LINEAR PROGRAMMING

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Written in accordance with the new syllabus of Honours and Masters of the National University, Dhaka University, Jahangirnagar University, Chittagong University, Rajshahi University for the students of the Department of Mathematics, Physics, Statistics, Economics and Business Administration.

Linear Programming

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PREFACE

Bismillahir Rahmanir Rahim.

The book titled *Linear Programming* has been written with the blessings of Almighty Allah. It is clear that the use of linear programming has been on the increase day by day in many fields of science, socio-economics & commerce and now-a-days linear programming is a compulsory course in all mathematics related disciplines. But the fact is that our students are always afraid of linear programming as well as mathematics and they need proper guidance and dependable books. There are textbooks available in the market on Linear Programming but most of them are inadequate in meeting the requirements of the students. So in order to gear up the students in Linear Programming and Operation Research I have attempted to write the book using easy language and techniques.

I have included in the book a large number of examples, mostly collected from question papers of different universities. I have verified every question and their solution minutely, and if there is any error yet, I beg apology for that.

I am grateful and indebted to my teachers Professor Dr. A.A.K. Majumdar, Ritsumeikan Asia-Pacific University, Japan; Professor Dr. Md. Abdul Hoque and Professor Dr. Sayed Sabbir Ahmed of the department of Mathematics, Jahangirnagar University and many of my colleagues who encouraged and helped me in writing the book. I am also grateful to the authors whose books I have freely consulted in the preparation of the manuscript. The authority of Kabir Publications deserves thanks for undertaking the responsibility to publish the book in these days of hardships in publication business.

Finally, special recognition should go to the endless co-operation of my wife without which the book could not have seen the light of the day.

Constructive criticisms, corrections, and suggestions towards the improvement of the book will be thankfully received.

S. M. Shahidul Islam

DEDICATION

To

All of my teachers,
I am ever indebted to them.

National University

Subject: Mathematics (Hon's.)

Course: MMT-406 *Linear Programming*

1. **Convex sets and related theorems:** Introduction to linear programming and related theorems of feasibility and optimality.
2. **Formulation and graphical solutions:** Formulation and graphical solutions.
3. **Simplex method:** Simplex methods.
4. **Duality:** Duality of linear programming and related theorems.
5. **Sensitivity analysis:** Sensitivity analysis in linear programming.

Evaluation: Final exam. is (Theoretical) of 50 marks. Four out of six questions are to be answered. Time – 3 hours.

Dhaka University

Subject: Mathematics (Hon's.)

Course: MTH-308 *Linear Programming*

1. Convex sets and related theorems
2. Introduction to linear programming.
3. Formulation of linear programming problem.
4. Graphical solutions.
5. Simplex method.
6. Duality of linear programming and related theorems.
7. Sensitivity.

Evaluation: Final exam. (Theory, 3 hours): 50 marks. SIX questions will be set, of which any FOUR are to be answered.

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Chapter

01

Introduction

Highlights:

1.1 Linear programming: (যোগাশ্রয়ী প্রোগ্রাম) Linear programming is a technique for determining an optimum schedule (such as maximizing profit or minimizing cost) of interdependent activities in view of the available resources. Programming is just another word for "Planning" and refers to the process of determining a particular plan of actions from amongst several alternatives. A linear programming (LP) problem with only two variables presents a simple case, for which the solution can be derived using a graphical method. We can solve any linear programming problem by simplex method. A linear programming problem may have a solution (a definite and unique solution or an infinite number of optimal solutions or an unbounded solution) or not. **[JU-01]**

(প্রাপ্ত সুযোগ সুবিধা ব্যবহর করে কোন সমস্যার চুড়ান্ড় (লাভ বাড়ানো বা ক্ষতি কমানো) মান বের করার একটি পদ্ধতিই হলো যোগাশ্রয়ী প্রোগ্রাম। অন্য কথায় প্রোগ্রামিং বলতে বুঝায় প্রাপ্ত সুযোগ সুবিধার মধ্যে সেরা পরিকল্পনা প্রনয়নের পদ্ধতি। দুই চলকের সহজ যোগশ্রয়ী প্রোগ্রামের সমাধান লেখ চিত্র পদ্ধতির মাধ্যমে করা যায়। তবে যে কোন যোগশ্রয়ী প্রোগ্রামের সমাধান সিমপেক্স পদ্ধতিতে করা যায়। একটা যোগশ্রয়ী প্রোগ্রামের সমাধান (নির্দ্দিষ্ট এবং একক সমাধান অথবা অসংখ্য সমাধান) থাকতেও পারে আবার নাও থাকতে পারে।)

<u>Example (1.1)</u>: A company uses three different resources for the manufacture of two different products 200 unites of the resource A, 120 units of B and 160 units of C being available. 1 unit of the first product requires 2, 2 and 4 units of the respective resources and 1 unit of the second product requires 4, 2 and 0 units of the respective resources. It is known that the first product gives a profit of 20 monetary units per unit and the second 30. Company likes to know how many units of each product should be manufactured for maximizing the profit.
(কোন কোম্পানী দুই ধরনের জিনিস তৈরী করার জন্য তিন ধরনের কাচামাল ক, খ এবং গ ব্যবহার করে। তাদের ক ধরনের কাচামাল আছে ২০০ একক, খ ধরনের কাচামাল আছে ১২০ একক এবং গ ধরনের কাচামাল আছে ১৬০ একক। প্রথম ধরনের একটি জিনিস তৈরী করতে ২ একক ক, ২ একক খ এবং ৪ একক গ লাগে, ঠিক তেমনি দ্বিতীয় ধরনের একটি জিনিস তৈরী করতে ৪ একক ক, ২ একক খ এবং ০ একক গ লাগে। প্রথম জিনিস থেকে লাভ আসে ২০ টাকা এবং দ্বিতীয় জিনিস থেকে লাভ আসে ৩০ টাকা। কোম্পানী জানতে চায় কোন ধরনের কতটি জিনিস তৈরী করলে তাদের লাভ সবচেয়ে বেশী হবে।)
If we consider x and y be the numbers of first and second products respectively to be produced for maximizing the profit then company's total profit $z = 20x + 30y$ is to be maximized.
Since 1 unit of the first product requires 2, 2 and 4 units, 1 unit of the second product requires 4, 2 and 0 units of the respective resources and the units available of the three resources are 20, 12 and 16 respectively, subject to the constraints are

$$
\begin{aligned}
2x + 4y &\le 200 \qquad \text{Or,} \qquad x + 2y \le 100\\
2x + 2y &\le 120 \qquad \text{Or,} \qquad x + y \le 60\\
4x + 0y &\le 160 \qquad \text{Or,} \qquad x \le 40
\end{aligned}
$$

Since it is not possible for the manufacturer to produce negative number of the products, it is obvious that $x \ge 0$, $y \ge 0$. So, we can summarize the above linguistic linear programming problem as the following mathematical form:

$$\begin{aligned}
&\text{Maximize } z = 20x + 30y \\
&\text{Subject to} \quad x + 2y \leq 100 \\
&\qquad\qquad\quad\; x + y \leq 60 \\
&\qquad\qquad\quad\; x \leq 40 \\
&\qquad\qquad\quad\; x \geq 0,\ y \geq 0
\end{aligned}$$

1.2 What is optimization: (চরম মান কি) The fundamental problem of optimization is to arrive at the 'best' possible decision under some given circumstances of some typical optimization problems, taken from a variety of practical field of interest. In linear programming problem, we have to optimize (maximize or minimize) certain functions under certain constraints. **[JU-94]**

1.3 Summation symbol: (প্রতীকের সাহায্যে যোগ) The Greek capital letter ∑ (sigma) is the mathematical symbol for summation. If *f*(i) denotes some quantity whose value depends on the value of i, the expression

$$\sum_{i=1}^{4} i$$

is read as 'sigma i, i going from 1 to 4' and means to insert 1 for i, then 2 for i, then 3 for i, 4 for i, and sum the results. Therefore,

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10.$$

In the above example, the i under the sigma symbol is called the index of the summation. In this way,

$$\sum_{i=1}^{3} (i^2 - 1) = (1^2 - 1) + (2^2 - 1) + (3^2 - 1) = 0 + 3 + 8 = 11$$

$$\sum_{i=1}^{4} p = p + p + p + p = 4p$$

$$\sum_{i=1}^{n} p = np$$

$$\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4.$$

1.4 <u>The general linear programming problem</u>: (সাধারন যোগাশ্রয়ী প্রোগ্রাম) The general linear programming problem is to find a vector (x_1, x_2, ..., x_j, ..., x_n) which minimizes (or maximizes) the linear form **[NUH-01, 04,07]**

$$z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n \qquad \ldots \ (1)$$

subject to the linear constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \geq (or = or \leq) b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \geq (or = or \leq) b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \geq (or = or \leq) b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \geq (or = or \leq) b_m \end{array}\right\} \ \ldots \ (2)$$

and $\quad x_j \geq 0;\ j = 1, 2, 3, \ldots, n. \qquad \ldots \ (3)$

where the a_{ij}, b_i and c_j are given constants and $m \leq n$. We shall always assume that the constraints equation have been multiplied by (–1) where necessary to make all $b_i \geq 0$, because of the variety of notations in common use.

The linear function $z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$ which is to be minimized (or maximized) is called the **objective function** of the general LP problem.

The inequalities (or equations) (2) are called the **constraints** of the general LP problem.

The set of inequalities (3) is usually known as the set of **non-negative restrictions** of the general LP problem.

One will find the general linear programming problem stated in many forms. The more common forms are the following:

(a) Minimize (or, maximize) $\sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j \geq (or = or \leq) b_i$; i = 1, 2, 3, . . ., m

and $x_j \geq 0$; j = 1, 2, 3, . . ., n

(b) Minimize (or, maximize) f(X) = $\underline{C}.\underline{X}$
subject to $\underline{A}.\underline{X} \geq$ (or, = or, $\leq$) $\underline{b}$
and $\underline{X} \geq \underline{0}$
where $\underline{A} = (a_{ij})_{m\times n}$, $m\times n$ matrix, $\underline{C} = (c_1, c_2, \ldots, c_n)$ is a row vector, $\underline{X} = (x_1, x_2, \ldots, x_n)$ is a column vector, $\underline{b} = (b_1, b_2, \ldots, b_m)$ is a column vector and $\underline{0}$ is an n-dimensional null column vector.

(c) Minimize (or, maximize) $\underline{C}.\underline{X}$
subject to $x_1\underline{P}_1 + x_2\underline{P}_2 + \ldots + x_n\underline{P}_n \geq$ (or, = or, $\leq$) $\underline{P}_o$
and $\underline{X} \geq \underline{0}$
where $\underline{P}_j$ for j = 1, 2, . . ., n is the jth column of the matrix $\underline{A} = (a_{ij})$ and $\underline{P}_o = \underline{b}$.

In some situations, it is convenient to define a new unrestricted variable, eg., x_0 or z, which is equal to the value of the objective. We would then have the following representation for the linear programming problem.

(d) Minimize (or, maximize) x_0

subject to $x_0 - \sum_{j=1}^{n} c_j x_j = 0$

$\sum_{j=1}^{n} a_{ij} x_j \geq (or = or \leq) b_i$; i = 1, 2, 3, . . ., m

and $x_j \geq 0$; j = 1, 2, 3, . . ., n

<u>Example (1.2)</u>: Minimize $z = 420x_1 + 300x_2$

Subject to $4x_1 + 3x_2 \geq 240$

$2x_1 + x_2 \leq 100$

$x_1, x_2 \geq 0$

is a linear programming problem.

1.5 <u>Standard form of LP problem</u>: (যোগাশ্রয়ী প্রোগ্রামের আদর্শ আকার)

The characteristics of the standard form of LP problem are **[NUH-00]**

(i) The objective function is of the minimization or maximization type.

(ii) All the constraints are expressed in the form of equations, except the non-negativity constraints which remain inequalities ($\geq$)

(a) If any constraint is of $\leq$ type (or, $\geq$ type) add slack variable s_i (or, subtract surplus variable s_i) to convert it an equality, where $s_i \geq 0$.

(b) If any constraint is of approximation type then introduce slack and surplus variables at a time. As for example $2x_1 + x_2 \approx 3$ to be converted $2x_1 + x_2 + s_1 - s_2 = 3$ where slack variable $s_1 \geq 0$ and surplus variable $s_2 \geq 0$.

(iii) The right hand side of each constraints equation is non-negative.

(iv) All the decision variables are non-negative.

(a) If $x_i \leq 0$ then put $x_i = -x_i^{/}$ where $x_i^{/} \geq 0$

(b) If x_i is unrestricted in sign then put $x_i = x_i^{/} - x_i^{//}$ where $x_i^{/} \geq 0$, $x_i^{//} \geq 0$.

(c) If $x_i \geq d_i$ (or $x_i \leq d_i$) then put $x_i = x_i^{/} + d_i$ (or, $x_i = d_i - x_i^{/}$) where $x_i^{/} \geq 0$.

That is, the following general linear programming (LP) problem

Minimize (or maximize) $z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$

subject to
$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n &= b_i \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n &= b_m \end{aligned}\right\}$$

and $x_j \geq 0$; $j = 1, 2, 3, \ldots, n$.

is in standard form.

<u>Example (1.3)</u>: Minimize (or maximize) $z = 420x_1 + 300x_2$

Subject to $4x_1 + 3x_2 = 240$

$2x_1 + x_2 = 100$

$x_1, x_2 \geq 0$

is a standard linear programming problem.

<u>Example (1.4)</u>: Convert the following linear programming problem into the standard form: **[NUH-07, JU-92, 95, DU-90, 99]**

Maximize $z = x_1 + 4x_2 + 3x_3$

Subject to $2x_1 + 2x_2 - 5x_3 \leq 2$

$3x_1 - x_2 + 6x_3 \geq 1$

$x_1 + x_2 + x_3 = 4$

$x_1 \geq 0$, $x_2 \leq 0$, x_3 unrestricted.

Solution: Putting $x_2 = -x_2'$, $x_3 = x_3' - x_3''$; $x_2' \geq 0$, $x_3' \geq 0$, $x_3'' \geq 0$ and introducing slack variable $s_1 \geq 0$ in 1st constraint and surplus variable $s_2 \geq 0$ in 2nd constraint, we get the problem as follows:

Maximize z = $x_1 - 4x_2^{/} + 3x_3^{/} - 3x_3^{//} + 0.s_1 + 0.s_2$

Subject to $2x_1 - 2x_2^{/} - 5x_3^{/} + 5x_3^{//} + s_1 = 2$

$3x_1 + x_2^{/} + 6x_3^{/} - 6x_3^{//} - s_2 = 1$

$x_1 - x_2^{/} + x_3^{/} - x_3^{//} = 4$

$x_1, x_2^{/}, x_3^{/}, x_3^{//}, s_1, s_2 \geq 0.$

This is the required standard form.

1.6 <u>Canonical form of LP problem</u>: (যোগাশ্রয়ী প্রোগ্রামের ক্যাননিক্যাল আকার) The characteristics of the canonical form of LP problem are **[NU-03, 05, 06]**

(i) The objective function is of the minimization or maximization type

(ii) All the constraints are expressed in the form of inequalities connected by the sign '$\geq$' when the objective function is of minimization type or connected by the sign '$\leq$' when the objective function is of maximization type.

(iii) All the decision variables are non-negative.

(iv) Constraints are in '=' type will be converted into two inequalities; one is '$\geq$' type and the other is '$\leq$' type.

That is, the following general linear programming (LP) problems

Minimize $z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$

subject to
$$\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \geq b_2 \\
\vdots \\
a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \geq b_i \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \geq b_m
\end{array}\right\}$$

$x_j \geq 0;\ j = 1, 2, 3, \ldots, n.$

And also

Maximize $z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$

subject to
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \le b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \le b_m \end{array}\right\}$$

$x_j \ge 0$; $j = 1, 2, 3, \ldots, n$.

are in the canonical form.

Example (1.5): The following linear programming problem

Minimize $z = 420x_1 + 300x_2$

Subject to $4x_1 + 3x_2 \ge 240$

$2x_1 + x_2 \ge 100$

$x_1, x_2 \ge 0$

is in the canonical form.

Example (1.6): The following linear programming problem

Maximize $z = 420x_1 + 300x_2$

Subject to $4x_1 + 3x_2 \le 240$

$2x_1 + x_2 \le 100$

$x_1, x_2 \ge 0$

is also in the canonical form.

Example (1.7): Convert the following linear programming problem in canonical form: **[DU-94, JU-01, NUH-98]**

Maximize $z = x_1 + 4x_2 + 3x_3$

Subject to $2x_1 + 2x_2 - 5x_3 \le 2$

$3x_1 - x_2 + 6x_3 \ge 1$

$x_1 + x_2 + x_3 = 4$

$x_1 \ge 3$, $x_2 \le 0$, x_3 unrestricted.

Solution: We can write the given problem as follows:

$$\left.\begin{array}{l} \text{Maximize } z = x_1 + 4x_2 + 3x_3 \\ \text{Subject to } 2x_1 + 2x_2 - 5x_3 \le 2 \\ \qquad 3x_1 - x_2 + 6x_3 \ge 1 \\ \qquad x_1 + x_2 + x_3 \ge 4 \\ \qquad x_1 + x_2 + x_3 \le 4 \\ x_1 \ge 3,\ x_2 \le 0,\ x_3 \text{ unrestricted.} \end{array}\right\} \begin{array}{l} \text{Or, Maximize } z = x_1 + 4x_2 + 3x_3 \\ \text{Subject to } 2x_1 + 2x_2 - 5x_3 \le 2 \\ \qquad -3x_1 + x_2 - 6x_3 \le -1 \\ \qquad -x_1 - x_2 - x_3 \le -4 \\ \qquad x_1 + x_2 + x_3 \le 4 \\ x_1 \ge 3,\ x_2 \le 0,\ x_3 \text{ unrestricted.} \end{array}$$

Putting $x_1 = x_1' + 3$, $x_2 = -x_2'$, $x_3 = x_3' - x_3''$; $x_1' \ge 0, x_2' \ge 0$, $x_3' \ge 0$, $x_3'' \ge 0$, we get problem as follows:

Maximize $z = x_1' - 4x_2' + 3x_3' - 3x_3'' + 3$

$$\begin{array}{ll} \text{Subject to} & 2x_1' - 2x_2' - 5x_3' + 5x_3'' \le -4 \\ & -3x_1' - x_2' - 6x_3' + 6x_3'' \le 8 \\ & -x_1' + x_2' - x_3' + x_3'' \le -1 \\ & x_1' - x_2' + x_3' - x_3'' \le 1 \\ & x_1', x_2', x_3', x_3'' \ge 0. \end{array}$$

This is the required canonical form.

1.7 <u>Advantages of linear programming</u>: (যোগাশ্রয়ী প্রোগ্রামের সুবিধা)

Following are the main advantages of linear programming (LP):

(i) It helps in attaining the optimum use of productive factors. Linear programming indicates how a manager can utilize his productive factors most effectively by a better selection and distribution of these elements. For example, more efficient use of manpower and machines can be obtained by the use of linear programming.

(ii) It improves the quantity of decisions. The individual who makes use of linear programming methods becomes more objective than subjective. The individual having a clear picture of the relationships within the basic equations, inequalities or constraints can have a better idea about the problem and its solution.

(iii) It can go a long way in improving the knowledge and skill of tomorrow's executives.

(iv) Although linear programming gives possible and practical solutions, there might be other constraints operating outside the problem, which must be taken into account, for example, sales, demand, etc. Just because we can produce so many units does not mean that they can be sold. Linear programming method can handle such situations also because it allows modification of its mathematical solutions.

(v) It highlights the bottleneck in the production processes. When bottlenecks occur, some machines cannot meet demand while others remain idle, at least part of the time. Highlighting of bottlenecks is one of the most significant advantages of linear programming.

1.8 <u>Limitations of linear programming</u>: (যোগাশ্রয়ী প্রোগ্রামের সীমাবদ্ধতা) Though having a wide field, linear programming has the following limitations:

(i) For large problems having many limitations and constraints, the computational difficulties are enormous, even when assistance of large digital computers is available.

(ii) According to the linear programming problem, the solution variables can have any value, where as sometimes it happens that some of the variables can have only integral values.

(iii) The model does not take into account the effect of time. The operation research team must define the objective function and constraints, which can change overnight due to internal as well as external factors.

(iv) Many times, it is not possible to express both the objective function and constraints in linear form.

(যোগশ্রয়ী প্রোগ্রামের যথেষ্ট সুবিধা থাকা সত্ত্বেও তার নিম্নোক্ত সীমাবদ্ধতা আছে।

(১) অনেক চলক ও শর্ত যুক্ত যোগশ্রয়ী প্রোগ্রামের সমাধান করা খুবই কঠিন।

(২) চলকের যে কোন মান হতে পারে কিন্তু কিছু কিছু সময় চলক নির্দ্দিষ্ট ভাবে পূর্ণ সংখ্যা।
(৩) এই মডেল কম্পুটেশনাল সময়ের সাথে সম্পর্কযুক্ত নয়।
(৪) অনেক সময় উদ্দেশ্যমুলক আপেক্ষক এবং শর্তগুলোকে যোগাশ্রয়ী আকরে প্রকাশ করা যায় না।)

1.9 <u>Some definitions</u>: Very important definitions related to the linear programming problem are discussed below:

1.9.1 <u>Objective function</u>: (উদ্দেশ্যমুলক আপেক্ষক) The linear function $z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$ in the general linear programming problem which is to be minimized (or maximized) is called the objective function. As for example, $4x_1 + 3x_2$ is the objective function of the following linear programming problem.

Minimize $z = 5x_1 + 3x_2$ **[JU-01]**

Subject to $2x_1 + 3x_2 \geq 10$

$2x_1 + x_2 \leq 6$

$x_1, x_2 \geq 0$

N.B: The objective functions "Maximize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$" and "Minimize $(-z) = -(c_1x_1 + c_2x_2 + \ldots + c_nx_n)$" are same. As for example "Minimize $z = 5x_1 + 3x_2$" has the same meaning as "Maximize $(-z) = -5x_1 - 3x_2$".

<u>Example (1.8)</u>: The following two linear programming problems are equivalent to each other:

Minimize z =	$2x_1 + 3x_2$	Minimize (–z) =	$-2x_1 - 3x_2$
Subject to	$2x_1 + x_2 \geq 11$	Subject to	$2x_1 + x_2 \geq 11$
	$3x_1 + 2x_2 \leq 20$		$3x_1 + 2x_2 \leq 20$
	$x_1, x_2 \geq 0$		$x_1, x_2 \geq 0$

1.9.2 <u>Constraints</u>: (শর্তগুলো বা বাধা সমুহ) The inequalities or equations in general linear programming problem **[DU-99]**

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \geq (or = or \leq) b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \geq (or = or \leq) b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \geq (or = or \leq) b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \geq (or = or \leq) b_m \end{array}\right\}$$

are called the constraints of the general linear programming problem. Or, shortly $\underline{A}.\underline{X} \geq$ (or, = or, $\leq$) $\underline{b}$ is called the constraints of the general linear programming problem. As for example, $5x_1 + 3x_2 \geq 15$, $2x_1 + 3x_2 \leq 6$ are the constraints of the following linear programming problem.

$$\begin{array}{ll} \text{Maximize } z = & 4x_1 + x_2 \\ \text{Subject to} & 5x_1 + 2x_2 \geq 15 \\ & 7x_1 + 3x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

1.9.3 <u>Non-negativity conditions</u>: (অঋনাত্মক শর্ত) The inequalities $x_j \geq 0$ (j = 1, 2, ..., n) are called the non-negativity conditions of the general linear programming problem. As for example, $x_1 \geq 0$, $x_2 \geq 0$ are the non-negativity conditions of the following linear programming problem

$$\begin{array}{ll} \text{Minimize } z = & 2x_1 + 8x_2 \\ \text{Subject to} & 7x_1 + 3x_2 \geq 25 \\ & 3x_1 + x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{array}$$

1.9.4 <u>Solution</u>: (সমাধান) An n-tuple $\underline{X} = (x_1, x_2, \ldots, x_n)$ of real numbers that satisfies all the constraints of the general linear programming problem is called the solution of the general linear programming problem. As for example, $\underline{X} = (x_1, x_2) = (4, 0)$ is a solution of the following linear programming problem.

Maximize $z = 3x_1 + 5x_2$ **[NUH-01, 04,07]**

Subject to $5x_1 + 3x_2 \geq 15$

$x_1 + 2x_2 \leq 7$

$x_1, x_2 \geq 0$

1.9.5 <u>Feasible solution</u>: (সকল শর্ত মান্যকারী সমাধন) A feasible solution to a linear programming (LP) problem is a solution, which satisfies the constraints (equality or inequality constraints and the non-negativity constraints). As for example, $\underline{X}$ = (1, 1, 0, 0) is a feasible solution of the following linear programming problem.

Maximize $z = x_1 + 2x_2 + x_3 + 2x_4$ **[NUH-01,02,04,05,07]**

Subject to $4x_1 + 2x_2 + 3x_3 - 8x_4 = 6$

$3x_1 + 5x_2 + 4x_3 - 6x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

1.9.6 <u>Basic solution</u>: (ভিত্তি সমাধান) A basic solution to a linear programming (LP) problem with m constraints in n variables is a solution obtained by setting (n – m) variables equal to zero and solving for the remaining m variables, provided that the determinant of the coefficients of these m variables are non-zero. The m variables are called <u>basic variables</u>. Basic solution may or may not be feasible solution and conversely feasible solution may or may not be a basic solution. As for example, $\underline{X}$ = (0, 1, 0, – ½) is a basic non- feasible solution of the following linear programming problem.

Maximize $z = x_1 + 2x_2 + x_3 + 2x_4$ **[NU-98, 02]**

Subject to $4x_1 + 2x_2 + 3x_3 - 8x_4 = 6$

$3x_1 + 5x_2 + 4x_3 - 6x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

Note: We consider a system of m simultaneous linear equations with n unknowns (m < n) AX = b, where A is an $m \times n$ matrix of rank m, X and b are $m \times 1$ column matrices. Let B be any $m \times m$ sub-matrix formed by m linearly independent columns of A. Then the solution of the resulting system found by taking (n – m) variables not associated with the columns of B equal to zero is called basic solution of the given system of simultaneous linear equations. The sub-matrix B is called the **basis matrix** of the system, the columns of B are **basis vectors** and other than basis vectors are **non-basis vectors**. The m variables associated with the columns of B are said to be **basic variables** and the others are **non-basic variables**.

Example (1.9): Find all basic solutions of the following system of simultaneous linear equations: **[RU-93, JU-89]**

$$4x_1 + 2x_2 + 3x_3 - 8x_4 = 6$$
$$3x_1 + 5x_2 + 4x_3 - 6x_4 = 8$$

Solution: We can write the given system in the matrix form as follows:

$$AX = b \text{ where, } A = \begin{pmatrix} 4 & 2 & 3 & -8 \\ 3 & 5 & 4 & -6 \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

The column vectors of the coefficient matrix of A are

$$a_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix},\ a_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix},\ a_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } a_4 = \begin{pmatrix} -8 \\ -6 \end{pmatrix}$$

Since the rank of A is 2, the given system is consistent. We find ${}^4c_2 = 6$ square sub matrices taking two at a time from a_1, a_2, a_3, a_4.

$$B_1=(a_1,a_2)=\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix},\ B_2=(a_1,a_3)=\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix},\ B_3=(a_1,a_4)=\begin{pmatrix} 4 & -8 \\ 3 & -6 \end{pmatrix},$$

$$B_4=(a_2,a_3)=\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix},\ B_5=(a_2,a_4)=\begin{pmatrix} 2 & -8 \\ 5 & -6 \end{pmatrix},\ B_6=(a_3,a_4)=\begin{pmatrix} 3 & -8 \\ 4 & -6 \end{pmatrix}.$$

The value of determinants formed by the above sub matrices are

$|B_1| = 14$, $|B_2| = 10$, $|B_3| = 0$, $|B_4| = -7$, $|B_5| = 28$, $|B_6| = 14$.

Since $|B_3| = 0$, B_3 is singular matrix. So, only five basic solutions will be found corresponding to five basis matrices B_1, B_2, B_4, B_5 and B_6.

(i) For basis matrix $B_1 = (a_1, a_2) = \begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$, basic variables are x_1, x_2 and non-basic variables are x_3, x_4. Putting $x_3 = x_4 = 0$, the system reduce to

$$4x_1 + 2x_2 = 6$$
$$3x_1 + 5x_2 = 8$$

Subtracting 2nd equation from 1st after multiplying first by 5 and 2nd by 2, we get $14x_1 = 14$. So, $x_1 = 1$.

Again subtracting 1st equation from 2nd after multiplying first by 3 and 2nd by 4, we get $14x_2 = 14$. So, $x_2 = 1$. Hence the basic solution is $X_1 = (1, 1, 0, 0)$.

(ii) For basis matrix $B_2 = (a_1, a_3) = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$, basic variables are x_1, x_3 and non-basic variables are x_2, x_4. Putting $x_2 = x_4 = 0$, the system reduce to

$$4x_1 + 3x_3 = 6$$
$$3x_1 + 4x_3 = 8$$

Solving this system we get, $x_1 = 0$, $x_3 = 2$. Hence the basic solution is $X_2 = (0, 0, 2, 0)$.

(iii) For basis matrix $B_4 = (a_2, a_3) = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$, basic variables are x_2, x_3 and non-basic variables are x_1, x_4. Putting $x_1 = x_4 = 0$, the system reduce to

$$2x_2 + 3x_3 = 6$$
$$5x_2 + 4x_3 = 8$$

Solving this system we get, $x_2 = 0$, $x_3 = 2$. Hence the basic solution is $X_3 = (0, 0, 2, 0)$.

(iv) For basis matrix $B_5 = (a_2, a_4) = \begin{pmatrix} 2 & -8 \\ 5 & -6 \end{pmatrix}$, basic variables are x_2, x_4 and non-basic variables are x_1, x_3. Putting $x_1 = x_3 = 0$, the system reduce to

$$2x_2 - 8x_4 = 6$$
$$5x_2 - 6x_4 = 8$$

Solving this system we get, $x_2 = 1$, $x_4 = -½$. Hence the basic solution is $X_4 = (0, 1, 0, - ½)$.

(v) For basis matrix $B_6 = (a_3, a_4) = \begin{pmatrix} 3 & -8 \\ 4 & -6 \end{pmatrix}$, basic variables are x_3, x_4 and non-basic variables are x_1, x_2. Putting $x_1 = x_2 = 0$, the system reduce to

$$3x_3 - 8x_4 = 6$$
$$4x_3 - 6x_4 = 8$$

Solving this system we get, $x_3 = 2$, $x_4 = 0$. Hence the basic solution is $X_5 = (0, 0, 2, 0)$.

Note: The solutions $X_1 = (1,1,0,0)$, $X_2 = (0,0,2,0)$, $X_3 = (0,0,2,0)$, $X_5 = (0,0,2,0)$ are basic feasible solutions and $X_4 = (0,1,0,- ½)$ basic non-feasible solution. Also (0,0,2,0) is degenerate basic feasible solution, (1,1,0,0) is non-degenerate basic feasible solution and (0,1,0,– ½) is non-degenerate basic solution.

1.9.7 <u>Degenerate basic solution</u>: A basic solution to a linear programming problem is called degenerate if one or more basic variable(s) vanishes. As for example, $\underline{X} = (0, 0, 2, 0)$ is a degenerate basic solution of the following linear programming problem.

Maximize $z = x_1 + 2x_2 + x_3 + 2x_4$

Subject to $4x_1 + 2x_2 + 3x_3 - 8x_4 = 6$ **[JU-87]**

$3x_1 + 5x_2 + 4x_3 - 6x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

1.9.8 <u>Non-degenerate basic solution</u>: A basic solution to a linear programming problem is called non-degenerate if no basic variable vanishes. As for example, $\underline{X} = (1, 1, 0, 0)$ is a non-degenerate basic solution of the following linear programming problem.

Maximize $z = x_1 + 2x_2 + x_3 + 2x_4$ **[JU-93, NUH-02]**

Subject to $4x_1 + 2x_2 + 3x_3 - 8x_4 = 6$

$3x_1 + 5x_2 + 4x_3 - 6x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

1.9.9 <u>Basic feasible solution</u>: A basic feasible solution is a basic solution of a linear programming problem, which also satisfies all basic variables are non-negative. That is a basic solution, which satisfies all the constraints (equality or inequality constraints and the non-negativity constraints) is called a basic feasible solution. As for example, $\underline{X} = (1, 1, 0, 0)$ is a basic feasible solution of the following linear programming problem. **[NU-98, 02]**

Maximize $z = x_1 + 2x_2 + x_3 + 2x_4$

Subject to $4x_1 + 2x_2 + 3x_3 - 8x_4 = 6$

$3x_1 + 5x_2 + 4x_3 - 6x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

1.9.10 <u>Degenerate basic feasible solution</u>: A degenerate basic solution to a linear programming problem is called degenerate basic feasible solution if it satisfies all the constraints (equality or inequality constraints and the non-negativity constraints) of that problem. As for example, $\underline{X} = (0, 0, 2, 0)$ is a degenerate basic feasible solution of the following linear programming problem.

Maximize $z = x_1 + 2x_2 + x_3 + 2x_4$ **[JU-88]**

Subject to $4x_1 + 2x_2 + 3x_3 - 8x_4 = 6$

$3x_1 + 5x_2 + 4x_3 - 6x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

1.9.11 <u>Non-degenerate basic feasible solution</u>: A non-degenerate basic solution to a linear programming problem is called non-degenerate basic feasible solution if it satisfies all the constraints (equality or inequality constraints and the non-negativity constraints) of that problem. As for example, $\underline{X} = (1, 1, 0, 0)$ is a non-degenerate basic feasible solution of the following linear programming problem. **[RU-90]**

$$\begin{aligned}
\text{Maximize } z = \ & x_1 + 2x_2 + x_3 + 2x_4 \\
\text{Subject to } \ & 4x_1 + 2x_2 + 3x_3 - 8x_4 = 6 \\
& 3x_1 + 5x_2 + 4x_3 - 6x_4 = 8 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

1.9.12 <u>Optimum solution</u>: (চরম সমাধন) Any feasible solution that optimizes (minimizes or maximizes) the objective function of a general LP problem is called an optimum solution of the general LP problem. The term optimal solution is also used for optimum solution. As for example, $\underline{X}$ = (0, 0, 2, 0) is an optimum solution of the following linear programming problem.**[NU-01, 02, 04, 05,07]**

$$\begin{aligned}
\text{Minimize } z = \ & x_1 + 2x_2 + x_3 + 2x_4 \\
\text{Subject to } \ & 4x_1 + 2x_2 + 3x_3 - 8x_4 = 6 \\
& 3x_1 + 5x_2 + 4x_3 - 6x_4 = 8 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

N.B: Mathematica Code: **ConstrainedMin[x_1 +2x_2 + x_3 +2x_4,{4x_1 + 2x_2 + 3x_3 – 8x_4 = = 6,3x_1 + 5x_2 + 4x_3 – 6x_4 = = 8},{x_1,x_2,x_3,x_4}]**
N.B: The optimum solution must be a basic feasible solution.

1.9.13 <u>Slack variables</u>: Let the constraints of a general LP problem be $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$; i = 1, 2, 3, . . ., m **[NU-01,03,04,05,07]**

Then, the non-negative variables x_{n+i} which satisfies

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \ ; \ i = 1, 2, 3, ..., m$$

are called slack variables. That is, when the constraints are inequalities connected by the sign '$\leq$', in each inequality an extra non-negative variable is to be added to the left hand side of the inequality to convert it into an equation. These variables are known as slack variables. As for example, we add a non-negative variable

x_3 to the left hand side of the inequality $2x_1 + 3x_2 \leq 4$ to convert it an equation $2x_1 + 3x_2 + x_3 = 4$. This 'x_3' is a slack variable.

Note: The variables are called slack variables, because
Slack = Requirement – Production

1.9.14 <u>Surplus variables</u>: Let the constraints of a general LP problem be $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$; i = 1, 2, 3, . . ., m **[NU-03, 05,07]**

Then, the non-negative variables x_{n+i} which satisfies

$$\sum_{j=1}^{n} a_{ij} x_j - x_{n+i} = b_i \text{ ; } i = 1, 2, 3, ..., m$$

are called surplus variables. That is, when the constraints are inequalities connected by the sign '$\geq$', in each inequality an extra non-negative variable is to be subtracted from the left hand side of the inequality to convert it into an equation. These variables are known as surplus variables. As for example, we subtract a non-negative variable x_3 from the left hand side of the inequality $2x_1 + 3x_2 \geq 4$ to convert it an equation $2x_1 + 3x_2 - x_3 = 4$. This 'x_3' is a surplus variable.

Note: The variables x_{n+i} are called surplus variables, because
Surplus = Production – Requirement

<u>Example (1.10)</u>: Express the following LP problem into standard form: Maximize $z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$

Solution: Introducing slack s_1 and surplus s_2 variables, the problem in the standard form can be expressed as

Maximize $z = 3x_1 + 2x_2$

Subject to $2x_1 + x_2 + s_1 = 2$

$3x_1 + 4x_2 - s_2 = 12$

$x_1, x_2, s_1, s_2 \geq 0$

Theorem (1.1): If a linear programming problem $\underline{A}.\underline{X} = \underline{b}$, $\underline{X} \geq \underline{0}$ (where $\underline{A}$ is a $m\times n$ matrix with rank m, $\underline{X}$, $\underline{b}$ and $\underline{0}$ are $m\times 1$ column vectors and $m < n$) has a feasible solution then it has at least one basic feasible solution. **[RU-97]**

Proof: Let $\underline{X} = (x_1, x_2, ..., x_n)$ be a feasible solution of the linear programming problem $\underline{A}.\underline{X} = \underline{b}$, $\underline{X} \geq \underline{0}$. Among the n components of the feasible solution, suppose first k ($k \leq n$) components be positive and the remaining $(n - k)$ components be zero. That is, $\underline{X} = (x_1, x_2, ..., x_k, 0, 0, ...,0)$ and if $\underline{a}_1, \underline{a}_2, ..., \underline{a}_k$ be the column vectors of the matrix $\underline{A}$ corresponding to the variables $x_1, x_2, ..., x_k$ then $x_1\underline{a}_1 + x_2\underline{a}_2 + ... + x_k\underline{a}_k = \underline{b}$ i.e, $\sum_{i=1}^{k} x_i \underline{a}_i = \underline{b}$ ... (i)

Case –1: If $k\leq m$ and column vectors $\underline{a}_1, \underline{a}_2, ..., \underline{a}_k$ are linearly independent then it is clear that the feasible solution is a basic feasible solution.

Case – 2: If $k > m$ then the column vectors $\underline{a}_1, \underline{a}_2, ..., \underline{a}_k$ are not linearly independent and hence the solution $\underline{X}=(x_1,x_2,...,x_k,0,0,...,0)$ is not a basic feasible solution. So there exist scalar $s_1, s_2, ..., s_k$ not all zero such that $s_1\underline{a}_1 + s_2\underline{a}_2 + ... + s_k\underline{a}_k = \underline{0}$; i.e, $\sum_{i=1}^{k} s_i \underline{a}_i = \underline{0}$ ... (ii)

Here, at least one s_i is positive (if necessary, multiply both sides of (ii) by –1 to make an s_i positive). Let $p = \underset{i}{Max}\left(\frac{s_i}{x_i}\right)$, $i = 1, 2, ..., k$ and then $p > 0$ as all $x_i > 0$ and at least one $s_i > 0$ for $i = 1, 2, ..., k$.

Doing (i) $-\frac{1}{p}$ (ii), we get $\sum_{i=1}^{k}(x_i - \frac{s_i}{p})\underline{a}_i = \underline{b}$ or $\sum_{i=1}^{k} x_i' \underline{a}_i = \underline{b}$ where $x_i' = x_i - \frac{s_i}{p}$. It implies that

$$\underline{X}' = \left(x_1 - \frac{s_1}{p}, x_2 - \frac{s_2}{p}, \ldots, x_k - \frac{s_k}{p}, 0, 0, \ldots, 0 \right) = (x_1', x_2', \ldots, x_k', 0, 0, \ldots, 0)$$

is a solution to the set of equations $\underline{A}.\underline{X} = \underline{b}$.

Since $p \geq \frac{s_i}{x_i}$ [at least for one i equality holds as $p = \underset{i}{Max}\left(\frac{s_i}{x_i}\right)$],

we have $x_i \geq \frac{s_i}{p} \Rightarrow x_i - \frac{s_i}{p} \geq 0 \Rightarrow x_i' \geq 0$ and for at least one i, $x_i' = 0$.

So, $\underline{X}' = (x_1', x_2', \ldots, x_k', 0, 0, \ldots, 0)$ with maximum (k –1) positive components is a feasible solution of the linear programming problem. Repeatedly applying this method, a feasible solution with only m positive components will be obtained finally. And then if the column vectors corresponding to the non-zero variables are linearly independent, the final solution will be non-degenerate basic feasible solution.

Case – 3: If $k \leq m$ and column vectors $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_k$ are not linearly independent then the feasible solution is not a basic feasible solution. Applying the above procedure given in case-2, the number of non-zero variables can be reduced to r (r < m) of which all the non-zero values are positive so that the associative column vectors $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_r$ are linearly independent and the solution is a basic feasible solution.

Example (1.11): If $x_1 = 2$, $x_2 = 3$, $x_3 = 1$ is a feasible solution of the system of linear equations $2x_1 + x_2 + 4x_3 = 11$ **[NU-98]**

$$3x_1 + x_2 + 5x_3 = 14.$$

Reduce the above feasible solution to a basic feasible solution.

Solution: We can write the given system in the matrix form as follows:

$$\underline{A}.\underline{X} = \underline{b} \text{ where, } \underline{A} = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 1 & 5 \end{pmatrix},\ \underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } \underline{b} = \begin{pmatrix} 11 \\ 14 \end{pmatrix}$$

The rank of $\underline{A}$ is 2 and hence the given equations are linearly independent. The given solution $(x_1, x_2, x_3) = (2, 3, 1)$ is feasible as all components are non-negative, but not basic feasible because basic feasible solution to a system of two equations cannot have more than 2 non-zero components.

Let the column vectors of the coefficient matrix of $\underline{A}$ are

$$\underline{a}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix},\ \underline{a}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \underline{a}_3 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

and they are not linearly independent because more than two vectors of two components cannot be independent. So, there exists three scalars s_1, s_2, s_3 not all zero such that

$$s_1\underline{a}_1 + s_2\underline{a}_2 + s_3\underline{a}_3 = \underline{0} \qquad \text{... (i)}$$

$$\Rightarrow s_1\begin{pmatrix} 2 \\ 3 \end{pmatrix} + s_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + s_3\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \underline{0} \quad \Rightarrow \begin{pmatrix} 2s_1 + s_2 + 4s_3 \\ 3s_1 + s_2 + 5s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2s_1 + s_2 + 4s_3 = 0 \\ 3s_1 + s_2 + 5s_3 = 0 \end{cases}$$

$$\Rightarrow \frac{s_1}{5-4} = \frac{s_2}{12-10} = \frac{s_3}{2-3} = \alpha \text{ (say)}$$

$$\Rightarrow \frac{s_1}{1} = \frac{s_2}{2} = \frac{s_3}{-1} = \alpha = 1 \text{ (say)}$$

$\Rightarrow$ $s_1 = 1$, $s_2 = 2$, $s_3 = -1$

So form (i) we get, $\underline{a}_1 + 2\underline{a}_2 - \underline{a}_3 = \underline{0}$ ... (ii)

Let p = $\underset{i}{Max}\left(\dfrac{s_i}{x_i}\right)$ for $x_i > 0$

$\Rightarrow$ p = max $\left(\frac{s_1}{x_1}, \frac{s_2}{x_2}, \frac{s_3}{x_3}\right) \Rightarrow$ p = max $\left(\frac{1}{2}, \frac{2}{3}, \frac{-1}{1}\right) = \frac{2}{3}$

Hence we get new feasible solution as follows:

$$\underline{X}_1 = \left(x_1 - \frac{s_1}{p}, x_2 - \frac{s_2}{p}, x_3 - \frac{s_3}{p}\right) = \left(2 - \frac{1}{\frac{2}{3}}, 3 - \frac{2}{\frac{2}{3}}, 1 - \frac{-1}{\frac{2}{3}}\right) = \left(\frac{1}{2}, 0, \frac{5}{2}\right)$$

The sub matrix formed by the column vectors of $\underline{A}$ corresponding to positive variables of the new solution is $B_1 = (\underline{a}_1, \underline{a}_3) = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$ and $|B_1| = -2 \neq 0$. So, $\underline{a}_1$, $\underline{a}_3$ are linearly independent and hence $\underline{X}_1 = \left(\frac{1}{2}, 0, \frac{5}{2}\right)$ is the required basic feasible solution corresponding the given feasible solution.

