ME 564

Q1 Let $(V, \langle \cdot \rangle)$ be an innerproduct space over the field $\mathbb{R}$. Let $x, y \in V$ be some fixed vectors. Prove that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in V$, then $x = y$.

Let's show that $x - y = 0$, which implies $x = y$.

IP: $\langle x-y, x-y \rangle = \langle x, x \rangle - \langle y, x \rangle - \langle x, y \rangle + \langle y, y \rangle$

Let z arbitrary vector

$\langle x-y, z \rangle = \langle x, z \rangle - \langle y, z \rangle$ since $\langle x, z \rangle = \langle y, z \rangle$ we get

$\langle x-y, z \rangle = 0$ $(x-y) \perp z$ ← every arbitrary vector. So $(x-y)$ must be the zero vector.

$x = y \leftarrow x - y = 0$ ←

So if $\langle x, z \rangle = \langle y, z \rangle \rightarrow x = y$ ✓

Q2 Let $U = \text{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$

a) Find a basis for $U^\perp$.
b) Find an orthonormal basis for $U^\perp$
c) " " " " for U via Gram-Schmidt, starting with vector $[1\ 1\ 0]^T$.

a) basis of U : $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

apply Gram-Schmidt to make an orthogonal basis $\{v_1, v_2\}$:

$$v_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$v_2 = \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{[0\ 1\ 1]\begin{bmatrix}1\\1\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\\0\end{bmatrix}\right\|^2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1\end{bmatrix}$$

$$\{v_1, v_2\} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1/2\\1/2\\1\end{bmatrix}\right\}$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \qquad u_2 = \frac{v_2}{\|v_2\|} = \sqrt{\frac{2}{3}}\begin{bmatrix}-1/2\\1/2\\1\end{bmatrix}$$

orthonormal basis for U : $\left\{\begin{bmatrix}0,707\\0,707\\0\end{bmatrix}, \begin{bmatrix}-0,408\\0,408\\0,816\end{bmatrix}\right\}$

a) $U^\perp \to$ we need to solve $u \cdot x = 0$ for $\forall u \in U$

$u_1 x = 0 \to \begin{bmatrix}1\\1\\0\end{bmatrix} x = 0 \to x + y = 0 \to x = -y$

$u_2 x = 0 = \begin{bmatrix}0\\1\\1\end{bmatrix} x = 0 \to y + z = 0 \to z = -y$

basis for $U^\perp$: $\left\{ \begin{bmatrix}-1\\1\\-1\end{bmatrix} \right\}$

b) basis vector: $v = \begin{bmatrix}-1\\1\\-1\end{bmatrix} \to$ normalize it: $\frac{v}{\|v\|} = \begin{bmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{bmatrix}$

Q3 Sol. to the optimization problem

minimize $\frac{1}{2}x^T x$ subject to $Ax = y$

is $x = A^+ y$, where $A^+ := A^T(AA^T)^{-1}$ is the unweighted pseudo-inverse. We now define the weighted pseudo-inverse $A_B^+ := BA^T(ABA^T)^{-1}$. What optimization problem does $x = A_B^+ y$ solve? In other words, find $f(x)$ s.t. $x = A_B^+ y$ is the solution to problem minimize $f(x)$, subject to $Ax = y$.

Hint: Make a guess at what $f(x)$ might be, then use Hamiltonian to check whether the sol. is indeed $x = A_B^+ y$.

guess: $f(x) = \frac{1}{2}x^T B x$

$H(x, \lambda) = f(x) + \lambda^T(Ax - y) = \frac{1}{2}x^T \bar{B} x + \lambda^T(Ax - y)$

$\nabla_x H = \bar{B}x + A^T\lambda = 0$

$\bar{B}x = -A^T\lambda$

$\to x = -B^{-1}A^T\lambda$

$Ax = y$

$-AB^{-1}A^T\lambda = y$

$\lambda = -(AB^{-1}A^T)^{-1}y$

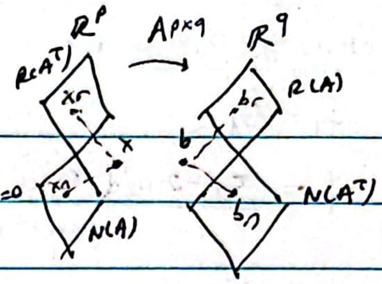
$\to x = \underbrace{B^{-1}A^T(AB^{-1}A^T)^{-1}}\cdot y \to f(x) = \frac{1}{2}x^T \bar{B} x$

