Rabia Konuk

Q1

Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined as $\underline{\underline{Ax_i = y_i}}$ for $i = 1, 2, 3$ and x_i, y_i defined below:

$$\{x_1, x_2, x_3\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

$$\{y_1, y_2, y_3\} = \left\{ \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \right\}$$

this is a basis for both domain & codomain

a) Find the matrix rep. of A using the standard basis for $\mathbb{R}^3$.

Hint: try writing the standard basis in terms of x_i.

Standard basis for $\mathbb{R}^3$: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

e_1 in terms of x_i: $e_1 = a_1 x_1 + a_2 x_2 + a_3 x_3 = x_3 - x_1$

e_2 " " ": $e_2 = b_1 x_1 + b_2 x_2 + b_3 x_3 = 2x_1 - x_2$

e_3 " " ": $e_3 = c_1 x_1 + c_2 x_2 + c_3 x_3 = x_1$

Since $Ax_i = y_i$:

$$Ae_1 = A(x_3 - x_1) = Ax_3 - Ax_1 = y_3 - y_1 = \begin{bmatrix} -1 \\ +2 \\ -1 \end{bmatrix}$$

$$Ae_2 = 2y_1 - y_2 = \begin{bmatrix} -5 \\ 7 \\ 2 \end{bmatrix}$$

$$Ae_3 = y_1 = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -5 & -2 \\ +2 & 7 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

b) Find the matrix representation of A in the $X = \{x_1, x_2, x_3\}$ basis.

$$A_X = X^{-1} . A . X$$

$$A = \begin{bmatrix} -1 & -5 & -2 \\ 2 & 7 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$ → I used online solver for the inverse.

$$X^{-1} = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_X = \begin{bmatrix} 9 & 2 & 5 \\ -2 & -2 & -3 \\ -2 & 1 & 0 \end{bmatrix}$$

Q2. Let $A: V \to U$ be a linear operator. Let $\mathcal{O}_V$ denote the zero vector in V, (i.e. additive identity)
Prove that: (1) A is one-to-one $\overset{\text{iff}}{\iff}$ $N(A) = \mathcal{O}_V$ (2)
by doing the following steps:
a) ($\rightarrow$) prove that if (1) holds, then (2) must be true.
b) ($\leftarrow$) prove that if (2) holds, then (1) must be true.

a) Assume A is one-to-one, v_1 & $v_2 \in V$ s.t. $A(v_1) = A(v_2)$

Since A is one-to-one $v_1 = v_2$

consider $v_3 \in N(A)$. by def. of the nullspace $A(v_3) = \mathcal{O}_U$ → any linear operator sends 0 vector to 0 vector (domain → codomain)

$A(v_3) = A(\mathcal{O}_V)$

$v_3 = \mathcal{O}_V$

So, the only vector in $N(A)$ is $\mathcal{O}_V$ → $N(A) = \mathcal{O}_V$.

b) Assume $N(A) = \mathcal{O}_V$, v_1 & v_2 & $v_3 \in V$ s.t. $A(v_1) = A(v_2)$.

let's say: $v_3 = v_1 - v_2 \rightarrow A(v_3) = A(v_1) - A(v_2) = \mathcal{O}_U$

This means v_3 is in the nullspace of A. For our assumption nullspace contains $\mathcal{O}_V$ only.

$v_3 = \mathcal{O}_V$ ← So ←

which implies (• $A(v_1) = A(v_2)$ • $v_3 = v_1 - v_2 = 0$) Therefore A is one-to-one.

Q3 For $x, y, z \in \mathbb{R}$, consider the lin. operator $\underbrace{A(x,y,z)}_{\mathbb{R}^3} = \underbrace{\begin{bmatrix} x+2y \\ x+z \\ 2x+4y \\ 2x+2y+z \end{bmatrix}}_{\mathbb{R}^4}$

a) Find the matrix rep. of A wrt the standard bases. Hint: Think about domain & co-dom

$$A : \underbrace{\mathbb{R}^3}_{\text{A takes 3 components}} \rightarrow \underbrace{\mathbb{R}^4}_{\text{4 comp.}}$$

The standard basis for $\mathbb{R}^4$: $e_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, e_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, e_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

$\mathbb{R}^3$: $\bar{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \bar{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \bar{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

So, $A(\bar{e}_1) \Rightarrow$ 1st col. of matrix $\rightarrow A(1,0,0) = \begin{bmatrix}1\\1\\2\\2\end{bmatrix}$

$A(\bar{e}_2) \Rightarrow$ 2nd " "

$A(\bar{e}_3) \Rightarrow$ 3rd " "

$A(0,1,0) = \begin{bmatrix}2\\0\\4\\2\end{bmatrix}$

$A(0,0,1) = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$

So matrix rep of $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 4 & 0 \\ 2 & 2 & 1 \end{bmatrix}$

b) Let A be the matrix you found from a. If A is a real matrix, then the adjoint of A, A^*, is the transpose of A. Find A^*

So, $A^* = A^T$

$$A^* = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 0 & 4 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

c) Find bases for N(A) and R(A).