Again we can write equation (ii) as follows: $-\underline{a}_1 - 2\underline{a}_2 + \underline{a}_3 = \underline{0}$ and so $s_1 = -1$, $s_2 = -2$, $s_3 = 1$. Let p = $\underset{i}{Max}\left(\frac{s_i}{x_i}\right)$ for $x_i > 0$

$\Rightarrow$ p = max $\left(\frac{s_1}{x_1}, \frac{s_2}{x_2}, \frac{s_3}{x_3}\right) \Rightarrow$ p = max $\left(\frac{-1}{2}, \frac{-2}{3}, \frac{1}{1}\right) = 1$

Hence we get another new feasible solution as follows:

$$\underline{X}_2 = \left(x_1 - \frac{s_1}{p}, x_2 - \frac{s_2}{p}, x_3 - \frac{s_3}{p}\right) = \left(2 - \frac{-1}{1}, 3 - \frac{-2}{1}, 1 - \frac{1}{1}\right) = (3,5,0)$$

The sub matrix formed by the column vectors of $\underline{A}$ corresponding to positive variables of solution (3,5,0) is $B_2 = (\underline{a}_1, \underline{a}_2) = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ and $|B_2| = -1 \neq 0$. So, $\underline{a}_1$, $\underline{a}_2$ are linearly independent and hence $\underline{X}_2$ = (3, 5, 0) is another required basic feasible solution corresponding the given feasible solution.

Therefore, $\left(\frac{1}{2}, 0, \frac{5}{2}\right)$ and (3, 5, 0) are required basic feasible solutions corresponding the given feasible solution.

1.10 Some done examples:

Example (1.12): Find the numerical values of

(i) $\sum_{i=1}^{5} 3i^2$ and (ii) $\sum_{x=1}^{4}(3+2x)$

Solution: (i) Here, $\sum_{i=1}^{5} 3i^2 = 3.1^2 + 3.2^2 + 3.3^2 + 3.4^2 + 3.5^2$

$= 3 + 12 + 27 + 48 + 75$

$= 165$

(ii) Here, $\sum_{x=1}^{4}(3+2x) = (3 + 2.1) + (3 + 2.2) + (3 + 2.3) + (3 + 2.4)$

$= 5 + 7 + 9 + 11 = 32$

Example (1.13): Write a linear programming problem that contains two variables.

Solution: The following is a linear programming problem which contains two variables namely x and y.

Minimize $z = 5x - 4y$

Subject to $3x + 2y \geq 10$

$6x - 3y \geq 2$

$x, y \geq 0$

Example (1.14): Convert the following minimization problem to maximization type. Minimize $z = 7x_1 - 3x_2 + 5x_3$

Subject to $2x_1 - 3x_2 + 2x_3 \leq -5$

$5x_1 + 2x_2 + 3x_3 \geq 2$

$3x_1 + 2x_3 = 20$

$x_1, x_2, x_3 \geq 0$

Solution: Changing the sign of the objective function, we get the given problem as maximization type

Maximize $(-z) = -7x_1 + 3x_2 - 5x_3$
Subject to $2x_1 - 3x_2 + 2x_3 \leq -5$
$5x_1 + 2x_2 + 3x_3 \geq 2$
$3x_1 + 2x_3 = 20$
$x_1, x_2, x_3 \geq 0$

Example (1.15): Transform the following linear programming problem to the standard form.

Minimize $2x + y + 4z$
Subject to $-2x + 4y \leq 5$
$x + 2y + z \geq 5$
$2x + 3z \leq 5$
$x, y, z \geq 0$

Solution: Adding slack variable $u \geq 0$, $v \geq 0$ to the left hand side of the first and third constraints respectively and also subtracting surplus variable $w \geq 0$ from the left hand side of the second constraint, we get the following standard linear programming problem of the given problem:

Minimize $2x + y + 4z + 0u + 0v + 0w$
Subject to $-2x + 4y + u = 5$
$x + 2y + z - w = 5$
$2x + 3z + v = 5$
$x, y, z, u, v, w \geq 0$

Example (1.16): Reduce the following linear program to standard form: Maximize $z = 3x_1 + 2x_2 + 5x_3$ **[NU-00]**
Subject to $2x_1 - 3x_2 \leq 3$
$x_1 + 2x_2 + 3x_3 \geq 5$
$3x_1 + 2x_3 \leq 2$
$x_1, x_2, x_3 \geq 0$

Solution: Adding slack variables $x_4 \geq 0$ and $x_5 \geq 0$ to the left hand sides of first and third constraints respectively and subtracting surplus variable $x_6 \geq 0$ from the left hand side of second constraint, we get,

Maximize $z = 3x_1 + 2x_2 + 5x_3 + 0x_4 + 0x_5 + 0x_6$
Subject to $2x_1 - 3x_2 + x_4 = 3$
$x_1 + 2x_2 + 3x_3 - x_6 = 5$
$3x_1 + 2x_3 + x_5 = 2$
$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

This is the required standard form.

Example (1.17): Convert the following maximization problem to minimization type then reduce it to standard form:

Maximize $z = 2x_1 + 2x_2 + 5x_3$ **[JU-88]**
Subject to $2x_1 - 3x_2 + x_3 \leq -3$
$x_1 + 2x_2 + 3x_3 \geq 5$
$5x_1 + 2x_3 = 8$
$x_1, x_2, x_3 \geq 0$

Solution: Changing the sign of the objective function, we get the given problem as minimization type

Minimize $(-z) = -2x_1 - 2x_2 - 5x_3$
Subject to $2x_1 - 3x_2 + x_3 \leq -3$
$x_1 + 2x_2 + 3x_3 \geq 5$
$5x_1 + 2x_3 = 8$
$x_1, x_2, x_3 \geq 0$

Adding slack variable $x_4 \geq 0$ to the LHS of the 1st constraint and subtracting surplus variable $x_5 \geq 0$ from the LHS of 2nd constraint, we get,

Minimize $(-z) = -2x_1 - 2x_2 - 5x_3$
Subject to $2x_1 - 3x_2 + x_3 + x_4 = -3$
$x_1 + 2x_2 + 3x_3 - x_5 = 5$
$5x_1 + 2x_3 = 8$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

Multiplying first constraint by -1, we get

Minimize $(-z) = -2x_1 - 2x_2 - 5x_3$
Subject to $-2x_1 + 3x_2 - x_3 - x_4 = 3$
$x_1 + 2x_2 + 3x_3 - x_5 = 5$
$5x_1 + 2x_3 = 8$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

This is the required standard form.

<u>Example (1.18)</u>: Reduce the following linear program to standard form: Maximize $z = 2x_1 + 3x_2 + 5x_3$ **[RU-90]**

Subject to $2x_1 - 5x_2 \leq 5$

$3x_1 + 2x_2 + 3x_3 \geq -3$

$5x_1 + x_2 + 2x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Solution: Adding slack variables $x_4 \geq 0$ and $x_5 \geq 0$ to the left hand sides of first and third constraints respectively and subtracting surplus variable $x_6 \geq 0$ from the left hand side of second constraint, we get Maximize $z = 2x_1 + 3x_2 + 5x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to $2x_1 - 5x_2 + x_4 = 5$

$3x_1 + 2x_2 + 3x_3 - x_6 = -3$

$5x_1 + x_3 + 2x_5 = 10$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Multiplying second constraint by –1, we get

Maximize $z = 2x_1 + 3x_2 + 5x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to $2x_1 - 5x_2 + x_4 = 5$

$-3x_1 - 2x_2 - 3x_3 + x_6 = 3$

$5x_1 + x_3 + 2x_5 = 10$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

This is the required standard form.

<u>Example (1.19)</u>: Convert the following linear programming problem into the standard form: **[DU-00]**

Maximize $z = 5x_1 + 4x_2 + 3x_3$

Subject to $2x_1 + 2x_2 - 5x_3 \leq 9$

$3x_1 - x_2 + 6x_3 \geq 2$

$x_1 + x_2 + 3x_3 = 5$

$x_1 \geq 0$, $x_2 \leq 0$, x_3 unrestricted.

Solution: Putting $x_2 = -x_2'$, $x_3 = x_3' - x_3''$; $x_2' \geq 0$, $x_3' \geq 0$, $x_3'' \geq 0$ and introducing slack variable $s_1 \geq 0$ in 1st constraint and surplus variable $s_2 \geq 0$ in 2nd constraint, we get the problem as follows:

Maximize z = $5x_1 - 4x_2' + 3x_3' - 3x_3'' + 0.s_1 + 0.s_2$

Subject to $2x_1 - 2x_2' - 5x_3' + 5x_3'' + s_1 = 9$

$3x_1 + x_2' + 6x_3' - 6x_3'' - s_2 = 2$

$x_1 - x_2' + 3x_3' - 3x_3'' = 5$

$x_1, x_2', x_3', x_3'', s_1, s_2 \geq 0.$

This is the required standard form.

Example (1.20): Convert the problem into the standard form:

Maximize z = $8x_1 + 4x_2 + 3x_3$ **[JU-01]**

Subject to $6x_1 + 2x_2 - 5x_3 \leq 3$

$9x_1 - x_2 + 6x_3 \geq 5$

$x_1 + 3x_2 + x_3 = 1$

$x_1 \geq 2$, $x_2 \leq 0$, x_3 unrestricted.

Solution: Putting $x_1 = x_1' + 2$, $x_2 = -x_2'$, $x_3 = x_3' - x_3''$; $x_1' \geq 0, x_2' \geq 0$, $x_3' \geq 0$, $x_3'' \geq 0$ and adding slack variable $s_1 \geq 0$ to the left hand side of 1st constraint and subtracting surplus variable $s_2 \geq 0$ from left hand side of 2nd constraint, we get the problem as follows:

Maximize z = $8(x_1' + 2) + 4(-x_2') + 3(x_3' - x_3'') + 0.s_1 + 0.s_2$

Subject to $6(x_1' + 2) + 2(-x_2') - 5(x_3' - x_3'') + s_1 = 3$

$9(x_1' + 2) - (-x_2') + 6(x_3' - x_3'') - s_2 = 5$

$(x_1' + 2) + 3(-x_2') + (x_3' - x_3'') = 1$

$x_1', x_2', x_3', x_3'', s_1, s_2 \geq 0.$

After simplifying, we get

Maximize z = $8x_1' - 4x_2' + 3x_3' - 3x_3'' + 0.s_1 + 0.s_2 + 16$

Subject to $6x_1' - 2x_2' - 5x_3' + 5x_3'' + s_1 = -9$

$9x_1' + x_2' + 6x_3' - 6x_3'' - s_2 = -12$

$x_1' - 3x_2' + x_3' - x_3'' = -1$

$x_1', x_2', x_3', x_3'', s_1, s_2 \geq 0.$

Multiplying the three constraints by – 1, we get the following form

Maximize z = $8x_1' - 4x_2' + 3x_3' - 3x_3'' + 0.s_1 + 0.s_2 + 16$

Subject to $-6x_1' + 2x_2' + 5x_3' - 5x_3'' - s_1 = 9$

$-9x_1' - x_2' - 6x_3' + 6x_3'' + s_2 = 12$

$-x_1' + 3x_2' - x_3' + x_3'' = 1$

$x_1', x_2', x_3', x_3'', s_1, s_2 \geq 0.$

This is the required standard form.

<u>Example (1.21)</u>: Transform the following linear programming problem to the standard form. **[JU-89]**

Minimize $5x + 2y + 4z$

Subject to $-2x + 3y = 7$

$3x + 2y + z = 8$

$2x + 5z \approx 11$

$x, y, z \geq 0$

Solution: Introducing both slack variable $u \geq 0$, and surplus variable $v \geq 0$ to the left hand side of the third constraints, we get the following standard linear programming problem of the given problem: Minimize $5x + 2y + 4z + 0u + 0v$

Subject to $-2x + 3y = 7$

$3x + 2y + z = 8$

$2x + 5z + u - v = 11$

$x, y, z, u, v \geq 0$

<u>Example (1.22)</u>: Convert the following linear program into the standard form: Minimize $z = 5x_1 + 4x_2 + x_3$ **[DU-93, JU-00]**

Subject to $6x_1 + 2x_2 - 5x_3 \leq 3$

$9x_1 - x_2 + 6x_3 \geq 5$

$2x_1 + 3x_2 + x_3 \approx 1$

$x_1 \geq 2$, $x_2 \leq 0$, x_3 unrestricted.

Solution: Putting $x_1 = x_1' + 2$, $x_2 = -x_2'$, $x_3 = x_3' - x_3''$; $x_1' \geq 0, x_2' \geq 0,$ $x_3' \geq 0$, $x_3'' \geq 0$ and adding slack variable $s_1 \geq 0$ to the left hand side of 1st constraint and subtracting surplus variable $s_2 \geq 0$ from left hand side of 2nd constraint and adding $s_3 - s_4$ (here, $s_3 \geq 0$ is slack variable and $s_4 \geq 0$ is surplus variable) to the left hand side of the third constraint, we get the problem as follows:

Minimize z = $5(x_1^{/} + 2) + 4(-x_2^{/}) + (x_3^{/} - x_3^{//}) + 0s_1 + 0s_2 + 0s_3 + 0s_4$

Subject to $6(x_1^{/} + 2) + 2(-x_2^{/}) - 5(x_3^{/} - x_3^{//}) + s_1 = 3$

$9(x_1^{/} + 2) - (-x_2^{/}) + 6(x_3^{/} - x_3^{//}) - s_2 = 5$

$2(x_1^{/} + 2) + 3(-x_2^{/}) + (x_3^{/} - x_3^{//}) + (s_3 - s_4) = 1$

$x_1^{/}, x_2^{/}, x_3^{/}, x_3^{//}, s_1, s_2, s_3, s_4 \geq 0.$

After simplifying, we get

Minimize z = $5x_1^{/} - 4x_2^{/} + x_3^{/} - x_3^{//} + 10$

Subject to $6x_1^{/} - 2x_2^{/} - 5x_3^{/} + 5x_3^{//} + s_1 = -9$

$9x_1^{/} + x_2^{/} + 6x_3^{/} - 6x_3^{//} - s_2 = -12$

$2x_1^{/} - 3x_2^{/} + x_3^{/} - x_3^{//} + s_3 - s_4 = -3$

$x_1^{/}, x_2^{/}, x_3^{/}, x_3^{//}, s_1, s_2, s_3, s_4 \geq 0.$

Multiplying the three constraints by – 1, we get the following form

Minimize z = $5x_1^{/} - 4x_2^{/} + x_3^{/} - x_3^{//} + 10$

Subject to $-6x_1^{/} + 2x_2^{/} + 5x_3^{/} - 5x_3^{//} - s_1 = 9$

$-9x_1^{/} - x_2^{/} - 6x_3^{/} + 6x_3^{//} + s_2 = 12$

$-2x_1^{/} + 3x_2^{/} - x_3^{/} + x_3^{//} - s_3 + s_4 = 3$

$x_1^{/}, x_2^{/}, x_3^{/}, x_3^{//}, s_1, s_2, s_3, s_4 \geq 0.$

This is the required standard form.

Example (1.23): Convert the following linear program into the standard form: Minimize z = $-3x_1 + 4x_2 - 2x_3 + 5x_4$ [NU-01,JU-94]

Subject to $4x_1 - x_2 + 2x_3 - x_4 = -2$

$x_1 + x_2 + 3x_3 - x_4 \leq 5$

$-2x_1 + 3x_2 - x_3 + 2x_4 \geq 1$

$x_1, x_2 \geq 0, x_3 \leq 0, x_4$ unrestricted in sign.

Solution: Putting $x_3 = -x_3^{/}$, $x_4 = x_4^{/} - x_4^{//}$; $x_3^{/} \geq 0$, $x_4^{/} \geq 0$, $x_4^{//} \geq 0$ and adding slack variable $s_1 \geq 0$ to the left hand side of 2nd constraint and subtracting surplus variable $s_2 \geq 0$ from left hand side of 3rd constraint, we get the problem as follows:

Minimize $z = -3x_1 + 4x_2 - 2(-x_3^{/}) + 5(x_4^{/} - x_4^{//}) + 0s_1 + 0s_2$

Subject to
$$4x_1 - x_2 + 2(-x_3^{/}) - (x_4^{/} - x_4^{//}) = -2$$
$$x_1 + x_2 + 3(-x_3^{/}) - (x_4^{/} - x_4^{//}) + s_1 = 5$$
$$-2x_1 + 3x_2 - (-x_3^{/}) + 2(x_4^{/} - x_4^{//}) - s_2 = 1$$
$$x_1, x_2, x_3^{/}, x_4^{/}, x_4^{//}, s_1, s_2 \geq 0$$

Multiplying the first constraint by –1, we get

Minimize $z = -3x_1 + 4x_2 + 2x_3^{/} + 5x_4^{/} - 5x_4^{//} + 0s_1 + 0s_2$

Subject to
$$-4x_1 + x_2 + 2x_3^{/} + x_4^{/} - x_4^{//} = 2$$
$$x_1 + x_2 - 3x_3^{/} - x_4^{/} + x_4^{//} + s_1 = 5$$
$$-2x_1 + 3x_2 + x_3^{/} + 2x_4^{/} - 2x_4^{//} - s_2 = 1$$
$$x_1, x_2, x_3^{/}, x_4^{/}, x_4^{//}, s_1, s_2 \geq 0$$

This is the required standard form.

<u>Example (1.24)</u>: Transform the following linear programming problem to the canonical form: Minimize $2x + 3y + 4z$

Subject to
$$-2x + 4y \leq 3$$
$$x + 2y + z \geq 4$$
$$2x + 3z \leq 5$$
$$x, y, z \geq 0$$

Solution: Multiplying the first and third constraints by – 1, we get

Minimize $2x + 3y + 4z$

Subject to
$$2x - 4y \geq -3$$
$$x + 2y + z \geq 4$$
$$-2x - 3z \geq -5$$
$$x, y, z \geq 0$$

This is the required canonical form.

Example (1.25): Transform the linear programming problem to the canonical form. Maximize $2x + 3y + 4z$

Subject to $-2x + 4y \le 3$

$x + 2y + z \ge 4$

$2x + 3z \le 5$

$x, y, z \ge 0$

Solution: Multiplying the second constraint by -1, we get

Maximize $2x + 3y + 4z$

Subject to $-2x + 4y \le 3$

$-x - 2y - z \le -4$

$2x + 3z \le 5$

$x, y, z \ge 0$

This is the required canonical form.

Example (1.26): Reduce the following LP problem into canonical form: Minimize $z = 2x_1 + x_2$

Subject to $3x_1 + 2x_2 = 15$

$-3x_1 + x_2 \ge 2$

$x_1, x_2 \ge 0$

Solution: Converting the first equality constraint into two inequalities, one is '$\ge$' type and the other is '$\le$' type, then we get

Minimize $z = 2x_1 + x_2$

Subject to $3x_1 + 2x_2 \ge 15$

$3x_1 + 2x_2 \le 15$

$-3x_1 + x_2 \ge 2$

$x_1, x_2 \ge 0$

Multiplying second constraint by -1, we get

Minimize $z = 2x_1 + x_2$

Subject to $3x_1 + 2x_2 \ge 15$

$-3x_1 - 2x_2 \ge -15$

$-3x_1 + x_2 \ge 2$

$x_1, x_2 \ge 0$

This is the required canonical form.

Example (1.27): Reduce the following LP problem into canonical form: Maximize $z = x_1 + x_2$ **[NU-03, 06]**

Subject to $x_1 + 2x_2 \le 5$

$-3x_1 + x_2 \ge 3$

$x_1 \ge 0$, x_2 is unrestricted in sign.

Solution: Taking $x_2 = x_2^{/} - x_2^{//}$; $x_2^{/}, x_2^{//} \ge 0$, we have

Maximize $z = x_1 + x_2^{/} - x_2^{//}$

Subject to $x_1 + 2(x_2^{/} - x_2^{//}) \le 5$

$-3x_1 + x_2^{/} - x_2^{//} \ge 3$

$x_1, x_2^{/}, x_2^{//} \ge 0$

Multiplying second constraint by –1, we have

Maximize $z = x_1 + x_2^{/} - x_2^{//}$

Subject to $x_1 + 2x_2^{/} - 2x_2^{//} \le 5$

$3x_1 - x_2^{/} + x_2^{//} \le -3$

$x_1, x_2^{/}, x_2^{//} \ge 0$

Example (1.28): Convert the linear programming problem in canonical form: Minimize $z = 2x_1 + 3x_2 + 4x_3$ **[DU-99]**

Subject to $2x_1 + 2x_2 - 5x_3 \le 2$

$3x_1 - x_2 + 6x_3 \ge 1$

$x_1 + x_2 + x_3 = 4$

$x_1 \ge -3$, $x_2 \le 0$, x_3 unrestricted.

Solution: We can write the given problem as follows:

Minimize $z = 2x_1 + 3x_2 + 4x_3$

Subject to $2x_1 + 2x_2 - 5x_3 \le 2$

$3x_1 - x_2 + 6x_3 \ge 1$

$x_1 + x_2 + x_3 \ge 4$

$x_1 + x_2 + x_3 \le 4$

$x_1 \ge -3$, $x_2 \le 0$, x_3 unrestricted.

Or, Minimize $z = 2x_1 + 3x_2 + 4x_3$

Subject to $-2x_1 - 2x_2 + 5x_3 \ge -2$

$3x_1 - x_2 + 6x_3 \ge 1$

$x_1 + x_2 + x_3 \ge 4$

$-x_1 - x_2 - x_3 \ge -4$

$x_1 \ge -3$, $x_2 \le 0$, x_3 unrestricted.

Putting $x_1 = x_1^{/} - 3$, $x_2 = -x_2^{/}$, $x_3 = x_3^{/} - x_3^{//}$; $x_1^{/} \ge 0, x_2^{/} \ge 0, x_3^{/} \ge 0, x_3^{//} \ge 0$, we get,

$$\text{Minimize } z = 2x_1' - 3x_2' + 4x_3' - 4x_3'' - 6$$

Subject to $\quad -2x_1' + 2x_2' + 5x_3' - 5x_3'' \geq -8$

$3x_1' + x_2' + 6x_3' - 6x_3'' \geq 10$

$x_1' - x_2' + x_3' - x_3'' \geq 7$

$-x_1' + x_2' - x_3' + x_3'' \geq -7$

$x_1', x_2', x_3', x_3'' \geq 0$

This is the required canonical form.

Example (1.29): Convert the following LP problem in canonical form: Maximize $z = 2x_1 + 3x_2 + 4x_3$ **[JU-02]**

Subject to $\quad 2x_1 + 2x_2 - 5x_3 \leq 2$

$3x_1 - x_2 + 6x_3 \geq 1$

$x_1 + x_2 + x_3 = 4$

$x_1 \geq 3$, $x_2 \leq 0$, x_3 unrestricted.

Solution: We can write the given problem as follows:

Maximize $z = 2x_1 + 3x_2 + 4x_3$

Subject to $2x_1 + 2x_2 - 5x_3 \leq 2$

$3x_1 - x_2 + 6x_3 \geq 1$

$x_1 + x_2 + x_3 \geq 4$

$x_1 + x_2 + x_3 \leq 4$

$x_1 \geq 3$, $x_2 \leq 0$, x_3 unrestricted.

Or, Maximize $z = 2x_1 + 3x_2 + 4x_3$

Subject to $2x_1 + 2x_2 - 5x_3 \leq 2$

$-3x_1 + x_2 - 6x_3 \leq -1$

$-x_1 - x_2 - x_3 \leq -4$

$x_1 + x_2 + x_3 \leq 4$

$x_1 \geq 3$, $x_2 \leq 0$, x_3 unrestricted.

Putting $x_1 = x_1' + 3$, $x_2 = -x_2'$, $x_3 = x_3' - x_3''$; $x_1' \geq 0, x_2' \geq 0$, $x_3' \geq 0$, $x_3'' \geq 0$, we get problem as follows:

$$\text{Maximize } z = 2x_1' - 3x_2' + 4x_3' - 4x_3'' + 6$$

Subject to $\quad 2x_1' - 2x_2' - 5x_3' + 5x_3'' \leq -4$

$-3x_1' - x_2' - 6x_3' + 6x_3'' \leq 8$

$-x_1' + x_2' - x_3' + x_3'' \leq -1$

$x_1' - x_2' + x_3' - x_3'' \leq 1$

$x_1', x_2', x_3', x_3'' \geq 0$

Example (1.30): Find basic feasible solutions to given system:

$4x_1 + 2x_2 - 3x_3 = 1$ **[NU-97, DU-96]**

$6x_1 + 4x_2 - 5x_3 = 1$

Solution: We can write the given system in the matrix form as follows:

AX = b where, $A = \begin{pmatrix} 4 & 2 & -3 \\ 6 & 4 & -5 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The column vectors of the coefficient matrix of A are

$a_1 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$, $a_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $a_3 = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$

Since the rank of A is 2, the given system is consistent. We find ${}^3c_2 = 3$ square sub matrices taking two at a time from a_1, a_2, a_3.

$B_1=(a_1,a_2)=\begin{pmatrix} 4 & 2 \\ 6 & 4 \end{pmatrix}$, $B_2=(a_1,a_3)=\begin{pmatrix} 4 & -3 \\ 6 & -5 \end{pmatrix}$, $B_3=(a_2,a_3)=\begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix}$.

The value of determinants formed by the above sub matrices are

$|B_1| = 4$, $|B_2| = -2$, $|B_3| = 2$.

Since all sub matrices are non-singular, we shall find three basic solutions corresponding to three basis matrices B_1, B_2 and B_3.

(i) For basis matrix $B_1 = (a_1, a_2) = \begin{pmatrix} 4 & 2 \\ 6 & 4 \end{pmatrix}$, basic variables are x_1, x_2 and non-basic variable is x_3. Putting $x_3 = 0$, the system reduce to

$4x_1 + 2x_2 = 1$

$6x_1 + 4x_2 = 1$

Subtracting 2nd equation from 1st after multiplying first by 4 and 2nd by 2, we get $4x_1 = 2$. So, $x_1 = ½$.

Putting $x_1 = ½$ in any of the above equation, we get $x_2 = -½$.

Hence the basic solution is $X_1 = (½, -½, 0)$ which is not feasible as it contains negative number.

(ii) For basis matrix $B_2 = (a_1, a_3) = \begin{pmatrix} 4 & -3 \\ 6 & -5 \end{pmatrix}$, basic variables are x_1, x_3 and non-basic variables are x_2. Putting $x_2 = 0$, the system reduce to

$$4x_1 - 3x_3 = 1$$
$$6x_1 - 5x_3 = 1$$

Solving this system we get, $x_1 = 1$, $x_3 = 1$. Hence the basic solution is $X_2 = (1, 0, 1)$ which is also feasible.

(iii) For basis matrix $B_3 = (a_2, a_3) = \begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix}$, basic variables are x_2, x_3 and non-basic variables are x_1. Putting $x_1 = 0$, the system reduce to

$$2x_2 - 3x_3 = 1$$
$$4x_2 - 5x_3 = 1$$

Solving this system we get, $x_2 = -1$, $x_3 = -1$. Hence the basic solution is $X_3 = (0, -1, -1)$ which is not feasible as it contains negative numbers.

Therefore, $X_2 = (1, 0, 1)$ is the only one basic feasible solution of the given system.

Example (1.31): Find all basic solutions of the following system of simultaneous linear equations: **[DU-93, RU-90]**

$$4x_1 + 2x_2 + 3x_3 - 8x_4 = 12$$
$$3x_1 + 5x_2 + 4x_3 - 6x_4 = 16$$

Solution: We can write the given system in the matrix form as follows:

$$AX = b \text{ where, } A = \begin{pmatrix} 4 & 2 & 3 & -8 \\ 3 & 5 & 4 & -6 \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

The column vectors of the coefficient matrix of A are

$$a_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix},\ a_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix},\ a_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ and } a_4 = \begin{pmatrix} -8 \\ -6 \end{pmatrix}$$

Since the rank of A is 2, the given system is consistent. We find ${}^4c_2 = 6$ square sub matrices taking two at a time from a_1, a_2, a_3, a_4.

$B_1=(a_1,a_2)=\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$, $B_2=(a_1,a_3)=\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$, $B_3=(a_1,a_4)=\begin{pmatrix} 4 & -8 \\ 3 & -6 \end{pmatrix}$,

$B_4=(a_2,a_3)=\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$, $B_5=(a_2,a_4)=\begin{pmatrix} 2 & -8 \\ 5 & -6 \end{pmatrix}$, $B_6= (a_3,a_4)=\begin{pmatrix} 3 & -8 \\ 4 & -6 \end{pmatrix}$.

The value of determinants formed by the above sub matrices are $|B_1| = 14$, $|B_2| = 10$, $|B_3| = 0$, $|B_4| = -7$, $|B_5| = 28$, $|B_6| = 14$.

Since $|B_3| = 0$, B_3 is singular matrix. So, only five basic solutions will be found corresponding to five basis matrices B_1, B_2, B_4, B_5 and B_6.

(i) For basis matrix $B_1= (a_1, a_2) =\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$, basic variables are x_1, x_2 and non-basic variables are x_3, x_4. Putting $x_3 = x_4 = 0$, the system reduce to

$$4x_1 + 2x_2 = 12$$
$$3x_1 + 5x_2 = 16$$

Subtracting 2nd equation from 1st after multiplying first by 5 and 2nd by 2, we get $14x_1 = 28$. So, $x_1 = 2$.

Again subtracting 1st equation from 2nd after multiplying first by 3 and 2nd by 4, we get $14x_2 = 28$. So, $x_2 = 2$. Hence the basic solution is $X_1 = (2, 2, 0, 0)$.

(ii) For basis matrix $B_2 = (a_1,a_3) =\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$, basic variables are x_1, x_3 and non-basic variables are x_2, x_4. Putting $x_2 = x_4 = 0$, the system reduce to

$$4x_1 + 3x_3 = 12$$
$$3x_1 + 4x_3 = 16$$

Solving this system we get, $x_1 = 0$, $x_3 = 4$. Hence the basic solution is $X_2 = (0, 0, 4, 0)$.

(iii) For basis matrix $B_4 = (a_2,a_3)=\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$, basic variables are x_2, x_3 and non-basic variables are x_1, x_4. Putting $x_1 = x_4 = 0$, the system reduce to

$$2x_2 + 3x_3 = 12$$
$$5x_2 + 4x_3 = 16$$

Solving this system we get, $x_2 = 0$, $x_3 = 4$. Hence the basic solution is $X_3 = (0, 0, 4, 0)$.

(iv) For basis matrix $B_5 = (a_2, a_4) = \begin{pmatrix} 2 & -8 \\ 5 & -6 \end{pmatrix}$, basic variables are x_2, x_4 and non-basic variables are x_1, x_3. Putting $x_1 = x_3 = 0$, the system reduce to

$$2x_2 - 8x_4 = 12$$
$$5x_2 - 6x_4 = 16$$

Solving this system we get, $x_2 = 2$, $x_4 = -1$. Hence the basic solution is $X_4 = (0, 2, 0, -1)$.

(v) For basis matrix $B_6 = (a_3, a_4) = \begin{pmatrix} 3 & -8 \\ 4 & -6 \end{pmatrix}$, basic variables are x_3, x_4 and non-basic variables are x_1, x_2. Putting $x_1 = x_2 = 0$, the system reduce to

$$3x_3 - 8x_4 = 12$$
$$4x_3 - 6x_4 = 16$$

Solving this system we get, $x_3 = 4$, $x_4 = 0$. Hence the basic solution is $X_5 = (0, 0, 4, 0)$.

Note: The solutions $X_1 = (2,2,0,0)$, $X_2 = (0,0,4,0)$, $X_3 = (0,0,4,0)$, $X_5 = (0,0,4,0)$ are basic feasible solutions and $X_4 = (0,2,0,-1)$ basic non-feasible solution. Also $(0,0,4,0)$ is degenerate basic feasible solution, $(2,2,0,0)$ is non-degenerate basic feasible solution and $(0,2,0,-1)$ is non-degenerate basic solution.

Example (1.32): Find all basic feasible solutions of the following system of simultaneous linear equations: **[NU-00, JU-99]**

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

Solution: We can write the given system in the matrix form as follows:

AX = b where, $A = \begin{pmatrix} 2 & 6 & 2 & 1 \\ 6 & 4 & 4 & 6 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

The column vectors of the coefficient matrix of A are

$a_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, $a_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, $a_3 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $a_4 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

Since the rank of A is 2, the given system is consistent. We find ${}^4c_2 = 6$ square sub matrices taking two at a time from a_1, a_2, a_3, a_4.

$B_1=(a_1,a_2)=\begin{pmatrix} 2 & 6 \\ 6 & 4 \end{pmatrix}$, $B_2=(a_1,a_3)=\begin{pmatrix} 2 & 2 \\ 6 & 4 \end{pmatrix}$, $B_3=(a_1,a_4)=\begin{pmatrix} 2 & 1 \\ 6 & 6 \end{pmatrix}$,

$B_4=(a_2,a_3)=\begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix}$, $B_5=(a_2,a_4)=\begin{pmatrix} 6 & 1 \\ 4 & 6 \end{pmatrix}$, $B_6= (a_3,a_4)=\begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix}$.

The value of determinants formed by the above sub matrices are

$|B_1| = -28$, $|B_2| = -4$, $|B_3| = 6$, $|B_4| = 16$, $|B_5| = 32$, $|B_6| = 8$.

Since all sub matrices are non-singular, we shall find six basic solutions corresponding to six basis matrices B_1, B_2, B_3, B_4, B_5 and B_6.

(i) For basis matrix $B_1=(a_1,a_2)=\begin{pmatrix} 2 & 6 \\ 6 & 4 \end{pmatrix}$, basic variables are x_1, x_2 and non-basic variables are x_3, x_4. Putting $x_3 = x_4 = 0$, the system reduce to

$$2x_1 + 6x_2 = 3$$
$$6x_1 + 4x_2 = 2$$

Subtracting 2nd equation from 1st after multiplying first by 4 and 2nd by 6, we get $-28x_1 = 0$ or, $x_1 = 0$.

Putting $x_1 = 0$ in any of the above equation, we get $x_2 = ½$. Hence the basic solution is $X_1 = (0, ½, 0, 0)$ which is also feasible.

(ii) For basis matrix $B_2=(a_1,a_3)=\begin{pmatrix} 2 & 2 \\ 6 & 4 \end{pmatrix}$, basic variables are x_1, x_3 and non-basic variables are x_2, x_4. Putting $x_2 = x_4 = 0$, the system reduce to

$$2x_1 + 2x_3 = 3$$
$$6x_1 + 4x_3 = 2$$

Solving this system we get, $x_1 = -2$, $x_3 = \frac{7}{2}$. Hence the basic solution is $X_2 = (-2, 0, \frac{7}{2}, 0)$ which is non-feasible.

(iii) For basis matrix $B_3=(a_1,a_4)=\begin{pmatrix} 2 & 1 \\ 6 & 6 \end{pmatrix}$, basic variables are x_1, x_4 and non-basic variables are x_2, x_3. Putting $x_2 = x_3 = 0$, the system reduce to

$$2x_1 + x_4 = 3$$
$$6x_1 + 6x_4 = 2$$

Solving this system we get, $x_1 = \frac{8}{3}$, $x_4 = -\frac{7}{3}$. Hence the basic solution is $X_3 = (\frac{8}{3}, 0, 0, -\frac{7}{3})$ which is non-feasible.

(iv) For basis matrix $B_4=(a_2,a_3)=\begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix}$, basic variables are x_2, x_3 and non-basic variables are x_1, x_4. Putting $x_1 = x_4 = 0$, the system reduce to

$$6x_2 + 2x_3 = 3$$
$$4x_2 + 4x_3 = 2$$

Solving this system we get, $x_2 = ½$, $x_3 = 0$. Hence the basic solution is $X_4 = (0, ½, 0, 0)$ which is also feasible.

(v) For basis matrix $B_5=(a_2,a_4)=\begin{pmatrix} 6 & 1 \\ 4 & 6 \end{pmatrix}$, basic variables are x_2, x_4 and non-basic variables are x_1, x_3. Putting $x_1 = x_3 = 0$, the system reduce to

$$6x_2 + x_4 = 3$$
$$4x_2 + 6x_4 = 2$$

Solving this system we get, $x_2 = ½$, $x_4 = 0$. Hence the basic solution is $X_5 = (0, ½, 0, 0)$ which is also feasible.

(vi) For basis matrix $B_6= (a_3,a_4)=\begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix}$, basic variables are x_3, x_4 and non-basic variables are x_1, x_2. Putting $x_1 = x_2 = 0$, the system reduce to

$$2x_3 + x_4 = 3$$
$$4x_3 + 6x_4 = 2$$

Solving this system we get, $x_3 = 2$, $x_4 = -1$. Hence the basic solution is $X_6 = (0, 0, 2, -1)$ which is not feasible.

Therefore, the system has only one basic feasible solution, which is (0, ½, 0, 0) and the other solutions are not basic feasible because they contains negative numbers.

<u>Example (1.33)</u>: $\underline{X}^{/} = (x_1, x_2, x_3) = (1, 3, 2)$ is a feasible solution of the system of linear equations

$$2x_1 + 4x_2 - 2x_3 = 10 \qquad \textbf{[RU-92]}$$
$$10x_1 + 3x_2 + 7x_3 = 33.$$

Reduce the above feasible solution to a basic feasible solution.

Solution: We can write the given system in the matrix form as follows:

$$\underline{A}.\underline{X} = \underline{b} \text{ where, } \underline{A} = \begin{pmatrix} 2 & 4 & -2 \\ 10 & 3 & 7 \end{pmatrix}, \underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } \underline{b} = \begin{pmatrix} 10 \\ 33 \end{pmatrix}$$

The rank of $\underline{A}$ is 2 and hence the given equations are linearly independent. The given solution $\underline{X}^{/} = (1, 3, 2)$ is feasible as all components are non-negative, but not basic feasible because basic feasible solution to a system of two equations cannot have more than 2 non-zero components.

Let the column vectors of the coefficient matrix of $\underline{A}$ are

$$\underline{a}_1 = \begin{pmatrix} 2 \\ 10 \end{pmatrix}, \underline{a}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ and } \underline{a}_3 = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$$

and they are not linearly independent because more than two vectors of two components cannot be independent. So, there exists three scalars s_1, s_2, s_3 not all zero such that

$$s_1\underline{a}_1 + s_2\underline{a}_2 + s_3\underline{a}_3 = \underline{0} \quad \ldots \text{ (i)}$$

$$\Rightarrow s_1\begin{pmatrix}2\\10\end{pmatrix} + s_2\begin{pmatrix}4\\3\end{pmatrix} + s_3\begin{pmatrix}-2\\7\end{pmatrix} = \underline{0} \quad \Rightarrow \begin{pmatrix}2s_1+4s_2-2s_3\\10s_1+3s_2+7s_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \begin{cases}2s_1+4s_2-2s_3=0\\10s_1+3s_2+7s_3=0\end{cases} \Rightarrow \frac{s_1}{28+6} = \frac{s_2}{-20-14} = \frac{s_3}{6-40} = \alpha \text{ (say)}$$

$$\Rightarrow \frac{s_1}{34} = \frac{s_2}{-34} = \frac{s_3}{-34} = \alpha = -\frac{1}{34} \text{ (say)} \Rightarrow s_1 = -1,\ s_2 = 1,\ s_3 = 1$$

So form (i) we get, $-\underline{a}_1 + \underline{a}_2 + \underline{a}_3 = \underline{0}$ ... (ii)

Let p = $\underset{i}{Max}\left(\frac{s_i}{x_i}\right)$ for $x_i > 0$

$$\Rightarrow p = \max\left(\frac{s_1}{x_1}, \frac{s_2}{x_2}, \frac{s_3}{x_3}\right) \Rightarrow p = \max\left(\frac{-1}{1}, \frac{1}{3}, \frac{1}{2}\right) = \frac{1}{2}$$

Hence we get new feasible solution as follows:

$$\underline{X}_1 = \left(x_1 - \frac{s_1}{p}, x_2 - \frac{s_2}{p}, x_3 - \frac{s_3}{p}\right) = \left(1 - \frac{-1}{\frac{1}{2}}, 3 - \frac{1}{\frac{1}{2}}, 2 - \frac{1}{\frac{1}{2}}\right) = (3,1,0)$$

The sub matrix formed by the column vectors of $\underline{A}$ corresponding to positive variables of the new solution is $B_1 = (\underline{a}_1, \underline{a}_2) = \begin{pmatrix}2 & 4\\10 & 3\end{pmatrix}$

and $|B_1| = -34 \neq 0$. So, $\underline{a}_1$, $\underline{a}_2$ are linearly independent and hence $\underline{X}_1 = (3, 1, 0)$ is the required basic feasible solution corresponding the given feasible solution.

Again we can write equation (ii) as follows: $\underline{a}_1 - \underline{a}_2 - \underline{a}_3 = \underline{0}$ and so $s_1 = 1$, $s_2 = -1$, $s_3 = -1$. Let p = $\underset{i}{Max}\left(\frac{s_i}{x_i}\right)$ for $x_i > 0$

$\Rightarrow$ p = max ($\frac{s_1}{x_1}, \frac{s_2}{x_2}, \frac{s_3}{x_3}$) $\Rightarrow$ p = max ($\frac{1}{1}, \frac{-1}{3}, \frac{-1}{2}$) = 1

Hence we get another new feasible solution as follows:

$\underline{X}_2 = \left(x_1 - \frac{s_1}{p}, x_2 - \frac{s_2}{p}, x_3 - \frac{s_3}{p} \right) = \left(1 - \frac{1}{1}, 3 - \frac{-1}{1}, 2 - \frac{-1}{1} \right) = (0,4,3)$

The sub matrix formed by the column vectors of $\underline{A}$ corresponding to positive variables of the new solution is $B_2 = (\underline{a}_2, \underline{a}_3) = \begin{pmatrix} 4 & -2 \\ 3 & 7 \end{pmatrix}$

and $|B_2| = 34 \neq 0$. So, $\underline{a}_2$, $\underline{a}_3$ are linearly independent and hence $\underline{X}_2$ = (0, 4, 3) is another required basic feasible solution corresponding the given feasible solution.

Therefore, (3, 1, 0) and (0, 4, 3) are required basic feasible solutions corresponding the given feasible solution.

Example (1.34): If $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$ is a feasible solution of the system of linear equations.

$$2x_1 + 2x_2 - x_3 + 4x_4 = 7$$
$$x_1 + 3x_2 - 2x_3 + 6x_4 = 8$$

Reduce the above feasible solution to basic feasible solutions.

Solution: We can write the given system in the matrix form as follows:

$$\underline{A}.\underline{X} = \underline{b} \text{ where, } \underline{A} = \begin{pmatrix} 2 & 2 & -1 & 4 \\ 1 & 3 & -2 & 6 \end{pmatrix}, \ \underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } \underline{b} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

The rank of $\underline{A}$ is 2 and hence the given equations are linearly independent. The given solution (1, 1, 1, 1) is feasible as all components are non-negative, but not basic feasible because basic feasible solution to a system of two equations cannot have more than 2 non-zero components.