$x = A_B^+ y \to$ indeed a sol. to the optimization problem

minimize $f(x) = \frac{1}{2}x^T \bar{B} x$ subject to $Ax = y$

Q4

Sys. of eqns $Ax=y$, $x\in\mathbb{R}^p$, $y\in\mathbb{R}^q$.
given: the nullity of A, $\nu(A)$, is equal to 0.



a) It is possible that $p<q$
b) It's possible that $p>q$
c) If $p\neq q$, it's possible that no sol. exists
d) If $p=q$, it's possible that no sol. exists

$$\left.\begin{array}{l} \rho(A)+\nu(A^T)=q \\ \rho(A^T)+\nu(A)=p \\ \cancel{\rho(A)+\nu(A)=p \text{ (rank-nullity)}} \\ \rho(A^T)+\nu(A^T)=q \end{array}\right\} \begin{array}{l} \nu(A)=0 \\ \rho(A^T)=p \\ \cancel{\rho(A)=p} \\ \nu(A^T)=q-p \end{array}$$

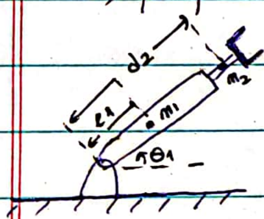
TRUE: a) $\nu(A)=0 \rightarrow$ A has full col. rank, p lin. indep. cols.
A must have at least as many rows as col.s $\rightarrow q\geq p$

FALSE: b) $p>q \rightarrow \nu(A^T)=q-p\geq 0 \rightarrow q\geq p$

TRUE: c) $p\neq q$ & $\nu(A)=0 \rightarrow$ possible that $Ax=y$ has no sol.
(e.g. $p>q$, $b\notin R(A)$)

FALSE: d) $p=q$ & $\nu(A)=0 \rightarrow$ A is a square matrix, full rank $\rightarrow$ makes it invertible.
There must be a sol.

Q5



robot arm manipulator.

km. eqns: $(m_1\ell_1^2+I_1+I_2+m_2d_2^2)\ddot{\theta}_1+2m_2d_2\dot{\theta}_1\dot{d}_2+(m_1\ell_1+m_2d_2)g\cos\theta_1=\tau_1$ (1a)

$m_2\ddot{d}_2-m_2d_2\dot{\theta}_1^2+m_2g\sin\theta_1=\tau_2$ (1b)

$\theta_1, d_2, \tau_1, \tau_2$: funcs of time/signals

State x & input u: $x:=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}:=\begin{bmatrix}\theta_1\\\dot{\theta}_1\\d_2\\\dot{d}_2\end{bmatrix}$ $\quad u:=\begin{bmatrix}u_1\\u_2\end{bmatrix}:=\begin{bmatrix}\tau_1\\\tau_2\end{bmatrix}$

a) Rearrange (1) into the form $\dot{x}=f(x,u)$, i.e. find f_1, f_2, f_3, f_4 s.t. $\dot{x}=\begin{bmatrix}\dot{\theta}_1\\\ddot{\theta}_1\\\dot{d}_2\\\ddot{d}_2\end{bmatrix}=\begin{bmatrix}f_1(\theta_1,\dot{\theta}_1,d_2,\dot{d}_2,\tau_1,\tau_2)\\f_2(\;"\;)\\f_3(\;"\;)\\f_4(\;"\;)\end{bmatrix}$

(1a) $\ddot{\theta}_1=\dfrac{\tau_1-2m_2d_2\dot{\theta}_1\dot{d}_2-(m_1\ell_1+m_2d_2)g\cos\theta_1}{(m_1\ell_1^2+I_1+I_2+m_2d_2^2)}$

(1b) $\ddot{d}_2=\dfrac{\tau_2+m_2d_2\dot{\theta}_1^2-m_2g\sin\theta_1}{m_2}$

$$\dot{x}=\begin{bmatrix}\dot{\theta}_1\\\ddot{\theta}_1\\\dot{d}_2\\\ddot{d}_2\end{bmatrix}=\begin{bmatrix}\dot{\theta}_1\\ \dfrac{\tau_1-2m_2d_2\dot{\theta}_1\dot{d}_2-(m_1\ell_1+m_2d_2)g\cos\theta_1}{(m_1\ell_1^2+I_1+I_2+m_2d_2^2)}\\ \dot{d}_2\\ \dfrac{\tau_2+m_2d_2\dot{\theta}_1^2-m_2g\sin\theta_1}{m_2}\end{bmatrix}$$

