $x = Qw$

$\dot{w} = Q^{-1}(AQw + Bu) = \underbrace{Q^{-1}AQ}_{\tilde{A}} w + \underbrace{Q^{-1}B}_{\tilde{B}} u$

b) Show original sys controllable iff second sys. controllable

original sys: Cont. matrix : $C = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$ — sys is controllable iff $\text{rank}(C) = n$

2nd sys " : $\tilde{C} = [\tilde{B} \ \tilde{A}\tilde{B} \ \tilde{A}^2\tilde{B} \ \dots \ \tilde{A}^{n-1}\tilde{B}]$ — substitute $\tilde{A}$ & $\tilde{B}$

$= [Q^{-1}B, \ Q^{-1}A\underbrace{QQ^{-1}}_{I}B, \ Q^{-1}A\underbrace{QQ^{-1}}_{I}A\underbrace{QQ^{-1}}_{I}B, \ \dots \ ]$

$= [Q^{-1}B, \ Q^{-1}AB, \ Q^{-1}A^2B \ \dots \ ]$

$\tilde{C} = Q^{-1}[B, AB, A^2B, \dots \ ] = Q^{-1}C$

$\tilde{C} = Q^{-1}C$

$\boxed{\text{invertible}} \rightarrow \text{rank}(\tilde{C}) = \text{rank}(C)$ — so statement is true.

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