Let the column vectors of the coefficient matrix of $\underline{A}$ are

$$\underline{a}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ \underline{a}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix},\ \underline{a}_3 = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \text{ and } \underline{a}_4 = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

and they are not linearly independent because more than two vectors with two components cannot be independent. So, there exists four scalars s_1, s_2, s_3, s_4 not all zero such that

$$s_1\underline{a}_1 + s_2\underline{a}_2 + s_3\underline{a}_3 + s_4\underline{a}_4 = \underline{0} \quad \ldots \text{ (i)}$$

$$\Rightarrow s_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + s_2\begin{pmatrix} 2 \\ 3 \end{pmatrix} + s_3\begin{pmatrix} -1 \\ -2 \end{pmatrix} + s_4\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \underline{0}$$

$$\Rightarrow \begin{pmatrix} 2s_1 + 2s_2 - s_3 + 4s_4 \\ s_1 + 3s_2 - 2s_3 + 6s_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{cases} 2s_1 + 2s_2 - s_3 + 4s_4 = 0 \\ s_1 + 3s_2 - 2s_3 + 6s_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2s_1 + 2s_2 - s_3 + 4s_4 = 0 \\ -2s_1 - 6s_2 + 4s_3 - 12s_4 = 0\ [\text{Multiplying by } -2] \end{cases}$$

Adding the two equations, we get

$-4s_2 + 3s_3 - 8s_4 = 0$. This equation has many solutions and we consider a particular solution $s_2 = 0$, $s_3 = 8$, $s_4 = 3$ and hence $s_1 = -2$.

Branch –1: So form (i) we get, $-2\underline{a}_1 + 8\underline{a}_3 + 3\underline{a}_4 = \underline{0}$ ... (ii)

Let p = $\underset{i}{Max}\left(\dfrac{s_i}{x_i}\right)$ for $x_i > 0$

$$\Rightarrow p = \max\left(\frac{s_1}{x_1}, \frac{s_2}{x_2}, \frac{s_3}{x_3}, \frac{s_4}{x_4}\right) \Rightarrow p = \max\left(\frac{-2}{1}, \frac{0}{1}, \frac{8}{1}, \frac{3}{1}\right) = 8$$

Hence we get new feasible solution as follows:

$$\underline{X}_1 = \left(x_1 - \frac{s_1}{p}, x_2 - \frac{s_2}{p}, x_3 - \frac{s_3}{p}, x_4 - \frac{s_4}{p}\right) = \left(1 - \frac{-2}{8}, 1 - \frac{0}{8}, 1 - \frac{8}{8}, 1 - \frac{3}{8}\right)$$

$= \left(\frac{5}{4}, 1, 0, \frac{5}{8}\right)$ which is not a basic solution.

Now the given system of linear equations becomes as follows:

$$2x_1 + 2x_2 + 4x_4 = 7$$
$$x_1 + 3x_2 + 6x_4 = 8$$

where $x_1 = 5/4$, $x_2 = 1$, $x_4 = 5/8$ is a feasible solution.
Now we apply the above procedure again.
There exists three scalars r_1, r_2, r_4 not all zero such that

$$r_1\underline{a}_1 + r_2\underline{a}_2 + r_4\underline{a}_4 = \underline{0} \quad \ldots \text{ (iii)}$$

$$\Rightarrow r_1\begin{pmatrix}2\\1\end{pmatrix} + r_2\begin{pmatrix}2\\3\end{pmatrix} + r_4\begin{pmatrix}4\\6\end{pmatrix} = \underline{0} \quad \Rightarrow \begin{pmatrix}2r_1+2r_2+4r_4\\ r_1+3r_2+6r_4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \begin{cases}2r_1+2r_2+4r_4 = 0\\ r_1+3r_2+6r_4 = 0\end{cases} \Rightarrow \frac{r_1}{12-12} = \frac{r_2}{4-12} = \frac{r_4}{6-2} = \alpha \text{ (say)}$$

$$\Rightarrow \frac{r_1}{0} = \frac{r_2}{-8} = \frac{r_4}{4} = \alpha = -\frac{1}{4} \text{ (say)} \Rightarrow r_1 = 0,\ r_2 = 2,\ r_4 = -1$$

So form (iii) we get, $0\underline{a}_1 + 2\underline{a}_2 - \underline{a}_4 = \underline{0}$ ... (iv)

Let $q = \underset{i}{Max}\left(\frac{r_i}{x_i}\right)$ for $x_i > 0$

$$\Rightarrow q = \max\left(\frac{r_1}{x_1}, \frac{r_2}{x_2}, \frac{r_4}{x_4}\right) \quad \Rightarrow q = \max\left(\frac{0}{\frac{5}{4}}, \frac{2}{1}, \frac{-1}{\frac{5}{8}}\right) = 2$$

Hence we get new feasible solution as follows:

$$(x_1,\ x_2,\ x_4) = \left(x_1 - \frac{r_1}{q}, x_2 - \frac{r_2}{q}, x_4 - \frac{r_4}{q}\right) = \left(\frac{5}{4} - \frac{0}{2}, 1 - \frac{2}{2}, \frac{5}{8} - \frac{-1}{2}\right) =$$

$\left(\frac{5}{4}, 0, \frac{9}{8}\right)$. That is, $(x_1, x_2, x_3, x_4) = \left(\frac{5}{4}, 0, 0, \frac{9}{8}\right)$ is a new feasible solution of the given system.

The sub matrix formed by the column vectors of $\underline{A}$ corresponding to positive variables of the new feasible solution is $B_1 = (\underline{a}_1, \underline{a}_4) = \begin{pmatrix}2 & 4\\ 1 & 6\end{pmatrix}$ and $|B_1| = 8 \neq 0$. So, $\underline{a}_1$, $\underline{a}_4$ are linearly independent and hence the solution $(x_1, x_2, x_3, x_4) = \left(\frac{5}{4}, 0, 0, \frac{9}{8}\right)$ is a required basic feasible solution corresponding the given feasible solution.

Again we can write equation (iv) as follows: $0\underline{a}_1 - 2\underline{a}_2 + \underline{a}_4 = \underline{0}$ and so $r_1 = 0$, $r_2 = -2$, $r_4 = 1$.

Again let q = $\underset{i}{Max}\left(\dfrac{r_i}{x_i}\right)$ for $x_i > 0$

$\Rightarrow$ q = max $(\dfrac{r_1}{x_1}, \dfrac{r_2}{x_2}, \dfrac{r_4}{x_4})$ $\Rightarrow$ q = max $(\dfrac{0}{\frac{5}{4}}, \dfrac{-2}{1}, \dfrac{1}{\frac{5}{8}}) = \dfrac{8}{5}$

Hence we get new feasible solution as follows:

$(x_1, x_2, x_4) = \left(x_1 - \dfrac{r_1}{q}, x_2 - \dfrac{r_2}{q}, x_4 - \dfrac{r_4}{q}\right) = \left(\dfrac{5}{4} - \dfrac{0}{\frac{8}{5}}, 1 - \dfrac{-2}{\frac{8}{5}}, \dfrac{5}{8} - \dfrac{1}{\frac{8}{5}}\right) =$ $(\dfrac{5}{4}, \dfrac{9}{4}, 0)$. That is, $(x_1, x_2, x_3, x_4) = (\dfrac{5}{4}, \dfrac{9}{4}, 0, 0)$ is a new feasible solution of the given system.

The sub matrix formed by the column vectors of $\underline{A}$ corresponding to positive variables of the new feasible solution is $B_2 = (\underline{a}_1, \underline{a}_2) = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ and $|B_2| = 4 \neq 0$. So, $\underline{a}_1$, $\underline{a}_2$ are linearly independent and hence the solution $(x_1, x_2, x_3, x_4) = (\dfrac{5}{4}, \dfrac{9}{4}, 0, 0)$ is another required basic feasible solution corresponding the given feasible solution.

Branch –2: We can write equation (ii) as follows:

$$2\underline{a}_1 - 8\underline{a}_3 - 3\underline{a}_4 = \underline{0} \quad \dots \text{(v)}$$

Let p = $\underset{i}{Max}\left(\dfrac{s_i}{x_i}\right)$ for $x_i > 0$

$\Rightarrow$ p = max $(\dfrac{s_1}{x_1}, \dfrac{s_2}{x_2}, \dfrac{s_3}{x_3}, \dfrac{s_4}{x_4})$ $\Rightarrow$ p = max $(\dfrac{2}{1}, \dfrac{0}{1}, \dfrac{-8}{1}, \dfrac{-3}{1}) = 2$

Hence we get new feasible solution as follows:

$\underline{X}_2 = \left(x_1 - \dfrac{s_1}{p}, x_2 - \dfrac{s_2}{p}, x_3 - \dfrac{s_3}{p}, x_4 - \dfrac{s_4}{p}\right) = \left(1 - \dfrac{2}{2}, 1 - \dfrac{0}{2}, 1 - \dfrac{-8}{2}, 1 - \dfrac{-3}{2}\right)$

= (0, 1, 5, $\frac{5}{2}$) which is not a basic solution.

Now the given system of linear equations becomes as follows:

$$2x_2 - x_3 + 4x_4 = 7$$
$$3x_2 - 2x_3 + 6x_4 = 8$$

where $x_2 = 1$, $x_3 = 5$, $x_4 = 5/2$ is a feasible solution.

Now we apply the above procedure again.

There exists three scalars r_2, r_3, r_4 not all zero such that

$$r_2\underline{a}_2 + r_3\underline{a}_3 + r_4\underline{a}_4 = \underline{0} \qquad \text{... (vi)}$$

$$\Rightarrow r_2\begin{pmatrix}2\\3\end{pmatrix} + r_3\begin{pmatrix}-1\\-2\end{pmatrix} + r_4\begin{pmatrix}4\\6\end{pmatrix} = \underline{0} \quad \Rightarrow \quad \begin{pmatrix}2r_2 - r_3 + 4r_4\\3r_2 - 2r_3 + 6r_4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \begin{cases}2r_2 - r_3 + 4r_4 = 0\\3r_2 - 3r_3 + 6r_4 = 0\end{cases} \Rightarrow \frac{r_2}{-6+12} = \frac{r_3}{12-12} = \frac{r_4}{-6+3} = \alpha \text{ (say)}$$

$$\Rightarrow \frac{r_2}{6} = \frac{r_3}{0} = \frac{r_4}{-3} = \alpha = -\frac{1}{3} \text{ (say)} \Rightarrow r_2 = -2,\ r_3 = 0,\ r_4 = 1$$

So form (vi) we get, $-2\underline{a}_2 + 0\underline{a}_3 + \underline{a}_4 = \underline{0}$ ... (vii)

Let q = $\underset{i}{Max}\left(\frac{r_i}{x_i}\right)$ for $x_i > 0$

$$\Rightarrow q = \max\left(\frac{r_2}{x_2}, \frac{r_3}{x_3}, \frac{r_4}{x_4}\right) \quad \Rightarrow q = \max\left(\frac{-2}{1}, \frac{0}{5}, \frac{1}{\frac{5}{2}}\right) = \frac{2}{5}$$

Hence we get new feasible solution as follows:

$$(x_2,\ x_3,\ x_4) = \left(x_2 - \frac{r_2}{q}, x_3 - \frac{r_3}{q}, x_4 - \frac{r_4}{q}\right) = \left(1 - \frac{-2}{\frac{2}{5}}, 5 - \frac{0}{\frac{2}{5}}, \frac{5}{2} - \frac{1}{\frac{2}{5}}\right) =$$

(6, 5, 0). That is, (x_1, x_2, x_3, x_4) = (0, 6, 5, 0) is a new feasible solution of the given system.

The sub matrix formed by the column vectors of $\underline{A}$ corresponding to positive variables of the new feasible solution is $B_3 = (\underline{a}_2, \underline{a}_3) = \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}$ and $|B_3| = -1 \neq 0$. So, $\underline{a}_2$, $\underline{a}_3$ are linearly independent and

hence the solution $(x_1, x_2, x_3, x_4) = (0, 6, 5, 0)$ is a required basic feasible solution corresponding the given feasible solution.
Again we can write equation (vii) as follows:

$$2\underline{a}_2 - 0\underline{a}_3 - \underline{a}_4 = \underline{0} \text{ and so } r_2 = 2,\ r_3 = 0,\ r_4 = -1.$$

Again let $q = \underset{i}{Max}\left(\dfrac{r_i}{x_i}\right)$ for $x_i > 0$

$$\Rightarrow q = \max\left(\frac{r_2}{x_2}, \frac{r_3}{x_3}, \frac{r_4}{x_4}\right) \quad \Rightarrow q = \max\left(\frac{2}{1}, \frac{0}{5}, \frac{-1}{\frac{5}{2}}\right) = 2$$

Hence we get new feasible solution as follows:

$$(x_2, x_3, x_4) = \left(x_2 - \frac{r_2}{q}, x_3 - \frac{r_3}{q}, x_4 - \frac{r_4}{q}\right) = \left(1 - \frac{2}{2}, 5 - \frac{0}{2}, \frac{5}{2} - \frac{-1}{2}\right) =$$

$(0, 5, 3)$. That is, $(x_1, x_2, x_3, x_4) = (0, 0, 5, 3)$ is a new feasible solution of the given system.
The sub matrix formed by the column vectors of $\underline{A}$ corresponding to positive variables of the new feasible solution is $B_4 = (\underline{a}_3, \underline{a}_4) = \begin{pmatrix} -1 & 4 \\ -2 & 6 \end{pmatrix}$ and $|B_4| = 2 \neq 0$. So, $\underline{a}_3$, $\underline{a}_4$ are linearly independent and hence the solution $(x_1, x_2, x_3, x_4) = (0, 0, 5, 3)$ is a required basic feasible solution corresponding the given feasible solution.

In the conclusion, $(x_1, x_2, x_3, x_4) = (\frac{5}{4}, 0, 0, \frac{9}{8})$, $(\frac{5}{4}, \frac{9}{4}, 0, 0)$, $(0, 6, 5, 0)$, $(0, 0, 5, 3)$ are the required basic feasible solutions of given system corresponding to the given feasible solution.

<u>Example (1.35)</u>: Find optimum solution of the following linear programming problem:

$$\text{Maximize } z = 2x_1 + 4x_2 - x_3 + 2x_4$$

$$\text{Subject to} \quad \begin{aligned} 2x_1 + 6x_2 + 2x_3 + x_4 &= 3 \\ 6x_1 + 4x_2 + 4x_3 + 6x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Solution: We can write the given constraints in the matrix form as follows:

$$AX = b \text{ where, } A = \begin{pmatrix} 2 & 6 & 2 & 1 \\ 6 & 4 & 4 & 6 \end{pmatrix},\ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

The column vectors of the coefficient matrix of A are

$$a_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix},\ a_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix},\ a_3 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \text{ and } a_4 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

Since the rank of A is 2, the given system is consistent. We find ${}^4c_2 = 6$ square sub matrices taking two at a time from a_1, a_2, a_3, a_4.

$$B_1=(a_1,a_2)=\begin{pmatrix} 2 & 6 \\ 6 & 4 \end{pmatrix},\ B_2=(a_1,a_3)=\begin{pmatrix} 2 & 2 \\ 6 & 4 \end{pmatrix},\ B_3=(a_1,a_4)=\begin{pmatrix} 2 & 1 \\ 6 & 6 \end{pmatrix},$$

$$B_4=(a_2,a_3)=\begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix},\ B_5=(a_2,a_4)=\begin{pmatrix} 6 & 1 \\ 4 & 6 \end{pmatrix},\ B_6=(a_3,a_4)=\begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix}.$$

The value of determinants formed by the above sub matrices are

$|B_1| = -28$, $|B_2| = -4$, $|B_3| = 6$, $|B_4| = 16$, $|B_5| = 32$, $|B_6| = 8$.

Since all sub matrices are non-singular, we shall find six basic solutions corresponding to six basis matrices B_1, B_2, B_3, B_4, B_5 and B_6.

(i) For basis matrix $B_1=(a_1,a_2)=\begin{pmatrix} 2 & 6 \\ 6 & 4 \end{pmatrix}$, basic variables are x_1, x_2 and non-basic variables are x_3, x_4. Putting $x_3 = x_4 = 0$, the system reduce to

$$2x_1 + 6x_2 = 3$$
$$6x_1 + 4x_2 = 2$$

Subtracting 2nd equation from 1st after multiplying first by 4 and 2nd by 6, we get $-28x_1 = 0$ or, $x_1 = 0$.

Putting $x_1 = 0$ in any of the above equation, we get $x_2 = ½$. Hence the basic solution is $X_1 = (0, ½, 0, 0)$ which is also feasible.

(ii) For basis matrix $B_2=(a_1,a_3)=\begin{pmatrix} 2 & 2 \\ 6 & 4 \end{pmatrix}$, basic variables are x_1, x_3 and non-basic variables are x_2, x_4. Putting $x_2 = x_4 = 0$, the system reduce to

$$2x_1 + 2x_3 = 3$$
$$6x_1 + 4x_3 = 2$$

Solving this system we get, $x_1 = -2$, $x_3 = \frac{7}{2}$. Hence the basic solution is $X_2 = (-2, 0, \frac{7}{2}, 0)$ which is non-feasible.

(iii) For basis matrix $B_3=(a_1,a_4)=\begin{pmatrix} 2 & 1 \\ 6 & 6 \end{pmatrix}$, basic variables are x_1, x_4 and non-basic variables are x_2, x_3. Putting $x_2 = x_3 = 0$, the system reduce to

$$2x_1 + x_4 = 3$$
$$6x_1 + 6x_4 = 2$$

Solving this system we get, $x_1 = \frac{8}{3}$, $x_4 = -\frac{7}{3}$. Hence the basic solution is $X_3 = (\frac{8}{3}, 0, 0, -\frac{7}{3})$ which is non-feasible.

(iv) For basis matrix $B_4=(a_2,a_3)=\begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix}$, basic variables are x_2, x_3 and non-basic variables are x_1, x_4. Putting $x_1 = x_4 = 0$, the system reduce to

$$6x_2 + 2x_3 = 3$$
$$4x_2 + 4x_3 = 2$$

Solving this system we get, $x_2 = ½$, $x_3 = 0$. Hence the basic solution is $X_4 = (0, ½, 0, 0)$ which is also feasible.

(v) For basis matrix $B_5=(a_2,a_4)=\begin{pmatrix} 6 & 1 \\ 4 & 6 \end{pmatrix}$, basic variables are x_2, x_4 and non-basic variables are x_1, x_3. Putting $x_1 = x_3 = 0$, the system reduce to

$$6x_2 + x_4 = 3$$
$$4x_2 + 6x_4 = 2$$

Solving this system we get, $x_2 = ½$, $x_4 = 0$. Hence the basic solution is $X_5 = (0, ½, 0, 0)$ which is also feasible.

(vi) For basis matrix $B_6 = (a_3, a_4) = \begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix}$, basic variables are x_3, x_4 and non-basic variables are x_1, x_2. Putting $x_1 = x_2 = 0$, the system reduce to

$$2x_3 + x_4 = 3$$
$$4x_3 + 6x_4 = 2$$

Solving this system we get, $x_3 = 2$, $x_4 = -1$. Hence the basic solution is $X_6 = (0, 0, 2, -1)$ which is not feasible.

Therefore, the program has only one basic feasible solution, which is (0, ½, 0, 0) and the other solutions are not basic feasible because they contains negative numbers. We know for the maximization linear programming problem, the basic feasible solution which maximizes the value of the objective function, is optimum solution. In this problem, we have only one basic feasible solution, hence this solution is the optimum solution. Therefore, the optimum solution is $x_1 = 0$, $x_2 = ½$, $x_3 = 0$, $x_4 = 0$ and $z_{max} = 2$.

N.B: Mathematica Code to solve this LP problem is
ConstrainedMax[2x$_1$ + 4x$_2$ – x$_3$ + 2x$_4$,{2x$_1$ + 6x$_2$ + 2x$_3$ + x$_4$ = = 3, 6x$_1$ + 4x$_2$ + 4x$_3$ + 6x$_4$ = = 2},{x$_1$, x$_2$, x$_3$, x$_4$}]

Example (1.36): Find optimum solution of the following linear programming problem: Minimize $z = 2x_1 + 5x_2 + x_3 + 2x_4$

Subject to
$$4x_1 + 2x_2 + 3x_3 - 8x_4 = 12$$
$$3x_1 + 5x_2 + 4x_3 - 6x_4 = 16$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Solution: We can write the given constraints as follows:

AX = b where, $A = \begin{pmatrix} 4 & 2 & 3 & -8 \\ 3 & 5 & 4 & -6 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and $b = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$

The column vectors of the coefficient matrix of A are

$a_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $a_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $a_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $a_4 = \begin{pmatrix} -8 \\ -6 \end{pmatrix}$

Since the rank of A is 2, the given system is consistent. We find ${}^4c_2 = 6$ square sub matrices taking two at a time from a_1, a_2, a_3, a_4.

$B_1=(a_1,a_2)=\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$, $B_2=(a_1,a_3)=\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$, $B_3=(a_1,a_4)=\begin{pmatrix} 4 & -8 \\ 3 & -6 \end{pmatrix}$,

$B_4=(a_2,a_3)=\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$, $B_5=(a_2,a_4)=\begin{pmatrix} 2 & -8 \\ 5 & -6 \end{pmatrix}$, $B_6= (a_3,a_4)=\begin{pmatrix} 3 & -8 \\ 4 & -6 \end{pmatrix}$.

The value of determinants formed by the above sub matrices are

$|B_1| = 14$, $|B_2| = 10$, $|B_3| = 0$, $|B_4| = -7$, $|B_5| = 28$, $|B_6| = 14$.

Since $|B_3| = 0$, B_3 is singular matrix. So, only five basic solutions will be found corresponding to five basis matrices B_1, B_2, B_4, B_5 and B_6.

(i) For basis matrix $B_1= (a_1, a_2) =\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$, basic variables are x_1, x_2 and non-basic variables are x_3, x_4. Putting $x_3 = x_4 = 0$, the system reduce to

$$4x_1 + 2x_2 = 12$$
$$3x_1 + 5x_2 = 16$$

Subtracting 2nd equation from 1st after multiplying first by 5 and 2nd by 2, we get $14x_1 = 28$. So, $x_1 = 2$.

Again subtracting 1st equation from 2nd after multiplying first by 3 and 2nd by 4, we get $14x_2 = 28$. So, $x_2 = 2$. Hence the basic solution is $X_1 = (2, 2, 0, 0)$.

(ii) For basis matrix $B_2 = (a_1,a_3) =\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$, basic variables are x_1, x_3 and non-basic variables are x_2, x_4. Putting $x_2 = x_4 = 0$, the system reduce to

$$4x_1 + 3x_3 = 12$$
$$3x_1 + 4x_3 = 16$$

Solving this system we get, $x_1 = 0$, $x_3 = 4$. Hence the basic solution is $X_2 = (0, 0, 4, 0)$.

(iii) For basis matrix $B_4 = (a_2,a_3)=\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$, basic variables are x_2, x_3 and non-basic variables are x_1, x_4. Putting $x_1 = x_4 = 0$, the system reduce to

$$2x_2 + 3x_3 = 12$$
$$5x_2 + 4x_3 = 16$$

Solving this system we get, $x_2 = 0$, $x_3 = 4$. Hence the basic solution is $X_3 = (0, 0, 4, 0)$.

(iv) For basis matrix $B_5 = (a_2, a_4) = \begin{pmatrix} 2 & -8 \\ 5 & -6 \end{pmatrix}$, basic variables are x_2, x_4 and non-basic variables are x_1, x_3. Putting $x_1 = x_3 = 0$, the system reduce to $\quad 2x_2 - 8x_4 = 12$

$$5x_2 - 6x_4 = 16$$

Solving this system we get, $x_2 = 2$, $x_4 = -1$. Hence the basic solution is $X_4 = (0, 2, 0, -1)$.

(v) For basis matrix $B_6 = (a_3, a_4) = \begin{pmatrix} 3 & -8 \\ 4 & -6 \end{pmatrix}$, basic variables are x_3, x_4 and non-basic variables are x_1, x_2. Putting $x_1 = x_2 = 0$, the system reduce to $\quad 3x_3 - 8x_4 = 12$

$$4x_3 - 6x_4 = 16$$

Solving this system we get, $x_3 = 4$, $x_4 = 0$. Hence the basic solution is $X_5 = (0, 0, 4, 0)$.

In the conclusion, the solutions (2, 2, 0, 0) and (0, 0, 4, 0) are basic feasible solutions and they give the values of the objective function 14 and 4 respectively. Since (0, 0, 4, 0) minimizes the objective function $z = 2x_1 + 5x_2 + x_3 + 2x_4$, it is the optimum solution.

N.B: Mathematica Code to solve this LP problem is
ConstrainedMin[2x₁ + 5x₂ + x₃ + 2x₄,{4x₁ + 2x₂+3x₃ – 8x₄ = =12, 3x₁ + 5x₂ + 4x₃ – 6x₄ = = 16},{x₁, x₂, x₃, x₄}]

<u>Example (1.37)</u>: Reduce the given LP problem into canonical form: Maximize $z = 2x_1 - 3x_2 + x_3$ **[NUH-05]**

Subject to $2x_1 + 3x_2 \leq 5$

$x_1 - x_3 \geq 3$

$x_1 + 2x_2 = 1$

$x_3 \geq 0$, x_1, x_2 are unrestricted in sign.

Solution: Expressing 3rd constraint by "$\geq$" and "$\leq$", we get the given problem as follows:

Maximize $z = 2x_1 - 3x_2 + x_3$

Subject to $2x_1 + 3x_2 \leq 5$

$x_1 - x_3 \geq 3$

$x_1 + 2x_2 \geq 1$

$x_1 + 2x_2 \leq 1$

$x_3 \geq 0$, x_1, x_2 are unrestricted in sign.

Multiplying 2nd and 3rd constraints by –1, we get

Maximize $z = 2x_1 - 3x_2 + x_3$

Subject to $2x_1 + 3x_2 \leq 5$

$-x_1 + x_3 \leq -3$

$-x_1 - 2x_2 \leq -1$

$x_1 + 2x_2 \leq 1$

$x_3 \geq 0$, x_1, x_2 are unrestricted in sign.

Putting $x_1 = x_1^{/} - x_1^{//}$, $x_2 = x_2^{/} - x_2^{//}$; $x_1^{/} \geq 0$, $x_1^{//} \geq 0$, $x_2^{/} \geq 0$, $x_2^{//} \geq 0$, we get

Maximize $z = 2x_1^{/} - 2x_1^{//} - 3x_2^{/} - 3x_2^{//} + x_3$

Subject to $2x_1^{/} - 2x_1^{//} + 3x_2^{/} - 3x_2^{//} \leq 5$

$-x_1^{/} + x_1^{//} + x_3 \leq -3$

$-x_1^{/} + x_1^{//} - 2x_2^{/} + 2x_2^{//} \leq -1$

$x_1^{/} - x_1^{//} + 2x_2^{/} - 2x_2^{//} \leq 1$

$x_1^{/}, x_1^{//}, x_2^{/}, x_2^{//}, x_3 \geq 0.$

This is the required canonical form.

1.11 <u>Exercises</u>:

1. What do you mean by optimization?
2. Define with example objective function, constraints and non-negativity condition of a linear programming problem.
3. Discuss the characteristics of the standard form and canonical form of linear programming problem.
4. What are the advantages and limitations of linear programming?
5. Define with examples feasible solution, basic solution, basic feasible solution, non-degenerate basic feasible solution and optimum solution of a linear programming problem.
6. Define slack and surplus variables with examples.
7. Find the numerical values of

   (i) $\sum_{a=1}^{5} a$ [Answer: 14] (ii) $\sum_{a=1}^{n} a$ if n is 6 [Answer: 20]

   (iii) $\sum_{i=1}^{5} 3i$ [Answer: 45] (iv) $\sum_{p=5}^{10} p^2$ [Answer: 355]

8. Write the expanded form of

   (i) $\sum_{a=1}^{5} x_a$ [Answer: $x_1 + x_2 + x_3 + x_4 + x_5$]

   (ii) $\sum_{i=1}^{n} a_i x_i = b$ [Answer: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$]

9. Express in compact summation form:

   (i) $y_1 - y_2 + y_3 - y_4 + y_5 - y_6$ [Answer: $\sum_{i=1}^{6} (-)^{i-1} y_i$]

   (ii) $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ [Answer: $\sum_{i=1}^{n} a_i x_i = c$]

10. Convert the following LP problem into standard form:

    i) Minimize $z = 3x_1 + 11x_2$

    Subject to $2x_1 + 3x_2 = 4$

    $-3x_1 + x_2 \geq 3$

    $x_1, x_2 \geq 0$

ii) Maximize $z = 3x_1 + x_2$
Subject to $8x_1 + 2x_2 \leq 4$
$-3x_1 + 2x_2 \geq 3$
$x_1 \geq 0$, x_2 is unrestricted in sign.

iii) Minimize $z = 9x_1 + 6x_2$
Subject to $5x_1 + 3x_2 + x_3 = 10$
$-3x_1 + x_2 + 2x_3 \geq 3$
$x_1, x_2, x_3 \geq 0$

iv) Maximize $z = 2x_1 + x_2$
Subject to $2x_1 + 3x_2 \leq 5$
$-3x_1 + x_2 \geq 3$
$x_1 \geq 0$, x_2 is unrestricted in sign.

v) Maximize $z = 3x_1 + 6x_2 + 2x_3$
Subject to $8x_1 + 3x_2 + x_3 = 4$
$-3x_1 + x_2 + 2x_3 \geq 3$
$x_1, x_2, x_3 \geq 0$

vi) Maximize $z = 3x_1 + x_2 + 3x_3$
Subject to $8x_1 + 2x_2 - x_3 \leq 4$
$-3x_1 + 2x_2 + 2x_3 \geq 3$
$x_1, x_2 \geq 0$, x_3 is unrestricted in sign.

vii) Maximize $z = 3x_1 + x_2 + 3x_3$
Subject to $8x_1 + 2x_2 - x_3 \leq 4$
$-3x_1 + 2x_2 + 2x_3 \geq 3$
$x_1 + 5x_2 - 3x_3 \leq 7$
$x_1, x_2 \geq 0$, x_3 is unrestricted in sign.

viii) Maximize $z = 2x_1 + 6x_2 + 5x_3$
Subject to $20x_1 + 3x_2 + x_3 = 4$
$-3x_1 + x_2 + 2x_3 \approx 8$
$x_1, x_2, x_3 \geq 0$

ix) Minimize $z = x_1 + 7x_2 + 5x_3$
Subject to $2x_1 + 3x_2 + x_3 = 4$
$-3x_1 + 8x_2 + 2x_3 \approx -3$
$x_1, x_2, x_3 \geq 0$

x) Minimize $z = 10x_1 + 11x_2$
Subject to $2x_1 + 3x_2 \leq 5$
$-3x_1 + x_2 \approx 3$
$x_1 \geq 0,\ x_2 \leq 0$

xi) Minimize $z = 10x_1 + 11x_2$
Subject to $2x_1 + 3x_2 \leq 5$
$-3x_1 + x_2 \geq 3$
$x_1 \geq 4$, x_2 is unrestricted in sign.

xii) Maximize $z = 3x_1 + x_2$
Subject to $8x_1 + 2x_2 \leq 4$
$-3x_1 + 2x_2 \geq 3$
$x_1 \geq 0$, x_2 is unrestricted in sign.

xiii) Maximize $z = 3x_1 + x_2 + 3x_3$
Subject to $8x_1 + 2x_2 - x_3 \leq 4$
$-3x_1 + 2x_2 + 2x_3 \geq 3$
$x_1 + 5x_2 - 3x_3 \leq 7$
$x_1 \geq 0,\ x_2 \leq 0$, x_3 is unrestricted in sign.

xiv) Maximize $z = 2x_1 + 8x_2 + 3x_3$
Subject to $2x_1 + 2x_2 - 9x_3 \leq 7$
$-3x_1 + x_2 + 2x_3 \geq 3$
$3x_1 + 5x_2 - 3x_3 \leq 6$
$x_1 \geq -2,\ x_2 \leq 0$, x_3 is unrestricted in sign.

xv) Minimize $z = x_1 + 9x_2 + 3x_3$
Subject to $7x_1 + 2x_2 - 9x_3 \leq 38$
$-4x_1 + 3x_2 + 2x_3 \geq -4$
$5x_1 + 5x_2 - 3x_3 \leq 6$
$x_1 \geq -2,\ x_2 \leq 7$, x_3 is unrestricted in sign.

xvi) Maximize $z = 5x_1 + 2x_2 + 3x_3 + x_4$
Subject to $5x_1 + 4x_2 - 9x_3 + 4x_4 \leq 10$
$-2x_1 + 3x_2 + 9x_3 + 2x_4 \geq 4$
$x_1 + 5x_2 - 2x_3 + 3x_4 \leq 56$
$x_1, x_2, x_3, x_4 \geq 0.$

xvii) Maximize $z = 7x_1 + 2x_2 + 3x_3 + 2x_4$

Subject to $2x_1 + 3x_2 - 9x_3 + 4x_4 = 10$

$2x_1 + 3x_2 - 9x_3 + 2x_4 \geq 4$

$3x_1 + 5x_2 - 2x_3 + 2x_4 \approx 25$

$x_1, x_2 \geq 0,\ x_3 \leq 0,\ x_4$ unrestricted

xviii) Maximize $z = 3x_1 + 9x_2 + 3x_3 + 3x_4$

Subject to $6x_1 + 3x_2 - 5x_3 + 4x_4 = 12$

$3x_1 + 8x_2 - 9x_3 + 2x_4 \leq 40$

$3x_1 + 5x_2 - 2x_3 + x_4 \approx 19$

$x_1, x_2 \geq 0,\ x_3 \leq 3,\ x_4$ unrestricted

xix) Minimize $z = x_1 + 9x_2 + 3x_3 + x_4$

Subject to $7x_1 + 2x_2 - 9x_3 + 4x_4 \approx 18$

$-4x_1 + 3x_2 + 2x_3 + 2x_4 \geq -14$

$5x_1 + 5x_2 - 3x_3 + 3x_4 \leq 6$

$x_1 \geq 0, x_2 \geq -2, x_3 \leq 1,\ x_4$ is unrestricted in sign.

xx) Minimize $z = x_1 + 7x_2 + 5x_3 + 2x_4 - 3x_5$

Subject to $3x_1 + 3x_2 + x_3 + x_4 - 3x_5 \leq -24$

$7x_1 + 3x_2 + 2x_3 + 2x_4 - x_5 = 20$

$2x_1 + 3x_2 + x_3 + 3x_4 - 3x_5 \geq 14$

$-3x_1 + 8x_2 + 2x_3 + 2x_4 - 7x_5 \approx -13$

$x_1, x_2 \geq 0,\ x_3 \geq 4,\ x_4 \leq 0,\ x_5$ unrestricted

11. Convert the following linear programming problem into the standard form: Maximize $z = 8x_1 + 4x_2 + 3x_3$ [RU-89]

Subject to $6x_1 + 2x_2 - 5x_3 \leq 3$

$9x_1 - x_2 + 6x_3 \geq 5$

$x_1 + 3x_2 + x_3 \approx 1$

$x_1 \geq 2,\ x_2 \leq 0,\ x_3$ unrestricted.

[Answer: Maximize $8x_1' - 4x_2' + 3x_3' - 3x_3'' + 16$

Subject to $-6x_1' + 2x_2' + 5x_3' - 5x_3'' - s_1 = 9$

$-9x_1' - x_2' - 6x_3' + 6x_3'' + s_2 = 12$

$-x_1' + 3x_2' - x_3' + x_3'' - s_3 + s_4 = 1$

$x_1', x_2', x_3', x_3'', s_1, s_2, s_3, s_4 \geq 0$.]

12. Convert the following LP problem into canonical form:

a. Maximize $z = 2x_1 + 5x_2$
Subject to $7x_1 + 2x_2 \leq 4$
$-3x_1 + 8x_2 \geq 3$
$x_1 \geq 0$, x_2 is unrestricted in sign.

b. Minimize $z = 3x_1 + 4x_2$
Subject to $5x_1 + 3x_2 + x_3 = 10$
$-3x_1 + 5x_2 + 2x_3 \geq 3$
$x_1, x_2, x_3 \geq 0$

c. Maximize $z = 2x_1 + 5x_2$
Subject to $3x_1 + 3x_2 \leq -5$
$-3x_1 + x_2 \geq 3$
$x_1 \geq 0$, x_2 is unrestricted in sign.

d. Maximize $z = 2x_1 + 7x_2 + 2x_3$
Subject to $3x_1 + 3x_2 + x_3 = 12$
$-3x_1 + x_2 + 2x_3 \geq 3$
$x_1, x_2, x_3 \geq 0$

e. Maximize $z = 3x_1 + 7x_2 + 3x_3$
Subject to $3x_1 + 2x_2 - x_3 \leq 14$
$-3x_1 + 4x_2 + 2x_3 \geq 3$
$x_1, x_2 \geq 0$, x_3 is unrestricted in sign.

f. Maximize $z = 13x_1 + 10x_2 + 3x_3$
Subject to $10x_1 + 2x_2 - x_3 \leq 44$
$-3x_1 + 2x_2 + 8x_3 \geq 3$
$9x_1 + 5x_2 - 3x_3 \leq 7$
$x_1, x_2 \geq 0$, x_3 is unrestricted in sign.

g. Maximize $z = 2x_1 + 6x_2 + 5x_3$
Subject to $20x_1 + 3x_2 + x_3 = 4$
$-3x_1 + x_2 + 2x_3 \approx 8$
$x_1, x_2, x_3 \geq 0$

h. Minimize $z = x_1 + 7x_2 + 5x_3$
Subject to $2x_1 + 3x_2 + x_3 = 4$
$-3x_1 + 8x_2 + 2x_3 \approx -3$
$x_1, x_2, x_3 \geq 0$

i. Minimize $z = 10x_1 + 11x_2$
Subject to $2x_1 + 3x_2 \leq 5$
$-3x_1 + x_2 \approx 3$
$x_1 \geq 0,\ x_2 \leq 0$

j. Minimize $z = 10x_1 + 11x_2$
Subject to $2x_1 + 3x_2 \leq 5$
$-3x_1 + x_2 \geq 3$
$x_1 \geq 4$, x_2 is unrestricted in sign.

k. Maximize $z = 3x_1 + 7x_2$
Subject to $9x_1 + 2x_2 \leq 24$
$-3x_1 + 5x_2 \geq 3$
x_1, x_2 are both unrestricted in sign.

l. Maximize $z = 3x_1 + x_2 + 3x_3$
Subject to $8x_1 + 2x_2 - x_3 \leq 4$
$-3x_1 + 2x_2 + 2x_3 \geq 3$
$x_1 + 5x_2 - 3x_3 \leq 7$
$x_1 \geq 0,\ x_2 \leq 0$, x_3 is unrestricted in sign.

m. Maximize $z = 7x_1 + 8x_2 + 13x_3$
Subject to $2x_1 + 2x_2 - 9x_3 \leq 17$
$-5x_1 + 3x_2 + 2x_3 \geq 3$
$3x_1 + 5x_2 - 3x_3 \leq 6$
$x_1 \geq -2,\ x_2 \leq 0$, x_3 is unrestricted in sign.

n. Minimize $z = x_1 + 9x_2 + 3x_3$
Subject to $7x_1 + 2x_2 - 9x_3 \leq 38$
$-4x_1 + 3x_2 + 2x_3 \geq -4$
$5x_1 + 5x_2 - 3x_3 \leq 6$
$x_1 \geq -2,\ x_2 \leq 7$, x_3 is unrestricted in sign.

o. Maximize $z = 5x_1 + 2x_2 + 3x_3 + x_4$
Subject to $5x_1 + 4x_2 - 9x_3 + 4x_4 \leq 10$
$-2x_1 + 3x_2 + 9x_3 + 2x_4 \geq 4$
$x_1 + 5x_2 - 2x_3 + 3x_4 \leq 56$
$x_1, x_2, x_3, x_4 \geq 0.$

p. Maximize $z = 7x_1 + 2x_2 + 3x_3 + 2x_4$
Subject to $2x_1 + 3x_2 - 9x_3 + 4x_4 = 13$

		$2x_1 + 3x_2 - 9x_3 + 2x_4 \geq 4$
		$3x_1 + 5x_2 - 2x_3 + 2x_4 \approx 25$
		$x_1, x_2 \geq 0,\ x_3 \leq 0,\ x_4$ unrestricted
q.	Maximize	$z = 3x_1 + 9x_2 + 3x_3 + 3x_4$
	Subject to	$6x_1 + 3x_2 - 5x_3 + 4x_4 = 12$
		$3x_1 + 8x_2 - 9x_3 + 2x_4 \leq 40$
		$3x_1 + 5x_2 - 2x_3 + x_4 \approx 19$
		$x_1 \geq 0,\ x_3 \leq 3,\ x_2$ and x_4 unrestricted
r.	Minimize	$z = 7x_1 + 9x_2 + 3x_3 + 7x_4$
	Subject to	$3x_1 + 2x_2 - 9x_3 + 4x_4 \approx 18$
		$-2x_1 + 3x_2 + 4x_3 + 2x_4 \geq -14$
		$5x_1 + 6x_2 - 3x_3 + 3x_4 \leq 45$
		$x_1 \geq 0, x_2 \geq -2, x_3 \leq 1,\ x_4$ is unrestricted in sign.
s.	Minimize	$z = 3x_1 + 7x_2 + 5x_3 + 3x_4 - 3x_5$
	Subject to	$4x_1 + 3x_2 + 4x_3 + x_4 - 3x_5 \leq 24$
		$3x_1 + 3x_2 + 2x_3 + 2x_4 - x_5 = 22$
		$2x_1 + 8x_2 + x_3 + 3x_4 - 3x_5 \geq -14$
		$-3x_1 + 5x_2 + 2x_3 + 2x_4 - 7x_5 \approx -13$
		$x_1, x_2 \geq 0,\ x_3 \geq 3,\ x_4 \leq 0,\ x_5$ unrestricted

13. Find all basic solutions of the system of linear equations:

a. $x_1 + 2x_2 + x_3 = 4$
$2x_1 + x_2 + 5x_3 = 5$

b. $x_1 + 2x_2 - x_3 = 4$
$2x_1 + 2x_2 + x_3 = 4$

c. $2x_1 + 4x_2 - x_3 = 3$
$2x_1 + 4x_2 + x_3 = 6$

d. $3x_1 + 4x_2 - x_3 = 6$
$2x_1 + 4x_2 + x_3 = 4$

e. $2x_1 + 3x_2 - x_3 = 3$
$4x_1 + 4x_2 + 2x_3 = 12$

f. $2x_1 - x_2 + 3x_3 + x_4 = 6$
$4x_1 - 2x_2 - x_3 + 2x_4 = 10$

g. $x_1 + 3x_2 + 2x_3 + 3x_4 = 10$
$2x_1 - x_2 + 4x_3 + 6x_4 = 16$

h. $2x_1 + 3x_2 + 6x_3 + 4x_4 = 1$
$3x_1 - x_2 + 9x_3 + 6x_4 = 1$

i. $3x_1 + 3x_2 + 9x_3 + 4x_4 = 8$
$3x_1 - 2x_2 + 9x_3 + 6x_4 = 4$

j. $x_1 + 3x_2 + 4x_3 - 4x_4 = 4$
$x_1 - 2x_2 + 9x_3 + 8x_4 = 9$

14. Find all basic feasible solutions of the system of linear equations:

a. $x_1 + 2x_2 + x_3 = 4$
$2x_1 + x_2 + 5x_3 = 5$

b. $2x_1 + 2x_2 - x_3 = -4$
$2x_1 + 2x_2 + x_3 = 4$

c. $3x_1 + 2x_2 - x_3 = 6$
$x_1 + 4x_2 + x_3 = 3$

d. $3x_1 + 4x_2 - x_3 = 8$
$3x_1 + 2x_2 + x_3 = 4$

e. $2x_1 + 4x_2 + x_3 = 3$
$4x_1 + 8x_2 + 2x_3 = 10$

f. $2x_1 - x_2 + 3x_3 + x_4 = 6$
$4x_1 - 2x_2 - x_3 + 2x_4 = 10$

g. $x_1 + 3x_2 + 2x_3 + 3x_4 = 10$
$2x_1 - x_2 + 4x_3 + 6x_4 = 16$

h. $3x_1 + 3x_2 + 6x_3 + 4x_4 = 3$
$3x_1 - 2x_2 + 9x_3 + 4x_4 = 3$

i. $2x_1 + 4x_2 + 7x_3 + 4x_4 = 8$
$3x_1 - 2x_2 + 7x_3 + 6x_4 = 4$

j. $x_1 + 2x_2 + 4x_3 - 4x_4 = 4$
$x_1 - 2x_2 + 6x_3 + 8x_4 = 6$

15. How many basic solutions are there in the following linearly independent set of equations? Find all of them

$$2x_1 - x_2 + 3x_3 + x_4 = 6$$
$$4x_1 - 2x_2 - x_3 + 2x_4 = 10$$

16. Show that $x_1 = 5$, $x_2 = 0$, $x_3 = -1$ is a basic solution of the system of linear equations.

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5$$

Find the other basic solutions, if there be any.