b) Find the Jacobian of $f(x,u)$ wrt x. Hint: the value of the i^{th} row j^{th} col. of this Jacobian is the der. of f_i wrt x_j

$$f_1 = \dot{\theta}_1 \qquad\qquad x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ d_2 \\ \dot{d}_2 \end{bmatrix}$$

$$f_2 = \frac{\tau_1 - 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 - (m_1 l_1 + m_2 d_2) g\cos\theta_1}{(m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2)} = \frac{N}{D}$$

$$f_3 = \dot{d}_2$$

$$f_4 = \frac{\tau_2 + m_2 d_2 \dot{\theta}_1^2 - m_2 g \sin\theta_1}{m_2}$$

$$\rightarrow J_{ij} = \frac{\partial f_i}{\partial x_j} = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \dot{\theta}_1} & \frac{\partial f_1}{\partial d_2} & \frac{\partial f_1}{\partial \dot{d}_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_4}{\partial \theta_1} & - & - & \frac{\partial f_4}{\partial \dot{d}_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \dot{\theta}_1} & \frac{\partial f_2}{\partial d_2} & \frac{\partial f_2}{\partial \dot{d}_2} \\ 0 & 0 & 0 & 1 \\ \frac{\partial f_4}{\partial \theta_1} & \frac{\partial f_4}{\partial \dot{\theta}_1} & \frac{\partial f_4}{\partial d_2} & 0 \end{bmatrix}$$

$$\frac{\partial f_2}{\partial \theta_1} = \frac{(m_1 l_1 + m_2 d_2) g \sin\theta_1}{(m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2)}$$

$$\frac{\partial f_2}{\partial \dot{\theta}_1} = -\frac{2m_2 d_2 \dot{d}_2}{(m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2)}$$

$$\frac{\partial f_2}{\partial d_2} = \frac{-D(2m_2 \dot{\theta}_1 \dot{d}_2) + N(2m_2 d_2)}{D^2}$$

$$\frac{\partial f_2}{\partial \dot{d}_2} = -\frac{D(2m_2 d_2 \dot{\theta}_1)}{D^2}$$

$$\frac{\partial f_4}{\partial \theta_1} = -g\cos\theta_1$$

$$\frac{\partial f_4}{\partial \dot{\theta}_1} = 2d_2\dot{\theta}_1$$

$$\frac{\partial f_4}{\partial d_2} = \dot{\theta}_1^2$$

$$\frac{\partial f_4}{\partial \dot{d}_2} = 0$$

c) Jacobian wrt u.

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$f_1 = \dot{\theta}_1$$

$$f_2 = \frac{\tau_1 - 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 - (m_1 l_1 + m_2 d_2) g \cos\theta_1}{(m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2)} = \frac{N}{D}$$

$$f_3 = \dot{d}_2$$

$$f_4 = \frac{\tau_2 + m_2 d_2 \dot{\theta}_1^2 - m_2 g \sin\theta_1}{m_2}$$

$$J_{ij} = \frac{\partial f_i}{\partial u_j} = \begin{bmatrix} \frac{\partial f_1}{\partial \tau_1} & \frac{\partial f_1}{\partial \tau_2} \\ \vdots & \vdots \\ \frac{\partial f_4}{\partial \tau_1} & \frac{\partial f_4}{\partial \tau_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1/D & 0 \\ 0 & 0 \\ 0 & 1/m_2 \end{bmatrix}$$

d) Consider operating pts $x_{op} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$, $u_{op} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Assume $m_1, m_2, g > 0$. Is (x_{op}, u_{op}) an eqm. pt?

$$\dot{x} = \begin{bmatrix} \dot{\theta}_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$\theta_1 = 0$, $\dot{\theta}_1 = 0$, $d_2 = 3$, $\dot{d}_2 = 0$