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 4 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

for N(A): find a basis first. For nullspace $Ax=0$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 4 & 0 \\ 2 & 2 & 1 \end{bmatrix} \xrightarrow[\substack{R_2-R_1 \\ R_3-2R_1 \\ R_4-2R_1}]{} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow[\substack{-\frac{1}{2}R_2 \\ R_4+2R_2}]{} \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{1} & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot entries

row reduced echelon form

no pivot entry

rank(A) = 2, dim(V) = 3

nullity(A) = 3 - 2 = 1 → nullspace 1 dim

So, to find N(A) is spanned by:

$$Ax = 0$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left.\begin{aligned} x_1 + 2x_2 &= 0 \\ x_2 - \frac{x_3}{2} &= 0 \end{aligned}\right) \begin{aligned} x_1 &= -2x_2 \\ x_2 &= \frac{1}{2} x_3 \end{aligned} \rightarrow \underbrace{\begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix}}_{} \cdot x$$

* basis for N(A)

for R(A): spanned by the lin. indep. col.s of A. which are 1st & 2nd (bcs we've found RREF)

$$\text{* basis for } R(A): \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \\ 2 \end{bmatrix} \right\}$$

d) Find bases for $N(A^*)$ & $R(A^*)$

$$A^* = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 0 & 4 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[2)\ -\frac{1}{2}R_2]{1)\ R_3 + R_2} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$A^{*\prime}: \text{rref of } A^* : \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_3 + R_2}$$

no pivot, in the nullspace

for $A^{*\prime} x = 0$:

$x_1 + x_2 + 2x_3 + 2x_4 = 0$

$x_2 + x_4 = 0$

- $x_2 = -x_4$

$x_1 - x_4 + 2x_3 + 2x_4 = 0$

$x_1 + 2x_3 + x_4 = 0$

- $x_1 = -2x_3 - x_4$

free var. ← $x_3 = -\frac{1}{2}x_1 - \frac{1}{2}x_4 = x_3$

free var. ← x_4

$$x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

system has 2 free variables. We can express basis for $N(A^*)$:

$$* \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

* basis for $R(A)$: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

e) $\dim(N(A))$, $\dim(R(A))$?

$\dim(N(A)) = 1$, $\dim(R(A)) = 2$ $\quad (\dim(A) = 3, ✓)$

f) $\dim(N(A^*))$, $\dim(R(A^*))$?

$\dim(N(A^*)) = 2$, $\dim(R(A)) = 2$ $\quad (\dim(\text{domain } A^*) = \dim(\text{codomain } A) = 4$ ✓$)$

4) Vector space $(\mathbb{R}^2, \mathbb{R})$. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be the operator that rotates a ccw about the origin by $90°$. Define vectors:

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix}$$

Let A denote the rep. of L wrt $\{x_1, x_2\}$, and let $\bar{A}$ denote the rep. L wrt $\{x_1, x_3\}$.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$$

Find the matrix P satisfies $\bar{A} = PAP^{-1}$

So,

→ P: change matrix from basis $\{x_1, x_2\}$ to $\{x_1, x_3\}$

for $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ the coor. wrt $\{x_1, x_2\}$ are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (since x_1 is the 1st vecto

for $x_3 = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ //

$\alpha_1 = 1.5$

$\alpha_2 = 0.5$ //

$$* \; P = \begin{bmatrix} 1 & 1.5 \\ 0 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$$

→ let's check $\bar{A} = PAP^{-1}$

for $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{0.5 - \cancel{1.5} \cdot 0} \begin{bmatrix} 0.5 & -1.5 \\ 0 & 1 \end{bmatrix} = \cancel{\begin{bmatrix} 0.5 & 1.5 \\ 0 & 1 \end{bmatrix}}$ (typo).

$$\bar{A} = PAP^{-1}$$

$$\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1.5 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cancel{\begin{bmatrix} 0.5 & 1.5 \\ 0 & 1 \end{bmatrix}}$$