17. Show that $x_1 = 2$, $x_2 = -1$, $x_3 = 0$ is not a basic solution of the system of linear equations.

$$3x_1 - 2x_2 + x_3 = 8$$
$$9x_1 - 6x_2 + 4x_3 = 24$$

Find all basic solutions, if there be.

18. Show that $x_1 = 1$, $x_2 = 1$, $x_3 = 2$ is a feasible solution of the system of linear equations.

$$x_1 + 2x_2 + 3x_3 = 9$$
$$2x_1 - x_2 + x_3 = 3$$

Reduce the above feasible solution to a basic feasible solution.

19. If $x_1 = 1$, $x_2 = 1$, $x_3 = 3$ be a feasible solution of the system of linear equations.

$$x_1 + 2x_2 + 3x_3 = 12$$
$$2x_1 - x_2 + x_3 = 4$$

Reduce the above feasible solution to a basic feasible solution.

20. If $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ be a feasible solution of the system of linear equations.

$$3x_1 + 2x_2 + 3x_3 = 8$$
$$2x_1 - 3x_2 + 3x_3 = 2$$

Reduce the above feasible solution to a basic feasible solution.

21. If $x_1 = 2$, $x_2 = 1$, $x_3 = 1$ be a feasible solution of the following system of linear equations.

$$3x_1 + 2x_2 + x_3 = 9$$
$$2x_1 - 3x_2 + 2x_3 = 3$$

Reduce the above feasible solution to a basic feasible solution.

22. Show that $x_1 = 2$, $x_2 = 1$, $x_3 = 2$ is a feasible solution of the system of linear equations.

$$x_1 + 2x_2 + 3x_3 = 10$$
$$2x_1 - x_2 + x_3 = 5$$

Reduce the above feasible solution to a basic feasible solution.

23. Show that $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 0$ is a feasible solution of the system of linear equations.

$$11x_1 + 2x_2 - 9x_3 + 4x_4 = 6$$
$$15x_1 + 3x_2 - 12x_3 + 6x_4 = 9$$

Reduce the above feasible solution to basic feasible solutions and also show that one of them is non-degenerate and the other is degenerate.

24. If $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 1$ is a feasible solution of the system of linear equations.

$$11x_1 + 2x_2 - 9x_3 + 4x_4 = 10$$
$$15x_1 + 3x_2 - 12x_3 + 6x_4 = 15$$

Reduce the above feasible solution to basic feasible solutions.

25. Find the optimum solution of the following programming problem:

$$\begin{aligned} \text{Minimize } & z = 2x_1 + 5x_2 \\ \text{Subject to } & 3x_1 + 2x_2 = 5 \\ & 5x_1 + x_2 = 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

26. Find the optimum solution of the following programming problem:

$$\begin{aligned} \text{Minimize } & z = x_1 + 2x_2 + 3x_3 \\ \text{Subject to } & x_1 + 2x_2 + x_3 = 3 \\ & 2x_1 + x_2 + 5x_3 = 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

27. Find the optimum solution of the following programming problem:

$$\begin{aligned} \text{Maximize } & z = 2x_1 + 5x_2 + 3x_3 \\ \text{Subject to } & x_1 + 2x_2 + x_3 = 3 \\ & 2x_1 + x_2 + 5x_3 = 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

28. Find the optimum solution of the following programming problem:

$$\begin{aligned} \text{Maximize } & z = 2x_1 - 3x_2 + x_4 \\ \text{Subject to } & 3x_1 + 2x_2 + x_3 = 15 \\ & 2x_1 + 4x_2 + x_4 = 8 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

29. Find the optimum solution of the following linear programming problem:

$$\begin{aligned} \text{Maximize } & z = 3x_1 + 2x_2 + x_3 \\ \text{Subject to } & x_1 + 2x_2 + x_3 = 4 \\ & 2x_1 + x_2 + 5x_3 = 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

30. Find the optimum solution of the following programming problem:

$$\begin{aligned} \text{Minimize } & z = x_1 - x_2 + 2x_3 + 3x_4 \\ \text{Subject to } & 2x_1 + x_2 + 3x_3 + 2x_4 = 11 \\ & 3x_1 - 3x_2 + 5x_3 + x_4 = 17 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Chapter

02

Convex sets

Highlights:

2.1 Line segment: For any two points x and y in $\mathbf{R}^n$, the set $[x:y] = \{u : u = \lambda x + (1 - \lambda)y \; ; \; 0 \le \lambda \le 1\}$ is called the line segment joining the points x and y. The points x and y are called the end points of this segment. For each λ, $0 < \lambda < 1$, the point $\lambda x + (1 - \lambda)y$ is called an in-between point (or internal point) of the line segment and so $(x:y) = \{ \lambda x + (1 - \lambda)y \; ; \; 0 < \lambda < 1\}$ is called open line segment joining the points x and y.

$$u = \lambda x + (1-\lambda)y; \; 0 \le \lambda \le 1$$

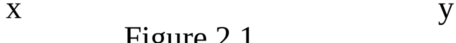

Figure 2.1

2.2 Convex set: A subset $S \subset \mathbf{R}^n$ is said to be convex if for each pair of points x, y in S, the line segment $[x:y] = \{u : u = \lambda x + (1 - \lambda)y \; ; \; 0 \le \lambda \le 1\}$ is contained in S. In symbols, a subset $S \subset \mathbf{R}^n$ is convex iff $x, y \in S \Rightarrow [x:y] \subset S$. **[NUH-00, 02, 04, 07, JU-91, 95]**

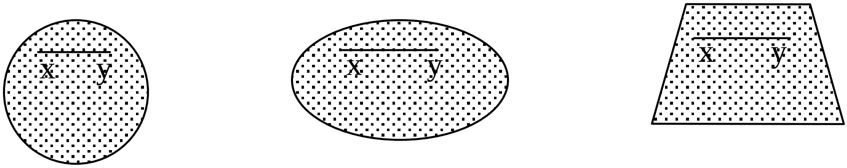

Figure 2.2: Convex set

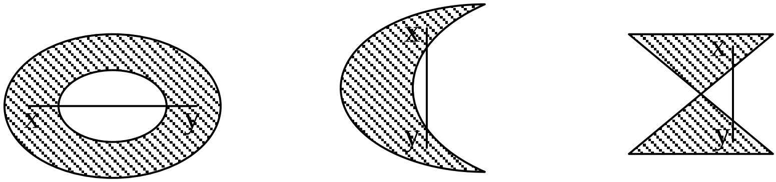

Figure 2.3: Non-convex set

2.3 <u>Convex combination of a set of vectors</u>: Let S = $\{x_1, x_2, ..., x_k\} \subset \mathbf{R}^n$. Then a linear combination $x = \lambda_1 x_1 + \lambda_2 x_2 + . . . + \lambda_k x_k$ is called a convex combination of the given vectors, if $\lambda_i \geq 0$, for i = 1, 2, 3, . . ., k; and $\sum_{i=1}^{k} \lambda_i = 1$ **[RU-90]**

Illustration (2.1): Let $\{x, y\} \subset \mathbf{R}^n$. Then $u = \lambda x + (1 - \lambda)y$ is a convex combination of x and y, since λ, $1 - \lambda$ are scalars such that $\lambda \geq 0$, $1 - \lambda \geq 0$ and $\lambda + (1 - \lambda) = 1$.

2.4 <u>Extreme point</u>: A point x in a convex set C is an extreme point, if it cannot be expressed as a convex combination of two distinct points in C. **[NUH-02]**

As for example the vertices of a triangle and all points lying on the circumference of a circle are extreme points.

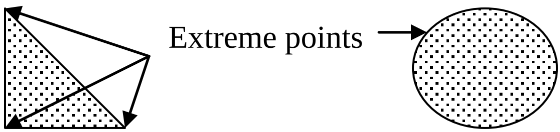

Figure 2.4

Example (2.1): Find the extreme points of $S = \{(x, y): |x| \le 2, |y| \le 2\}$

Solution: Given that $|x| \le 2$, $|y| \le 2$. This implies that, $-2 \le x \le 2$ and $-2 \le y \le 2$. So the set S represents the region of the square bounded by the lines $x = 2$, $x = -2$, $y = 2$ and $y = -2$ as shown in the figure.

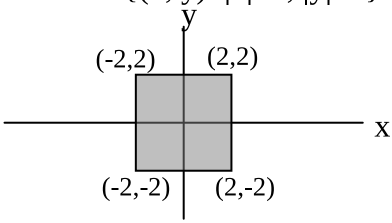

Figure 2.5

Therefore, the extreme points of this convex set are (2, 2), (2, –2), (–2, –2) and (–2, 2).

2.5 Convex cone: A non empty subset $C \subset \mathbf{R}^n$ is called a cone if for each x in C, and $\lambda \ge 0$, the vector λx is also in C. A cone is called a convex cone if it is a convex set. **[DU-01]**

Illustration (2.2): If A be an $m \times n$ matrix, then the set of n vectors x satisfying $Ax \ge 0$ is a convex cone in $\mathbf{R}^n$. It is a cone, because if $Ax \ge 0$, then $A\lambda x \ge 0$ for non-negative λ. It is convex because if $Ax^{(1)} \ge 0$ and $Ax^{(2)} \ge 0$, then $A[\lambda x^{(1)} + (1 - \lambda)x^{(2)}] \ge 0$.

2.6 Convex hull: A convex hull of a set of points S is denoted by C(S) is the set of all convex combination of the sets of points in S. A convex hull is a convex set. **[NU-99, 01, 00, 03, 04, 07]**

Let, S = { (0,0), (1,0), (0,1)}. Then C(S) is the set of all points in and on the boundary of the triangle with (0,0), (1,0) and (0,1) as the extreme points.

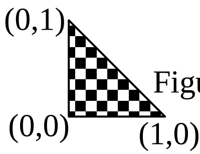

Figure 2.6

2.7 Convex polyhedron: If S be a set of a finite number of points, then the set of all convex combinations C(S) of the sets of points in S is called a convex polyhedron. A convex polyhedron always is a convex set. As for example, **[NUH-99, 01, 00, 04, 07]**

$S = \{(0,0), (0, 1), (1, 1), (1, 0)\}$ is a finite set. Then the convex polyhedron C(S) is the set of all points in and on the boundary of the square with (0,0), (0, 1), (1, 1) and (1, 0) as the extreme points.

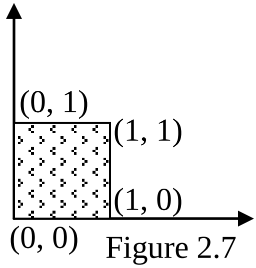

Figure 2.7

2.8 <u>Simplex</u>: A simplex in n dimension is a convex polyhedron having exactly (n + 1) vertices. A simplex in zero dimensions is a point, in one dimension it is a line segment, in two dimensions it is a triangle, in three dimensions it is a tetrahedron and so on.

2.9 <u>Hyper plane</u>: Let any point, $\underline{x} = (x_1, x_2, x_3, \ldots, x_n) \in X \subset \mathbf{R}^n$, $\underline{A} = (a_1, a_2, a_3, \ldots, a_n)$ and $\underline{A}.\underline{x} = a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$, then $H(\underline{A}, b) = \{X \mid \forall\, \underline{x} \in X, \underline{A}.\underline{x} = b\}$ is called a hyper plane for a given vector $\underline{A}$ and a given scalar b. That means, the set X, which any element $\underline{x}$ satisfies the condition $\underline{A}.\underline{x} = b$, is the hyper plane with respect to vector $\underline{A}$ and a scalar b. A hyper plane in two dimension, is a line and in three dimension, is a plane. **[NU-01, 00, 02, 04,07]**

As for example, let $\underline{A} = (2, 3)$, b = 5 then the set of solutions of the equation $2x_1 + 3x_2 = 5$ is the hyper plane with respect to vector (2, 3) and scalar 5, which is a straight line. That is, $X = \{(x_1, x_2) \mid 2x_1 + 3x_2 = 5\}$ is the hyper plane.

<u>Theorem (2.1)</u>: A hyper plane in $\mathbf{R}^n$ is a convex set. **[NU-98, 02]**
Proof: Let $H(\underline{A}, b) = \{\underline{x} \in X \mid \underline{A}.\underline{x} = b\}$ be a hyper plane in $\mathbf{R}^n$ and let $\underline{x}$, $\underline{y}$ be two points in this hyper plane X. Then $\underline{A}.\underline{x} = b$ and $\underline{A}.\underline{y} = b$. Let $\underline{u}$ be any point of line segment $[\underline{x}:\underline{y}] = \{\lambda\underline{x} + (1 - \lambda)\underline{y} : 0 \le \lambda \le 1\}$, then

$$\underline{A}.\underline{u} = \underline{A}.[\lambda\underline{x} + (1 - \lambda)\underline{y}]$$
$$= \lambda(\underline{A}.\underline{x}) + (1 - \lambda)(\underline{A}.\underline{y})$$

$$= b\lambda + (1 - \lambda)b$$
$$= b$$

i.e., $\underline{u}$ is a point of the hyper plane X.
Thus, for $\underline{x}$, $\underline{y} \in$ X implies that the line segment $[\underline{x}:\underline{y}] \subset$ X.
Hence, the hyper plane X in $\mathbf{R}^n$ is a convex set.

Example (2.2): Show that (hyper plane), $S = \{(x_1, x_2): 2x_1+3x_2=7\} \subset \mathbf{R}^2$ is a convex set. **[DU-88]**

Proof: Let any points x, y $\in$ S, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The line segment joining x and y is the set $\{u : u = \lambda x + (1 - \lambda)y ; 0 \le \lambda \le 1\}$. For some λ, $0 \le \lambda \le 1$, let $u = (u_1, u_2)$ be a point of this set, so that, $u_1 = \lambda x_1 + (1 - \lambda)y_1$ and $u_2 = \lambda x_2 + (1 - \lambda)y_2$.

Since, x, y $\in$ S, $2x_1 + 3x_2 = 7$ and $2y_1 + 3y_2 = 7$

Now, $2u_1 + 3u_2 = 2[\lambda x_1 + (1 - \lambda)y_1] + 3[\lambda x_2 + (1 - \lambda)y_2]$
$= \lambda(2x_1 +3x_2) +(1-\lambda)(2y_1 + 3y_2) = 7\lambda+ 7(1- \lambda) = 7$

So, $u = (u_1, u_2)$ is a point of S. Since u is any point of the line segment [x:y], for x, y $\in$ S, $[x:y] \subset$ S. Hence S is a convex set.

Theorem (2.2): Any point of a convex polyhedron can be expressed as a convex combination of its extreme points.

Proof: We know that a convex polyhedron contains a finite number of extreme points. So, we let $x_1, x_2, ..., x_n$ be the extreme points of the polyhedron. Let x be any point of the polyhedron. If x be an extreme point then there is nothing to prove. So we let x is not an extreme point.

Now if x lies on the boundary hyper-plane or line, then it can be expressed as the convex combination of the end points of that boundary hyper-plane or line. That is, $x = \sum_{i=j}^{k} \mu_i x_i \; ; \mu_i \ge 0, \sum_{i=j}^{k} \mu_i = 1$, $j<k\le n$ and $x_j, x_{j+1}, x_{j+2}, ..., x_k$ are end points of the boundary hyper-plane or line. So the theorem is proved for the boundary points.

Now if x does not lie on the boundary hyper-plane or line, then we take any extreme point x_1 (say) and join it with x by a line segment and extend it up to a boundary surface or line. Let the point of intersection of the line segment and the boundary surface or line be x^*. So, we can write $x^* = \sum_{i=p}^{q} \delta_i x_i$; $\delta_i \geq 0$, $\sum_{i=p}^{q} \delta_i = 1$, j < q $\leq$ n and $x_p, x_{p+1}, x_{p+2}, \ldots, x_q$ are end points of the boundary surface or line. And hence the line segment, $x = \lambda_1 x_1 + \lambda_2 x^*$; $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$

$$\Rightarrow x = \lambda_1 x_1 + \lambda_2 \sum_{i=p}^{q} \delta_i x_i$$

$$\Rightarrow x = \lambda_1 x_1 + \lambda_2 \delta_p x_p + \lambda_2 \delta_{p+1} x_{p+1} + \ldots + \lambda_2 \delta_q x_q$$

$$\Rightarrow x = \lambda_1 x_1 + 0x_2 + \ldots + 0x_{p-1} + \lambda_2 \delta_p x_p + \lambda_2 \delta_{p+1} x_{p+1} + \ldots + \lambda_2 \delta_q x_q + 0x_{q+1} + \ldots + 0x_n$$

where $\lambda_1, \lambda_2 \delta_i \geq 0$; i =1,2,...,q and $\lambda_1 + \lambda_2 \delta_p + \ldots + \lambda_2 \delta_{p+1} + \lambda_2 \delta_q = 1$.

Therefore, x can be expressed as the convex combination of all the extreme points. (Hence the theorem)

2.10 <u>Half space</u>: In $\mathbf{R}^n$, let$\{\underline{x} \in X \mid \underline{A}.\underline{x} = b\}$ be a hyper plane then the two sets of points $H^+(\underline{A}, b) = \{\underline{x} \mid \underline{A} . \underline{x} \geq b\}$ and $H^-(\underline{A}, b) = \{\underline{x} \mid \underline{A} . \underline{x} \leq b\}$ are called closed half spaces determined by $\underline{A}.\underline{x} = b$. Also the corresponding two sets of points $H^+(\underline{A}, b) = \{\underline{x} \mid \underline{A} . \underline{x} > b\}$ and $H^-(\underline{A}, b) = \{\underline{x} \mid \underline{A} . \underline{x} < b\}$ are called open half spaces determined by $\underline{A} . \underline{x} = b$. **[NUH-99, 01, 04,07]**

<u>Theorem (2.3)</u>: The closed half spaces $H^+(\underline{A}, b) = \{\underline{x} \mid \underline{A} . \underline{x} \geq b\}$ and $H^-(\underline{A}, b) = \{\underline{x} \mid \underline{A} . \underline{x} \leq b\}$ are convex sets. **[NU-01, 02, JU-99]**

Proof: Let $\underline{x}$, $\underline{y}$ be any two points of H^+. Then $\underline{A}.\underline{x} \geq b$ and $\underline{A}.\underline{y} \geq b$. Let $\underline{u}$ be any point of line segment $[\underline{x}:\underline{y}] = \{\lambda\underline{x} + (1-\lambda)\underline{y} : 0 \leq \lambda \leq 1\}$, then $\underline{A}.\underline{u} = \underline{A}.[\lambda\underline{x} + (1-\lambda)\underline{y}]$

$$= \lambda(\underline{A}.\underline{x}) + (1-\lambda)(\underline{A}.\underline{y})$$
$$\geq b\lambda + (1-\lambda)b$$
$$\geq b$$

i.e., $\underline{u}$ is a point of the closed half space H^+.
Thus, for $\underline{x}$, $\underline{y} \in H^+$ implies that the line segment $[\underline{x}:\underline{y}] \subset H^+$.
Hence, the closed half space H^+ is a convex set.
Similarly, taking '$\leq$ ' in places of ' $\geq$', we can show that the closed half space, H^- is also a convex set.

<u>Corollary (2.1)</u>: The open half spaces, $H^+(\underline{A}, b) = \{\underline{x} \mid \underline{A} \cdot \underline{x} > b\}$ and $H^-(\underline{A}, b) = \{\underline{x} \mid \underline{A} \cdot \underline{x} < b\}$ are convex sets. **[NUH-02, DU-98]**
Proof: Let $\underline{x}$, $\underline{y}$ be any two points of H^+. Then $\underline{A}.\underline{x} > b$ and $\underline{A}.\underline{y} > b$.
Let $\underline{u}$ be any point of line segment $[\underline{x}:\underline{y}] = \{\lambda\underline{x}+(1-\lambda)\underline{y} : 0 \leq \lambda \leq 1\}$,

then $\underline{A}.\underline{u} = \underline{A}.[\lambda\underline{x} + (1-\lambda)\underline{y}]$
$$= \lambda(\underline{A}.\underline{x}) + (1-\lambda)(\underline{A}.\underline{y})$$
$$> b\lambda + (1-\lambda)b$$
$$> b$$

i.e., $\underline{u}$ is a point of the open half space H^+.
Thus, for $\underline{x}$, $\underline{y} \in H^+$ implies that the line segment $[\underline{x}:\underline{y}] \subset H^+$.
Hence, the open half space H^+ is a convex set.
Similarly, taking '$<$ ' in places of '$>$', we can show that the open half space, H^- is also a convex set.

<u>Example (2.3)</u>: Show that (the half space), $S = \{(x_1, x_2, x_3) \mid 2x_1 - x_2 + x_3 \leq 4\} \subset \mathbf{R}^3$, is a convex set. **[JU-94, NUH-01]**
Proof: Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two points of S.
Then $2x_1 - x_2 + x_3 \leq 4$ . . . (1)
and $2y_1 - y_2 + y_3 \leq 4$ . . . (2)
Let $u = (u_1, u_2, u_3)$ be any point of $[x:y]$ so that $0 \leq \lambda \leq 1$
$u_1 = \lambda x_1+(1-\lambda)y_1$, $u_2 = \lambda x_2+(1-\lambda)y_2$ and $u_3 = \lambda x_3+(1-\lambda)y_3$ . . . (3)
$\therefore$ $2u_1-u_2+u_3 = 2[\lambda x_1+(1-\lambda)y_1] - [\lambda x_2+(1-\lambda)y_2] + [\lambda x_3+(1-\lambda)y_3]$
Or, $2u_1 - u_2 + u_3 = \lambda(2x_1 - x_2 + x_3) + (1-\lambda)(2y_1 - y_2 + y_3)$

Or, $2u_1 - u_2 + u_3 \leq 4\lambda + 4(1-\lambda)$ [Using (1) and (2)]
Or, $2u_1 - u_2 + u_3 \leq 4$
i.e., $u = (u_1, u_2, u_3)$ is a point of S.
Thus, for x, y $\in$ S implies that the line segment $[x{:}y] \subset S$.
Hence, S is a convex set.

Example (2.4): Show that the open negative half space, $S=\{(x_1,x_2,x_3) \mid 2x_1 + 2x_2 - x_3 < 9\} \subset \mathbf{R}^3$, is a convex set.

Proof: Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two points of S.
Then $2x_1 + 2x_2 - x_3 < 9$ . . . (1)
and $2y_1 + 2y_2 - y_3 < 9$ . . . (2)
Let $u = (u_1, u_2, u_3)$ be any point of $[x{:}y]$ so that $0 \leq \lambda \leq 1$
$u_1 = \lambda x_1+(1-\lambda)y_1$, $u_2 = \lambda x_2+(1-\lambda)y_2$ and $u_3 = \lambda x_3+(1-\lambda)y_3$ . . . (3)
$\therefore 2u_1+2u_2-u_3 = 2[\lambda x_1+(1-\lambda)y_1] + 2[\lambda x_2+(1-\lambda)y_2] - [\lambda x_3+(1-\lambda)y_3]$
Or, $2u_1 + 2u_2 - u_3 = \lambda(2x_1 + 2x_2 - x_3) + (1-\lambda)(2y_1 + 2y_2 - y_3)$
Or, $2u_1 + 2u_2 - u_3 < 9\lambda + 9(1-\lambda)$ [Using (1) and (2)]
Or, $2u_1 + 2u_2 - u_3 < 9$
i.e., $u = (u_1, u_2, u_3)$ is a point of S. Thus, for x, y $\in$ S implies that the line segment $[x{:}y] \subset S$. Hence, S is a convex set.

2.11 Supporting hyper plane: Let $S \subset \mathbf{R}^n$ be a non-empty closed convex set and $\underline{x}_0 \in S$ be a boundary point of S. Then the hyper

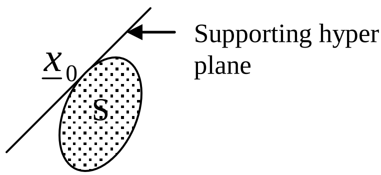

Figure 2.8

plane $\underline{A}.\underline{x} = b$ is called a supporting hyper plane of S at $\underline{x}_0$, if it satisfies the following two conditions: **[NUH-03, 04]**

(i) $\underline{A}.\underline{x}_0 = b$ and
(ii) $S \subset H^+(\underline{A},b) = \{\underline{x} \mid \underline{A}.\underline{x} \geq b\}$ or $S \subset H^-(\underline{A},b) = \{\underline{x} \mid \underline{A}.\underline{x} \leq b\}$

Example (2.5): we know that S ={(x_1,x_2): $x_1^2+x_2^2 \le 1$}is a convex set and let $\underline{A}$= (1,0), b=1 and $\underline{x}$ =(x_1, x_2) so that H={(x_1,x_2):x_1+0x_2 =1} is a hyper plane. Also we know that $\underline{x}_0$= (1, 0) is a boundary point of S. Here, $\underline{A}.\underline{x}_0$= b and S$\subset$ H^- = {(x_1,x_2):x_1+0x_2 ≤1}, hence H is a supporting hyper plane of S at $\underline{x}_0$ = (1, 0).

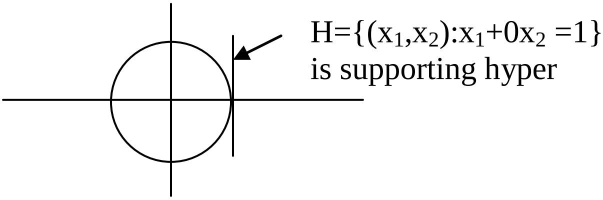

Figure 2.9

2.12 <u>Separating hyper plane</u>: Let S$\subset$ $\mathbf{R}^n$ be a non-empty closed convex set and the point $\underline{x}_0 \notin$S. Then the hyper plane $\underline{A}.\underline{x}$ = b containing $\underline{x}_0$ is called a separating hyper plane of S, if it satisfies the following two conditions: **[NUH-03, 04]**

(i) $\underline{A}.\underline{x}_0$ = b and

(ii) S$\subset H^+(\underline{A}$,b) ={$\underline{x}$ |$\underline{A}.\underline{x}$ > b} or S$\subset H^-(\underline{A}$,b) = {$\underline{x}$ |$\underline{A}.\underline{x}$ < b}

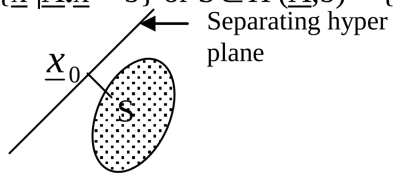

Figure 2.10

Example (2.6): we know that S ={(x_1,x_2): $x_1^2+x_2^2 \le 1$}is a convex set and let $\underline{A}$= (1, 0), b = 2 and $\underline{x}$ =(x_1, x_2) so that H ={(x_1, x_2):x_1 + 0x_2 = 2} is a hyper plane. Also we know that $\underline{x}_0$= (2, 1)$\notin$S. Here, $\underline{A}.\underline{x}_0$= b and S$\subset$ H^- = {(x_1,x_2):x_1+0x_2 < 2}, hence H is a separating hyper plane of S containing $\underline{x}_0$ = (2, 1).

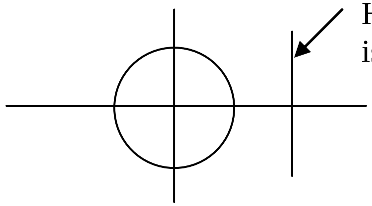

Figure 2.11

2.13 <u>Convex function</u>: Let $S \subset \mathbf{R}^n$ be a non-empty convex set. A function f(x) defined on S is said to be convex if for any two points x_1, x_2 in S, $f[\lambda x_1+(1-\lambda)x_2] \leq \lambda f(x_1) + (1-\lambda)f(x_2)$; $0 \leq \lambda \leq 1$.
The function f(x) is said to be strictly convex if
$f[\lambda x_1+(1-\lambda)x_2] < \lambda f(x_1)+(1-\lambda)f(x_2)$; $0 \leq \lambda \leq 1$.

<u>Theorem (2.4)</u>: The intersection of the members of any family of convex sets is again a convex set. **[NUH-98, RU-99]**

Proof: Let $F = \{S_\alpha : \alpha \in Z^+\}$ be a family of convex sets and $I = \bigcap_{\alpha \in Z^+} S_\alpha$ be the intersection of the members, where Z^+ is the set of positive integers. Let $x, y \in I$, so that $x, y \in S_\alpha$ for all $\alpha \in Z^+$. Since S_α is a convex set, x & $y \in S_\alpha$ implies that the line segment $[x:y] \subset S_\alpha$. Thus $[x:y] \subset S_\alpha$ for all $\alpha \in Z^+$ and, therefore, $[x:y] \subset I$. Since $x, y \in I$ implies $[x:y] \subset I$, hence, I is a convex set. Thus prove the theorem.

<u>Example (2.7)</u>: Show that the set, $S = \{(x_1, x_2, x_3) | 2x_1 - x_2 + x_3 \leq 4, x_1 + 2x_2 - x_3 \leq 1\}$ is a convex set. **[NUH-01]**

Proof: The set is the intersection of two half spaces, viz.,

$H_1=\{(x_1, x_2, x_3)|2x_1-x_2+x_3 \leq 4\}$ and

$H_2=\{(x_1, x_2, x_3)|x_1+2x_2-x_3 \leq 1\}$

$\because$ H_1 and H_2 are convex sets, and $S = H_1 \cap H_2$, therefore S is a convex set.

Example (2.8): Show that intersection of two convex sets is also a convex set.

Proof: Let S_1 and S_2 be two convex sets and $S = S_1 \cap S_2$. Let $x, y \in S$, then $x, y \in S_1$ and $x, y \in S_2$ implies $\lambda x + (1 - \lambda)y \in S_1$ and $\lambda x + (1 - \lambda)y \in S_2; \forall \lambda, 0 \leq \lambda \leq 1$. Hence $\lambda x + (1-\lambda)y \in S; \forall \lambda, 0 \leq \lambda \leq 1$, that is, for $x, y \in S$ implies the line segment $[x:y] \subset S$. Therefore, S is a convex set. So, the intersection of two convex sets is also a convex set.

Example (2.9): Let A be an $m \times n$ matrix, and b be an m-vector, then show that $\{x \in \mathbf{R}^n \mid Ax \leq b\}$ is a convex set.

Proof: Let $x = \{x_1, x_2, ..., x_n\}$, $b = \{b_1, b_2, ..., b_m\}$ and $A = (a_{ij})_{m \times n}$, then the set $S = \{x \in \mathbf{R}^n \mid Ax \leq b\}$ is described by the m inequalities:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\
&\ldots \qquad\qquad \ldots \qquad\qquad \ldots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m
\end{aligned}
$$

Thus, the set S is the intersection of m half spaces
$H_i = \{(x_1, x_2, ..., x_n) \mid a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \leq b_i\}$, $i = 1, 2, ..., m$

Since each half space is a convex set, therefore, $S = \bigcap_{i=1}^{m} H_i$, is also a convex set.

2.14 Convex hull of a set: Let $A \subset \mathbf{R}^n$, then the intersection of all convex sets, containing A, is called the convex hull of A and denoted by $\langle A \rangle$. In symbols, if $A \subset \mathbf{R}^n$, then $\langle A \rangle = \cap W_i$, where for each i, $W_i \supset A$ and W_i is a convex set. **[NUH-03, 06]**

Theorem (2.5): The convex hull $\langle A \rangle$ of a set A is the smallest convex set containing A. **[JU-89]**

Proof: We know that the convex hull of a set A is <A> = $\cap W_i$, where for each i, $W_i \supset A$ and w_i is a convex set. Since, the intersection of the members of any family of convex sets is convex, it follows that <A> is a convex set.
Now, for any set $A \subset \mathbf{R}^n$, we have

(i) <A> is a convex set and $A \subset$ <A> &

(ii) if $W \supset A$, be a smallest convex set, then <A> $\subset W$.

Thus, the convex hull of a set $A \subset \mathbf{R}^n$, is the smallest convex set containing A.

2.15 <u>Closure, Interior and Boundary of a convex set</u>: Students should understand ε-neihbourhood and limit point before interring into the closure, interior and boundary of a convex set. We can define these as follows: **[NUH-03, 05, 06]**

(i) ε-neihbourhood of a point: Let $a \in \mathbf{R}^n$ and very small positive number $\varepsilon > 0$, then the open ball with centre a and radius ε is called the ε -neihbourhood of a, is denoted by $N_\varepsilon(a)$ and is defined by $N_\varepsilon(a) = \{x: |x - a| < \varepsilon\}$

In this point of view, we can say that $N_2(5)=\{x: |x - 5| < 2\}$, i.e., (3, 7) is the 2-Neihbourhood of 5.

(ii) Closure of a convex set: Let $S \subset \mathbf{R}^n$ is a convex set. A point x is said to be a limit point (or accumulation point) of S, if every ε-neihbourhood of x contains at least one point of S other than x.

The set of all limit points of S is called the <u>derived set</u> of S and is denoted by S'. The union of the convex set S and the derived set S' is called the closure of S and is denoted by $\overline{S} = S \cup S'$.

(iii) Interior of a convex set: Let $S \subset \mathbf{R}^n$ be a convex set. Then a point $x \in S$ is said to be an interior point of the convex set S, if there exist $\varepsilon > 0$ such that $N_\varepsilon(x) \subset S$.

The set of all interior points of S is called the interior of S and is denoted by Int(S).

(iv) Boundary of a convex set: Let $S \subset \mathbf{R}^n$ be a convex set. A point x is said to be a boundary point of S, if every ε-neihbourhood of x contains at least two points, one from S and the other from outside of S.

The set of all boundary points of S is called the boundary of S.

Theorem (2.6): Let S and T be two convex sets in $\mathbf{R}^n$. Then for any scalars α, β; $\alpha S + \beta T$ is also convex.**[NUH-03, 04, 06 DU-02]**

Proof: Let $S \subset \mathbf{R}^n$ and $T \subset \mathbf{R}^n$ be two convex sets and $\alpha, \beta \in \mathbf{R}$. Let x, y be two points of $\alpha S + \beta T$.

Then, $x = \alpha s_1+\beta t_1$ and $y = \alpha s_2+\beta t_2$ where $s_1,s_2 \in S$ and $t_1,t_2 \in T$ ... (i)

For any scalar λ, $0 \le \lambda \le 1$; $\lambda x+(1-\lambda)y = \lambda(\alpha s_1+\beta t_1)+(1-\lambda)(\alpha s_2+\beta t_2)$

Or, $\lambda x + (1 - \lambda)y = \alpha[\lambda s_1 + (1 - \lambda)s_2] + \beta[\lambda t_1 + (1 - \lambda)t_2]$ ... (ii)

Since S is a convex set, $s_1,s_2 \in S \Rightarrow \lambda s_1+(1-\lambda)s_2 \in S$, $0 \le \lambda \le 1$ ... (iii)

Since T is a convex set, $t_1,t_2 \in S \Rightarrow \lambda t_1+(1-\lambda)t_2 \in T$, $0 \le \lambda \le 1$ ... (iv)

Using (iii) and (iv) in (ii), we have, for any λ in [0, 1]

$$\lambda x + (1 - \lambda)y \in \alpha S + \beta T$$

Thus, $x, y \in \alpha S + \beta T \Rightarrow [x:y] \subset \alpha S + \beta T$

Hence, $\alpha S + \beta T$ is a convex set.

Corollary (2.2): If S and T be two convex sets in $\mathbf{R}^n$, then S+T, S–T and $\alpha S - \beta T$ are convex sets.

Proof: Prove yourselves.

Theorem (2.7): The set of all convex combinations of a finite number of vectors (or points) $x_1, x_2, ..., x_k$ in $\mathbf{R}^n$, is a convex set.

Or, A convex polyhedron is a convex set. **[JU-90, 00 NU-00, 04]**

Proof: Let, $S = \{x \mid x = \sum_{i=1}^{k} \lambda_i x_i$, $\lambda_i \ge 0$, $\sum_{i=1}^{k} \lambda_i = 1\}$. We shall show that S is a convex set. Let, $x^/$ and $x^{//}$ be in S, so that we have,

$x^{/} = \sum_{i=1}^{k} \lambda_i^{/} x_i$, $\lambda_i^{/} \geq 0$, $\sum_{i=1}^{k} \lambda_i^{/} = 1$ and $x^{//} = \sum_{i=1}^{k} \lambda_i^{//} x_i$, $\lambda_i^{//} \geq 0$, $\sum_{i=1}^{k} \lambda_i^{//} = 1$.

For any scalar t, $0 \leq t \leq 1$,

let $u = tx^{/}+(1-t)x^{//} = t\sum_{i=1}^{k} \lambda_i^{/} x_i + (1-t) \sum_{i=1}^{k} \lambda_i^{//} x_i = \sum_{i=1}^{k} [t\lambda_i^{/} + (1-t)\lambda_i^{//}]x_i$

$= \sum_{i=1}^{k} \mu_i x_i$,   where $\mu_i = t\lambda_i^{/} + (1-t)\lambda_i^{//}$; i = 1, 2, . . ., k.

Since, $\lambda_i^{/} \geq 0$, $\lambda_i^{//} \geq 0$, $\sum_{i=1}^{k} \lambda_i^{/} = 1 = \sum_{i=1}^{k} \lambda_i^{//}$ $\therefore$ $\mu_i \geq 0$ and

$\sum_{i=1}^{k} \mu_i = \sum_{i=1}^{k} [t\lambda_i^{/} + (1-t)\lambda_i^{//}] = t\sum_{i=1}^{k} \lambda_i^{/} + (1-t) \sum_{i=1}^{k} \lambda_i^{//} = t + (1-t) = 1$

Thus $u = \sum_{i=1}^{k} \mu_i x_i$, $\mu_i \geq 0$ and $\sum_{i=1}^{k} \mu_i = 1$

So, $u \in S$, therefore the line segment $[x^{/}:x^{//}] \subset S$.
Hence, S is a convex set.

<u>Theorem (2.8)</u>: A finite set $S \subseteq \mathbf{R}^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S. **[NUH-98, 00, 01, 04, 07, JU-02, DU-01]**

Proof: <u>Necessary condition</u>: Suppose every convex combination of any finite number of points of S is contained in S, we have to prove that S is a convex set. Since S contains every convex combination of its points, we let $S = \{x \mid x = \sum_{i=1}^{k} \lambda_i x_i$, $\lambda_i \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1\}$, $\forall x_i \in S$. Again let, $x^{/}$ and $x^{//}$ be in S, so that we have,

$x^{/} = \sum_{i=1}^{k} \lambda_i^{/} x_i$, $\lambda_i^{/} \geq 0$, $\sum_{i=1}^{k} \lambda_i^{/} = 1$ and $x^{//} = \sum_{i=1}^{k} \lambda_i^{//} x_i$, $\lambda_i^{//} \geq 0$, $\sum_{i=1}^{k} \lambda_i^{//} = 1$.

For any scalar t, $0 \leq t \leq 1$,

let u = $tx^{/}+(1-t)x^{//}$ = $t\sum_{i=1}^{k}\lambda_i^{/}x_i$ +(1–t) $\sum_{i=1}^{k}\lambda_i^{//}x_i = \sum_{i=1}^{k}[t\lambda_i^{/}+(1-t)\lambda_i^{//}]x_i$

$= \sum_{i=1}^{k}\mu_i x_i$, where $\mu_i = t\lambda_i^{/}+(1-t)\lambda_i^{//}$; i = 1, 2, . . ., k.

Since, $\lambda_i^{/} \geq 0$, $\lambda_i^{//} \geq 0$, $\sum_{i=1}^{k}\lambda_i^{/} = 1 = \sum_{i=1}^{k}\lambda_i^{//}$ $\therefore$ $\mu_i \geq 0$ and

$\sum_{i=1}^{k}\mu_i = \sum_{i=1}^{k}[t\lambda_i^{/}+(1-t)\lambda_i^{//}] = t\sum_{i=1}^{k}\lambda_i^{/} + (1-t)\sum_{i=1}^{k}\lambda_i^{//} = t + (1-t) = 1$

Thus u = $\sum_{i=1}^{k}\mu_i x_i$, $\mu_i \geq 0$ and $\sum_{i=1}^{k}\mu_i = 1$

So, u $\in$ S, therefore the line segment $[x^{/}:x^{//}] \subset$ S.
Hence, S is a convex set. Thus the necessary condition is satisfied.

Sufficient condition: Let S be a convex set. We have to prove that every convex combination of any finite number of points of S is contained in S. We prove it by the mathematical induction method. Let x = $\lambda x_1 + (1-\lambda)x_2$ is a line segment as well as a convex combination for $x_1, x_2 \in$ S, then x $\in$ S because S is a convex set.
Now we assume that the convex combination of any k points of S is a point of S. So let y = $\sum_{i=1}^{k}\lambda_i x_i$, $\lambda_i \geq 0$, $\sum_{i=1}^{k}\lambda_i = 1$, $\forall\, x_i \in$ S, y $\in$ S.

Again let z = $\sum_{i=1}^{k+1}\lambda_i x_i$, where $x_i \in$ S, $x_i \in$ S, $\sum_{i=1}^{k+1}\lambda_i = 1$

Then without loss of generality, we can assume $\lambda_{k+1} \neq 1$ and

$$z = (1-\lambda_{k+1})\sum_{i=1}^{k}\left\{\lambda_i\left(\frac{1}{1-\lambda_{k+1}}\right)x_i\right\} + \lambda_{k+1}x_{k+1}$$

$$= (1-\lambda_{k+1})\sum_{i=1}^{k}\mu_i x_i + \lambda_{k+1}x_{k+1} \text{ where } \mu_i = \frac{\lambda_i}{1-\lambda_{k+1}} \text{ for i = 1, 2, ...,k}$$

Now since $\lambda_i \geq 0$ and $\lambda_{k+1} \neq 1$ so, $\mu_i = \dfrac{\lambda_i}{1-\lambda_{k+1}} \geq 0$ and $1- \lambda_{k+1} > 0$.

Hence, $\sum_{i=1}^{k} \mu_i = 1$. We have by the hypothesis $\sum_{i=1}^{k} \mu_i x_i \in S$.

So, $z \in S$ because $z = (1-\lambda_{k+1}) \sum_{i=1}^{k} \mu_i x_i + \lambda_{k+1} x_{k+1}$ is a line segment as well as a convex combination of two points $\sum_{i=1}^{k} \mu_i x_i$ and x_{k+1} of S where the sum of scalars $(1-\lambda_{k+1})$ and λ_{k+1} is 1.

Thus by the mathematical induction method it is clear that every convex combination of any finite number of points of S is contained in S. Hence the necessary condition is satisfied.

<u>Example (2.10)</u>: Show that in $\mathbf{R}^3$, the closed ball $x_1^2 + x_2^2 + x_3^2 \leq 1$, is a convex set. **[JU-92, NUH-02]**

Proof: Suppose the set of points of the given closed ball be

$$S = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$$

Let $x, y \in S$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then we have $x_1^2 + x_2^2 + x_3^2 \leq 1$ . . . (1) and $y_1^2 + y_2^2 + y_3^2 \leq 1$ . . . (2)

Let $u = (u_1, u_2, u_3)$ be any point of line segment

$[x:y] = \{u : u = \lambda x + (1 - \lambda)y \,;\, 0 \leq \lambda \leq 1\}$

So, $u_1 = \lambda x_1 + (1 - \lambda)y_1$, $u_2 = \lambda x_2 + (1 - \lambda)y_2$, $u_3 = \lambda x_3 + (1 - \lambda)y_3$

And we know from linear algebra, norm, $\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

Or, $\|u\|^2 = u_1^2 + u_2^2 + u_3^2$

Or, $\|u\|^2 = [\lambda x_1 + (1-\lambda)y_1]^2 + [\lambda x_2 + (1-\lambda)y_2]^2 + [\lambda x_3 + (1-\lambda)y_3]^2$

Or, $\|u\|^2 = \lambda^2(x_1^2 + x_2^2 + x_3^2) + (1 - \lambda)^2(y_1^2 + y_2^2 + y_3^2) + 2\lambda(1 - \lambda)$

$(x_1y_1 + x_2y_2 + x_3y_3)$ ... (3)

By Cauchy – Schwarz's inequality, we get

$$||x.y|| \le ||x||.||y||$$

Or, $(x_1y_1 + x_2y_2 + x_3y_3) \le \sqrt{x_1^2 + x_2^2 + x_3^2} \cdot \sqrt{y_1^2 + y_2^2 + y_3^2}$

Or, $(x_1y_1 + x_2y_2 + x_3y_3) \le 1$ [Using (1) and (2)] . . . (4)

Using (1), (2) and (4) in (3), we get,

$$||u||^2 \le \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda)$$

Or, $||u||^2 \le 1$

Or, $u_1^2 + u_2^2 + u_3^2 \le 1$

i.e., $u = (u_1, u_2, u_3)$ is a point of S.