$\tau_1 = 0$, $\tau_2 = 0$

(if (x_{op}, u_{op}) is an eqm. pt $\longleftrightarrow$ $\dot{x}\big|_{x_{op}, u_{op}} = 0$)

let's check → $f_1\big|_{x_{op}, u_{op}} = 0$

$$f_2\big|_{x_{op}, u_{op}} = \frac{0 - 2m_2 d_2 \cdot 0 \cdot 0 - (m_1 l_1 + m_2 \cdot 3) g \cdot \cos(0)}{(m_1 l_1^2 + I_1 + I_2 + m_2 9)} = 0 \rightarrow \text{true only if: } (m_1 l_1 + m_2 3) g = 0$$

if $m_1, m_2, g, (l_1) > 0$ this cannot be true. distance (≥ 0) ~~contradicts~~

$f_3\big|_{x_{op}, u_{op}} = 0$

$$f_4\big|_{x_{op}, u_{op}} = \frac{0 + m_2 3 \, 0^2 - m_2 g \sin(0)}{m_2} = 0$$

* So, since f_2 @ $(x_{op}, u_{op}) \neq 0$ → (x_{op}, u_{op}) is not eqm. pt.

e) Linearise the sys. about x_{op}, u_{op}. Define $\tilde{x} = x - x_{op}$, $\tilde{u} = u - u_{op}$ & write the sol's in the form of:

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u} + c$$

Jacobians

from b) $A = \frac{\partial f}{\partial x}\Big|_{x_0, u_0}$, $x_{op} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$ $\rightarrow$ for c: eqn 1a, 1b $\Big|_{x_{op}, u_{op}}$

from c) $B = \frac{\partial f}{\partial u}\Big|_{x_0, u_0}$, $u_{op} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

1a) $(m_1 l_1^2 + I_1 + I_2 + m_2 3^2) \cdot 0 + 2m_2 3 \cdot 0 0 + (m_1 l_1 + m_2 3) g \cos(0)$
$\hookrightarrow (m_1 l_1 + m_2 3) g$

1b) $m_2 0 - m_2 3 0^2 + m_2 g \sin(0)) = 0 \rightarrow 0$

$$c = \begin{bmatrix} (m_1 l_1 + m_2 3) g \\ 0 \end{bmatrix}$$

Jacobian $A\Big|_{x_0, u_0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$B\Big|_{x_0, u_0} = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1 l_1^2 + I_1 + I_2 + m_2 9} & 0 \\ 0 & 0 \\ 0 & 1/m_2 \end{bmatrix}$$

$$\Rightarrow \dot{\tilde{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1 l_1^2 + I_1 + I_2 + 9 m_2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \tilde{u} + \begin{bmatrix} (m_1 l_1 + 3 m_2) g \\ 0 \end{bmatrix}$$

f) Define $x_{eq} = x_{op}$ and $u_{eq} = \begin{bmatrix} \bar{\tau}_1 \\ \bar{\tau}_2 \end{bmatrix}$. Find cons. values $\bar{\tau}_1, \bar{\tau}_2$ s.t. (x_{eq}, u_{eq}) is an eqm pt. for the sys.

if (x_{eq}, u_{eq}) is an eqm pt for the sys. $\rightarrow \dot{x}\Big|_{x_{eq}, u_{eq}} = 0$

$$\dot{x}_{eq} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{d}_2 \\ \ddot{d}_2 \end{bmatrix}, \quad u_{eq} = \begin{bmatrix} \bar{\tau}_1 \\ \bar{\tau}_2 \end{bmatrix}$$

eqn 1a) $(m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2)\ddot{\theta}_1 + 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 + (m_1 l_1 + m_2 d_2) g \cos\theta_1 = \tau_1$

$$x_{op1} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ d_2 \\ \dot{d}_2 \end{bmatrix}$$

$\hookrightarrow (m_1 l_1 + m_2 d_2) g \cos\theta_1 = \bar{\tau}_1 = \underline{(m_1 l_1 + 3m_2) \cdot g}$

eqn 1b) $m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g \sin\theta_1 = \tau_2$

$\hookrightarrow m_2 g \sin(0) = \underline{\bar{\tau}_2 = 0}$

g) Linearize the sys. about the eqm. pt. from part e). Define $x^* = x - x_{eq}$, $u^* = u - u_{eq}$ & write your sol.s in the form of:

↳ need to use Taylor's expansion around that pt.