✓

?, 5) Let $A = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} \cdot \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} \cdot \\ 1 \\ 1 \\ 1 \end{bmatrix}$

The sets $S_1 = \{b, Ab, A^2b, A^3b\}$ & $S_2 = \{\bar{b}, A\bar{b}, A^2\bar{b}, A^3\bar{b}\}$ are both lin. indep. in $(\mathbb{R}^4, \mathbb{R})$. Now think A as a lin. operator from $\mathbb{R}^4 \to \mathbb{R}^4$.

a) Find the matrix rep. of A wrt S_1.

b) Find the " " " wrt S_2

) Should be the same due to Cayley-Hamilton theorem.

a) vectors in S_1:

$v_1 = b \longrightarrow Av_1 = Ab \rightarrow$ this is v_2: 1st col: $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$v_2 = Ab \longrightarrow Av_2 = A^2b \rightarrow v_3$: 2nd col: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$v_3 = A^2b \longrightarrow Av_3 = A^3b \rightarrow v_4$: 3rd col: $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$v_4 = A^3b \longrightarrow Av_4 = A^4b = A \cdot (A^3b)$

$Ab = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$ $A^2b = A(Ab) = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$

$A^3b = A(A^2b) = \begin{bmatrix} 3 \\ 4 \\ 6 \\ 1 \end{bmatrix}$ $A^4b = A(A^3b) = \begin{bmatrix} 4 \\ 5 \\ 8 \\ 1 \end{bmatrix}$

$\alpha_1 b + \alpha_2 Ab + \alpha_3 A^2b + \alpha_4 A^3b = \begin{bmatrix} 4 \\ 5 \\ 8 \\ 1 \end{bmatrix}$

$\alpha_1 = -8$
$\alpha_2 = 20$
$\alpha_3 = -18$
$\alpha_4 = 7$

matrix rep. of A wrt S1:

4 $\begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

I lost in calculation, and used online solver, don't know if the result true but i believe this is the method

b) vectors in S_2:

$z_1 = \bar{b} \longrightarrow Az_1 = A\bar{b} \rightarrow z_2 \rightarrow$ 1st col $\begin{bmatrix} \cdot \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$z_2 = A\bar{b} \longrightarrow Az_2 = A^2\bar{b} \rightarrow z_3$: 2nd col: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$z_3 = A^2\bar{b} \longrightarrow Az_3 = A^3\bar{b} \rightarrow z_4$: 3rd col: $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$z_4 = A^3\bar{b}$

$Az_4 = A^4\bar{b} = A(A(A(A\bar{b})))$

I guess? :(

matrix rep. of A wrt S2:

$\begin{bmatrix} 0 & 0 & 0 & -8 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 18 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

6. Let (X, F) and (Y, F) be vector spaces, & $x, z \in X$. Let $L: X \to Y$ be a linear operator.

a) Prove that if the set $\{L(x), L(z)\}$ is lin. indep., then $\{x, z\}$ is also linearly indep.

b) Is the converse true? (if $\{x, z\}$ lin. indep $\rightarrow$ $\{L(x), L(z)\}$ lin. indep.)

a) Assume $\{L(x), L(z)\}$ lin. indep. (1) let's say $a, b \in$ scalars.

$a \cdot L(x) + b \cdot L(z) = 0$

$a = b = 0$ is the only solution.

Now suppose $\{x, z\}$ is lin. dependent (for contradiction) (2)

Then $\rightarrow$ $\alpha_1 x + \alpha_2 z = 0$ both α_1 and α_2 are both not zero.

this means (apply lin. operator L to both sides

$L(\alpha_1 x + \alpha_2 z) = L(0)$

$\alpha_1 \cdot L(x) + \alpha_2 \cdot L(z) = 0$ $\rightarrow$ this contradicts our assumption (1).

This means our assumption (2) is not true.

if $\{L(x), L(z)\}$ lin. indep. $\leftarrow$ Then:

True then $\{x, z\}$ is also lin. indep.

b) Assume $\{x, z\}$ is lin. independent. Let's define lin. operator L: (from $\mathbb{R}^2 \to \mathbb{R}^2$)

$L\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\rightarrow$ for any vectors x & z $L(x)$ & $L(z)$ will be zero vector

Therefore $L(x)$ & $L(z)$ are lin. indep.

let's choose $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\rightarrow$ lin. indep.

Now the lin oper. L is not zero vector for them.

So the converse is <u>not</u> true.