Thus, for x, y $\in$ S implies that the line segment $[x:y] \subset S$.

Hence, S is a convex set, i.e., in $\mathbf{R}^3$, the closed ball $x_1^2 + x_2^2 + x_3^2 \le 1$ is a convex set.

Theorem (2.9): The set of all feasible solutions to the linear programming problem is a convex set. **[JU-87, DU-89, NUH-02]**

Or, Let K be the set of all points $X \ge 0$ such that $AX = b$. Then K is a convex set.

Proof: Let K be the set of all feasible solutions. If there is no feasible solution, then K is empty and an empty set is always convex.

If K is non empty, then at least one $X \ge 0$ such that $AX = b$

$\therefore$ $\lambda X + (1 - \lambda)X = X$ for $0 \le \lambda \le 1$.

So the set K is convex.

If more than one points belong to K, then let $X_1, X_2 \in K$.

So, $X_1 \ge 0$, $X_2 \ge 0$ [since every feasible solution is non-negative] and $AX_1 = b$, $AX_2 = b$.

We need to show that for any $\lambda \ge 0$ and $0 \le \lambda \le 1$

$$\lambda X_1 + (1 - \lambda)X_2 \in K.$$

Now, $A\{\lambda X_1 + (1 - \lambda)X_2\} = \lambda AX_1 + (1 - \lambda)AX_2$

$= \lambda b + (1 - \lambda) b$

$= b$

$\therefore$ $\lambda X_1 + (1 - \lambda)X_2 \in K$ and hence K is a convex set.

Similarly, we can prove that K is convex for more than two points. (Hence proved the theorem)

Note: Taking $AX \geq b$ or $AX \leq b$ as the constraints, we can prove the above theorem in the similar manner.

<u>Theorem (2.10)</u>: If the set of constraints T of a linear program (LP) is non-empty closed and bounded then an optimal (minimum or maximum) solution to the LP exists and it is attained at a vertex of T. **[NU-98, 01, 02]**

Proof: Since the set of constraints T of a linear program (LP) is non-empty closed and bounded, it has finite number of vertices or extreme points. Hence the set of constraints T is a convex polyhedron, which is the set of all feasible solutions. A linear function is always continuous, so the linear objective function f(X) is continuous and hence an optimal solution of the linear program must exist.

To prove the second part of the theorem let us denote the vertices by $X_1, X_2, ..., X_p$ and the optimum solution by X_0. This means that $f(X_0) \leq [or \geq]\ f(X)$ for all X in T. If X_0 is a vertex, the second part of the theorem is true. (In two dimensions T might look like figure)

Suppose X_0 is not a vertex (as indicated in figure). We can then write X_0 as a convex combination of the vertices of T, that is,

$$X_0 = \sum_{i=1}^{p} \alpha_i X_i \ ; \quad \text{for} \quad \alpha_i \geq 0, \quad \sum_i \alpha_i = 1$$

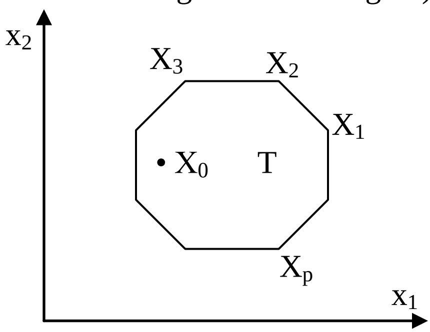

Figure 2.12

Then, since f(X) is a linear function, we have

$$f(X_0) = f\left(\sum_{i=1}^{p} \alpha_i X_i\right) = f(\alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_p X_p)$$

$$= \alpha_1 f(X_1) + \alpha_2 f(X_2) + ... + \alpha_p f(X_p) = m \qquad . . . \text{ (i)}$$

where m is the optimum value of f(X) for all X in T.
Since all $\alpha_i \geq 0$, we do not increase [or decrease] the sum (i) if we substitute for each $f(X_i)$ the optimum of the values $f(X_i)$. Let $f(X_m) = \underset{i}{optimum}\, f(X_i)$, substituting in (i) we have, since $\sum_i \alpha_i = 1$,

$$f(X_0) \geq [\text{or} \leq]\ \alpha_1 f(X_m) + \alpha_2 f(X_m) + \ldots + \alpha_p f(X_m) = f(X_m)$$

Since we assumed $f(X_0) \leq$ [or $\geq$] f(X) for all X in T, we must have $f(X_0) = f(X_m) = m$. Therefore, there is a vertex, X_m, at which the objective function assumes its optimum value.
Hence an optimal (minimum or maximum) solution to the LP exists and it is attained at a vertex of T.

Theorem (2.11): The objective function assures its minimum at an extreme point of the convex set K generated by the set of feasible solutions to the linear programming problem. If it assures its minimum at more than one extreme point, then it takes same value for every convex combination of those particular points.

Proof: The set of all feasible solutions to the linear programming problem, K is a convex polyhedron. So, K has a finite number of extreme points. In two dimensions K might look like figure. Let us denote the objective function by f(X), the extreme points by $X_1, X_2, \ldots, X_p$ and the minimum feasible solution by X_0. This means that $f(X_0) \leq f(X)$ for all X in K. If X_0 is an extreme point, the first part of the theorem is true.

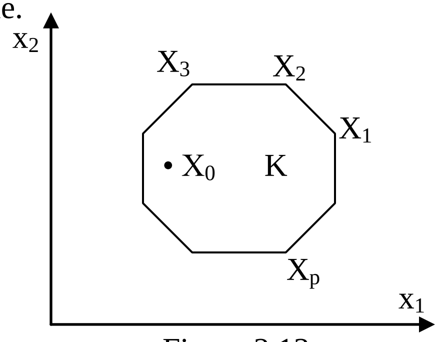

Figure 2.13

Suppose X_0 is not an extreme point (as indicated in figure). We can then write X_0 as a convex combination of the extreme points of K, that is,

$$X_0 = \sum_{i=1}^{p} \alpha_i X_i\ ; \quad \text{for} \quad \alpha_i \geq 0,$$

$$\sum_i \alpha_i = 1$$

Then, since f(X) is a linear function, we have

$$f(X_0) = f(\sum_{i=1}^{p} \alpha_i X_i) = f(\alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_p X_p)$$
$$= \alpha_1 f(X_1) + \alpha_2 f(X_2) + ... + \alpha_p f(X_p) = m \quad ... \quad (i)$$

where m is the minimum of f(X) for all X in K.

Since all $\alpha_i \geq 0$, we do not increase the sum (i) if we substitute for each $f(X_i)$ the minimum of the values $f(X_i)$. Let $f(X_m) = \min_i f(X_i)$ substituting in (i) we have, since $\sum_i \alpha_i = 1$,

$$f(X_0) \geq \alpha_1 f(X_m) + \alpha_2 f(X_m) + ... + \alpha_p f(X_m) = f(X_m)$$

Since we assumed $f(X_0) \leq f(X)$ for all X in K, we must have $f(X_0) = f(X_m) = m$.

Therefore, there is an extreme point, X_m, at which the objective function assumes its minimum value.

To prove the second part of the theorem, let f(X) assumes its minimum at more than one extreme point, say at $X_1, X_2, ..., X_q$. Here we have $f(X_1) = f(X_2) = ... = f(X_q) = m$. If X is any convex combination of the above X_i, say

$$X = \sum_{i=1}^{q} \alpha_i X_i \text{ ; for } \alpha_i \geq 0, \sum_i \alpha_i = 1$$

$$\text{Then } f(X) = f(\sum_{i=1}^{q} \alpha_i X_i) = f(\alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_q X_q)$$
$$= \alpha_1 f(X_1) + \alpha_2 f(X_2) + ... + \alpha_p f(X_p)$$
$$= \alpha_1 m + \alpha_2 m + ... + \alpha_p m$$
$$= \sum_i \alpha_i m = m$$

That is, f(X) assumes its minimum at every convex combination of extreme points $X_1, X_2, ..., X_q$. The proof is now complete.

Note: By making the obvious changes, the theorem can be proved for the case where the objective function is to be maximized.

Theorem (2.12): The set of all optimal solutions to a linear programming (LP) problem is a convex set. **[NU-98]**

Proof: If there is only one optimal solution to the linear programming problem then it is a convex set of one point.

If there are more than one optimal solutions to the linear programming problem then these optimal solutions must lie on the boundary line segment of the convex set of all feasible solutions. Let two optimal solutions X_1, X_2 be the end points of the boundary line segment of the convex set of all feasible solutions. That is, $Z_{optimum} = f(X_1) = f(X_2)$, where f(X) is the linear objective function. Then the convex line segment $S = [X_1:X_2] = \{U : U = \lambda X_1 + (1-\lambda)X_2\}$ represents all the optimal solutions as $f(U) = f[\lambda X_1 + (1-\lambda)X_2] = \lambda f(X_1) + (1-\lambda)f(X_2) = \lambda Z_{optimum} + (1-\lambda)Z_{optimum} = Z_{optimum}$. Therefore, the set of all optimal solutions to a linear programming (LP) problem is a convex set.

Theorem (2.13): Every extreme point of the convex set of all feasible solutions to a linear programming (LP) problem is a basic feasible solution and vice versa.

Proof: Let us consider a linear programming (LP) problem with m constraints and n variables (m < n), which we can write as follows:

$$\underline{A}.\underline{X} = \underline{b} \text{ (or, } x_1\underline{P}_1 + x_2\underline{P}_2 + \ldots + x_n\underline{P}_n = \underline{P}_o\text{)}, \ \underline{X} \geq 0$$

Necessary condition: Suppose $\underline{X} = (x_1, x_2, ..., x_n)$ be any extreme point of the set of feasible solutions. We have to prove that at most m components of $\underline{X}$ are positive and the column vectors $\underline{P}_i$s associated with positive x_is are linearly independent, i.e., $\underline{X}$ is a basic feasible solution. To prove this let the first k ($k \leq m$) components of $\underline{X} = (x_1, x_2, ..., x_n)$ are positive, i.e., $\underline{X} = (x_1, x_2, ..., x_k, 0, ..., 0)$, $x_i > 0$; $i = 1, 2, ..., k$.

So, $\quad x_1\underline{P}_1 + x_2\underline{P}_2 + \ldots + x_k\underline{P}_k = \underline{P}_o \quad \ldots \quad$ (i)

Assume that $\underline{P}_1, \underline{P}_2, ..., \underline{P}_k$ are linearly dependent. Then there exists a linear combination of these vectors with at least one scalar $d_i \neq 0$

$$d_1\underline{P}_1 + d_2\underline{P}_2 + \ldots + d_k\underline{P}_k = 0 \quad \ldots \quad \text{(ii)}$$

For some d > 0, we multiply (ii) by d and then add and subtract the result from (i) to obtain the two equations:

$$\begin{cases}(x_1 + dd_1)\underline{P}_1 + (x_2 + dd_2)\underline{P}_2 + \ldots + (x_k + dd_k)\underline{P}_k = \underline{P}_0 \\ (x_1 - dd_1)\underline{P}_1 + (x_2 - dd_2)\underline{P}_2 + \ldots + (x_k - dd_k)\underline{P}_k = \underline{P}_0\end{cases}$$

We then have the two solutions (note that they might not be feasible solutions)

$\underline{X}_1 = (x_1+dd_1, x_2+dd_2, \ldots, x_k+dd_k, 0, 0, \ldots, 0)$

and $\underline{X}_2 = (x_1-dd_1, x_2-dd_2, \ldots, x_k-dd_k, 0, 0, \ldots, 0)$

Since all $x_i > 0$, we can let d be as small as necessary, but still positive, to make the first k components of both $\underline{X}_1$ and $\underline{X}_2$ positive. Then $\underline{X}_1$ and $\underline{X}_2$ are feasible solutions and $\underline{X} = \frac{1}{2}\underline{X}_1 + \frac{1}{2}\underline{X}_2$ which contradicts the hypothesis that $\underline{X}$ is an extreme point (because extreme point can not be expressed as the convex combination of two other points of that set). The assumption of linear dependence for the vectors $\underline{P}_1, \underline{P}_2, \ldots, \underline{P}_k$ has led to a contradiction and hence must be false, i.e., the vectors $\underline{P}_1, \underline{P}_2, \ldots, \underline{P}_k$ associated with positive x_is of $\underline{X}$ are linearly independent.

Since every set of more than m vectors in m-dimensional space is necessarily linearly dependent, we cannot have more than m positive components in $\underline{X}$.

So every extreme point $\underline{X} = (x_1, x_2, \ldots, x_n)$ of the set of feasible solutions, has at most m positive components and the column vectors $\underline{P}_i$s associated with positive x_is are linearly independent, i.e., $\underline{X}$ is a basic feasible solution.

Sufficient condition: Suppose $\underline{X} = (x_1, x_2, \ldots, x_k, 0, \ldots, 0_n)$ is a basic feasible solution ($k \le m$). Then $x_i > 0$; i = 1, 2, ..., k and the column vectors $\underline{P}_1, \underline{P}_2, \ldots, \underline{P}_k$ associated with positive x_is are linearly independent. We have to prove that $\underline{X}$ is an extreme point of the set of feasible solutions. Suppose $\underline{X}$ is not an extreme point. Since $\underline{X}$ is a feasible solution, it can be written as a convex combination of two other points $\underline{X}_1$ and $\underline{X}_2$ of the set of feasible solutions, i.e., $\underline{X} = \lambda\underline{X}_1 + (1 - \lambda)\underline{X}_2$ for $0 < \lambda < 1$. Since all the components x_i of $\underline{X}$ are non-negative and $0 < \lambda < 1$, the last n – k components of $\underline{X}_1$ and $\underline{X}_2$ must also equal to zero. That is,

$$\underline{X}_1 = \left(x_1^{(1)}, x_2^{(1)}, ..., x_k^{(1)}, 0, ..., 0\right) \text{ and } \underline{X}_2 = \left(x_1^{(2)}, x_2^{(2)}, ..., x_k^{(2)}, 0, ..., 0\right)$$

Since $\underline{X}_1$ and $\underline{X}_2$ are feasible solutions, we have

$$x_1^{(1)}\underline{P}_1 + x_2^{(1)}\underline{P}_2 + ... + x_k^{(1)}\underline{P}_k = \underline{P}_0 \quad ... \quad \text{(iii)}$$

and $$x_1^{(2)}\underline{P}_1 + x_2^{(2)}\underline{P}_2 + ... + x_k^{(2)}\underline{P}_k = \underline{P}_0 \quad ... \quad \text{(iv)}$$

Subtracting (iv) from (iii), we have

$$(x_1^{(1)} - x_1^{(2)})\underline{P}_1 + (x_2^{(1)} - x_2^{(2)})\underline{P}_2 + ... + (x_k^{(1)} - x_k^{(2)})\underline{P}_k = 0$$

Since $\underline{P}_1, \underline{P}_2, ..., \underline{P}_k$ are linearly independent vectors, we have

$$x_1^{(1)} - x_1^{(2)} = 0,\ x_2^{(1)} - x_2^{(2)} = 0,\ ...,\ x_k^{(1)} - x_k^{(2)} = 0$$

Or, $$x_1^{(1)} = x_1^{(2)},\ x_2^{(1)} = x_2^{(2)},\ ...,\ x_k^{(1)} = x_k^{(2)}.$$

This implies that, $x_i = x_i^{(1)} = x_i^{(2)}$ for i = 1, 2, ..., k; i.e., $\underline{X} = \underline{X}_1 = \underline{X}_2$. Therefore, $\underline{X}$ cannot be expressed as a convex combination of two distinct points of the set of feasible solutions. Hence, $\underline{X}$ must be an extreme point, i.e., every basic feasible solution is an extreme point of the set of all feasible solutions to a linear programming problem.

2.16 <u>Some done examples</u>:

<u>Example (2.11)</u>: Show that (hyper plane), $S=\{(x_1, x_2):2x_1+3x_2 =5\} \subset \mathbf{R}^2$ is a convex set. **[JU-88]**

Proof: Let any points $x, y \in S$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The line segment joining x and y is the set $\{u : u = \lambda x + (1 - \lambda)y ;\ 0 \le \lambda \le 1\}$. For some λ, $0 \le \lambda \le 1$, let $u = (u_1, u_2)$ be a point of this set, so that, $u_1 = \lambda x_1 + (1 - \lambda)y_1$ and $u_2 = \lambda x_2 + (1 - \lambda)y_2$.

Since, $x, y \in S$, $2x_1 + 3x_2 = 5$ and $2y_1 + 3y_2 = 5$

Now,
$$\begin{aligned} 2u_1 + 3u_2 &= 2[\lambda x_1 + (1 - \lambda)y_1] + 3[\lambda x_2 + (1 - \lambda)y_2] \\ &= \lambda(2x_1 + 3x_2) + (1 - \lambda)(2y_1 + 3y_2) \\ &= 5\lambda + 5(1 - \lambda) \end{aligned}$$

$= 5$

So, $u = (u_1, u_2)$ is a point of S. Since u is any point of the line segment [x:y], for $x, y \in S$, $[x:y] \subset S$. Hence S is a convex set.

Example (2.12): Show that the hyper plane, $S = \{(x_1, x_2, x_3): 2x_1+3x_2 + 4x_3 = 9\} \subset \mathbf{R}^3$ is a convex set. **[RU-87]**

Proof: Let any points $x, y \in S$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The line segment joining x and y is the set $\{u : u = \lambda x + (1 - \lambda)y \; ; 0 \leq \lambda \leq 1\}$. For some λ, $0 \leq \lambda \leq 1$, let $u = (u_1, u_2, u_3)$ be a point of this set, so that, $u_1 = \lambda x_1 + (1 - \lambda)y_1$, $u_2 = \lambda x_2 + (1 - \lambda)y_2$ and $u_3 = \lambda x_3 + (1 - \lambda)y_3$. Since, $x, y \in S$, $2x_1 + 3x_2 + 4x_3 = 9$ and $2y_1 + 3y_2 + 4y_3 = 9$. Now,

$$\begin{aligned}2u_1+3u_2+4u_3 &= 2[\lambda x_1+(1-\lambda)y_1]+3[\lambda x_2+(1-\lambda)y_2]+4[\lambda x_3 +(1-\lambda)y_3]\\ &= \lambda(2x_1 + 3x_2 + 4x_3) + (1 - \lambda)(2y_1 + 3y_2 + 4y_3)\\ &= 9\lambda + 9(1 - \lambda)\\ &= 9\end{aligned}$$

So, $u = (u_1, u_2, u_3)$ is a point of S. Since u is any point of the line segment [x:y], for $x, y \in S$, $[x:y] \subset S$. Hence S is a convex set.

Example (2.13): Show that the half space, $S = \{(x_1, x_2, x_3) \mid 3x_1 - x_2 + x_3 \leq 5\} \subset \mathbf{R}^3$, is a convex set. **[DU-94]**

Proof: Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two points of S.

Then $3x_1 - x_2 + x_3 \leq 5$ . . . (1)

and $3y_1 - y_2 + y_3 \leq 5$ . . . (2)

Let $u = (u_1, u_2, u_3)$ be any point of [x:y] so that $0 \leq \lambda \leq 1$

$u_1 = \lambda x_1+(1-\lambda)y_1$, $u_2 = \lambda x_2+(1-\lambda)y_2$ and $u_3 = \lambda x_3+(1-\lambda)y_3$ . . . (3)

$\therefore \quad 3u_1-u_2+u_3 = 3[\lambda x_1+(1-\lambda)y_1] - [\lambda x_2+(1-\lambda)y_2] + [\lambda x_3+(1-\lambda)y_3]$

Or, $3u_1 - u_2 + u_3 = \lambda(3x_1 - x_2 + x_3) + (1 - \lambda)(3y_1 - y_2 + y_3)$

Or, $3u_1 - u_2 + u_3 \leq 5\lambda + 5(1 - \lambda)$ [Using (1) and (2)]

Or, $3u_1 - u_2 + u_3 \leq 5$

i.e., $u = (u_1, u_2, u_3)$ is a point of S.

Thus, for $x, y \in S$ implies that the line segment $[x:y] \subset S$.

Hence, S is a convex set.

Example (2.14): Show that the closed half space, $S = \{(x_1, x_2, x_3) \mid x_1 + 2x_2 + x_3 \leq 10\} \subset \mathbf{R}^3$, is a convex set. **[JU-98]**

Proof: Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two points of S.

Then $\quad x_1 + 2x_2 + x_3 \leq 10 \quad \ldots \quad (1)$

and $\quad y_1 + 2y_2 + y_3 \leq 10 \quad \ldots \quad (2)$

Let $u = (u_1, u_2, u_3)$ be any point of [x:y] so that $0 \leq \lambda \leq 1$

$u_1 = \lambda x_1+(1-\lambda)y_1$, $u_2 = \lambda x_2+(1-\lambda)y_2$ and $u_3 = \lambda x_3+(1-\lambda)y_3 \quad \ldots \quad (3)$

$\therefore \quad u_1+2u_2+u_3 = [\lambda x_1+(1-\lambda)y_1] + 2[\lambda x_2+(1-\lambda)y_2] + [\lambda x_3+(1-\lambda)y_3]$

Or, $u_1 + 2u_2 + u_3 = \lambda(x_1 + 2x_2 + x_3) + (1 - \lambda)(y_1 + 2y_2 + y_3)$

Or, $u_1 + 2u_2 + u_3 \leq 10\lambda + 10(1 - \lambda)$ [Using (1) and (2)]

Or, $u_1 + 2u_2 + u_3 \leq 10$

i.e., $u = (u_1, u_2, u_3)$ is a point of S.

Thus, for $x, y \in S$ implies that the line segment $[x:y] \subset S$.

Hence, S is a convex set.

Example (2.15): Show that the closed half space, $S = \{(x_1, x_2, x_3) \mid x_1 + 2x_2 - x_3 \geq 1\} \subset \mathbf{R}^3$, is a convex set. **[JU 93]**

Proof: Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two points of S.

Then $\quad x_1 + 2x_2 - x_3 \geq 1 \quad \ldots \quad (1)$

and $\quad y_1 + 2y_2 - y_3 \geq 1 \quad \ldots \quad (2)$

Let $u = (u_1, u_2, u_3)$ be any point of [x:y] so that $0 \leq \lambda \leq 1$

$u_1 = \lambda x_1+(1-\lambda)y_1$, $u_2 = \lambda x_2+(1-\lambda)y_2$ and $u_3 = \lambda x_3+(1-\lambda)y_3 \ldots \quad (3)$

$\therefore u_1+2u_2-u_3 = [\lambda x_1+(1-\lambda)y_1] + 2[\lambda x_2+(1-\lambda)y_2] - [\lambda x_3+(1-\lambda)y_3]$

Or, $u_1 + 2u_2 - u_3 = \lambda(x_1 + 2x_2 - x_3) + (1 - \lambda)(y_1 + 2y_2 - y_3)$

Or, $u_1 + 2u_2 - u_3 \geq \lambda + (1 - \lambda)$ [Using (1) and (2)]

Or, $u_1 + 2u_2 - u_3 \geq 1$

i.e., $u = (u_1, u_2, u_3)$ is a point of S.

Thus, for $x, y \in S$ implies that the line segment $[x:y] \subset S$.

Hence, S is a convex set.

<u>Example (2.16)</u>: Show that the open positive half space, $S=\{(x_1,x_2,x_3) \mid x_1 + 2x_2 - x_3 > 3\} \subset \mathbf{R}^3$, is a convex set.

Proof: Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two points of S.

Then $\quad x_1 + 2x_2 - x_3 > 3 \quad \ldots \quad (1)$

and $\quad y_1 + 2y_2 - y_3 > 3 \quad \ldots \quad (2)$

Let $u = (u_1, u_2, u_3)$ be any point of [x:y] so that $0 \le \lambda \le 1$

$u_1 = \lambda x_1+(1-\lambda)y_1$, $u_2 = \lambda x_2+(1-\lambda)y_2$ and $u_3 = \lambda x_3+(1-\lambda)y_3 \ldots$ (3)

$\therefore u_1+2u_2-u_3 = [\lambda x_1+(1-\lambda)y_1] + 2[\lambda x_2+(1-\lambda)y_2] - [\lambda x_3+(1-\lambda)y_3]$

Or, $u_1 + 2u_2 - u_3 = \lambda(x_1 + 2x_2 - x_3) + (1 - \lambda)(y_1 + 2y_2 - y_3)$

Or, $u_1 + 2u_2 - u_3 > 3\lambda + 3(1 - \lambda)$ $\qquad$ [Using (1) and (2)]

Or, $u_1 + 2u_2 - u_3 > 3$

i.e., $u = (u_1, u_2, u_3)$ is a point of S.

Thus, for $x, y \in S$ implies that the line segment $[x:y] \subset S$.

Hence, S is a convex set.

<u>Example (2.17)</u>: Show that intersection of three convex sets is also a convex set.

Proof: Let S_1, S_2 and S_3 be three convex sets and $S = S_1 \cap S_2 \cap S_3$. Let $x, y \in S$, then $x, y \in S_1$; $x, y \in S_2$ and $x, y \in S_3$ implies $\lambda x+(1-\lambda)y \in S_1$; $\lambda x+(1-\lambda)y \in S_2$ and $\lambda x + (1-\lambda)y \in S_3$; $\forall \lambda$, $0 \le \lambda \le 1$. Hence $\lambda x + (1 - \lambda)y \in S$; $\forall \lambda$, $0 \le \lambda \le 1$, that is, for $x, y \in S$ implies the line segment $[x:y] \subset S$. Therefore, S is a convex set. Hence the intersection of three convex sets is also a convex set.

<u>Example (2.18)</u>: Define hyper sphere. Show that in $\mathbf{R}^3$, the closed ball $x_1^2 + x_2^2 + x_3^2 \le 4$, is a convex set. **[JU-90, NU-02]**

Solution: <u>First part</u>: A hyper sphere in $\mathbf{R}^n$ with centre at $a = (a_1, a_2, \ldots, a_n)$ and radius $r > 0$ is defined by the set X of points given by

$X = \{x = (x_1, x_2, \ldots, x_n) : |x - a| = r\}$

$\quad = \{(x_1, x_2, \ldots, x_n) : (x_1 - a_1)^2 + (x_2 - a_2)^2 + \ldots + (x_n - a_n)^2 = r^2\}$

[So, the hyper sphere in two dimensions is a circle and in three dimensions is a sphere.

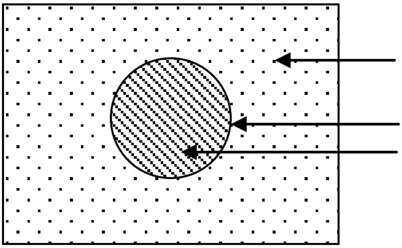

Figure 2.14

From the Figure 2.14, it is clear that inner part of a hyper sphere is a convex set but the outer part is not convex and the inner part with the hyper sphere makes a closed ball.]

Second part: Suppose the set of points of the given closed ball be

$$S = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \le 4\}$$

Let x, y $\in$ S, where x = (x_1, x_2, x_3) and y = (y_1, y_2, y_3). Then we have $x_1^2 + x_2^2 + x_3^2 \le 4$ . . . (1) and $y_1^2 + y_2^2 + y_3^2 \le 4$ . . . (2)

Let u = (u_1, u_2, u_3) be any point of line segment

$[x:y] = \{u : u = \lambda x + (1 - \lambda)y ; 0 \le \lambda \le 1\}$

So, $u_1 = \lambda x_1 + (1 - \lambda)y_1$, $u_2 = \lambda x_2 + (1 - \lambda)y_2$, $u_3 = \lambda x_3 + (1 - \lambda)y_3$

And we know from linear algebra, norm, $||u|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

Or, $||u||^2 = u_1^2 + u_2^2 + u_3^2$

Or, $||u||^2 = [\lambda x_1 + (1-\lambda)y_1]^2 + [\lambda x_2 + (1-\lambda)y_2]^2 + [\lambda x_3 + (1-\lambda)y_3]^2$

Or, $||u||^2 = \lambda^2(x_1^2 + x_2^2 + x_3^2) + (1 - \lambda)^2(y_1^2 + y_2^2 + y_3^2) + 2\lambda(1 - \lambda)$

$(x_1y_1 + x_2y_2 + x_3y_3)$ ... (3)

By Cauchy – Schwarz's inequality, we get

$$||x.y|| \le ||x||.||y||$$

Or, $(x_1y_1 + x_2y_2 + x_3y_3) \leq \sqrt{x_1^2 + x_2^2 + x_3^2} \cdot \sqrt{y_1^2 + y_2^2 + y_3^2}$

Or, $(x_1y_1 + x_2y_2 + x_3y_3) \leq 4$ [Using (1) and (2)] . . . (4)

Using (1), (2) and (4) in (3), we get,

$$\|u\|^2 \leq 4\lambda^2 + 4(1-\lambda)^2 + 8\lambda(1-\lambda)$$

Or, $\|u\|^2 \leq 4$

Or, $u_1^2 + u_2^2 + u_3^2 \leq 4$

i.e., $u = (u_1, u_2, u_3)$ is a point of S.

Thus, for x, y $\in$ S implies that the line segment [x:y] $\subset$ S.

Hence, S is a convex set, i.e., in $\mathbf{R}^3$, the closed ball $x_1^2 + x_2^2 + x_3^2 \leq 4$ is a convex set.

<u>Example (2.19)</u>: Show that in $\mathbf{R}^2$, the set S = $\{(x_1, x_2)$: $x_1^2 + x_2^2 \leq 4\}$ is a convex set. **[JU-91, NU-02]**

Proof: Suppose the set of points of the given closed ball be

$$S = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 4\}$$

Let x, y $\in$ S, where x = (x_1, x_2) and y = (y_1, y_2). Then we have $x_1^2 + x_2^2 \leq 4$ . . . (1) and $y_1^2 + y_2^2 \leq 4$ . . . (2)

Let u = (u_1, u_2) be any point of line segment

$[x:y] = \{u : u = \lambda x + (1-\lambda)y \,;\, 0 \leq \lambda \leq 1\}$

So, $u_1 = \lambda x_1 + (1-\lambda)y_1$, $u_2 = \lambda x_2 + (1-\lambda)y_2$.

And we know from linear algebra, norm, $\|u\| = \sqrt{u_1^2 + u_2^2}$

Or, $\|u\|^2 = u_1^2 + u_2^2$

Or, $\|u\|^2 = [\lambda x_1 + (1-\lambda)y_1]^2 + [\lambda x_2 + (1-\lambda)y_2]^2$

Or, $\|u\|^2 = \lambda^2(x_1^2 + x_2^2) + (1-\lambda)^2(y_1^2 + y_2^2) + 2\lambda(1-\lambda)(x_1y_1 + x_2y_2)$ ... (3)

By Cauchy – Schwarz's inequality, we get

$$\|x.y\| \leq \|x\|.\|y\|$$

Or, $(x_1y_1 + x_2y_2) \leq \sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2}$

Or, $(x_1y_1 + x_2y_2) \leq 4$ [Using (1) and (2)] . . . (4)

Using (1), (2) and (4) in (3), we get,

$$||u||^2 \leq 4\lambda^2 + 4(1-\lambda)^2 + 8\lambda(1-\lambda)$$

Or, $||u||^2 \leq 4$

Or, $u_1^2 + u_2^2 \leq 4$

i.e., $u = (u_1, u_2)$ is a point of S.

Thus, for x, y $\in$ S implies that the line segment $[x:y] \subset S$.

Hence, S is a convex set, i.e., in $\mathbf{R}^2$, the closed ball $x_1^2 + x_2^2 \leq 4$ is a convex set.

Example (2.20): Show that in $\mathbf{R}^2$, the set $S = \{(x_1, x_2): x_2^2 \leq x_1\}$ is a convex set. **[DU-90]**

Proof: Given that, $S = \{(x_1, x_2): x_2^2 \leq x_1\}$

Let x, y $\in$ S, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then we have

$$x_2^2 \leq x_1 \quad \ldots (1) \quad \text{and} \quad y_2^2 \leq y_1 \quad \ldots (2)$$

Let $u = (u_1, u_2)$ be any point of line segment

$$[x:y] = \{u : u = \lambda x + (1-\lambda)y \;;\; 0 \leq \lambda \leq 1\}$$

So, $u_1 = \lambda x_1 + (1-\lambda)y_1$, $u_2 = \lambda x_2 + (1-\lambda)y_2$.

Now $u_2^2 = [\lambda x_2 + (1-\lambda)y_2]^2 = \lambda^2 x_2^2 + (1-\lambda)^2 y_2^2 + 2\lambda(1-\lambda)x_2y_2$

Or, $u_2^2 \leq \lambda^2 x_2^2 + (1-\lambda)^2 y_2^2 + \lambda(1-\lambda)(x_2^2 + y_2^2)$ $[\because 2x_2y_2 \leq x_2^2 + y_2^2]$

Or, $u_2^2 \leq y_2^2 - 2\lambda y_2^2 + \lambda x_2^2 + \lambda y_2^2 = \lambda x_2^2 + (1-\lambda) y_2^2$

Or, $u_2^2 \leq \lambda x_1 + (1-\lambda)y_1$ $[\because x_2^2 \leq x_1,\ y_2^2 \leq y_1]$

Or, $u_2^2 \leq u_1$

i.e., $u = (u_1, u_2)$ is a point of S.

Thus, for x, y $\in$ S implies that the line segment $[x:y] \subset S$.

Hence, S is a convex set.

Example (2.21): Express any internal point of a triangle as a convex combination of its vertices.

Solution: Let $P(\underline{u})$ be any point inside the triangle ABC, whose vertices are $A(\underline{x}_1)$, $B(\underline{x}_2)$ and $C(\underline{x}_3)$. Draw the line segment AP and extend it to meet BC at $D(\underline{x}_4)$. Since $D(\underline{x}_4)$ is a point on the line segment BC, it can be written as a convex combination of $\underline{x}_2$ and $\underline{x}_3$. So, $\underline{x}_4 = \lambda\underline{x}_2 + (1-\lambda)\underline{x}_3$, $0 \le \lambda \le 1$. Now $P(\underline{u})$ is a point on the line segment

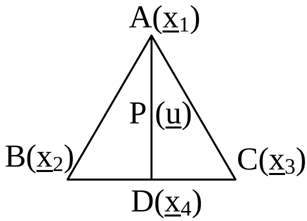

Figure 2.15

AD. So $\underline{u} = a\underline{x}_1 + (1-a)\underline{x}_4$, $0 \le a \le 1$.
Or $\underline{u} = a\underline{x}_1 + (1-a)[\lambda\underline{x}_2 + (1-\lambda)\underline{x}_3] = a\underline{x}_1 + \lambda(1-a)\underline{x}_2 + (1-\lambda)(1-a)\underline{x}_3$
$= \mu_1\underline{x}_1 + \mu_2\underline{x}_2 + \mu_3\underline{x}_3$, where, $\mu_1 = a$, $\mu_2 = \lambda(1-a)$, $\mu_3 = (1-\lambda)(1-a)$
Since $0 \le \lambda \le 1$ and $0 \le a \le 1$ so, $0 \le \mu_i \le 1$, $i = 1, 2, 3$ and $\mu_1 + \mu_2 + \mu_3 = a + \lambda(1-a) + (1-\lambda)(1-a) = 1$. So, $\underline{u} = \mu_1\underline{x}_1 + \mu_2\underline{x}_2 + \mu_3\underline{x}_3$ is a convex combination of vertices.

Example (2.22): Show that the set, $S = \{(x_1, x_2, x_3) \mid 2x_1 + x_2 - x_3 \le 1, x_1 - 2x_2 + x_3 \le 4\}$ is a convex set. **[NUH-06]**

Proof: The set is the intersection of two half spaces, viz.,

$H_1 = \{(x_1, x_2, x_3) \mid 2x_1 + x_2 - x_3 \le 1\}$ and

$H_2 = \{(x_1, x_2, x_3) \mid x_1 - 2x_2 + x_3 \le 4\}$

We know that half spaces are always convex set
So, H_1 and H_2 are convex sets, and $S = H_1 \cap H_2$
Since the intersection of any number of convex sets is convex set. Hence, S is a convex set.

2.17 Exercises:

31. Discuss line segment and convex set with examples.
32. Is line segment a convex combination?
33. Define with examples extreme point of a convex set, convex cone, convex hull and convex polyhedron.
34. What do you mean by hyper plane and half space? Show that hyper plane and half space are convex sets. [JU-93]
35. Discuss with examples supporting and separating hyper plane.

6. Show that the intersection of two convex sets is a convex set.
7. Show that intersection of four convex sets is also a convex set.
8. Is the intersection of two hyper planes in $\mathbf{R}^n$ a convex set?
9. Show that the intersection of two convex cones is a convex set.
10. Which of the following sets are convex?
    a. $S = \{(x_1, x_2): x_1 \geq 0, x_2 \geq 0\}$ [Answer: Yes]
    b. $S = \{(x_1, x_2): x_1 \geq 5, x_2 \geq 7\}$ [Answer: Yes]
    c. $S = \{(x_1, x_2): x_1.x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$ [Answer: No]
    d. $S = \{(x_1, x_2): x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$ [Answer: Yes]
11. Show that (hyper plane), $S = \{(x_1, x_2): 3x_1 + 5x_2 = 10\} \subset \mathbf{R}^2$ is a convex set.
12. Show that, $S = \{(x_1, x_2): 3x_1 + x_2 = 1\} \subset \mathbf{R}^2$ is a convex set.
13. Show that (hyper plane), $S = \{(x_1, x_2, x_3): 3x_1 + 5x_2 + 2x_3 = 7\} \subset \mathbf{R}^3$ is a convex set.
14. Show that $\{(x_1, x_2, x_3): x_1 + 4x_2 + 2x_3 = 3\}$ is a convex set.
15. Show that, $\{(x_1, x_2, x_3): 10x_1 + 5x_2 + 12x_3 = 50\} \subset \mathbf{R}^3$ is a convex set.
16. Show that, $\{(x_1, x_2, x_3, x_4): 3x_1 + 5x_2 + 10x_3 - 5x_4 = 50\} \subset \mathbf{R}^4$ is a convex set.
17. Show that, $\{(x_1, x_2): 3x_1 + x_2 = 1, 3x_1 + 5x_2 = 10\}$ is a convex set. [Hints: It is intersection of two convex sets, so is convex.]
18. Find the extreme points of $S = \{(x, y): x^2 + y^2 \leq 16\}$ [Answer: Each point on its circumference is an extreme point]
19. Find the extreme points, if any of $S = \{(x, y): |x| \leq 1, |y| \leq 1\}$.
20. Show that (the half space), $S = \{(x_1, x_2, x_3) \mid 2x_1 - x_2 + 2x_3 \leq 8\} \subset \mathbf{R}^3$, is a convex set.
21. Show that (the half space), $S = \{(x_1, x_2, x_3) \mid 2x_1 - 3x_2 + x_3 \geq 1\} \subset \mathbf{R}^3$, is a convex set.
22. Show that the open half space, $S = \{(x_1, x_2, x_3) \mid 2x_1 - x_2 + 7x_3 > 2\} \subset \mathbf{R}^3$, is a convex set.
23. Show that (the half space), $S = \{(x_1, x_2, x_3) \mid x_1 + 3x_2 - 4x_3 \leq 7\} \subset \mathbf{R}^3$, is a convex set.

24. Show that, $T = \{(x_1, x_2, x_3) \mid 6x_1 + 5x_2 - x_3 \leq 3\}$ is a convex set.
25. Show that closed half space, $S = \{(x_1, x_2, x_3, x_4) \mid 3x_1 - x_2 + 2x_3 + 5x_4 \leq 4\} \subset \mathbf{R}^4$, is a convex set.
26. Show that, $\{(x_1, x_2, x_3) \mid x_1 + 3x_2 - 4x_3 \leq 7, 6x_1 + 5x_2 - x_3 \leq 3\}$ is a convex set. [JU-05]
27. Show that, $\{(x_1, x_2, x_3) \mid 3x_1 + 3x_2 - x_3 \geq 5, 3x_1 + 5x_2 - x_3 \geq 2\}$ is a convex set. [DU-01]
28. Show that in $\mathbf{R}^2$, the set $S = \{(x_1, x_2)\text{: } x_1^2 \leq x_2\}$ is a convex set.
29. Show that the set $S = \{(x_1, x_2)\text{: } x_1^2 + x_2^2 \leq 1\}$ is a convex set.
30. Show that the set $S = \{(x_1, x_2)\text{: } x_1^2 + x_2^2 \leq 4\}$ is a convex set.
31. Is the set $S = \{(x_1, x_2)\text{: } x_1^2 + x_2^2 \leq 1,\ x_1^2 + x_2^2 \leq 4\}$ a convex set? If your answer is "Yes", prove your answer. [Hints: Yes, because S is intersection of two convex sets.]
32. Is the complement of the set $S = \{(x_1, x_2)\text{: } x_1^2 + x_2^2 \leq 4\}$ a convex set? If the answer is "No", explain why.
33. Show that the closed ball $x_1^2 + x_2^2 + x_3^2 \leq 2$, is a convex set.

Chapter

03

Formulation and Graphical Methods

Highlights:

3.1 Formulation: (গানিতিক রূপান্ড়র) As any other planning problem, the operations researcher must analyze the goals and the system in which the solution must operate. The complex of inter related components in a problem area, referred to by operations researchers, as a ‘system’ is the environment of a decision, and it represents planning premises. To solve any linear programming problem, we have to transfer it as mathematical problem. This problem transformation is known as formulation.

3.2 Algorithm to formulate a linear programming problem: (যোগাশ্রয়ী প্রোগ্রামের গানিতিক রূপান্ড়রের ধাপসমুহ) The following is the step-by-step algorithm to formulate any linear programming problem as mathematical (or symbolic) problem:

Step-1: Study the given problem and find key decision, i.e., find out what will be determined.

Step-2: Select symbols for variable quantities identified in step-1.

Step-3: Set all variables greater than or equal to zero.

Step-4: Identify the objective quantitatively and express it as a linear function of variables, which will be maximized for profit or minimized for cost.

Step-5: Express the constraints (or restrictions) as linear equalities or inequalities in terms of the variables.

Then we shall find the symbolic or mathematical representation of a linear programming problem.

(যোগাশ্রয়ী প্রোগ্রামের গানিতিক রূপান্ড়রের ধাপসমুহ নিম্নরূপঃ
ধাপ-১ঃ কি নির্ণয় করতে হবে তা ঠিক করতে হভে।
ধাপ-২ঃ নির্নীতব্য বিষয়ের জন্য চলক ঠিক করতে হবে।
ধাপ-৩ঃ সকল চলকের মান অঋনাত্বক ধরতে হবে।
ধাপ-৪ঃ চলকের মাধ্যমে উদ্দেশ্যমুলক অপেক্ষক ঠিক করতে হবে এবং তা সর্বোচ্চকরণ না সর্বোনিম্নকরণ নির্নয় করতে হবে।
ধাপ-৫ঃ শর্ত গুলোকে সমীকরন বা অসমতার সাহায্যে প্রকাশ করতে হবে।
তাহলে আমরা যোগাশ্রয়ী প্রোগ্রামের গানিতিক রূপান্ড়র পেয়ে যাব।)

<u>Example (3.1)</u>: (Production planning problem)
A firm manufactures 3 products A, B and C. The profit per unit sold of each product is Tk.3, Tk.2 and Tk.4 respectively. The time required for manufacture one unit of each of the three products and the daily capacity of the two machines P and Q is given in the table below:

Type of machine	Required time (minutes) to produce per unit product			Machine capacity (minutes/day)
	A	B	C	
P	4	3	5	2000
Q	2	2	4	2500

It is required to determine the daily number of units to be manufactured for each product, so as to maximize the profit. However, the firm must manufacture at least 100 A's, 200 B's and 50 C's but no more that 150 A's. It is assumed that all the amounts produced are consumed in the market. Formulate this problem as linear programming problem. [JU-93]

Solution: <u>Step–1</u>: Here, the key decision is that how many products of A, B and C type are to be made for maximizing the total profit.