form of the lin. sys. ⟵ $\dot{x}^* = Fx^* + Gu^* + h$

$$f_1 = \dot{\theta}_1$$

$$f_2 = \frac{\tau_1 - 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 - (m_1 l_1 + m_2 d_2) g\cos\theta_1}{m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2}$$

$$f_3 = \dot{d}_2$$

$$f_4 = \frac{\tau_2 + m_2 d_2 \dot{\theta}_1^2 - m_2 g \sin\theta_1}{m_2}$$

$$F = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \dot{\theta}_1} & \frac{\partial f_1}{\partial d_2} & \frac{\partial f_1}{\partial \dot{d}_2} \\ \vdots & & & \\ \vdots & & & \frac{\partial f_4}{\partial \dot{d}_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \dot{\theta}_1} & \frac{\partial f_2}{\partial d_2} & \frac{\partial f_2}{\partial \dot{d}_2} \\ 0 & 0 & 0 & 1 \\ \frac{\partial f_4}{\partial \theta_1} & \frac{\partial f_4}{\partial \dot{\theta}_1} & \frac{\partial f_4}{\partial d_2} & \frac{\partial f_4}{\partial \dot{d}_2} \end{bmatrix}$$

$$\frac{\partial f_2}{\partial \theta_1} = \frac{(m_1 l_1 + m_2 d_2) g \sin\theta_1}{m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2}$$

$$\frac{\partial f_2}{\partial \dot{\theta}_1} = -\frac{2 m_2 d_2 \dot{d}_2}{m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2}$$

$$\frac{\partial f_2}{\partial d_2} = \frac{-2 m_2 \dot{\theta}_1^2 + m_2 g \cos\theta_1}{m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2}$$

$$\frac{\partial f_2}{\partial \dot{d}_2} = -\frac{2 m_2 \dot{\theta}_1 d_2}{m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2}$$

$$\frac{\partial f_4}{\partial \theta_1} = -g\cos\theta_1$$

$$\frac{\partial f_4}{\partial \dot{\theta}_1} = 2 d_2 \dot{\theta}_1$$

$$\frac{\partial f_4}{\partial d_2} = \dot{\theta}_1^2$$

$$\frac{\partial f_4}{\partial \dot{d}_2} = 0$$

Similarly

$$G = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}$$

$$h: f(x,u)\Big|_{\substack{x_{eq} \\ u_{eq}}} = \begin{bmatrix} 0 \\ (m_1 l_1 + m_2 d_2) g \cos\theta_1 - \bar{\tau}_1 \\ 0 \\ m_2 g \sin\theta_1 - \bar{\tau}_2 \end{bmatrix}$$

So $\dot{x}^* = Fx^* + Gu^* + h$

h) $m_1 = m_2 = I_1 = I_2 = l_1 = 1$, $g = 0$ → Compute F and G.

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 \\ 1/12 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Q7 The Lotka-Volterra model → A population of two species (hunter h & prey p) in an ecosys. The model is:

$$\dot{h} = Ahp - Mh \qquad A, M, G, B \rightarrow \text{pos. const.}$$
$$\dot{p} = Gp - Bhp$$

↳ the effect of both pop's on h: (eating and reproducing)
↳ death rate of h.
↳ growth rate of p.
↳ the effect of both pop's on p. (getting eaten)

a) this sys. has 2 eqm pts. One is $(h_0, p_0) = (0,0)$
Find the other (h_e, p_e)

for eqm: $\dot{h} = 0$, $\dot{p} = 0$

$Ahp - Mh = 0 = h(Ap - M) \rightarrow h = 0$ or $Ap = M$
$p = \frac{M}{A}$ (when $h \neq 0$)

$Gp - Bhp = 0 = p(G - Bh) \rightarrow p = 0$ or $G = Bh$
$h = \frac{G}{B}$ (when $p \neq 0$)

$(h_e, p_e) = \left(\frac{G}{B}, \frac{M}{A}\right)$

b) Linearize sys. about (h_e, p_e). $x = \begin{bmatrix} h \\ p \end{bmatrix}$, $\tilde{x} = x - \begin{bmatrix} h_e \\ p_e \end{bmatrix}$, sol → $\dot{\tilde{x}} = A\tilde{x}$

$\dot{\tilde{x}} = A\tilde{x}$ where $\tilde{x} = x - \begin{bmatrix} h_e \\ p_e \end{bmatrix}$

$\dot{h} = Ahp - Mh$
$\dot{p} = Gp - Bhp$

$$A = \left.\begin{bmatrix} \frac{\partial \dot{h}}{\partial h} & \frac{\partial \dot{h}}{\partial p} \\ \frac{\partial \dot{p}}{\partial h} & \frac{\partial \dot{p}}{\partial p} \end{bmatrix}\right|_{h_e, p_e} = \begin{bmatrix} A\overset{M/A}{p} - M & A\overset{G/B}{h} \\ -B\underset{M/A}{p} & G - B\underset{G/B}{h} \end{bmatrix}_{(h_e, p_e)} = \begin{bmatrix} 0 & \frac{AG}{B} \\ -\frac{BM}{A} & 0 \end{bmatrix}$$

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & \frac{AG}{B} \\ -\frac{BM}{A} & 0 \end{bmatrix} \tilde{x}$$

$(h_1, p_1) = (1, 1)$

c) from part (a) we know that there're only two eqm. pts. $(0,0)$ & $\left(\frac{G}{B}, \frac{M}{A}\right)$. Whether or not finding a matrix F s.t. we can write $\dot{x}^* = Fx^*$, depends on if $(1,1)$ is an eqm. pt.

we need to check if this is a possible scenario

$\frac{G}{B} = 1 \rightarrow G = B$

$\frac{M}{A} = 1 \rightarrow M = A$

So it's possible to find matrix F, only if $G = B$ and $M = A$ & $(1,1)$ can be an eqm. pt

d) $u = \#$ of predators $= p$

u_e : eqm pt

$\tilde{u} = u - u_e$

$$\begin{bmatrix} \dot{h} \\ \dot{p} \end{bmatrix} = w + A\tilde{x} + B\tilde{u}$$

eqm. pt → $(h_e, p_e) = \left(\frac{G}{B}, \frac{M}{A}\right) \rightarrow u_e = \frac{M}{A}$

@ eqm → $w = 0$

$$\tilde{u} = u - u_e$$
$$= p - \frac{M}{A}$$
$$\tilde{x} = x - \begin{bmatrix} h_e \\ p_e \end{bmatrix} \text{ (from (b))}$$
$$= \begin{bmatrix} h - h_e \\ p - p_e \end{bmatrix}$$

$\dot{h} = Ahp - Mh$

$\dot{p} = Gp - Bhp$

$$A \text{ (from part (b))} = \begin{bmatrix} 0 & \frac{AG}{B} \\ -\frac{BM}{A} & 0 \end{bmatrix}$$

Since $u = p$, the effect of u is directly on $\dot{p}$, not on $\dot{h}$.

B → which shows how the input u enters the system → $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} \dot{h} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & AG/B \\ -\frac{BM}{A} & 0 \end{bmatrix} \begin{bmatrix} h - h_e \\ p - p_e \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(p - \frac{M}{A}\right)$$

e) Define output y to be equal to the # of prey, $y = g(h,p)$. Then linearize it by writing it in the form of $y = k + C\tilde{x}$ where k & C are constants.

$$y = g(h,p) = p$$

around $(h_e, p_e) = \left(\frac{G}{B}, \frac{M}{A}\right)$ we can write it in the form $y = k + C\tilde{x}$