Step-2:Let x_1, x_2, x_3 numbers of product A, B, C are to be made respectively for maximizing the total profit.

Step-3 Since it is not possible to manufacture any negative quantities, it is quite obvious that in the present situation feasible alternatives are sets of variables of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step-4: The objective here is to maximize the profit. In view of the assumption that all the units produced are consumed in that market, it is given by the linear function

$$z = 3x_1 + 2x_2 + 4x_3$$

Step-5: Here in order to produce x_1 units of product A, x_2 units of product B and x_3 units of product C, the total times needed on machines P and Q are given by

$4x_1 + 3x_2 + 5x_3$ and $2x_1 + 2x_2 + 4x_3$ respectively.

Since the manufacturer does not have more than 2000 minutes available on machine P and 2500 minutes available on machine Q, we must have

$4x_1 + 3x_2 + 5x_3 \leq 2000$ and $2x_1 + 2x_2 + 4x_3 \leq 2500$.

Also, additional restrictions are

$100 \leq x_1 \leq 150$, $x_2 \geq 200$, $x_3 \geq 50$.

Hence, the manufacturer's problem can be put in the following mathematical form:

$$\begin{array}{ll} \text{Maximize} & z = 3x_1 + 2x_2 + 4x_3 \\ \text{Subject to} & 4x_1 + 3x_2 + 5x_3 \leq 2000 \\ & 2x_1 + 2x_2 + 4x_3 \leq 2500 \\ & 100 \leq x_1 \leq 150,\ x_2 \geq 200,\ x_3 \geq 50 \end{array}$$

Example (3.2): (Blending problem)

A firm produces an alloy having the following specifications

(i) specific gravity ≤ 0.98

(ii) chromium $\geq 8\%$

(iii) melting point $\geq 450^0$ C

Raw materials A, B and C having the properties shown in the table can be used to make the alloy:

Property	Raw material		
	A	B	C
Specific gravity	0.92	0.97	1.04
Chromium	7%	13%	16%
Melting point	440^0 C	490^0 C	480^0 C

Cost of the various raw materials per unit ton are: Tk. 90 for A, Tk. 280 for B and Tk. 40 for C. Find the proportion in which A, B and C be used to obtain an alloy of desired properties while the cost of raw materials is minimum. **[DU-91]**

Solution: Step-1: Key decision to be made is how much (percentage) of raw materials A, B and C be used for making the alloy.

Step-2: Let the percentage contents of A, B and C be x_1, x_2 and x_3 respectively.

Step-3: Feasible alternatives are sets of values of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step-4: Objective is to minimize the cost, i.e.,

Minimize $z = 90x_1 + 280x_2 + 40x_3$

Step-5: Constraints are imposed by the specifications required for the alloy. They are

$$0.92x_1 + 0.97x_2 + 1.04x_3 \leq 0.98$$
$$7x_1 + 13x_2 + 16x_3 \geq 8$$
$$440x_1 + 490x_2 + 480x_3 \geq 450$$

and $x_1 + x_2 + x_3 = 100$

Hence, the blending problem can be put in the following mathematical form:

Minimize $Z = 90x_1 + 280x_2 + 40x_3$

Subject to
$$0.92x_1 + 0.97x_2 + 1.04x_3 \leq 0.98$$
$$7x_1 + 13x_2 + 16x_3 \geq 8$$
$$440x_1 + 490x_2 + 480x_3 \geq 450$$
$$x_1 + x_2 + x_3 = 100$$
$$x_1, x_2, x_3 \geq 0$$

3.3 <u>Graphical method</u>: (লেখ চিত্র পদ্ধতি) The solution of any linear programming problem with only two variables can be derived using a graphical method. This method consists of the following steps:

(i) Represent the given problem in mathematical from, i.e., formulate an LP model for the given problem.

(ii) Represent the given constraints as equalities on x_1x_2 co-ordinates plane and find the convex region formed by them.

(iii) Find the vertices of the convex region and also the value of the objective function at each vertex. The vertex that gives the optimum value of the objective function gives the optimal solution to the problem.

<u>Note</u>: In general, a linear programming problem may have
(i) a definite and unique optional solution,
(ii) an infinite number of optimal solutions,
(iii) an unbounded solution, and
(iv) no solution.

<u>Example (3.3)</u>: (Product Allocation Problem): A manufacturer uses three different resources for the manufacture of two different products 20 unites of the resource A, 12 units of B and 16 units of C being available. 1 unit of the first product requires 2, 2 and 4 units of the respective resources and 1 unit of the second product requires 4, 2 and 0 units of the respective resources. It is known that the first product gives a profit of two monetary units per unit and the second 3. Formulate the linear programming problem. How many units of each product should be manufactured for maximizing the profit? Solve it graphically. **[DU-95, JU-00]**

Solution: Mathematical formulation of the problem:

<u>Step-1</u>: The key decision is to determine the number of units of the two products.

<u>Step-2</u>: Let x_1 units of the first product and x_2 units of the second product be manufactured for maximizing the profit.

Step-3: Feasible alternatives are the sets of the values of x_1 and x_2 satisfying $x_1 \geq 0$ and $x_2 \geq 0$, as negative number of production runs a meaningless situation (and thus not feasible).
Step-4: The objective is to maximize the profit realized from both the products, i.e., to maximize $z = 2x_1 + 3x_2$
Step-5: Since 1 unit of the first product requires 2, 2 and 4 units, 1 unit of the second product requires 4, 2 and 0 units of the respective resources and the units available of the three resources are 20, 12 and 16 respectively, the constraints (or restrictions) are

$$2x_1 + 4x_2 \leq 20 \quad \text{Or,} \quad x_1 + 2x_2 \leq 10$$
$$2x_1 + 2x_2 \leq 12 \quad \text{Or,} \quad x_1 + x_2 \leq 6$$
$$4x_1 + 0x_2 \leq 16 \quad \text{Or,} \quad x_1 \leq 4$$

Hence the manufacturer's problem can be put in the following mathematical form:

$$\boxed{\begin{aligned} &\text{Maximize } z = 2x_1 + 3x_2 \\ &\text{Subject to } x_1 + 2x_2 \leq 10 \\ &\qquad\qquad\quad x_1 + x_2 \leq 6 \\ &\qquad\qquad\quad x_1 \leq 4 \\ &\qquad\qquad\quad x_1, x_2 \geq 0 \end{aligned}}$$

Graphical solution of the problem:
Step-1: Consider a set of rectangular Cartesian axes OX_1 and OX_2 in the plane. Each point has coordinates of the type (x_1, x_2) and conversely every ordered pair (x_1, x_2) of real numbers determines a point in the plane. It is clear that any point which satisfies the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant and conversely for any point (x_1, x_2) in the first quadrant, $x_1 \geq 0$ and $x_2 \geq 0$. Thus our search for the number pair (x_1, x_2) is restricted to the points of the first quadrant only.
Step-2: To graph each constraint in the first quadrant satisfying the constraints, we first treat each inequality as equation and then find the set of points in the first quadrant satisfying the constraints.

Considering, the first constraint as an equation, we get $x_1+2x_2=10$. Clearly, this is the equation for a straight line and any point on the straight line satisfies the inequality also. Taking into consideration the point (0, 0), we observe that $0 + 2(0) = 0 < 10$, i.e., origin also satisfies the inequality. This indicates that any point satisfying the inequality $x_1 + 2x_2 \leq 10$ lies in the first quadrant on that side of the line, $x_1 + 2x_2 = 10$ which contains the origin. In a similar way, we see that all points satisfying the constraint $x_1 + x_2 \leq 6$ are the points in the first quadrant lying on or below the line $x_1 + x_2 = 6$.

The set of points satisfying the inequality $x_1 \leq 4$ lies on or towards the left of the line $x_1 = 4$.

Step-3: All points in the area shown shaded in figure satisfy al the three constraints $x_1 + 2x_2 \leq 10$, $x_1 + x_2 \leq 6$ and $x_1 \leq 4$ and also the non-negativity restrictions $x_1 \geq 0$ and $x_2 \geq 0$.

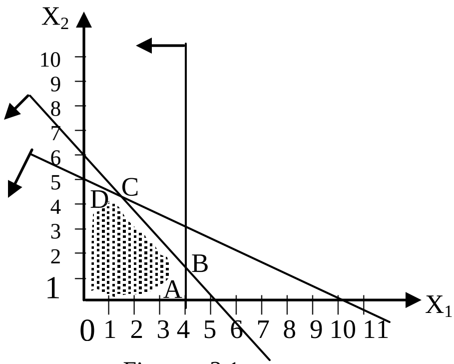

Figure – 3.1

This shaded area is called the convex region or the solution space or the region of feasible solutions. Any point in this region is a feasible solution of the given problem. The convex region OABCD is bounded by the lines $x_1 = 0$, $x_2 = 0$, $x_1 + 2x_2 = 10$, $x_1 + x_2 = 6$ and $x_1 = 4$. The five vertices of the convex region are O(0, 0), A(4, 0), B(4, 2), C(2, 4) and D(0, 5). Putting O(0, 0), A(4, 0), B(4,2), C(2, 4) and D(0, 5) in the objective function $2x_1 + 3x_2$, we have 0, 8, 14, 16 and 15 respectively. Here, 16 is the maximum value. Hence, the solution of the problem is $x_1 = 2$, $x_2 = 4$ and the maximum value of the objective function is $z = 16$, i.e., to gain maximum profit 16 monetary units the manufacturer should manufacture 2 units of first product and 4 units of second product.

Example (3.4): A furniture company makes tables and chairs. Each table takes 4 hours of carpentry and 2 hours in painting and varnishing shop. Each chair requires 3 hours in carpentry and 1 hour in painting and varnishing. During the current production period, 240 hours of carpentry and 100 hours of painting and varnishing time are available. Each table sold yields a profit of Tk.420 and each chair yields a profit of Tk.300. Determine the number of tables and chairs to be made to maximize the profit. **[JU-93, DU-90, AUB-03]**

Solution: Let x_1 = the number of tables and x_2 = the number of chairs. So, the profit from the tables is $420x_1$ and the profit from the chairs is $300x_2$. That is, the total profit, $420x_1 + 300x_2$ which is the objective function. We have to maximize the objective function $z = 420x_1 + 300x_2$. Required carpentry hours for tables are $4x_1$ and required carpentry hours for chairs are $3x_2$. So, the required total carpentry hours $4x_1 + 3x_2$. Since 240 carpentry hours are available, so, $4x_1 + 3x_2 \leq 240$. Similarly, for painting and varnishing, we have $2x_1 + x_2 \leq 100$. The non-negativity condition is $x_1, x_2 \geq 0$. So, the LP problem becomes:

Maximize $z = 420x_1 + 300x_2$

Subject to the constraints:

$$4x_1 + 3x_2 \leq 240$$
$$2x_1 + x_2 \leq 100$$
$$x_1, x_2 \geq 0$$

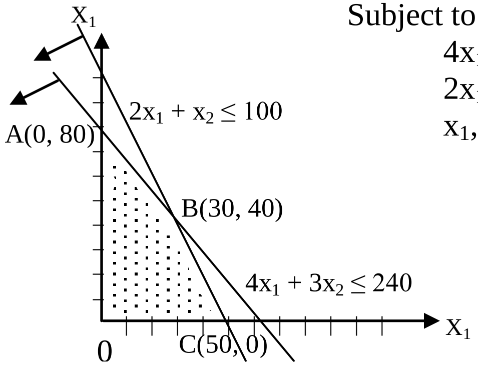

Figure – 3.2

The solution space, satisfying the given constraints are shown shaded in the figure-3.2. Any point in the shaded region is a feasible solution to the given problem. The coordinates of the four vertices of the bounded convex region are A(0,80), B(30,40),C(50,0) and O(0,0). The values of the objective function $z = 420x_1 + 300x_2$ at four vertices are 24000 at A, 24600 at B, 21000 at C and 0 at O. Since the maximum value of the objective function is 24600 which occurs at the vertices B(30,40). Hence, the solution to the given problem is $x_1 = 30$, $x_2 = 40$ and the maximum value = 24600. Therefore, to maximize the profit, the furniture company should make 30 tables and 40 chairs.

<u>Example (3.5)</u>: Solve by graphical method: Maximize $3x_1 + 2x_2$

Subject to $2x_1 - x_2 \geq -2$

$x_1 + 2x_2 \leq 8$

$x_1, x_2 \geq 0$

Solution: The graph of the problem is as follows:

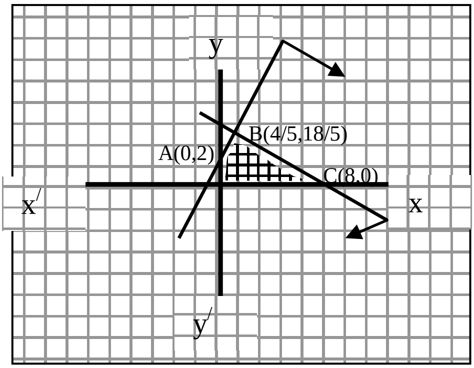

Figure – 3. 3

From the graph we get the vertices A(0,2), B(4/5,18/5), C(8,0) and O(0,0). The values of the objective function at these points are 4 at A, 48/5 at B, 24 at C and 0 at O. Here, 24 is the maximum value of the objective function which occurs at the vertices C(8,0). Hence, the solution of the given problem is $x_1 = 8$, $x_2 = 0$ and max. value of z = 24

3.4 <u>Limitations of the graphical method</u>: (লেখচিত্র পদ্ধতির সীমাবদ্ধতা)

Graphical method can be used to solve the linear programming problems involving only two variables while most of the practical situations do involve more than two variables. This is why, it is not a powerful tool to solve linear programming problems.

3.5 Some done examples:

Example (3.6): A farmer has 100 acres of land. He produces tomato, lettuce and radish and can sell them all. The price he can obtain is Tk.1 per kg. for tomato, Tk.0.75 a head for lettuce and Tk.2 per kg. for radish. The average yield per acre is 2000 kg. of tomato, 3000 heads of lettuce and 1000 kg. of radish. Fertilizer is available at Tk.5 per kg. and the amount required per acre is 100 kg. each for tomato and lettuce and 50 kg. for radish. Labour required for sowing, cultivation and harvesting per acre is 5 man-days for tomato and radish and 6 man-days for lettuce. The farmer has 400 man-days of labour are available at Tk.20 per man-day. Formulate a linear program for this problem to maximize the farmer's total profit. **[NU-00, 01,07]**

Solution: The mathematical formulation of the farmer's problem:

Step-1: How many acres of land to be selected for tomato, lettuce and radish is our key decision.

Step-2: Let x_1, x_2 and x_3 acres of land are selected for tomato, lettuce and radish respectively.

Step-3: Feasible alternatives are the set of the values of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step-4: The objective is to maximize the profit growing tomato, lettuce and radish. Here,

profit = selling price – producing cost (fertilizer cost +labour cost).

So, profit for tomato = Tk.$[1\times2000 - (5\times100 + 20\times5)]$ = Tk.1400 per acre, profit for lettuce = Tk.$[0.75\times3000 - (5\times100 + 20\times6)]$ = Tk.1630 per acre and profit for radish = Tk.$[2\times1000 - (5\times50 + 20\times5)]$ = Tk.1650 per acre.

Therefore, the total profit = $1400x_1 + 1630x_2 + 1650x_3$. So, the object function is $z = 1400x_1 + 1630x_2 + 1650x_3$ which will be maximized.

Step-5: The farmer has 100 acres of land and total 400 man-days of labour so, $x_1 + x_2 + x_3 \leq 100$ and $5x_1 + 6x_2 + 5x_3 \leq 400$.

Conclusion: Hence the farmer's problem can be put in the following mathematical form:

$$\begin{aligned}
\text{Maximize } z = \ & 1400x_1 + 1630x_2 + 1650x_3 \\
\text{Subject to, } \ & x_1 + x_2 + x_3 \leq 100 \\
& 5x_1 + 6x_2 + 5x_3 \leq 400 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

Example (3.7): A farmer has 100 acres of land. He produces tomato, lettuce and radish and can sell them all. The price he can obtain is Tk.8 per kg. for tomato, Tk.1.50 a head for lettuce and Tk.5 per kg. for radish. The average yield per acre is 2000 kg. of tomato, 8000 heads of lettuce and 3000 kg. of radish. Fertilizer is available at Tk.5 per kg. and the amount required per acre is 100 kg. each for tomato and lettuce and 50 kg. for radish. Labour required for sowing, cultivation and harvesting per acre is 10 man-days for tomato and lettuce and 5 man-days for radish. The farmer has 400 man-days of labour are available at Tk.50 per man-day. Formulate a linear program for this problem to maximize the farmer's total profit. **[NUH-98]**

(একজন কৃষকের ১০০ একর জমি আছে। তিনি ঐ জমিতে টমেটো, লেটুস ও মুলার চাষ করেন এবং তা বিক্রি করেন। প্রতি কেজি টমেটো, লেটুস ও মুলা থেকে তিনি যথাক্রমে ৮ টাকা, ১.৫০ টাকা ও ৫ টাকা পান। প্রতি একর থেকে তিনি ২০০০ কেজি টমেটো, ৪০০০ কেজি লেটুস এবং ৩০০০ কেজি মুলা পান। প্রতি কেজি সারের মুল্য ৫ টাকা। প্রতি একর জমিতে টমেটো ও লেটুস চাষের জন্য দেন ১০০ কেজি সার এবং মুলা চাষের জন্য দেন ৫০ কেজি। প্রতি একর জমিতে টমেটো ও লেটুস চাষের জন্য দরকার ১০ জনের দিন মুজুরী এবং মুলা চাষের জন্য দরকার ৫জনের দিন মুজুরী। প্রতি দিন মুজুরীর মুল্য ৫০ টাকা এবং সর্বোচ্চ ৪০০ দিন মুজুরী পাওয়া যায়। কৃষকের লাভ সর্বোচ্চ করণ করার জন্য একটি যোগাশ্রয়ী প্রোগ্রাম তৈরী কর।)

Solution: The mathematical formulation of the farmer's problem:

Step-1: How many acres of land to be selected for tomato, lettuce and radish is our key decision.

Step-2: Let x_1, x_2 and x_3 acres of land are selected for tomato, lettuce and radish respectively.

Step-3: Feasible alternatives are the set of the values of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step-4: The objective is to maximize the profit growing tomato, lettuce and radish. Here,
profit = selling price – producing cost (fertilizer cost +labour cost).
So, profit for tomato = Tk.[$8\times 2000 - (5\times 100 + 50\times 10)$] = Tk.15000 per acre, profit for lettuce = Tk.[$1.50\times 8000 - (5\times 100 + 50\times 10)$] = Tk.11000 per acre and profit for radish = Tk.[$5\times 3000 - (5\times 50 + 50\times 5)$] = Tk.14500 per acre.
Therefore, the total profit = $15000x_1 + 11000x_2 + 14500x_3$. So, the object function is $z = 15000x_1 + 11000x_2 + 14500x_3$ which will be maximized.
Step-5: The farmer has 100 acres of land and total 400 man-days of labour so, $x_1 + x_2 + x_3 \le 100$ and $10x_1 + 10x_2 + 5x_3 \le 400$.
Conclusion: Hence the farmer's problem can be put in the following mathematical form:

$$\begin{aligned} \text{Maximize } z = \ & 15000x_1 + 11000x_2 + 14500x_3 \\ \text{Subject to, } \ & x_1 + x_2 + x_3 \le 100 \\ & 10x_1 + 10x_2 + 5x_3 \le 400 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Example (3.8): A dairy firm has two milk plants with dairy milk production of 6 million litres and 9 million litres respectively. Each day the firm must fulfill the needs of three distribution centres which have milk requirement of 7, 5 and 3 million litres respectively. Cost of shipping one million litres of milk from each plant to each distribution centre is given in hundreds of taka below:

Plants	Distribution centres			Supply
	1	2	3	
1	2	3	11	6
2	1	9	6	9
Demands	7	5	3	

Formulate the linear programming model to minimize the transportation cost. **[NUH-00]**

Solution: The mathematical formulation of the problem:

<u>Step-1</u>: Our key decision is how many litres of milk will be shifted from one milk plant to a distribution centre.

<u>Step-2</u>: Let x_{ij} (x_{11}, x_{12}, x_{13}, x_{21}, x_{22} and x_{23}) represents the number of litres of milk to be shifted from ith milk plant to jth distribution centre.

<u>Step-3</u>: Feasible alternatives are $x_{ij} \geq 0$; i = 1, 2 and j = 1, 2, 3.

<u>Step-4</u>: The objective is to minimize the shifting cost, i.e.,

$$\text{minimize } z = \sum_{i=1}^{2}\sum_{j=1}^{3} x_{ij} = x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23}$$

<u>Step-5</u>: Since it is necessary to meet the demand and supply, the constraints are

$$\sum_{j=1}^{3} x_{1j} = 6 \qquad \text{or, } x_{11} + x_{12} + x_{13} = 6$$

$$\sum_{j=1}^{3} x_{2j} = 9 \qquad \text{or, } x_{21} + x_{22} + x_{23} = 9$$

$$\sum_{i=1}^{2} x_{i1} = 7 \qquad \text{or, } x_{11} + x_{21} = 7$$

$$\sum_{i=1}^{2} x_{i2} = 5 \qquad \text{or, } x_{12} + x_{22} = 5$$

$$\sum_{i=1}^{2} x_{i3} = 3 \qquad \text{or, } x_{13} + x_{23} = 3$$

<u>Conclusion</u>: Hence the transportation problem can be put in the following mathematical form:

$$\begin{aligned}
\text{Minimize } z = \ & x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} \\
\text{Subject to, } \ & x_{11} + x_{12} + x_{13} = 6 \\
& x_{21} + x_{22} + x_{23} = 9 \\
& x_{11} + x_{21} = 7, \\
& x_{12} + x_{22} = 5 \\
& x_{13} + x_{23} = 3 \\
& x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0
\end{aligned}$$

N.B: This type of problem is known as transportation problem.

<u>Example (3.9)</u>: A company sells two products A and B. The company makes profit Tk.40 and Tk.30 per unit of each product respectively. The two products are produced in a common process. The production process has capacity 30,000 man-hours. It takes 3 hours to produce one unit of A and one hour per unit of B. The market has been surveyed and it feels that A can be sold 8,000 units, B of 12,000 units. Subject to above limitations form LP problem, which maximizes the profit. **[NUH-04]**

Solution: Let x_1 and x_2 be the number of product A and B respectively to be produced for maximizing company's total profit satisfying demands and limitations. So, company's total profit is $z = 40x_1 + 30x_2$, limitations are $3x_1 + x_2 \leq 30000$, demands are $x_1 \leq 8000$, $x_2 \leq 12000$ and the feasibilities are $x_1 \geq 0$, $x_2 \geq 0$. Therefore, the LP form of the given problem is

$$\begin{aligned} \text{Maximize } z = \ & 40x_1 + 30x_2 \\ \text{Subject to } \ & 3x_1 + x_2 \leq 30000 \\ & x_1 \leq 8000 \\ & x_2 \leq 12000 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

<u>Example (3.10)</u>: A company is manufacturing two products A and B. The manufacturing times required to make them, the profit and capacity available at each work centre are given by the following table:

Products	Work centres			Profit per unit (in $)
	Matching (in hours)	Fabrication (in hours)	Assembly (in hours)	
A	1	5	3	80
B	2	4	1	100
Total Capacity	720	1800	900	

Company likes to maximize their profit making their products A and B. Formulate this linear programming problem.

Solution: If we consider x and y be the numbers of products A and B respectively to be produced for maximizing the profit. Then company's total profit $z = 80x + 100y$ is to be maximized. And subject to the constraints are $x + 2y \leq 720$, $5x + 4y \leq 1800$ and $3x + y \leq 900$. Since it is not possible for the manufacturer to produce negative number of the products, it is obvious that $x \geq 0$, $y \geq 0$. So, we can summarize the above linguistic linear programming problem as the following mathematical form:

$$\begin{aligned} \text{Maximize } Z &= 80x + 100y \\ \text{Subject to} \quad & x + 2y \leq 720 \\ & 5x + 4y \leq 1800 \\ & 3x + y \leq 900 \\ & x \geq 0,\ y \geq 0 \end{aligned}$$

Example (3.11): A farmer has 20 acres of land. He produces tomato and potato and can sell them all. The price he can obtain is Tk.8 per kg. for tomato and Tk.12 per kg. for potato. The average yield per acre is 2000 kg. of tomato and 1500 kg. of potato. Fertilizer is available at Tk.30 per kg. and the amount required per acre is 100 kg. each for tomato and 50 kg. for potato. Labour required for sowing, cultivation and harvesting per acre is 20 man-days for tomato and 15 man-days for potato. The farmer has 180 man-days of labour are available at Tk.80 per man-day. Formulate a linear program for this problem to maximize the farmer's total profit and then solve it graphically. **[JU-99]**

Solution: The mathematical formulation of the above problem:

Step-1: How many acres of land to be selected for tomato and potato is our key decision.

Step-2: Let x_1, and x_2 acres of land are selected for tomato and potato respectively.

Step-3: Feasible alternatives are the set of the values of x_1 and x_2 satisfying $x_1 \geq 0$ and $x_2 \geq 0$.

Step-4: The objective is to maximize the profit growing tomato and potato. Here, profit = selling price – fertilizer & labour cost.

So, profit for tomato = Tk.[$8\times2000 - (30\times100 + 80\times20)$] = Tk.11400 per acre and profit for potato = Tk.[$12\times1500 - (30\times50 + 80\times15)$] = Tk.15300 per acre.

Therefore, the total profit = $11400x_1 + 15300x_2$. So, the object function is $z = 11400x_1 + 15300x_2$ which will be maximized.

Step-5: The farmer has 20 acres of land and total 180 man-days of labour so, $x_1 + x_2 \le 20$ and $20x_1 + 15x_2 \le 180$. Hence the farmer's problem can be put in the following mathematical form:

$$\begin{aligned} \text{Maximize } z = \; & 11400x_1 + 15300x_2 \\ \text{Subject to,} \quad & x_1 + x_2 \le 20 \\ & 20x_1 + 15x_2 \le 180 \\ & x_1, x_2 \ge 0 \end{aligned}$$

The graphical solution: Drawing the constraints in the graph paper we find the shaded feasible solution space OABO. The vertices O(0, 0), A(9, 0) and B(12, 0) are basic feasible solution of the given problem. And the value of the objective function at O is 0, at A is 1,02,600 and at B is 1,83,600. Here the maximum value is 1,83,600 and attain at B(12, 0). Therefore, the farmer will plough his 12 acres of land to grow tomato for maximizing his profit.

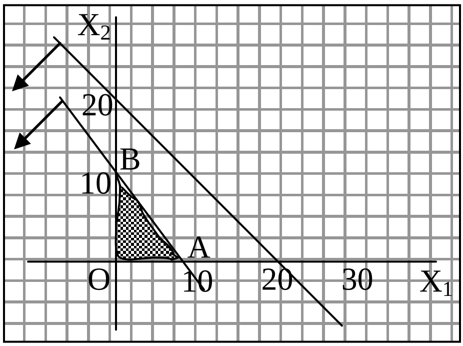

Figure 3.4

Example (3.12): A company produces AM and AM-FM radios. A plant of the company can be operated 48 hours per week. Production of an AM radio will require 2 hours and production of AM-FM radio will require 3 hours each. An AM radio yields Tk.40 as profit and an AM-FM radio yields Tk.80. The marketing department determined that a maximum of 15 AM and 10 AM-FM radios can be sold in a week. Formulate the problem as linear programming problem and solve it graphically. **[NU-02, 05]**

Solution: Mathematical formulation of the problem:

Step-1: The key decision is to determine the number of AM and AM-FM radios to be made.

Step-2: Let x_1 units of AM radio and x_2 units of AM-FM radios be manufactured for maximizing the profit.

Step-3: Feasible alternatives are the sets of the values of x_1 and x_2 satisfying $x_1 \geq 0$ and $x_2 \geq 0$, as negative number of production runs a meaningless situation (and thus not feasible).

Step-4: The objective is to maximize the profit realized from both radios, i.e., to maximize $z = 40x_1 + 80x_2$

Step-5: Since each AM radio requires 2 hours and each AM-FM radio requires 3 hours and the available time is 48 hours in a week, the first constraint (or restriction) is $2x_1 + 3x_2 \leq 48$. Since at most 15 AM and 10 AM-FM radios can be sold in a week, so $x_1 \leq 15$ and $x_2 \leq 10$

Hence the manufacturer's problem can be put in the following mathematical form:

$$\boxed{\begin{aligned} &\text{Maximize } z = 40x_1 + 80x_2 \\ &\text{Subject to } 2x_1 + 3x_2 \leq 48 \\ &\qquad\qquad x_1 \leq 15 \\ &\qquad\qquad x_2 \leq 10 \\ &\qquad\qquad x_1, x_2 \geq 0 \end{aligned}}$$

Graphical solution of the problem**:**

Step-1: Consider a set of rectangular Cartesian axes OX_1 and OX_2 in the plane. Each point has coordinates of the type (x_1, x_2) and conversely every ordered pair (x_1, x_2) of real numbers determines a point in the plane. It is clear that any point which satisfies the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant and conversely for any point (x_1, x_2) in the first quadrant, $x_1 \geq 0$ and $x_2 \geq 0$. Thus our search for the number pair (x_1, x_2) is restricted to the points of the first quadrant only.

Step-2: To graph each constraint in the first quadrant satisfying the constraints, we first treat each inequality as equation and then find the set of points in the first quadrant satisfying the constraints. Taking, the first constraint as an equation, we get $2x_1+3x_2 = 48$. Clearly, this is the equation for a straight line and any point on the straight line satisfies the inequality also. Taking into consideration the point (0, 0), we observe that $2(0) + 3(0) = 0 < 48$, i.e., origin also satisfies the inequality. This indicates that any point satisfying the inequality $2x_1 + 3x_2 \leq 48$ lies in the first quadrant on that side of the line, $2x_1 + 3x_2 = 48$ which contains the origin. In a similar way, we see that all points satisfying the constraint $x_1 \leq 15$ are the points in the first quadrant lying on or below the line $x_1 = 15$.

The set of points satisfying the inequality $x_2 \leq 10$ lies on or towards the left of the line $x_2 = 10$.

Step-3: All points in the area shown shaded in figure satisfy al the three constraints $2x_1 + 3x_2 \leq 48$, $x_1 \leq 15$ and $x_2 \leq 10$ and also the non-negativity restrictions $x_1 \geq 0$ and $x_2 \geq 0$.

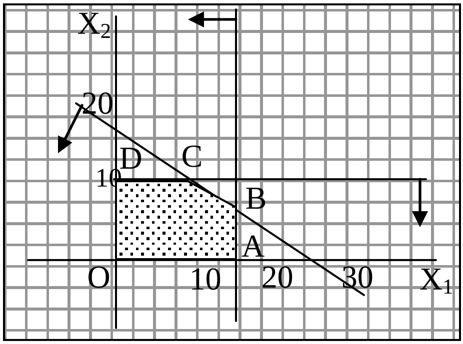

Figure 3.5

This shaded area is called the convex region or the solution space or the region of feasible solutions. Any point in this region is a feasible solution of the given problem. The convex region OABCD is bounded by the lines $x_1 = 0$, $x_2 = 0$, $2x_1 + 3x_2 = 48$, $x_1 = 15$ and $x_2 = 10$. The five vertices of the convex region are O(0, 0), A(15, 0), B(15, 6), C(9, 10) and D(0, 10). Putting O(0, 0), A(15,0), B(15, 6), C(9, 10) and D(0, 10) in the objective function $40x_1 + 80x_2$, we have 0, 600, 1080, 1160 and 800 respectively. Here, 1160 is the maximum value. Hence, the solution of the problem is $x_1 = 9$, $x_2 = 10$ and the maximum value of the objective function is $z = 1160$, i.e., to gain maximum profit Tk.1160 the manufacturer should manufacture 9 AM radios and 10 AM-FM radios in a week.

<u>Example (3.13)</u>: A firm manufacturing two types of electrical items A and B can make a profit Tk.170 per unit of A, Tk.250 per unit of B. Both A and B uses motors and transformers. Each unit of A requires 2 motors and 4 transformers; each unit of B requires 3 motors and 2 transformers. The total supply of components per month is 210 motors and 300 transformers. Type B requires a voltage stabilizer, which has a supply restricted to 56 units per month. How many of A and B should firm manufacture to maximize the profit? Formulate the problem as LP problem and solve it graphically. **[NU-97]**

Solution: <u>LP formulation</u>: Let the firm produce x_1 ($x_1 \geq 0$) number of product A and x_2 ($x_2 \geq 0$) number of product B to maximize their profit satisfying limitations. Then objective is maximizing $z = 170x_1 + 250x_2$ and limitations are $2x_1 + 3x_2 \leq 210$ (motors), $4x_1 + 2x_2 \leq 300$ (transformers), $x_2 \leq 56$ (voltage stabilizer). Therefore, the LP form of the given problem is as follows:

Maximize $z = 170x_1 + 250x_2$

Subject to
$$2x_1 + 3x_2 \leq 210$$
$$4x_1 + 2x_2 \leq 300$$
$$x_2 \leq 56$$
$$x_1, x_2 \geq 0$$

<u>Graphical solution</u>: Drawing the constraints in the graph paper we find the shaded feasible solution space OABCDO. The vertices O(0, 0), A(75, 0), B(60,30), C(21,56) and D(0,56) are basic feasible solution of the given problem. And the value of the objective function at O is 0, at A is 12750, at B is 17700, at C is 17570 and at D is 14000. Here the maximum value is 17700 and attain at B(60,30)

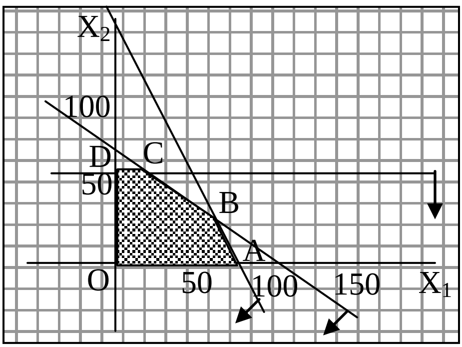

Figure 3.6

Therefore, the firm will produce 60 product A, 30 product B and will gain maximum profit Tk.17,700 per month.

<u>Example (3.14)</u>: Solve the LP problem graphically: **[NU-97]**

$$\begin{aligned} \text{Maximize} \quad & z = 2x_1 + 5x_2 \\ \text{Subject to} \quad & 4x_2 - 3x_1 \le 12 \\ & x_1 + x_2 \ge 2 \\ & x_1 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Solution: Drawing the constraints in the graph paper we find the shaded feasible solution space ABCDEA. The vertices A(2, 0), B(4,0), C(4,6), D(0,3) and E(0,2) are basic feasible solution of the given problem. And the value of the objective function at A is 4, at B is 8, at C is 38, at D is 15 and at E is 10. Here the maximum value is 38 and attain at C(4,6). Therefore, the optimal solution is $(x_1, x_2) = (4, 6)$ and $z_{max} = 38$.

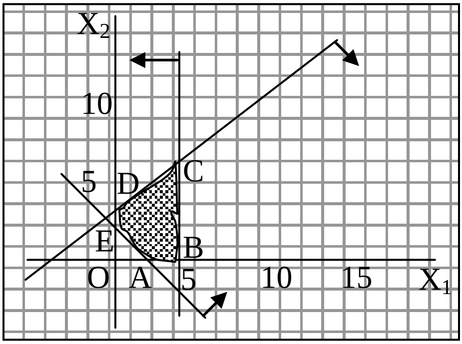

Figure 3.7

<u>Example (3.15)</u>: Solve the LP problem graphically: **[JU-97]**

$$\begin{aligned} \text{Maximize} \quad & z = 10x_2 + 2x_1 \\ \text{Subject to} \quad & x_1 - x_2 \ge 0 \\ & -x_1 + 5x_2 \ge 5 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Solution: Drawing the graph of constraints, we get an unbounded solution space and the problem is of maximization type with no negative sign in the objective function. Hence, it has an unbounded solution, i.e., it has a maximum at infinity.

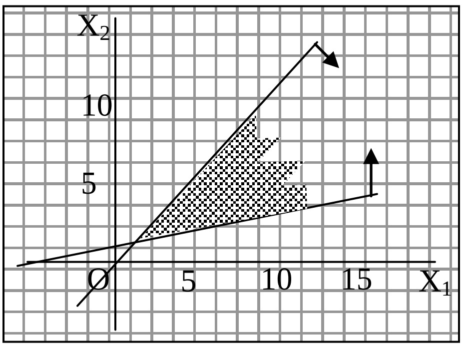

Figure 3.8

Example (3.16): Solve the LP problem graphically:

Minimize $z = 10x_2 - 2x_1$ **[DU-96]**

Subject to $x_1 - x_2 \geq 0$

$-x_1 + 5x_2 \geq 5$

$x_1, x_2 \geq 0$

Solution: Drawing the graph of constraints, we get an unbounded solution space and the problem is of minimization type. Hence, it has a bounded solution at an existing vertex. Here, the only one existing vertex is $A(\frac{5}{4}, \frac{5}{4})$. Therefore, the optimal solution of the given problem is $x_1 = \frac{5}{4}$, $x_2 = \frac{5}{4}$ and $z_{min} = 10$.

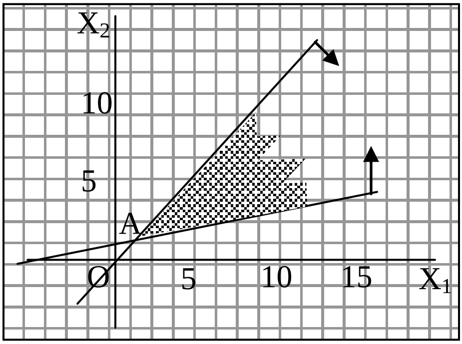

Figure 3.9

Example (3.17): Solve the LP problem graphically:

Maximize $z = 5x_2 + 2x_1$ **[JU-95]**

Subject to $-5x_1 + 4x_2 \geq 20$

$x_1 - x_2 \leq 0$

$-x_1 + 5x_2 \leq 5$

$x_1, x_2 \geq 0$

Solution: Drawing the graph of constraints, we do not get unique feasible solution space. Hence, the problem has no solution.

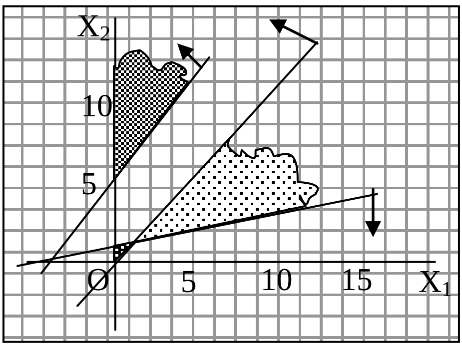

Figure 3.10

<u>Example (3.18)</u>: Solve the following linear programming problem with two variables by graphical method.

$$\begin{aligned} &\text{Minimize} \quad 5x_1 - 10x_2 \\ &\text{Subject to} \quad 2x_1 - x_2 \geq -2 \\ &\qquad\qquad\quad x_1 + 2x_2 \leq 8 \\ &\qquad\qquad\quad x_1, x_2 \geq 0 \end{aligned}$$

Solution: The graph of the problem is as follows:

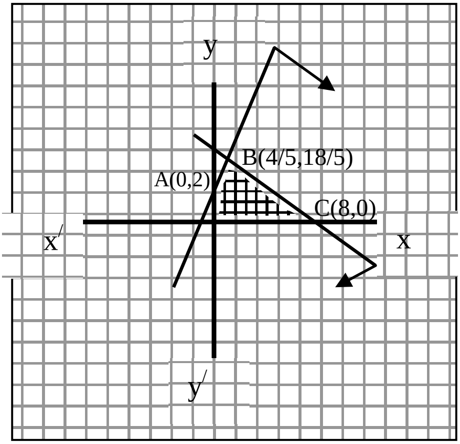

Figure – 3.11

From the graph we get the vertices A(0,2), B(4/5,18/5), C(8,0) and O(0,0) of the solution space. The values of the objective function at these points are – 20 at A, – 32 at B, 40 at C and 0 at O. Here, – 32 is the minimum value of the objective function which occurs at the vertices B(4/5,18/5). Hence, the solution of the given problem is $x_1 = 4/5$, $x_2 = 18/5$ and minimum value of z = –32

<u>Example (3.19)</u>: Solve graphically the following linear program.

$$\begin{aligned} &\text{Maximize} \quad z = 20x_1 + 10x_2 \qquad \textbf{[JU-00]} \\ &\text{Subject to} \quad x_1 + 2x_2 \leq 40 \\ &\qquad\qquad\quad 3x_1 + x_2 \leq 30 \\ &\qquad\qquad\quad 4x_1 + 3x_2 \leq 60 \\ &\qquad\qquad\quad x_1, x_2 \geq 0 \end{aligned}$$

Solution: Drawing the constraints in the graph paper as follows we find the shaded feasible solution space OABCO. The vertices O(0, 0), A(10, 0), B(6, 12) and C(0, 20) are basic feasible solution of the given problem. And the value of the objective function at O is 0, at A is 200, at B is 240 and at C is 200. Here the maximum value is 240 and attain at B(6,12). Therefore, the optimal solution is $(x_1, x_2) = (6, 12)$ and $z_{max} = 240$.

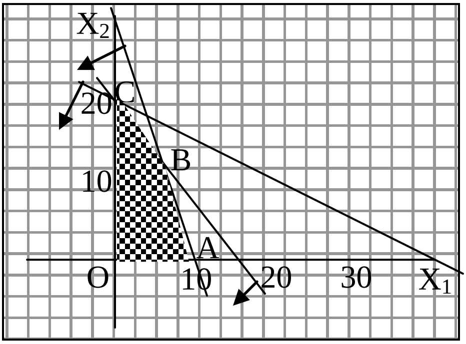

Figure 3.12

<u>Example (3.20)</u>: Solve graphically the following linear program.

Minimize $z = 20x_1 + 10x_2$ **[NUH-00]**

Subject to
$$x_1 + 2x_2 \leq 40$$
$$3x_1 + x_2 \geq 30$$
$$4x_1 + 3x_2 \geq 60$$
$$x_1, x_2 \geq 0$$

Solution: Drawing the constraints in the graph paper we find the shaded feasible solution space ABCDA. The vertices A(15,0), B(40,0), C(4, 18) and D(6, 12) are basic feasible solution of the given problem. And the value of the objective function at A is 300, at B is 800, at C is 260 and at D is 240. Here the minimum value is 240 and attain at D(6, 12). Therefore, the optimal solution is $(x_1, x_2) = (6, 12)$ and $z_{min} = 240$.

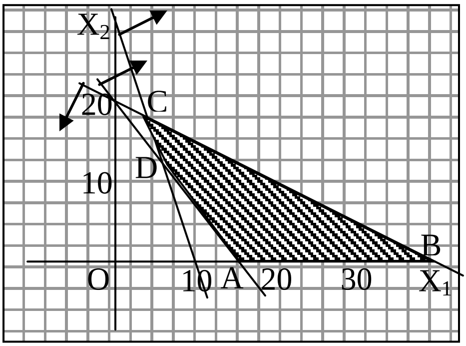

Figure 3.13

<u>Example (3.21)</u>: Solve graphically the following linear program.

Minimize $z = 3x_1 + 2x_2$ **[NUH-98, 00, 01, 04,07]**

Subject to
$$x_1 + 2x_2 \geq 4$$
$$2x_1 + x_2 \geq 4$$
$$x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

Solution: Drawing the constraints in the graph paper we find the shaded feasible solution space ABCDEA. The vertices A(4/3, 4/3), B(4, 0), C(5, 0), D(0, 5) and E(0, 4) are basic feasible solution of the given problem. And the value of the objective function at A is 20/3, at B is 12, at C is 15, at D is 10 and at E is 8. Here the minimum value is 20/3 and attained at A(4/3,4/3). Therefore, the optimal solution is $(x_1, x_2) = (4/3,4/3)$ and $z_{min}=20/3$.