$$\tilde{x} = \begin{bmatrix} h - h_e \\ p - p_e \end{bmatrix}$$

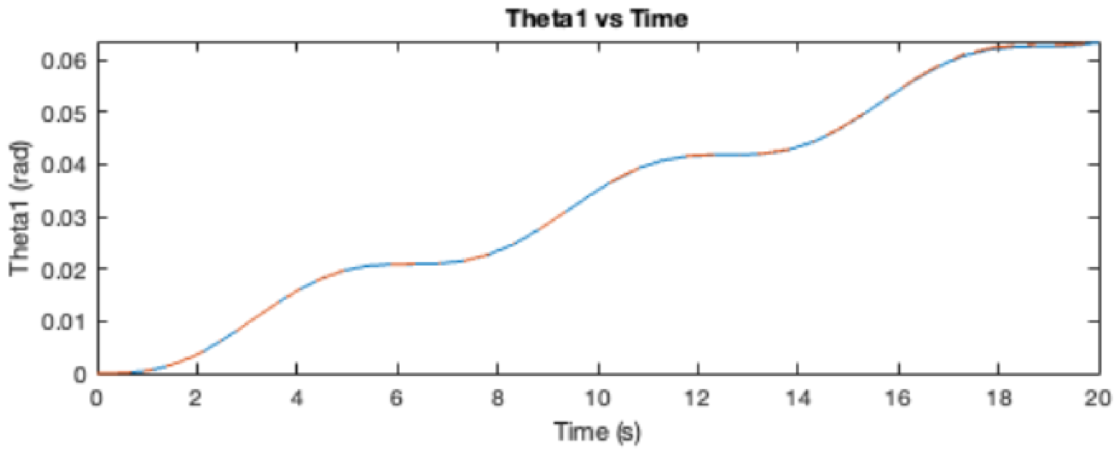
$$k = p_e = \frac{M}{A}$$

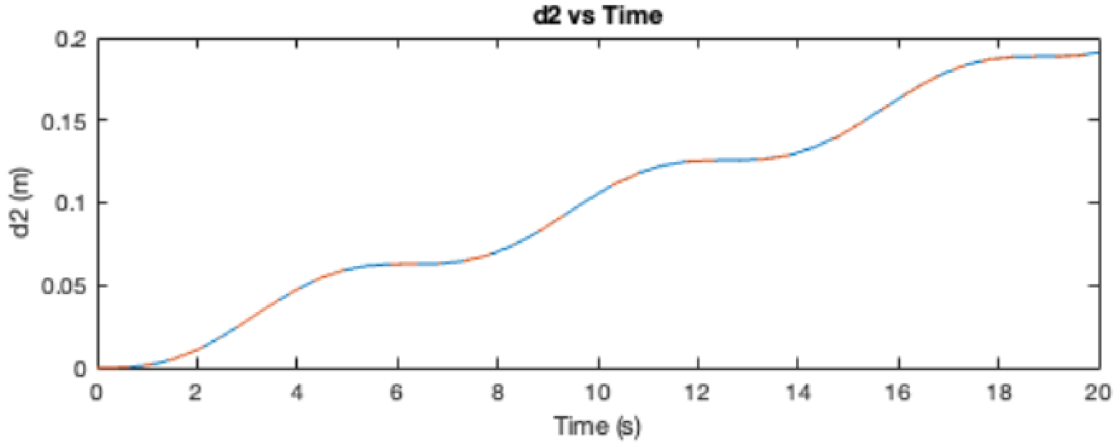
$$\frac{\partial g}{\partial h} = 0, \quad \frac{\partial g}{\partial p} = 1 \rightarrow C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y = \frac{M}{A} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} h - h_e \\ p - p_e \end{bmatrix}$$
$$= \frac{M}{A} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} h - \frac{G}{B} \\ p - \frac{M}{A} \end{bmatrix}$$

```
clear all
close all

% Define Parameters
m1 = 1;
m2 = 1;
I1 = 1;
I2 = 1;
l1 = 1;
g = 0;

% Time span
tspan = [0 20];

% Initial condition (equilibrium point)
x0 = [0; 0; 0; 0];

% Solve nonlinear system
[tn, xn] = ode45(@(t, x) nonlinear_system(t, x, m1, m2, I1, I2, l1, g), tspan,
 x0);

% Solve linear system
[t, xl] = ode45(@(t, x) linear_system(t, x, m1, m2, I1, I2, l1, g), tspan,
 x0);

% Plot Results
subplot(2,1,1);
plot(tn, xn(:,1), '-', t, xl(:,1), '-.');
title('Theta1 vs Time');
xlabel('Time (s)');
ylabel('Theta1 (rad)');

subplot(2,1,2);
plot(tn, xn(:,3), '-', t, xl(:,3), '-.');
title('d2 vs Time');
xlabel('Time (s)');
ylabel('d2 (m)');

function dx = nonlinear_system(t, x, m1, m2, I1, I2, l1, g)
    theta1 = x(1);
    theta1_dot = x(2);
    d2 = x(3);
    d2_dot = x(4);

    tau1 = 0.01 * sin(t);
    tau2 = 0.01 * sin(t);

    theta1_ddot = (tau1 - 2*m2*d2*theta1_dot*d2_dot - (m1*l1 +
 m2*d2)*g*cos(theta1)) / (m1*l1^2 + I1 + I2 + m2*d2^2);
    d2_ddot = (tau2 + m2*d2*theta1_dot^2 - m2*g*sin(theta1)) / m2;

    dx = [theta1_dot; theta1_ddot; d2_dot; d2_ddot];
```

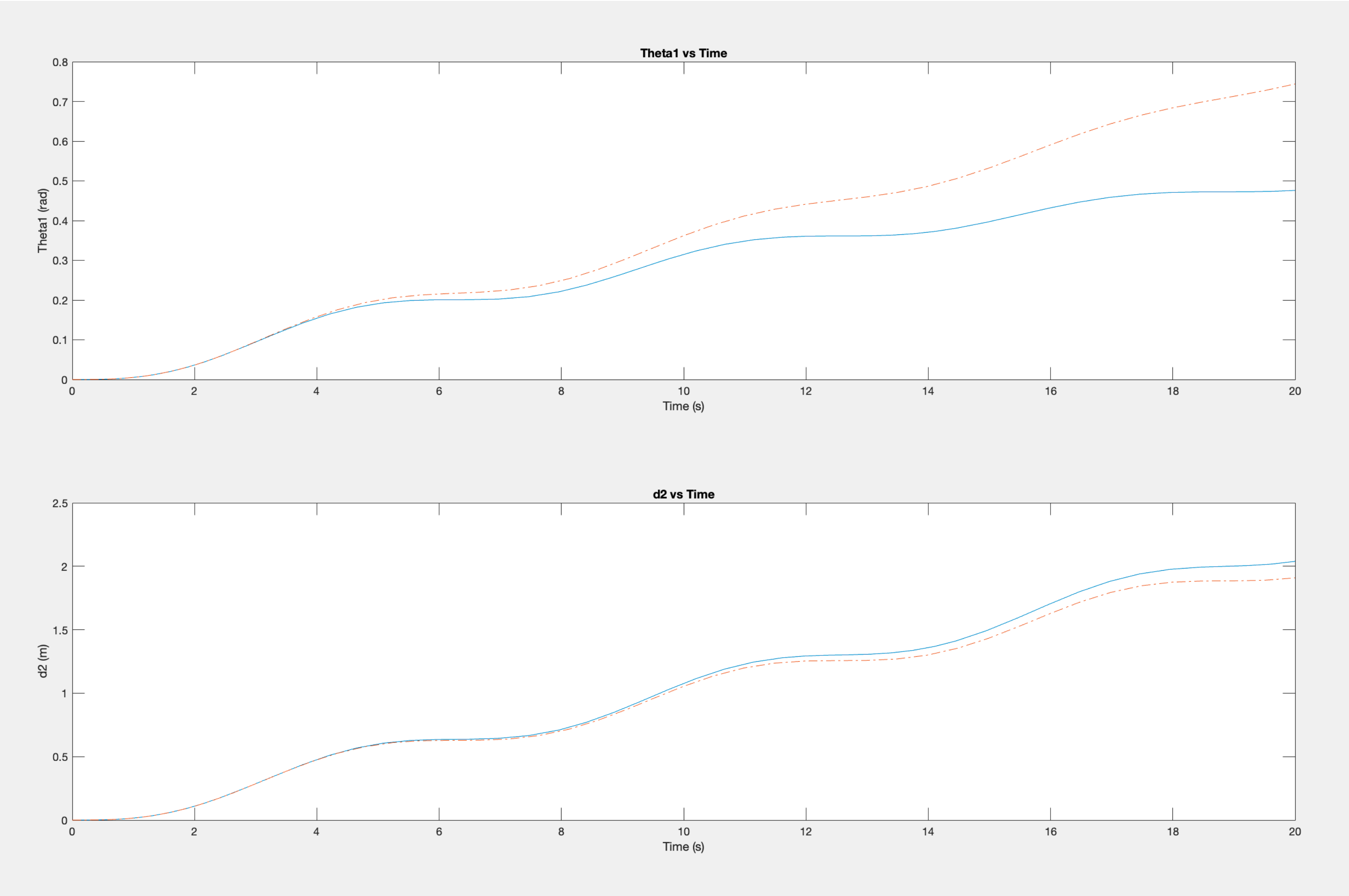
```
end

function dx_tilda = linear_system(t, x_tilda, m1, m2, I1, I2, l1, g)
    % Define A, B, and c matrices
    A = [0, 1, 0, 0;
         0, -g, 0, 0;
         0, 0, 0, 1;
         0, 0, 0, 0];

    B = [0, 0;
         1/(m1*l1^2 + I1 + I2 + g*m2), 0;
         0, 0;
         0, 1/m2];

    c = [(m1*l1 + 3*m2)*g; 0; 0; 0];

    % Define u_tilda (deviation from the operating point)
    % Assuming u_op = 0 for simplicity
    u_tilda = [0.01 * sin(t); 0.01 * sin(t)] - [0; 0];

    % Compute dx_tilda
    dx_tilda = A * x_tilda + B * u_tilda + c;
end
```





0.01sin(t)

0.1sin(t)



Theta1 vs Time
Theta1 (rad)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12
14
16
18
20
Time (s)
d2 vs Time
d2 (m)
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
12
14
16
18
20
Time (s)