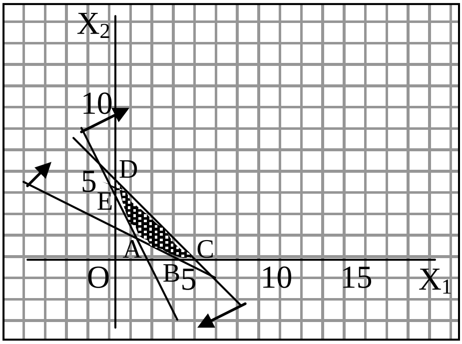

Figure 3.14

<u>Example (3.22)</u>: Solve graphically the following linear program.

Optimize (minimize or maximize) $z = 3x_1 + 2x_2$ **[JU-89]**

Subject to

$$x_1 + 2x_2 \geq 4$$
$$2x_1 + x_2 \geq 4$$
$$x_1 + x_2 \leq 8/3$$
$$x_1, x_2 \geq 0$$

Solution: Drawing the constraints in the graph paper we find only one basic feasible solution of the problem and that is A(4/3, 4/3). And hence, it gives optimum (minimum or maximum) value of the objective function 20/3. Therefore, the optimal solution is $x_1 = 4/3$, $x_2 = 4/3$ and $z_{optimum} = 20/3$.

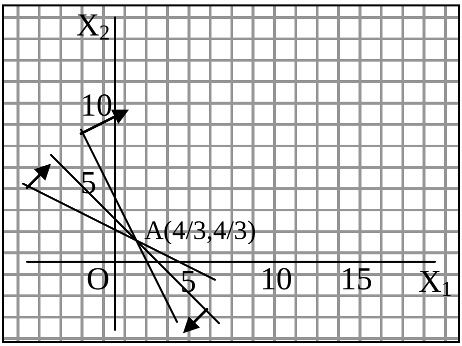

Figure 3.15

Example (3.23): Solve graphically the following linear program.

Maximize $z = 6x_1 + 9x_2$

Subject to $x_1 + 2x_2 \geq 4$

$2x_1 + x_2 \geq 4$

$x_1 + x_2 = 8/3$

$x_1, x_2 \geq 0$

Solution: Drawing the constraints in the graph paper we find only one basic feasible solution of the problem and that is A(4/3, 4/3). And hence, it gives maximum value 20 of the objective function $6x_1 + 9x_2$. So, the optimal solution is $x_1 = 4/3$, $x_2 = 4/3$ and $z_{max} = 20$.

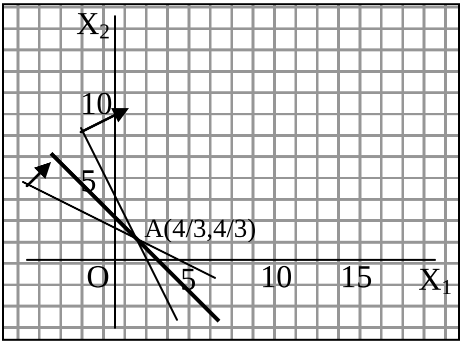

Figure 3.16

Example (3.24): Show graphically that the following LP problem has no solution. State the reason.

Maximize $z = x_1 + \frac{1}{2}x_2$ **[NUH-01]**

Subject to $3x_1 + 2x_2 \leq 12$

$5x_1 \leq 10$

$x_1 + x_2 \geq 8$

$-x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$

Solution: Drawing the graph of constraints, we do not get a feasible solution space that satisfies all the constraints. Hence the problem has no solution.

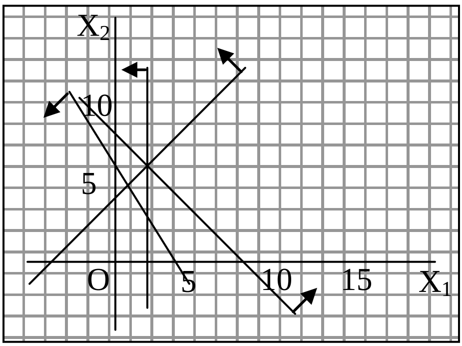

Figure 3.17

Example (3.25): Solve the problem graphically.

Maximize $z = 4x_1 + 3x_2$ **[NUH-03]**

Subject to $x_1 + x_2 \leq 50$

$x_1 + 2x_2 \leq 80$

$2x_1 + x_2 \geq 20$

$x_1 \leq 40$

$x_1, x_2 \geq 0$

Solution: Drawing the constraints in the graph paper we find the shaded feasible solution space ABCDEF. The vertices A(10, 0), B(40, 0), C(40, 10), D(20, 30), E(0, 40) and F(0, 20) are basic feasible solution of the given problem. And the value of the objective function at A is 40, at B is 160, at C is 190, at D is 170, at E is 120 and at F is 60. Here the maximum value is 190 and attained at C(40, 10). Therefore, the optimal solution is $(x_1, x_2) = (40, 10)$ and $z_{max} = 190$.

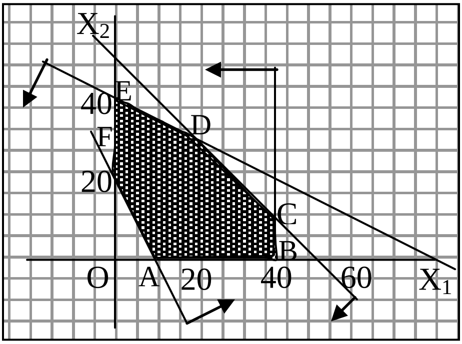

Figure 3.18

Example (3.26): Solve the problem graphically.

Maximize $z = x_1 + 5x_2$ **[NUH-03]**

Subject to $3x_1 + 4x_2 \leq 6$

$x_1 + 3x_2 \geq 2$

$x_1, x_2 \geq 0$

Solution: Drawing the graph, we find the shaded feasible solution space ABCA, where, A(2,0), B(0,3/2) and C(0,2/3). Value of the objective function at A is 2, at B is 15/2 and at C is 10/3. Therefore, $z_{max} = 15/2$ and $x_1 = 0$, $x_2 = 3/2$.

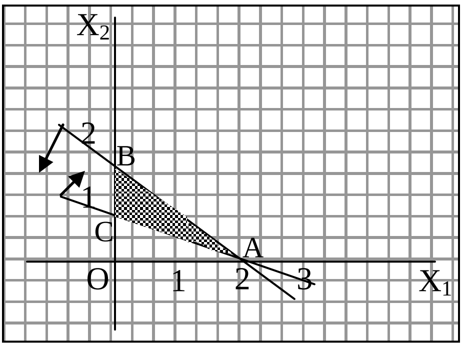

Figure 3.19

Example (3.27): Solve the problem graphically.

Minimize $z = 2x_1 + 4x_2$ **[JU-94]**

Subject to $x_1 - 3x_2 \leq 6$

$x_1 + 2x_2 \geq 4$

$x_1 - 3x_2 \geq -6$

$x_1, x_2 \geq 0$

Solution: The solution space satisfying the given constraints are shown shaded on the graph. Any point in the shaded region is a feasible solution to the given problem. The co-ordinates of the three vertices of the unbounded convex region are A(0, 2), B(4, 0) and C(6, 0). Values of the objective function $2x_1+4x_2$ at A is 8, at B is 8 and at C is 12. Since the minimum value of the objective function is 8 which occurs at the vertices A(0, 2) and B(4, 0), the solution to the given problem are $(x_1, x_2) = (0,2), (4,0)$ and $z_{min} = 8$.

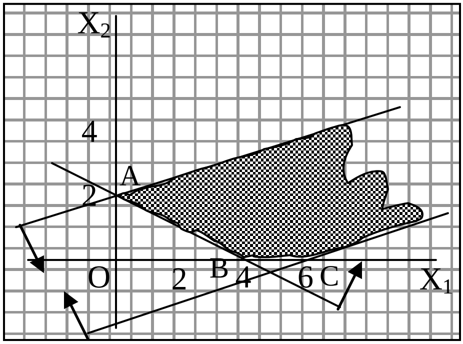

Figure 3.20

Having two solutions implies that the problem has many solutions and all the points on the line segment $\lambda(0,2) + (1-\lambda)(4,0)$; $0 \leq \lambda \leq 1$ are solutions of the problem.

Example (3.28): Solve the problem by cutting plane method.

Minimize $z = 2x_1 + 4x_2$

Subject to $x_1 - 3x_2 \leq 6$

$x_1 + 2x_2 \geq 4$

$x_1 - 3x_2 \geq -6$

$x_1, x_2 \geq 0$

Solution: We are going to solve the problem by alternative graphical method (cutting plane method). First we draw the constraints and find the solution space as shaded in the graph. We consider $2x_1 + 4x_2 = 12$ ... (i)

We draw a dotted straight line by (i) and observe that the line passing through the solution region. Then changing the right hand constant of (i) we draw many new straight lines parallel to (i). When we take the constant less than 8, the straight lines do not touch the solution region. Hence, we find the minimum value of the objective function is 8 and the co-ordinates of all points on the line $2x_1 + 4x_2 = 8$ & the solution region are the solutions of the problem.

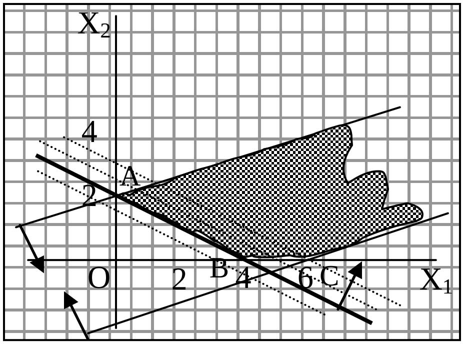

Figure 3.21

Example (3.29): Solve the problem by cutting plane method.

Minimize $z = 2x_1 + 2x_2$

Subject to $x_1 - 3x_2 \leq 6$

$x_1 + 2x_2 \geq 4$

$x_1 - 3x_2 \geq -6$

$x_1, x_2 \geq 0$

Solution: We are going to solve the problem by cutting plane method. First we draw the constraints and find the solution space as shaded in the graph. We consider $2x_1 + 2x_2 = 8$ ... (i)

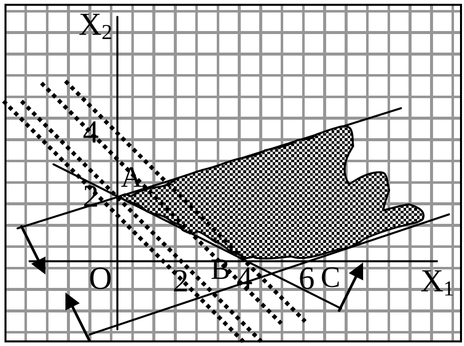

Figure 3.22

We draw a dotted straight line by (i) and observe that the line passing through the solution region. Then changing the right hand constant of (i) we draw many new straight lines parallel to (i). When we take the constant less than 4, the straight lines do not touch the solution region. When the constant is 4, the line touch only the point A(0, 2). Hence the solution is $(x_1, x_2) = (0, 2)$ and the minimum value of the objective function is 4, i.e., $z_{min} = 4$.

Example (3.30): Solve the LP problem graphically:

Maximize $z = 10x_2 - 2x_1$ **[NUH-97]**

Subject to $x_1 - x_2 \geq 0$

$-x_1 + 5x_2 \geq 5$

$x_1, x_2 \geq 0$

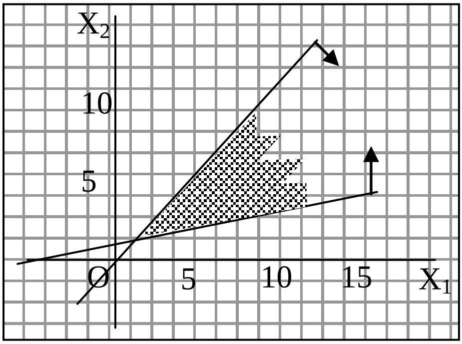

Figure 3.23

Solution: Drawing the graph of constraints, we get an unbounded solution space and the problem is of maximization type with negative sign in the objective function. The negative sign does not have any influence to decrease the value of the objective function when we go outward the origin through the upper line of the convex region. But the value of the objective function increases when we go outward the origin through the upper line of the convex solution space. Hence, it has an unbounded solution, i.e., it has a maximum at infinity.

Example (3.31): Solve the LP problem graphically: **[DU-99]**

Maximize $z = 6x_1 - 2x_2$

Subject to $2x_1 - x_2 \leq 2$

$x_1 \leq 3$

$x_1, x_2 \geq 0$

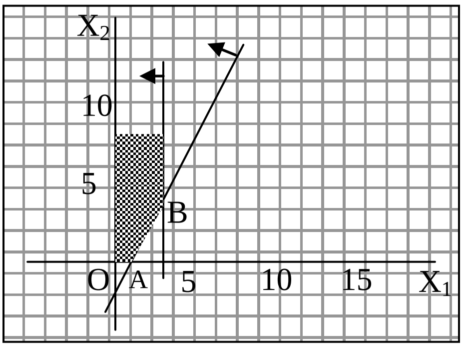

Figure 3.24

Solution: Drawing the graph of constraints, we get an unbounded solution space with 3 vertices O(0,0), A(1,0) & B(3,4) and the problem is of maximization type with negative sign in the objective function. The value of the objective function at O is 0, at A is 6 and at B is 10. Taking $x_1=3$ as fixed, any value $x_2>4$ decreases the value of the objective function. Optimum solution $(x_1,x_2)=(3,4)$, $z_{max}=10$.

Example (3.32): A company sells two products A and B. The company makes profit Tk.40 and Tk.30 per unit of each product. The two products are produced in a common process. The production process has capacity 30,000 man hours. It takes 3 hours to produce one unit of A and one hour per unit of B. The market has been surveyed and if feels that A can be sold 8,000 units, B of 12,000 units. Subject to above limitations form LP problem which maximizes the profit. Formulate the problem. **[NUH-04]**

Solution: The mathematical formulation of the company's problem as follows:

Step-1: How many products A and B to be produced is our key decision.

Step-2: Let x_1 and x_2 be the number of product A and B respectively.

Step-3: Feasible alternatives are the set of the values of x_1 and x_2 satisfying $x_1 \geq 0$ and $x_2 \geq 0$.

Step-4: The objective is to maximize the profit producing product A and B. Here, the objective function is $z = 40x_1 + 30x_2$ which will be maximized.

Step-5: Subject to the constraints are $3x_1 + x_2 \leq 30000$, $x_1 \leq 8000$ and $x_2 \leq 12000$.

Conclusion: Hence the company's problem can be put in the following mathematical form:

$$\begin{aligned} \text{Maximize } z = \ & 40x_1 + 30x_2 \\ \text{Subject to,} \quad & 3x_1 + x_2 \leq 30000 \\ & x_1 \leq 8000 \\ & x_2 \leq 12000 \\ & x_1, x_2 \geq 0 \end{aligned}$$

<u>Example (3.33)</u>: Vitamin A and B are found in two different Foods F_1 and F_2. One unit of Food F_1 contains 2 units of vitamin A and 5 units of vitamin B. One unit of Food F_2 contains 4 units of vitamin A and 2 units of vitamin B. One unit of Food F_1 and F_2 costs Tk.5 and Tk.3 respectively. The minimum daily requirements (for a man) of vitamin A and B are 40 and 50 units respectively. Assuming that anything is excess of daily minimum requirement of vitamin A and B is not harmful. Find out the optimum mixture of Food F_1 and F_2 at minimum cost which meets the daily minimum requirement of vitamin A and B. Formulate this as linear programming problem and solve it graphically. **[NUH-05]**

Solution: (Mathematical formulation) Let x_1 units of Food F_1 and x_2 units of Food F_2 are required to get the minimum amount of vitamin A and B; then the mathematical formulation is

$$\begin{aligned} &\text{Minimize } f(x) = 5x_1 + 3x_2 \\ &\text{Subject to } 2x_1 + 4x_2 \geq 40 \\ &\qquad\qquad 5x_1 + 2x_2 \geq 50 \\ &\qquad\qquad\ \ x_1, x_2 \geq 0 \end{aligned}$$

Graphical solution of the primal problem: Drawing the constraints in the graph paper we find the shaded unbounded feasible solution space X_1ABCX_2.

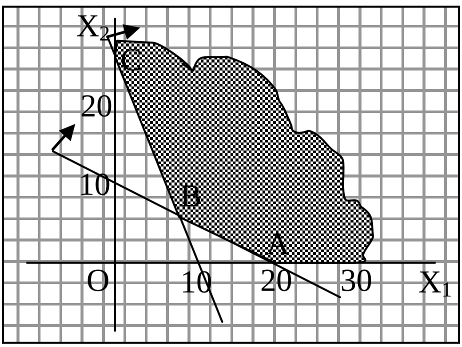

Figure 3.25

The vertices A(20, 0), B(15/2, 25/4) and C(0, 25) are basic feasible solution of the given problem. And the value of the objective function at A is 100, at B is 225/4 and at C is 75. Here the minimum value is 225/4 and attain at B(15/2, 25/4). So, Min. f(x) = 225/4. Therefore, cost will be minimum if 15/2 units of Food F_1 and 25/4 units of Food F_2 are taken.

Example (3.34): Show graphically that the following LP problem has no solution. State the reason.

Maximize $z = x_1 + \frac{1}{2}x_2$ **[NUH-05]**

Subject to $3x_1 + 2x_2 \leq 12$

$5x_1 \geq 10$

$x_1 + x_2 \geq \frac{8}{3}$

$-x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

Solution:

Drawing the constraints in the graph, we do not get a feasible solution space that satisfies all the constraints. Hence the problem has no solution.

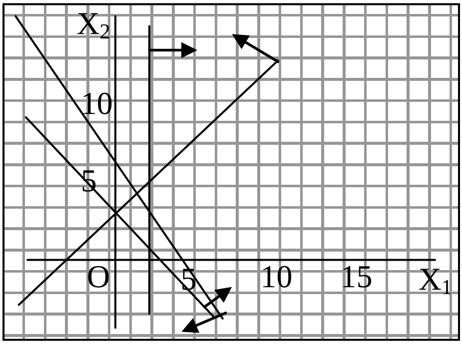

Figure 3.26

Example (3.35): Show graphically that the following LPP has no solution. Give the reason:-

Max. $Z = 3x + 2y$ **[NUH-06]**

Subject to $-2x + 3y \leq 9$

$3x - 2y \leq -20$

$x, y \geq 0$

Solution:

Drawing the constraints in the graph paper, we do not get a feasible solution space that satisfies all the constraints. Hence the problem has no solution.

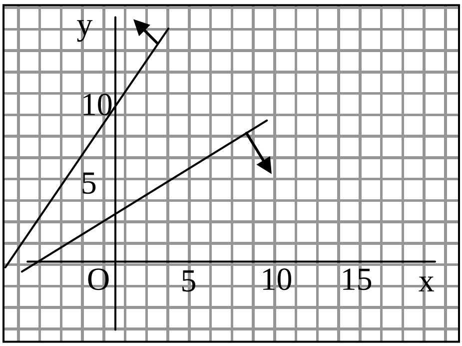

Figure 3.27

Example (3.36): A television manufacturer is concerned about what types of portable television sets should be produced during the next time period to maximize the profit. Because on past demands, a minimum of 200, 250 and 100 units of type I, II and III respectively are required. In addition the manufacturer has available maximum of 1000 units of time and 2000 units of raw materials during the next time period. Table below gives the essential data:

Types	Raw materials	Time	Minimum requirement (unit)	Profits per unit
I	1.0	2.0	200	10
II	1.5	1.2	250	14
III	4.0	1.0	100	12
Available	2000	1000		

Formulate a LPP model and solve it graphically. **[NUH-06]**
(একটি টেলিভিশন প্রস্তুতকারী প্রতিষ্ঠান সর্বোচ্চ লাভের জন্য সহজে বহনীয় কতকগুলো টেলিভিশন পরবর্তী সময়ের জন্য প্রস্তুত করতে চায়। কেননা পূর্বের চাহিদা মতে যথাক্রমে ২০০, ২৫০ এবং ১০০ টি টাইপ-১, টাইপ-২ এবং টাইপ-৩ টেলিভিশন দরকার। পরবর্তী সময়ের জন্য প্রস্তুতকারী প্রতিষ্ঠানের সর্বোচ্চ ১০০০ একক সময় এবং ২০০০ একক কাঁচামাল বরাদ্দ আছে। উপরের উপাত্ত থেকে একটি লিনিয়ার প্রোগ্রাম গঠন কর এবং লেখচিত্রের সাহায্যে সমাধান কর।)

Solution: The mathematical formulation of the manufacturer's problem as follows:

Step-1: How many type-I, type-II and type-III televisions to be manufactured is our key decision.

Step-2: Let x_1, x_2 and x_3 be the number of type-I, type-II and type-III televisions respectively.

Step-3: Feasible alternatives are the set of the values of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step-4: The objective is to maximize the profit producing type-I, type-II and type-III televisions. Here, the objective function is

$z = 10x_1 + 14x_2 + 12x_3$ which will be maximized.

Step-5: Subject to the constraints are

$1.0x_1 + 1.5x_2 + 4.0x_3 \leq 2000,$

$2.0x_1 + 1.2x_2 + 1.0x_3 \leq 1000,$

$x_1 \geq 200$, $x_2 \geq 250$ and $x_3 \geq 100$.

Conclusion: Hence the company's problem can be put in the following mathematical form:

$$\begin{aligned} \text{Maximize } z &= 10x_1 + 14x_2 + 12x_3 \\ \text{Subject to, } \quad & 1.0x_1 + 1.5x_2 + 4.0x_3 \leq 2000 \\ & 2.0x_1 + 1.2x_2 + 1.0x_3 \leq 1000 \\ & x_1 \geq 200 \\ & x_2 \geq 250 \\ & x_3 \geq 100 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Or,

$$\begin{aligned} \text{Maximize } z &= 10x_1 + 14x_2 + 12x_3 \\ \text{Subject to, } \quad & 1.0x_1 + 1.5x_2 + 4.0x_3 \leq 2000 \\ & 2.0x_1 + 1.2x_2 + 1.0x_3 \leq 1000 \\ & x_1 \geq 200,\ x_2 \geq 250,\ x_3 \geq 100 \end{aligned}$$

This is the required LP model.

Since the problem contains three variables, it can not be solve by the graphical method.

3.6 Exercises:

36. What do you mean by formulation? Discuss the algorithm to formulate a linear programming problem. **[DU-95, 99]**
37. Discuss the algorithm of graphical method. What is the limitation of this method? **[JU-98]**
38. At a cattle preceding firm, it is prescribed that the food ration for one animal must contain at least 14, 22 and 11 units of nutrients A, B and C respectively. Two different kinds of food are available. Each kilogram of food-1 contains 2, 2 and 1 units

of A,B and C and food-2 contains 1,3 and 1 units of A,B and C. Cost of food-1 is Tk.5.00 per kg and food-2 is Tk.4.00 per kg. To minimize the total cost, formulate the problem as a linear programming problem.

39. A farmer has 100 acres of land. He produces tomato, lettuce and radish and can sell them all. The price he can obtain is Tk.10 per kg. for tomato, Tk.6 a head for lettuce and Tk.5 per kg. for radish. The average yield per acre is 2000 kg. of tomato, 1500 heads of lettuce and 2100 kg. of radish. Fertilizer is available at Tk.30 per kg. and the amount required per acre is 125 kg. each for tomato and lettuce and 75 kg. for radish. Labour required for sowing, cultivation and harvesting per acre is 10 man-days for tomato and radish and 8 man-days for lettuce. The farmer has 300 man-days of labour are available at Tk.80 per man-day. Formulate a linear program for this problem to maximize the farmer’s total profit.
40. A company produces AM and AM-FM radios. A plant of the company can be operated 24 hours per week. Production of an AM radio will require 2 hours and production of AM-FM radio will require 3 hours each. An AM radio yields Tk.50 as profit and an AM-FM radio yields Tk.100. The marketing department determined that a maximum of 15 AM and 10 AM-FM radios can be sold in a week. Formulate the problem as linear programming problem and solve it graphically.
41. A farmer has 30 acres of land. He produces rice, wheat and potato and can sell them all. The price he can obtain is Tk.11 per kg. for rice, Tk.14 per kg. for wheat and Tk.12 per kg. for potato. The average yield per acre is 1600 kg. of rice, 1200 kg. of wheat and 2000 kg. of potato. Fertilizer is available at Tk.35 per kg. and the amount required per acre is 40 kg. each for rice and wheat and 55 kg. for potato. Labour required for sowing, cultivation and harvesting per acre is 8 man-days for rice and potato and 7 man-days for wheat. The farmer has 200 man-days of labour are available at Tk.80 per man-day. Formulate a

linear program for this problem to maximize the farmer's total profit.

42. A dairy firm has two milk plants with dairy milk production of 10 million litres and 12 million litres respectively. Each day the firm must fulfill the needs of three distribution centres which have milk requirement of 9, 6 and 7 million litres respectively. Cost of shipping one million litres of milk from each plant to each distribution centre is given in hundreds of taka below:

Plants	Distribution centres			Supply
	1	2	3	
1	5	3	7	10
2	3	8	5	12
Demands	9	6	7	

Formulate the linear programming model to minimize the transportation cost.

43. A firm manufactures three products – A, B and C. Time to manufacture product A is twice that for B and thrice that for C and they are to be produced in the ratio 3 : 4 : 5. The relevant data are in the given table. If the whole labour is engaged in manufacturing product A, 1600 units of this product can be produced. There are demand for at least 300, 250 and 200 units of products A, B and C; and the profit earned per unit is Rs.59, Rs.40 and Rs.70 respectively. Formulate the problem as a linear programming problem and solve the problem by any method.

Raw materials	Requirement per unit of product (Kg)			Total availability (Kg)
	A	B	C	
P	6	5	9	5,000
Q	4	7	8	6,000

44. A dietitian is planning the menu for the evening meal at a university dining hall. Three main items will be served, all having different nutritional content. The dietitian is interested to providing at least the minimum daily requirement of each of three vitamins in this one meal. The following table summarizes the vitamin content per ounce of each type of food, the cost per ounce of each food, and minimum daily requirements (MDR) for the three vitamins. Any combination of the three foods may be selected as long as the total serving size is at least 9 ounces.

Food	Vitamins (per mg)			Cost per ounce, $
	1	2	3	
1	50	20	10	0.10
2	30	10	50	0.15
3	20	30	20	0.12
MDR	290	200	210	

The problem is to determine the number of ounces of each food to be included in the meal. The objective is to minimize the cost of each meal subject to satisfying minimum daily requirements of the three vitamins as well as the restriction on minimum serving size. Give the formulation of the problem.

[Answer: Minimize $z = 0.10x_1 + 0.15x_2 + 0.12x_3$

Subject to
$$50x_1 + 30x_2 + 20x_3 \geq 290$$
$$20x_1 + 10x_2 + 30x_3 \geq 200$$
$$10x_1 + 50x_2 + 20x_3 \geq 210$$
$$x_1 + x_2 + x_3 \geq 9$$
$$x_1, x_2, x_3 \geq 0]$$

45. Solve the following linear programming problems using graphical method:

(i) Maximize $z = 2x_1 + 3x_2$
Subject to $x_1 + x_2 \leq 30$
$x_2 \geq 3$
$x_2 \leq 12$
$x_1 - x_2 \geq 0$
$0 \leq x_1 \leq 20$
[Answer: $x_1 = 18$, $x_2 = 12$, $z = 72$]

(ii) Maximize $z = 2x_1 - 6x_2$
Subject to $3x_1 + 2x_2 \leq 6$
$x_1 - x_2 \geq -1$
$-x_1 - 2x_2 \geq 1$
$x_1, x_2 \geq 0$
[Answer: No solution]

(iii) Maximize $z = 2x_1 + 3x_2$
Subject to $-x_1 + 2x_2 \leq 4$
$x_1 + x_2 \leq 6$
$x_1 + 2x_2 \leq 9$
$x_1, x_2 \geq 0$
[Answer: $x_1 = 9/2$, $x_2 = 3/2$, and $z = 27/2$]

(iv) Minimize $z = x_1 + 2x_2$
Subject to $x_1 - 3x_2 \leq 6$
$2x_1 + 4x_2 \geq 8$
$x_1 - 3x_2 \geq -6$
$x_1, x_2 \geq 0$
[Answer: Many solutions. In particular $(x_1, x_2) = (0,2), (4,0)$ and min. value = 4]

(v) Minimize $z = 2x_1 + x_2$
Subject to $3x_1 + x_2 \geq 3$
$4x_1 + 3x_2 \geq 6$
$x_1 + 3x_2 \leq 3$
$x_1, x_2 \geq 0$
[Answer: $x_1 = 1$, $x_2 = 2/3$, and $z = 8/3$]

(vi) Maximize $z = x_1 + x_2$
Subject to $x_1 + x_2 \geq 1$
$x_1 - x_2 \leq 1$
$-x_1 + x_2 \leq 1$
$x_1, x_2 \geq 0$
[Answer: Unbounded solution]

Chapter

04

Simplex Methods

Highlights:

4.1 Simplex: (সিমপে-ক্স) A simplex is an n dimensional convex polyhedron with exactly (n +1) extreme points,

(i) If n = 0, then the convex polyhedron is a point.
(ii) If n = 1, then the convex polyhedron is a straight line.
(iii) If n = 2, then the convex polyhedron is a triangle.
(iv) If n = 3, then the convex polyhedron is tetrahedron and so on.

4.2 Simplex method: (সিমপে-ক্স পদ্ধতি) The simplex method developed by G. B. Dantzig in 1947 is a simple iterative technique or algorithm to find an optimum basic feasible solution starting with an initial basic feasible solution by a finite number of iterations. That is, this method provides an algorithm, which consists in moving from one vertex of the region of feasible solutions to another in such a manner that the value of the objective function at the succeeding vertex is improved (less for minimization type problem or more for maximization type problem) than the preceding vertex. The iteration is continued until an optimum basic feasible solution is obtained or there is an indication of an unbounded solution or no solution. **[NUH-02]**

Theorem (4.1): If a set of $k \le m$ vectors $\underline{P}_1, \underline{P}_2, ..., \underline{P}_k$ can be found that is linearly independent and such that $x_1\underline{P}_1 + x_2\underline{P}_2 + ... + x_k\underline{P}_k = \underline{P}_0$ and all $x_i \ge 0$, then the point $\underline{X} = (x_1, x_2, ..., x_k, 0, ..., 0)$ is an extreme point of the convex set of feasible solutions. Here $\underline{X}$ is an n-dimensional row vector whose last $n - k$ elements are zero and $\underline{P}_i$'s are m-dimensional column vectors, that is, there are m constraint equations in the linear program. **[JU-88]**

Proof: $x_1\underline{P}_1 + x_2\underline{P}_2 + ... + x_k\underline{P}_k = \underline{P}_0$ implies that $\underline{X} = (x_1, x_2, ..., x_k, 0, ..., 0)$ is a feasible solution. Suppose $\underline{X}$ is not an extreme point. Then $\underline{X}$ can be written as a convex combination of two other points $\underline{X}_1$ and $\underline{X}_2$ in K, the set of feasible solutions. We have $\underline{X} = \alpha\underline{X}_1 + (1-\alpha)\underline{X}_2$ for $0<\alpha<1$. Since all the elements x_i of $\underline{X}$ are non-negative and since $0<\alpha<1$, the last $n - k$ elements of $\underline{X}_1$ and $\underline{X}_2$ must also equal zero. That is,

$\underline{X}_1 = (x_1^{(1)}, x_2^{(1)}, ..., x_k^{(1)}, 0, ..., 0)$, $\underline{X}_2 = (x_1^{(2)}, x_2^{(2)}, ..., x_k^{(2)}, 0, ..., 0)$

Since $\underline{X}_1$ and $\underline{X}_2$ are feasible solutions, we have

$$x_1^{(1)}\underline{P}_1 + x_2^{(1)}\underline{P}_2 + ... + x_k^{(1)}\underline{P}_k = \underline{P}_0 \quad ... \quad \text{(i)}$$
$$x_1^{(2)}\underline{P}_1 + x_2^{(2)}\underline{P}_2 + ... + x_k^{(2)}\underline{P}_k = \underline{P}_0 \quad ... \quad \text{(ii)}$$

Subtracting (ii) from (i), we have,

$$(x_1^{(1)} - x_1^{(2)})\underline{P}_1 + (x_2^{(1)} - x_2^{(2)})\underline{P}_2 + ... + (x_k^{(1)} - x_k^{(2)})\underline{P}_k = \underline{0}$$

Since $\underline{P}_1, \underline{P}_2, ..., \underline{P}_k$ are linearly independent vectors, hence,

$$(x_1^{(1)} - x_1^{(2)}) = 0,\ (x_2^{(1)} - x_2^{(2)}) = 0,\ ...,\ (x_k^{(1)} - x_k^{(2)}) = 0$$

So, $x_1^{(1)} = x_1^{(2)}$, $x_2^{(1)} = x_2^{(2)}$, ..., $x_k^{(1)} = x_k^{(2)}$ which implies that $x_i = x_i^{(1)} = x_i^{(2)}$

Therefore, $\underline{X}$ cannot be expressed as a convex combination of two distinct points in K and must be an extreme point of K.

Theorem (4.2): If $\underline{X} = (x_1, x_2, ..., x_n)$ be an extreme point of K (the set of feasible solutions), then the vectors associated with positive

x_i form a linearly independent set. From this it follows that, at most m number of the x_i's are positive, where m is the number of constraint equations in the linear program. **[CU-85]**

Proof: Let the positive coefficients be the first k coefficients, so that $\sum_{i=1}^{k} x_i \underline{P}_i = \underline{P}_0$; $x_i>0$ where $\underline{P}_i$'s are m-dimensional column vectors. We prove the main part of the theorem by contradiction. Assume that $\{\underline{P}_1, \underline{P}_2, ..., \underline{P}_k\}$is a linearly dependent set of vectors. Then there exists a linear combination of these vectors, which equals the zero vector, that is,

$$d_1\underline{P}_1 + d_2\underline{P}_2 + ... + d_k\underline{P}_k = \underline{0} \quad ... \text{ (i)} \quad \text{with at least one } d_i \neq 0.$$

From the hypothesis of the theorem, we have,

$$x_1\underline{P}_1 + x_2\underline{P}_2 + ... + x_k\underline{P}_k = \underline{P}_0 \quad ... \text{ (ii) with all } x_i > 0.$$

For some d>0, we multiply (i) by d then add and subtract the result from (ii) to obtain the two equations

$$\sum_{i=1}^{k} x_i \underline{P}_i + d\sum_{i=1}^{k} d_i \underline{P}_i = \underline{P}_0 \text{ and } \sum_{i=1}^{k} x_i \underline{P}_i - d\sum_{i=1}^{k} d_i \underline{P}_i = \underline{P}_0\text{, that is,}$$

$$(x_1 + dd_1)\underline{P}_1 + (x_2 + dd_2)\underline{P}_2 + ... + (x_k + dd_k)\underline{P}_k = \underline{P}_0$$

and $(x_1 - dd_1)\underline{P}_1 + (x_2 - dd_2)\underline{P}_2 + ... + (x_k - dd_k)\underline{P}_k = \underline{P}_0$

We then have the two feasible solutions, namely

$$\underline{X}_1 = (x_1 + dd_1, x_2 + dd_2, ..., x_k + dd_k, 0, ...,0)$$

and $\underline{X}_2 = (x_1 - dd_1, x_2 - dd_2, ..., x_k - dd_k, 0, ...,0)$

Since all $x_i > 0$, we can let d be as small as necessary, but still positive, to make the first k components of both $\underline{X}_1$ and $\underline{X}_2$ positive. $\underline{X}_1$ and $\underline{X}_2$ are feasible solutions, but $\underline{X} = \frac{1}{2}\underline{X}_1 + \frac{1}{2}\underline{X}_2 =$

$\frac{1}{2}\underline{X}_1 + (1 - \frac{1}{2})\underline{X}_2 = \lambda\underline{X}_1 + (1-\lambda)\underline{X}_2$ for $0 < \lambda < 1$, which contradicts the hypothesis that $\underline{X}$ is an extreme point [since, extreme point can not be expressed as the linear combination of two other points]. The assumption of linear dependence for the vectors $\underline{P}_1, \underline{P}_2, ..., \underline{P}_k$ has thus led to a contradiction and hence must be false. That is, the set of vectors $\underline{P}_1, \underline{P}_2, ..., \underline{P}_k$ is linearly independent. Since every set of m+1 vectors in m-dimensional space is necessarily linearly dependent, we cannot have more than m positive x_i.

Observations:

(i) The set of feasible solutions to an LP problem is a convex set.

(ii) The optimum value of the objective function is at an extreme point.

(iii) Every basic feasible solution corresponds to a set of linearly independent vectors.

(iv) Every set of linear independent vectors corresponds to an extreme point.

4.3 <u>Generating extreme point solution</u>: (চরম বিন্দু সমাধান উন্নয়ন)

Here we assume that an extreme point solution in terms of m vectors $\underline{P}_j$ of the original set of n vectors is known. We can let this set of m linearly independent vectors be the first m, and let $\underline{X} = (x_1, x_2, ..., x_m, 0, ..., 0)$ be the basic feasible solution vector. We then have

$$x_1\underline{P}_1 + x_2\underline{P}_2 + ... + x_m\underline{P}_m = \underline{P}_0 \quad ... \quad (1)$$

where all $x_i \geq 0$. With this information, the problem is to determine a new extreme point solution in a computationally efficient manner. Since the vector $\underline{P}_1, \underline{P}_2, ..., \underline{P}_m$ are linearly independent, so, we can write

$$\sum_{i=1}^{m} x_{ij} \underline{P}_i = \underline{P}_j \ ; \ \ j = 1, 2, ..., n$$

Assume that some vector not in the given independent basis vector, say $\underline{P}_{m+1}$, has at least one element $x_{i,m+1} > 0$ in the expression

$$x_{1,m+1}\underline{P}_1 + x_{2,m+1}\underline{P}_2 + ... + x_{m,m+1}\underline{P}_m = \underline{P}_{m+1} \quad ... \quad (2)$$

Let θ be any number and multiply (2) by θ and subtract the result from (1) to obtain

$$(x_1-\theta x_{1,m+1})\underline{P}_1 + (x_2 - \theta x_{2,m+1})\underline{P}_2 + ... + (x_m - \theta x_{m,m+1})\underline{P}_m + \theta \underline{P}_{m+1} = \underline{P}_0 \quad ... \quad (3)$$

The vector $\underline{X}^/ = (x_1-\theta x_{1,m+1}, x_2 - \theta x_{2,m+1}, ..., x_m-\theta x_{m,m+1}, \theta)$ is a solution to the problem, and if all the elements of $\underline{X}^/$ are non-negative, $\underline{X}^/$ is a feasible solution. Since we want $\underline{X}^/$ to be a feasible solution different from $\underline{X}$, we restrict θ to be greater than zero. With this restriction, all the elements of $\underline{X}^/$ that have a negative or zero $x_{i,m+1}$ with also have a non-negative $x_i-\theta x_{i,m+1}$. We need only concern ourselves with those elements having a positive $x_{i,m+1}$. We wish to find a $\theta > 0$ such that

$$x_i-\theta x_{i,m+1} \geq 0 \text{ for all } x_{i,m+1} > 0 \quad . \ . \ . \ (4)$$

Or, $\theta \leq \dfrac{x_i}{x_{i,m+1}}$ and hence any θ for which $0 < \theta \leq \dfrac{x_i}{x_{i,m+1}}$ will give a feasible solution for (3). However, as we are looking for an extreme point solution, we know by our theorem that we cannot have all the m + 1 elements of $\underline{X}^/$ positive. We then must force at least one of the elements of $\underline{X}^/$ to be exactly equal to zero. We see that, if we let $\theta = \theta_0 = \min\limits_i \dfrac{x_i}{x_{i,m+1}}$ for $x_{i,m+1} > 0$, then the element in $\underline{X}^/$ for which this minimum is attained will reduce to zero. Let this element be the first, that is,

$$\theta_0 = \min_i \frac{x_i}{x_{i,m+1}} = \frac{x_1}{x_{1,m+1}}$$

We have now obtained a new feasible solution $\underline{X}^/ = (x_2 - \theta_0 x_{2,m+1},\ldots,x_m - \theta_0 x_{m,m+1}, \theta_0) = (x_2^/, x_3^/, \ldots, x_m^/, x_{m+1}^/)$ (say), so that $x_2^/ \underline{P}_2 + x_3^/ \underline{P}_3 + \ldots + x_m^/ \underline{P}_m + x_{m+1}^/ \underline{P}_{m+1} = \underline{P}_0$; where $x_i^/ = x_i - \theta_0 x_{i,m+1}$; i = 2, 3, ..., m and $x_{m+1}^/ = \theta_0$

[If all the $x_{i,m+1}$ had been equal to or less than zero, then we would not have been able to select a positive θ that would have eliminated at least one of the vectors $\underline{P}_1$, $\underline{P}_2$, ..., $\underline{P}_m$ from the basis. For this situation we obtain for any $\theta > 0$ a non-extreme point feasible solution associated with the m + 1 vectors $\underline{P}_1$, $\underline{P}_2$, ..., $\underline{P}_m$, $\underline{P}_{m+1}$. For this case we can say that the problem does not have a finite minimum (or maximum) solution.]

To show that $\underline{X}^/ = (x_2^/, x_3^/, \ldots, x_m^/, x_{m+1}^/)$ is an extreme point, we have to prove that the set {$\underline{P}_2$, $\underline{P}_3$, ..., $\underline{P}_m$, $\underline{P}_{m+1}$} is linearly independent. Assume it is linearly dependent. We can then find

$$d_2\underline{P}_2 + d_3\underline{P}_3 + \ldots + d_m\underline{P}_m + d_{m+1}\underline{P}_{m+1} = \underline{0} \quad \ldots \ (5)$$

where not all $d_i = 0$. Since any subset of a set of linearly independent vectors is also a set of linearly independent vectors, the set {$\underline{P}_2$, $\underline{P}_3$, ..., $\underline{P}_m$} is linearly independent. This implies that $d_{m+1} \neq 0$. Therefore, from (5) we have,

$$e_2\underline{P}_2 + e_3\underline{P}_3 + \ldots + e_m\underline{P}_m = \underline{P}_{m+1} \quad \ldots \ (6)$$

where $e_i = -\dfrac{d_i}{d_{m+1}}$; i = 2, 3, ..., m. Subtracting (6) from (2), we have

$$x_{1,m+1}\underline{P}_1 + (x_{2,m+1} - e_2)\underline{P}_2 + \ldots + (x_{m,m+1} - e_m)\underline{P}m = \underline{0} \quad \ldots \ (7)$$

Since $\underline{P}_1$, $\underline{P}_2$, ..., $\underline{P}_m$ are linearly independent, all the coefficients in (7) must equal to zero. But $x_{1,m+1}$ was assumed to be positive, hence the assumption of linear dependence for $\underline{P}_2$, $\underline{P}_3$, ..., $\underline{P}_m$, $\underline{P}_{m+1}$ has led to a contradiction. So, $\{\underline{P}_2, \underline{P}_3, ..., \underline{P}_m, \underline{P}_{m+1}\}$ must be linearly independent and hence, $\underline{X}^/ = (x_2^/, x_3^/, ..., x_m^/, x_{m+1}^/)$ is an extreme point (or basic feasible) solution.

In order to continue this process of obtaining new extreme feasible solution, we need the representation of any vector not in the new basis $\underline{P}_2$, $\underline{P}_3$, ..., $\underline{P}_m$, $\underline{P}_{m+1}$ in terms of these basis. From (2) we have,

$$\underline{P}_1 = \frac{1}{x_{1,m+1}}(\underline{P}_{m+1} - x_{2,m+1}\underline{P}_2 - ... - x_{m,m+1}\underline{P}_m) \quad ... \quad (8)$$

Let $\underline{P}_j = x_{1j}\underline{P}_1 + x_{2j}\underline{P}_2 + ... + x_{mj}\underline{P}_m \quad ... \quad (9)$

be any vector not in the new basis. Substitute the expression (8) for $\underline{P}_1$ in (9) to obtain

$$\underline{P}_j = (x_{2j} - \frac{x_{1j}}{x_{1,m+1}} x_{2,m+1})\underline{P}_2 + (x_{3j} - \frac{x_{1j}}{x_{1,m+1}} x_{3,m+1})\underline{P}_3 + ...$$

$$+ (x_{mj} - \frac{x_{1j}}{x_{1,m+1}} x_{m,m+1})\underline{P}_m + \frac{x_{1j}}{x_{1,m+1}}\underline{P}_{m+1} \quad \dots \quad (10)$$

This is the formula for complete elimination.

<u>Illustration (4.1)</u>: We are given the following set of equations:

$$3x_1 - x_2 + 2x_3 \le 7$$
$$2x_1 - 4x_2 \le 12$$
$$-4x_1 - 3x_2 + 8x_3 \le 10$$
$$\underline{X} = (x_1, x_2, x_3) \ge \underline{0}$$

Find the other new extreme point.

Solution: We can rewrite the constraints as follows to obtain standard basis (or 3 independent vectors):

$$3x_1 - x_2 + 2x_3 + x_4 = 7$$
$$2x_1 - 4x_2 + x_5 = 12$$
$$-4x_1 - 3x_2 + 8x_3 + x_6 = 10$$
$$\underline{X} = (x_1, x_2, x_3, x_4, x_5, x_6) \ge \underline{0}$$

So, $\underline{P}_0 = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix}$, $\underline{P}_1 = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$, $\underline{P}_2 = \begin{pmatrix} -1 \\ -4 \\ -3 \end{pmatrix}$, $\underline{P}_3 = \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix}$,

$\underline{P}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{P}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{P}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Here, $\underline{P}_4$, $\underline{P}_5$ and $\underline{P}_6$ are linearly independent and we can write

$$7\underline{P}_4 + 12\,\underline{P}_5 + 10\,\underline{P}_6 = \underline{P}_0 \qquad \ldots \quad (1)$$

Or, $\quad 0\,\underline{P}_1 + 0\,\underline{P}_2 + 0\,\underline{P}_3 + 7\underline{P}_4 + 12\,\underline{P}_5 + 10\,\underline{P}_6 = \underline{P}_0$

$\therefore \quad x_1 = 0, x_2, = 0, x_3 = 0, x_4 = 7, x_5 = 12, x_6 = 10$

So, $\underline{X}_1 = (0, 0, 0, 7, 12, 10)$ is a given extreme point solution. We wish to introduce vector $\underline{P}_1$ to obtain another extreme point solution. The representation of P_1 in terms of the basis vectors is simply $\quad 3\underline{P}_4 + 2\,\underline{P}_5 - 4\,\underline{P}_6 = \underline{P}_1 \quad \ldots \quad (2)$

That is, $x_{41} = 3$, $x_{51} = 2$ and $x_{61} = -4$. If we multiply (2) by θ and subtract the result from (1), we have

$$(7 - 3\theta)\,\underline{P}_4 + (12 - 2\theta)\,\underline{P}_5 + (10 + 4\theta)\,\underline{P}_6 + \theta\,\underline{P}_1 = \underline{P}_0 \qquad \ldots \quad (3)$$

Since $x_{41} = 3$ and $x_{51} = 2$ are both positive, we determine θ_0 by evaluating for these positive x_{i1} as follows: $\theta = \theta_0 = \min \dfrac{x_i}{x_{i1}} = \dfrac{7}{3}$.

Substituting this value in (3), we eliminate $\underline{P}_4$ from the basis and we obtain $\dfrac{22}{3}\,\underline{P}_5 + \dfrac{58}{3}\,\underline{P}_6 + \dfrac{7}{3}\,\underline{P}_1 = \underline{P}_0 \qquad \ldots \qquad (4)$

Hence, $\underline{X}_2 = (7/3, 0, 0, 0, 22/3, 58/3)$ is a new extreme point solution.

Note 1: If, instead of $\underline{P}_1$, we tried in a similar manner to obtain an extreme point solution with $\underline{P}_2$, we get $-\underline{P}_4 - 4\underline{P}_5 - 3\underline{P}_6 = \underline{P}_2$ where, $x_{42} = -1$, $x_{52} = -4$, $x_{62} = -3$ and all are negative. We would have developed the following expression for $\underline{P}_0$ in terms of $\underline{P}_4$, $\underline{P}_5$, $\underline{P}_6$, $\underline{P}_2$:

$$(7 + \theta)\,\underline{P}_4 + (12 + 4\theta)\,\underline{P}_5 + (10 + 3\theta)\,\underline{P}_6 + \theta\,\underline{P}_2 = \underline{P}_0 \quad \ldots \quad (5)$$

From (5) we see that any $\theta > 0$ yields a feasible solution

$$\underline{X}_\theta = (0, \theta, 0, 7 + \theta, 12 + 4\theta, 10 + 3\theta).$$

Here, since all x_{i2} $(i = 4, 5, 6) < 0$, we do not obtain a new extreme point solution.

Note 2: A more efficient way of interpreting the elimination procedure of the problem is the following tabular system. Here we detach the coefficients of the equations and set up the following tableau.

Basis	$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	θ	
$\underline{P}_4$	7	3 Pivot	–1	2	1	0	0	$7/3 = \theta_o$	Row 1
$\underline{P}_5$	12	2	– 4	0	0	1	0	$12/2 = 6$	Row 2
$\underline{P}_6$	10	–4	–3	8	0	0	1		Row 3

Table 1

$\underline{P}_0$ vector tells us $\underline{X}_1 = (0, 0, 0, 7, 12, 10)$ is the given extreme point solution.

Using the complete elimination formula or doing the following instructions, we can get the following tableau. Firstly, we divide all elements of row 1 by pivot element 3 to obtain 1 as the first component of $\underline{P}_1$ and put it in the first row. Secondly, subtract 2 times of new first row from the old second row to obtain 0 as the second component of $\underline{P}_1$ and put it in the second row. Thirdly, add

4 times of new first row with the old third row to obtain 0 as the third component of $\underline{P}_1$ and put it in the third row.

Basis	$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	θ
$\underline{P}_1$	7/3	1	– 1/3	2/3	1/3	0	0	
$\underline{P}_5$	22/3	0	–10/3	– 4/3	–2/3	1	0	
$\underline{P}_6$	58/3	0	–13/3	–32/3	4/3	0	1	

Table 2

Last $\underline{P}_0$ vector tells us $\underline{X}_2$ = (7/3, 0, 0, 0, 22/3, 58/3) is the new extreme point solution. If we like to obtain an extreme point solution with $\underline{P}_3$ in the basis, we could start with table 2 or table 1 and determine θ_o as before and transform this table as above.

<u>Illustration (4.2)</u>: Minimize $z = 2x_1 - x_2 + x_3 - 5x_4 + 22x_5$

$$\begin{aligned} \text{Subject to: } x_1 \quad\quad - 2x_4 + x_5 &= 6 \quad \ldots \quad (1)\\ x_2 \quad + x_4 - 4x_5 &= 3 \quad \ldots \quad (2)\\ x_3 + 3x_4 + 2x_5 &= 10 \quad \ldots \quad (3)\\ x_j \geq 0;\ j = 1, 2, 3, 4, 5 \end{aligned}$$

Find the optimal solution.

Solution: The initial basic feasible solution is $x_1 = 6$, $x_2 = 3$, $x_3 = 10$, $x_4 = 0$, $x_5 = 0$, (x_1, x_2, x_3 are basic variables and x_4, x_5 are non-basic variables) with the value of the objective function for this solution by the unrestricted variable $z = 2x_1 - x_2 + x_3 = 19$. We would like to determine if a different basic feasible solution would yield a smaller value of the objective function or whether the current solution is the optimum. Note that this is equivalent to asking whether one of the non-basic variable, here x_4 and x_5, which are now set equal to zero, should be allowed to take on a positive value, if possible. From the above equation we solve for the current basic variable in terms of the non-basic variable to obtain

$$x_1 = 6 + 2x_4 - x_5 \quad --- \quad (4)$$
$$x_2 = 3 - x_4 + 4x_5 \quad --- \quad (5)$$
$$x_3 = 10 - 3x_4 - 2x_5 \quad --- \quad (6)$$

We next rewrite the objective function in terms of only the non-basic variables by substituting for x_1, x_2 and x_3 the corresponding right hand side expressions above to obtain

$z = 2(6 + 2x_4 - x_5) - (3 - x_4 + 4x_5) + (10 - 3x_4 - 2x_5) - 5x_4 + 22x_5$
Or, $z = 19 - 3x_4 + 14x_5$
Or, $z = 19 - (3x_4) - (- 14x_5)$

With $x_4 = 0$ and $x_5 = 0$, $z = 19$, which is the value for the current basic feasible solution. We see from the last transformed expression of z that if x_4 can be made positive, the objective will decrease 3 units for each unit increase of x_4: while any positive unit increase to x_5 will increase the value of the objective function by 14 units. Since we are minimizing, it would appear to be appropriate to determine a new basic feasible solution, i.e., an extreme point solution, involving x_4 at positive level, if possible. We next generate a new extreme point solution replacing x_2 by x_4 as follows:
As outgoing variable x_2 and necessary non-basic variable x_4 both are in equation (2), it is unchanged and we eliminate x_4 from equations (1) and (3) as follows:

$$
\begin{array}{llr}
x_1 + 2x_2 \qquad\qquad - 7x_5 = 12 & & [(1) + 2\times(2)] \\
\qquad x_2 \quad + x_4 - 4x_5 = 3 & & \\
\quad -3x_2 + x_3 \quad +14x_5 = 1 & & [(3) -3\times(2)]
\end{array}
$$

Or, $x_1 = 12 - 2x_2 + 7x_5$
$x_4 = 3 - x_2 + 4x_5$
$x_3 = 1 + 3x_2 - 14x_5$

The new basic feasible solution is $x_1 = 12$, $x_2 = 0$, $x_3 = 1$, $x_4 = 3$, $x_5 = 0$. Substituting the expressions of the basic variables x_1, x_3, x_4 in terms of the non-basic variables x_2 and x_5 into the objective function, now we have

$z = 2(12 - 2x_2 + 7x_5) - x_2 + (1 + 3x_2 - 14x_5) - 5(3 - x_2 + 4x_5) + 22x_5$
Or, $z = 10 - (-3x_2) - (-2x_5)$

From this last expression we see that any increase in the values of the non-basic variable x_2 and x_5 would increase the value of the objective function. We thus conclude that the new basic feasible solution is an optimum, with the value of the objective function of z = 10. Since z = 10 is less than z = 19. Hence the minimum value of the objective function is 10.

The above process is just the direct application of the elimination procedure on the linear programming problem.

Theorem (4.3): (Fundamental theorem of linear programming) If the feasible solution region of an LP problem is a convex polyhedron, then there exists an optimal (minimum or maximum) solution to the LP problem and at least one basic feasible solution must be optimal. **[JU-89]**

Proof: Let S be the feasible solution region of a linear programming (LP) problem whose linear objective function is z = f(X). Since S is a convex polyhedron it is non-empty closed and bounded. We know that a linear function is always continuous, so the linear objective function z = f(X) is continuous on S and hence an optimal solution of the linear program must exist on S. This proves the existence of an optimal solution.

To prove the second part of the theorem let us denote the vertices of the polyhedron by $X_1, X_2, ..., X_p$ and the optimum solution by X_0. This means that $f(X_0) \leq$ [*or* $\geq$ *for maximization*] f(X) for all X in S. If X_0 is a vertex, the second part of the theorem is true. (In two dimensions S might look like figure). Suppose X_0 is not a vertex (as indicated in figure). We can then write X_0 as a convex combination of the vertices of S, that is,

$$X_0 = \sum_{i=1}^{p} \alpha_i X_i \text{ ; for } \alpha_i \geq 0, \ \sum_i \alpha_i = 1$$

Then, since f(X) is a linear function, we have

$$f(X_0) = f\left(\sum_{i=1}^{p} \alpha_i X_i\right) = f(\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_p X_p)$$

$$= \alpha_1 f(X_1) + \alpha_2 f(X_2) + \ldots + \alpha_p f(X_p) = m \qquad \ldots \text{ (i)}$$

where m is the optimum value of f(X) for all X in S.

Since all $\alpha_i \geq 0$, we do not increase [*or decrease*] the sum (i) if we substitute for each $f(X_i)$ the optimum of the values $f(X_i)$. Let $f(X_m) = \underset{i}{optimum}\, f(X_i)$, substituting in (i) we have, since $\sum_i \alpha_i = 1$,

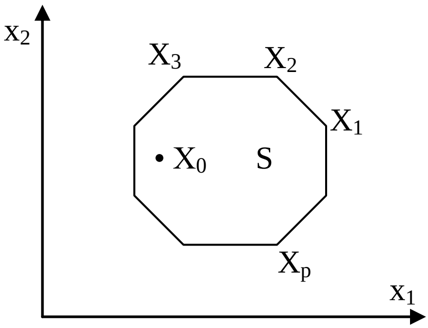

Figure 4.1

$$f(X_0) \geq [or \leq]\ \alpha_1 f(X_m) + \alpha_2 f(X_m) + \ldots + \alpha_p f(X_m) = f(X_m)$$

Since we assumed $f(X_0) \leq$ [*or* $\geq$] f(X) for all X in S, we must have $f(X_0) = f(X_m) = m$. Therefore, there is a vertex, X_m, at which the objective function assumes its optimum value. By Theorem (2.13), we know that each vertex of S is a basic feasible solution to the LP problem. This proves that at least one basic feasible solution must be optimal.
Hence an optimal (minimum or maximum) solution to the LP exists and at least one basic feasible solution must be optimal.

Note: If the optimal value of the objective function attains at more than one extreme points of S. then every convex combination of such extreme points also provides an optimal solution to the LP problem. Thus the optimal solution of an LP problem is either unique or infinite in number.

<u>Theorem (4.4)</u>: (Mini-Max theorem of linear programming) The objective function of a linear programming (LP) problem z is a homogeneous linear function of the variables $\underline{X} = (x_1, x_2, ..., x_n)$ then Min z = – Max (– z) with the same solution set.

Proof: Let $z = \underline{C}.\ \underline{X}$ and z attain its minimum z^* at $\underline{X} = \underline{X}^*$. Then

$$\text{Min } z = z^* = \underline{C}.\underline{X}^* \text{ implies } \underline{C}.\underline{X} \geq \underline{C}.\underline{X}^*$$
$$\Leftrightarrow -\underline{C}.\underline{X} \leq -\underline{C}.\underline{X}^*$$
$$\Leftrightarrow \text{Max}(-\underline{C}.\underline{X}) = -\underline{C}.\underline{X}^*$$
$$\Leftrightarrow -\text{Max}(-\underline{C}.\underline{X}) = \underline{C}.\underline{X}^*$$
$$\Leftrightarrow -\text{Max}(-\underline{C}.\underline{X}) = z^*$$
$$\Leftrightarrow -\text{Max}(-z) = \text{Min } z$$

So, Min z = – Max (– z) with the same solution set.

<u>Example (4.1)</u>: Convert the given minimization type LP problem as maximization type.

$$\begin{aligned} &\text{Minimize } z = 2x_1 + 3x_2 - x_3 \\ &\text{Subject to} \quad 2x_1 + 5x_2 + 2x_3 = 10 \\ &\qquad\qquad\quad\ 3x_1 - 4x_2 + 5x_3 \geq 5 \\ &\qquad\qquad\quad\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: The mini-max theorem shows that the equivalent maximization type problem of the given minimization type problem is as follows:

$$\begin{aligned} &-\text{Maximize } (-z) = -2x_1 - 3x_2 + x_3 \\ &\text{Subject to} \quad 2x_1 + 5x_2 + 2x_3 = 10 \\ &\qquad\qquad\quad\ 3x_1 - 4x_2 + 5x_3 \geq 5 \\ &\qquad\qquad\quad\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

<u>Example (4.2)</u>: Convert the given maximization type LP problem as minimization type.

$$\begin{aligned} &\text{Maximize } z = 3x_1 + 3x_2 - 2x_3 \\ &\text{Subject to} \quad 7x_1 + 5x_2 + 2x_3 = 8 \\ &\qquad\qquad\quad\ 3x_1 - 4x_2 + 2x_3 \geq 2 \\ &\qquad\qquad\quad\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: The mini-max theorem shows that the equivalent minimization type problem of the given maximization type problem is as follows: – Minimize $(-z) = -3x_1 - 3x_2 + 2x_3$

$$\text{Subject to} \quad 7x_1 + 5x_2 + 2x_3 = 8$$
$$3x_1 - 4x_2 + 2x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

4.4 <u>Formation of an initial simplex table</u>: (আদি সিমপে-ক্স তালিকা গঠন) Let us consider the general linear programming problem as follows:

Minimize (or maximize)

$$z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$$

Subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \leq b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \leq b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \leq b_m \end{array}\right\}$$

And $\quad x_j \geq 0; \; j = 1, 2, 3, \ldots, n.$

where the a_{ij}, b_i and c_j are given constants and $m \leq n$. We shall always assume that the constraints equation have been multiplied by (–1) where necessary to make all $b_i \geq 0$.

Introduce slack variables $x_{n+i} \geq 0$, i = 1, 2, ..., m to to left hand sides of each constraint to convert the problem into standard form as well as to have m independent coefficient vectors for making standard basis and also add the slack variables to objective function with coefficients zero. Thus we have a new LP problem as follows: Minimize (or maximize)

$$z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n + 0x_{n+1} + \ldots + 0x_{n+i} + \ldots + 0x_{n+m}$$

Subject to

$$\left.\begin{array}{lr} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n + x_{n+1} & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \quad + x_{n+2} & = b_2 \\ \vdots & \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \qquad + x_{n+i} & = b_i \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \qquad + x_{n+m} & = b_m \end{array}\right\}$$

And $x_j \geq 0$; $j = 1, 2, 3, \ldots, n, n+1, \ldots, n+m$.

First we detach the coefficients of x_j and c_j ($j = 1, 2, \ldots, n$) and then we put them into the following table as follows.

Basis	$\underline{C}_B^t$	$\underline{P}_0$	c_1 c_2 ... c_n 0 ... 0 ... 0 $\underline{P}_1$ $\underline{P}_2$ ... $\underline{P}_n$ $\underline{P}_{n+1}$...$\underline{P}_{n+i}$ ... $\underline{P}_{n+m}$	Min. ratio
$\underline{P}_{n+1}$ $\underline{P}_{n+2}$ $\vdots$ $\underline{P}_{n+i}$ $\vdots$ $\underline{P}_{n+m}$	0 0 $\vdots$ 0 $\vdots$ 0	b_1 b_2 $\vdots$ b_i $\vdots$ b_m	a_{11} a_{12} ... a_{1n} 1 ... 0 ... 0 a_{21} a_{22} ... a_{2n} 0 ... 0 ... 0 $\vdots$ a_{i1} a_{i2} ... a_{in} 0 ... 1 ... 0 $\vdots$ a_{m1} a_{m2} ... a_{mn} 0 ... 0 ... 1	
$z_j - c_j$		$\underline{C}_B.\underline{P}_0$	z_1-c_1 z_2-c_2... z_n-c_n 0 ... 0 ... 0	

The $\underline{P}_j$'s are the column coefficient vectors of x_j ($j = 1, 2, \ldots, n$) into the constraints, $\underline{P}_0$ is the column vector formed by right hand side constants and $\underline{C}_B$ is the row vector formed by the coefficients of basic variables into the objective function. Here, x_{n+i} are basic variables as P_{n+i} ($i = 1, 2, \ldots, m$) are independent vectors and so $\{P_{n+i} : i = 1, 2, \ldots, m\}$ is a basis. And $z_j = \underline{C}_B.P_j$ ($j = 1, 2, \ldots, n$). From the above table, setting non-basic variables $x_1 = x_2 = \ldots = x_n = 0$, we get the initial basic feasible solution $x_{n+i} = b_i$ ($i =1,2, \ldots, m$) with the value of the objective function $\underline{C}_B.\underline{P}_0 = 0$.

Example (4.3): Find the initial simplex table of the following LP problem: Minimize $z = 2x_1 + 2x_2 - 3x_3$

Subject to $2x_1 + x_3 = 10$

$3x_1 + x_2 = 5$

$x_1, x_2, x_3 \geq 0$

Solution: The given LP problem is in the standard form and we have two independent coefficient vectors. So, the initial simplex table is as follows:

Basis	$\underline{C}_B^t$	$c_j \longrightarrow$	2	2	-3	Min. ratio
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	θ
$\underline{P}_3$	-3	10	2	0	1	
$\underline{P}_2$	2	5	3	1	0	
$z_j - c_j$		-20	-2	0	0	

Example (4.4): Find the initial simplex table of the following LP problem: Minimize $z = x_1 + 2x_2 - 5x_3$

Subject to $3x_1 + 2x_2 + 4x_3 \leq 12$

$2x_1 + 3x_2 + 2x_3 \leq 20$

$x_1, x_2, x_3 \geq 0$

Solution: Adding slack variables x_4 and x_5 to the first and second constraints respectively, we get the problem in standard form as follows: Minimize $z = x_1 + 2x_2 - 5x_3 + 0x_4 + 0x_5$

Subject to $3x_1 + 2x_2 + 4x_3 + x_4 + 0x_5 = 12$

$2x_1 + 3x_2 + 2x_3 + 0x_4 + x_5 = 20$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

The standard form contains so many independent coefficient vectors as the number of constraints equations. So, the initial simplex table is as follows:

Basis	$\underline{C}_B^t$	c_j	1	2	-5	0	0	Min. ratio
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_4$	0	12	3	2	4	1	0	
$\underline{P}_5$	0	20	2	3	2	0	1	
$z_j - c_j$		0	-1	-2	5	0	0	

Example (4.5): Find the initial simplex table of the following LP problem: Maximize $z = 3x_1 - 2x_2 + 7x_3$

$$\text{Subject to } 3x_1 - 2x_2 + 4x_3 = 5$$
$$-2x_1 + 3x_2 \le 28$$
$$x_1, x_2, x_3 \ge 0$$

Solution: Adding slack variables $x_4 \ge 0$ to the left hand side of second constraint, we get the problem in standard form as follows.

$$\text{Maximize } z = 3x_1 - 2x_2 + 7x_3 + 0x_4$$
$$\text{Subject to } 3x_1 - 2x_2 + 4x_3 + 0x_4 = 5$$
$$-2x_1 + 3x_2 + 0x_3 + x_4 = 28$$
$$x_1, x_2, x_3, x_4 \ge 0$$

The rank of the constraints system is 2 but the standard form does not contain identity (or basis) matrix of rank 2. To find the required identity matrix, we divide first constraint by 4 and get

$$\text{Maximize } z = 3x_1 - 2x_2 + 7x_3 + 0x_4$$
$$\text{Subject to } \tfrac{3}{4}x_1 - \tfrac{1}{2}x_2 + x_3 + 0x_4 = \tfrac{5}{4}$$
$$-2x_1 + 3x_2 + 0x_3 + x_4 = 28$$
$$x_1, x_2, x_3, x_4 \ge 0$$

The initial table is as follows:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	3	-2	7	0	Min. ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	θ
$\underline{P}_3$	7	5/4	3/4	-1/2	1	0	
$\underline{P}_4$	0	28	-2	3	0	1	
$z_j - c_j$		35/4	9/4	-3/4	0	0	

If we like to solve the problem as minimization type, first we convert it as minimization type problem as follows:

$$-\text{Minimize } -z = -3x_1 + 2x_2 - 7x_3 + 0x_4$$
$$\text{Subject to } \frac{3}{4}x_1 - \frac{1}{2}x_2 + x_3 + 0x_4 = \frac{5}{4}$$
$$-2x_1 + 3x_2 + 0x_3 + x_4 = 28$$
$$x_1, x_2, x_3, x_4 \ge 0$$

Then the initial simplex table is as follows:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-3	2	-7	0	Min. ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	θ
$\underline{P}_3$	-7	5/4	3/4	-1/2	1	0	
$\underline{P}_4$	0	28	-2	3	0	1	
$z_j - c_j$		-35/4	-9/4	3/4	0	0	

4.5 <u>Some definitions</u>: (কিছু গুরুত্বপূর্ণ সংগা) Very important definitions related to the simplex algorithm are discussed below:

4.5.1 <u>Standard basis matrix (or identity matrix)</u>: The identity matrix in the initial simplex table represented by the coefficients of the slack variables that have been added to the original inequalities of constraints to make them equations is called standard basis matrix. Each simplex table must have an identity (or standard basis) matrix under the basic variables. The matrix under the non-basic variables in the simplex table is called the <u>coefficient matrix</u>.

4.5.2 <u>Net evaluation row</u>: The last row containing $z_j - c_j$ of a simplex table is called the net evaluation row, which indicates the solution to be optimum or not.

(a) If $z_k - c_k > 0$ largest positive for minimization problem (or $z_k - c_k < 0$ most negative for maximization problem) but all elements of that column are less than or equal to zero, then the problem has an unbounded optimum solution.

(b) If all $z_j - c_j \leq 0$ for minimization problem or all $z_j - c_j \geq 0$ for maximization problem then the table is optimal. In the optimal table if

- (i) none of $z_j - c_j = 0$ for any non-basic variable then the problem has a unique optimum solution.
- (ii) at least one $z_j - c_j = 0$ for non-basic variable then the problem has many optimum solutions.

4.5.3 <u>Pivot column (or key column)</u>: The column with largest positive $z_j - c_j$ (or most negative $c_j - z_j$) for minimization problem and most negative $z_j - c_j$ (or largest positive $c_j - z_j$) for maximization problem is the pivot column. It indicates that the variable represents the column enters into the next basic solution and this variable is called the <u>entering variable</u>.

4.5.4 <u>Pivot row (or key row)</u>: Let the column having $z_k - c_k$ be the pivot column. Then the row with the minimum ratio of

$$\left\{\frac{b_i}{a_{ik}},\ for\ \ a_{ik} > 0;\ \ i = 1,\ 2, ..., m\right\} = \frac{b_r}{a_{rk}} = \theta_o \text{ (say)}$$

is the pivot row. It indicates that the vector P_{n+r}, which is on the pivot row into the basis, will leave the basis. That is, the variable x_{n+r} will leave the basis in order to make room for the entering variable x_k. The leaving variable is called the <u>departing variable</u>.

4.5.5 <u>Pivot element (or key element)</u>: The element (a_{rk}) at the position of the intersection of the pivot column and pivot row is called the pivot element. In simplex table it plays a vital role. Pivot element is a must to make the corresponding simplex table from the given simplex table. In the simplex method it is always strictly positive and in the dual simplex method it is negative. If there is no positive element to take as pivot element in the key column, then the problem has an unbounded solution. **[HUH-03, 05]**

4.5.6 <u>Unbounded solution</u>: The LP problem han an unbounded solution if the value of the objective function increases upto infinite for maximizing problem (or decreases to negative infinite for minimizing prolem). In the simplex method, if there is no positive quantity to take as pivot element in the key column, then we say that, the problem has an unbounded solution.**[HUH-01, 02]**

4.5.7 <u>Alternative optimum solution</u>: Some LP problems have more than one optimum solution. If an LP problem has many solutions then the solutions are called alternative solution to each others. In the optimum simplex table, if at least one $z_j - c_j = 0$ for non-basic variable then the problem has alternative optimum solutions. **[HUH-01, 02]**

4.6 <u>Development of simplex algorithm for solving an LPP</u>: (যোগাশ্রয়ী প্রোগ্রাম সমাধানের সিমপে- ক্স এ্যালগরিদম গঠন) We assume that the linear programming problem is feasible so that every basic feasible solution is non-degenerate, and we are given a basic feasible solution. These assumptions, as will be discussed later, are made without any loss in generality. Let the given basic feasible solution be $\underline{X}_0 = (x_{10}, x_{20}, ..., x_{mo})$ and associated set of linearly independent vectors be $\underline{P}_1, \underline{P}_2, ..., \underline{P}_m$, we then have

$$x_{10}\underline{P}_1 + x_{20}\underline{P}_2 + ... + x_{mo}\underline{P}_m = \underline{P}_0 \qquad --- \ (1)$$
$$x_{10}c_1 + x_{20}c_2 + ... + x_{mo}c_m = z_0 \qquad --- \ (2)$$

Where all $x_{i0} > 0$, the c_j are the cost coefficients of the objective function, and z_0 is the corresponding value of the objective function for the given solution. Since the set $\underline{P}_1, \underline{P}_2, ..., \underline{P}_m$ is linearly independent and thus forms a basis, we can express any vector from the set $\underline{P}_1, \underline{P}_2, ..., \underline{P}_n$ in terms of $\underline{P}_1, \underline{P}_2, ..., \underline{P}_m$. Let $\underline{P}_j$ be given by

$$x_{1j}\,\underline{P}_1 + x_{2j}\,\underline{P}_2 + ... + x_{mj}\,\underline{P}_m = P_j \ ; \quad j = 1, 2, ..., n \qquad --- \ (3)$$

and we define

$$x_{1j}\,c_1 + x_{2j}\,c_2 + ... + x_{mj}\,c_m = z_j \ ; \quad j = 1, 2, ..., n \qquad --- \ (4)$$

where the c_j are the cost coefficients corresponding to the $\underline{P}_j$.

<u>Theorem (4.5)</u>: If for any fixed j, the condition $z_j - c_j > 0$ holds, then a set of feasible solutions can be constructed such that $z < z_0$ for any member of the set, where the lower bound of z is either finite or infinite (z is the value of the objective function for a particular member of the set of feasible solutions).

Case-1: If the lower bound is finite, a new feasible solution consisting of exactly m positive variables can be constructed whose value of the objective function is less than the value for the preceding solution.

Case-2: If the lower bound is infinite, a new feasible solution consisting of exactly m +1 positive variables can be constructed whose value of the objective function can be made arbitrarily small.

Proof: The following analysis applies to the proof of both cases: Multiplying (3) by some number θ and subtracting from (1), and similarly multiplying (4) by the some number θ and subtracting from (2), for j = 1, 2, ..., n, we get

$$(x_{10} - \theta x_{1j})\underline{P}_1 + (x_{20} - \theta x_{2j})\underline{P}_2 + \dots + (x_{m0} - \theta x_{mj})\underline{P}_m + \theta\underline{P}_j = \underline{P}_0 \quad \dots (5)$$

$$(x_{10}-\theta x_{1j})c_1 + (x_{20}-\theta x_{2j})c_2 + \dots + (x_{m0}-\theta x_{mj})c_m + \theta c_j = z_0-\theta(z_j-c_j) \quad ..(6)$$

where θc_j has been added to both sides of (6). If all the coefficients of the vectors $\underline{p}_1, \underline{p}_2, \dots, \underline{p}_m, \underline{p}_j$ in (5) are non negative, then we have determined a new feasible solution whose value of the objective function is by (6) $z = z_0 - \theta(z_j - c_j)$. Since the variables $x_{10}, x_{20}, \dots, x_{m0}$ in (5) are all positive, it is clear that there is a value of $\theta > 0$ (either finite or infinite) for which the coefficients of the vectors in (5) remain positive. From the assumption that, for a fixed j, $z_j - c_j > 0$, we have

$$z = z_0 - \theta\,(z_j - c_j) < z_0\,, \quad \text{for } \theta > 0.$$

Proof of case-1: If, for the fixed j, at least one $x_{ij} > 0$ in (3) for i = 1, 2, ..., m, the largest value of θ for which all coefficients of (5) remain non negative is given by

$$\theta_0 = \min_i \frac{x_{i0}}{x_{ij}} > 0 \qquad \text{for } x_{ij} > 0 \qquad \text{--- } (7)$$

Since we assumed that the problem is non-degenerate, i.e., all basic feasible solutions have m positive elements, the minimum in (7) will be obtained for a unique i. If θ_0 is substituted for θ in (5) and (6), the coefficient corresponding to this unique i will vanish. We have then constructed a new basic feasible solution consisting of $\underline{p}_j$ and m -1 vectors of the original basis. The new basis can be used as the previous one. If a new $z_j - c_j > 0$ and a corresponding $x_{ij} > 0$, another solution can be obtained which has a smaller value of the objective function. This process will continue either until all $z_j - c_j \le 0$, or until, for some $z_j - c_j > 0$, all $x_{ij} \le 0$. If $z_j - c_j \le 0$, for all j; the process is terminated.

<u>Proof of case-2</u>: If at any stage we have, for some j, $z_j - c_j > 0$ and all $x_{ij} \le 0$, then there is no upper bound to θ and the objective function has a lower bound of $-\infty$. We see for this case that, for any $\theta > 0$, all the coefficients of (5) are positive. We then have a feasible solution consisting of m + 1 positive elements. Hence, by taking θ large enough, the corresponding value of the objective function given by the right hand side of (6) can be made arbitrarily small.

<u>Illustration (4.3)</u>: Find the optimum solution of the following linear programming problem:

$$\begin{aligned} &\text{Minimize } -3x_1 - 2x_2 + 0x_3 + 0x_4 \\ &\text{Subject to } -2x_1 + x_2 + x_3 = 2 \\ &\qquad\qquad\quad x_1 + 2x_2 + x_4 = 8 \\ &\qquad\qquad\quad x_j \ge 0, \quad j = 1, 2, 3, 4 \end{aligned}$$

Solution: Our initial basis consists of vectors $\underline{P}_3$, $\underline{P}_4$ and the corresponding non-degenerate basic feasible solution is $\underline{X}_0 = (x_{30}, x_{40}) = (2, 8)$ with the initial value of the objective function $z_0 = 0$.

Basis	$\underline{C}_B^t$	$c_j \longrightarrow$	-3	-2	0	0	Min. ratio
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	θ
$\underline{P}_3$	0	2	-2	1	1	0	
$\underline{P}_4$	0	8	1 Pivot	2	0	1	$8/1=8=\theta_o$
$z_j - c_j$		0	3 Largest	2	0	0	

$\underline{P}_1$ is selected to go into the basis, because $\max\limits_j (z_j - c_j) = z_1 - c_1 = 3 > 0$. θ is the minimum of x_{i0}/x_{i1} for $x_{i1} > 0$, that is, only positive $x_{21} = 1$ implies that $\theta_0 = 8/1 = 8$ and hence $\underline{P}_4$ is eliminated. We transform the tableau as follows: Consider the second row as new second row because it contains 1 at pivot position. Add double of second row with the first row to obtain 0 at other than pivot position of entering vector P_1 and set it as first row.

Basis	$\underline{C}_B^t$	$c_j \longrightarrow$	-3	-2	0	0	Min. ratio
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	θ
$\underline{P}_3$	0	18	0	5	1	2	
$\underline{P}_1$	-3	8	1	2	0	1	
$z_j - c_j$		-24	0	-4	0	-3	

From the new table, we obtain a new non-degenerate basic feasible solution. $X_0^/ = (x_1, x_3) = (8, 18)$ and the value of the objective function is – 24. Since in the second table $\max\left(z_j - c_j\right) \not> 0$, the last solution $X_0^/ = (x_1, x_3) = (8, 18)$ is the minimum feasible solution with $z_{min} = -24$.

Illustration (4.4): Find the optimum solution of the following linear programming problem: Minimize $-3x_1 + 2x_2$

$$\begin{aligned} \text{Subject to } & -2x_1 + x_2 \le 2 \\ & -x_1 + 2x_2 \le 8 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Solution: Adding slack variables $x_3 \geq 0$ and $x_4 \geq 0$, we rewrite the given problem in standard form.

$$\begin{array}{ll} \text{Minimize} & -3x_1 + 2x_2 + 0x_3 + 0x_4 \\ \text{Subject to} & -2x_1 + x_2 + x_3 \qquad = 2 \\ & -\ x_1 + 2x_2 \qquad + x_4 = 8 \\ & x_j \geq 0, \quad j = 1, 2, 3, 4 \end{array}$$

Our initial basis consists of vectors $\underline{P}_3$, $\underline{P}_4$ and the corresponding non-degenerate basic feasible solution is $\underline{X}_0 = (x_{30}, x_{40}) = (2, 8)$ with the initial value of the objective function $z_0 = 0$.

Basis	$\underline{C}_B^t$	$c_j \longrightarrow$ $\underline{P}_0$	-3 $\underline{P}_1$	2 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min. ratio θ
$\underline{P}_3$	0	2	-2	1	1	0	
$\underline{P}_4$	0	8	-1	2	0	1	
$z_j - c_j$		0	3	-2	0	0	

$\underline{P}_1$ is selected to go into the basis, because $\max\limits_j (z_j - c_j) = z_1 - c_1 = 3 > 0$.

But all $x_{i1} \leq 0$ implies that the problem has an unbounded solution.

4.7 <u>Simplex Algorithm</u>: (সিমপে- ক্স এ্যালগরিদম) The following step by step algorithm to solve a linear programming (LP) problem is known as simplex algorithm. **[NUH-02, 04,07]**

Step-1: If the problem is of maximization type, convert the problem into a minimization problem (though, maximization problem can be solved as a maximization problem and in this case the vector corresponding to the most negative $z_j - c_j$, to be entered into the basis). GOTO Step-2.

Step-2: Convert the LP problem into the standard form. GOTO Step-3.

Step-3: Construct the first simplex table. GOTO Step-4.

Step-4: (Optimality test) If all $z_j - c_j \leq 0$ (or all $z_j - c_j \geq 0$ for maximization problem) then the table is optimal. In the optimal table if

(i) none of $z_j - c_j = 0$ for any non-basic variable then the problem has a unique optimum solution. GOTO Step-8.

(ii) at least one $z_j - c_j = 0$ for non-basic variable then the problem has many optimum solutions and introducing this vector into the basis we shall get an alternative optimum solution. If two optimum solutions exist, then any convex combination of these two optimum solutions is an optimum solution. GOTO Step-8.

If $z_k - c_k > 0$ largest positive (or $z_k - c_k < 0$ most negative for maximization problem) but all elements of that column are less than or equal to zero, then the problem has an unbounded optimum solution. STOP.

Otherwise GOTO Step-5.

Step-5: Find the pivot column with largest positive $z_j - c_j$ (or most negative $z_j - c_j$ for maximization problem). Let the largest positive be $z_k - c_k$. It indicates that the vector $\underline{P}_k$ will enter into the basis and the variable x_k will be a basic variable in the next iteration. If largest positive $z_j - c_j$ is not unique, choose any one arbitrarily but a decision variable will get priority. GOTO Step-6.

Step-6: Let the column having $z_k - c_k$ be the pivot column. Then find the pivot row with the minimum ratio of $\left\{\frac{b_i}{a_{ik}},\ for\ \ a_{ik} > 0;\ \ i = 1,\ 2,...,m\right\} = \frac{b_r}{a_{rk}} = \theta_o$ (say). It indicates that the departing vector, which is on the pivot row into the basis, will leave the basis and the vector $\underline{P}_k$ will take the place. If the minimum ratio is not unique, choose any one arbitrarily but be careful of cycling as it some times makes cycling. GOTO Step-7.

Step-7: Mark the pivot element, which is at the place of the intersection of pivot column and row. Using the complete elimination formula find the next iterative table or find the next iterative table taking the row operations as to make 1 at the pivot position of $\underline{P}_k$ and other components of $\underline{P}_k$ are zero. GOTO Step-4.

Step-8: Find the optimum solution as follows: Set respective components of $\underline{P}_0$ as the value of the respective basic variables and zero for all non-basic variables. And $z_{min} = \underline{C}_B.\underline{P}_0$. STOP.

<u>Example (4.6)</u>: Solve the following LP problem.

Minimize $-3x_1 - 5x_2 - 4x_3$ **[DU-88]**

Subject to $2x_1 + 3x_2 \le 8$

$3x_1 + 2x_2 + 4x_3 \le 15$

$2x_2 + 5x_3 \le 10$

$x_1, x_2, x_3 \ge 0$

Solution: Adding slack variables $x_4, x_5, x_6 \ge 0$ to the left hand side of first, second and third constraints respectively, we get the standard form: Minimize $-3x_1 - 5x_2 - 4x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to $2x_1 + 3x_2 + 0x_3 + x_4 + 0x_5 + 0x_6 = 8$

$3x_1 + 2x_2 + 4x_3 + 0x_4 + x_5 + 0x_6 = 15$

$0x_1 + 2x_2 + 5x_3 + 0x_4 + 0x_5 + x_6 = 10$

$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$

Using coefficients of x_i, we get the following initial simplex table:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-3	-5	-4	0	0	0	Min. ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	θ
$\underline{P}_4$	0	8	2	3	0	1	0	0	$8/3 = \theta_o$
$\underline{P}_5$	0	15	3	2	4	0	1	0	15/2
$\underline{P}_6$	0	10	0	2	5	0	0	1	10/2
$z_j - c_j$		0	3	5 Largest	4	0	0	0	

Since not all $z_j - c_j \le 0$, the above table is not optimal. So, we need next iterative table. To do so, first we find out the largest positive $z_j - c_j$, which is 5 at the pivot column $\underline{P}_2$. And then we find the minimum ratio $\frac{b_i}{a_{i2}}$ for $a_{i2} > 0$, which is 8/3 at the pivot row. So the pivot element is 3, we mark it by a circle. Now we divide the pivot row by the pivot element 3 and put it in the following table.

Subtracting 2 times of the new row from the second and third row, we put in the second and third row respectively in the following table. After then calculating the $z_j - c_j$ row, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-3	-5	-4	0	0	0	Min. ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_2$	-5	8/3	2/3	1	0	1/3	0	0	
$\underline{P}_5$	0	29/3	5/3	0	4	-2/3	1	0	(29/3)/4
$\underline{P}_6$	0	14/3	-4/3	0	5	-2/3	0	1	$(14/3)/5=\theta_o$
$z_j - c_j$		-40/3	-1/3	0	4 Largest	-5/3	0	0	

The above table is not optimal as not all are $z_j - c_j \leq 0$. Apply the above procedure again, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-3	-5	-4	0	0	0	Min. ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_2$	-5	8/3	2/3	1	0	1/3	0	0	(8/3)/(2/3)
$\underline{P}_5$	0	89/15	41/15	0	0	-2/15	1	-4/5	$89/41 = \theta_o$
$\underline{P}_3$	-4	14/15	-4/15	0	1	-2/15	0	1/5	
$z_j - c_j$		-256/15	11/15 Largest	0	0	-17/15	0	-4/5	

The above table is not optimal again, as not all $z_j - c_j \leq 0$. Applying the above procedure again, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-3	-5	-4	0	0	0	Min. ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_2$	-5	50/41	0	1	0	15/41	-10/40	8/81	
$\underline{P}_1$	-3	89/41	1	0	0	-2/41	15/41	-12/41	
$\underline{P}_3$	-4	62/41	0	0	1	-6/41	4/41	5/41	
$z_j - c_j$		-765/41	0	0	0	-45/41	-11/41	-24/41	

This is the optimal table as all $z_j - c_j \leq 0$. Hence the optimal solution is $x_1 = 89/41$, $x_2 = 50/41$, $x_3 = 62/41$ and $z_{min} = -765/41$.

Example (4.7): Find the optimum solution of the following linear programming problem: Minimize $z = -3x_1 - 5x_2$ **[DU-85]**

$$\text{Subject to } 3x_1 + 2x_2 \le 18$$
$$x_1 \le 4$$
$$x_2 \le 6$$
$$x_1, x_2 \ge 0$$

Solution: Introducing slack variables $x_3, x_4, x_5 \ge 0$, we get

$$\text{Minimize } z = -3x_1 - 5x_2$$
$$\begin{aligned} \text{Subject to } 3x_1 + 2x_2 + x_3 \quad &= 18 \\ x_1 \quad + x_4 \quad &= 4 \\ x_2 \quad + x_5 &= 6 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

Since the above standard form contains 3 independent vectors, i.e., identity matrix of rank 3, the initial simplex table is as follows:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-3 $\underline{P}_1$	-5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min. ratio θ	
$\underline{P}_3$	0	18	3	2	1	0	0	18/2 = 9	r_1
$\underline{P}_4$	0	4	1	0	0	1	0		r_2
$\underline{P}_5$	0	6	0	1	0	0	1	$6/1=6=\theta_o$	r_3
$z_j - c_j$		0	3	5 Largest	0	0	0		

The above table is not optimal because its last row contains positive number. The largest positive is 5, which is under the $\underline{P}_2$ column. We find the ratios of the elements of $\underline{P}_0$ with the corresponding positive elements of $\underline{P}_2$. The minimum ratio is 6 and

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-3 $\underline{P}_1$	-5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min. ratio θ	
$\underline{P}_3$	0	6	3	0	1	0	-2	$6/3 = \theta_o$	$r_1' = r_1 - 2r_3'$
$\underline{P}_4$	0	4	1	0	0	1	0	4/1 = 4	$r_2' = r_2$
$\underline{P}_2$	-5	6	0	1	0	0	1		$r_3' = r_3/1$
$z_j - c_j$		-30	3 Largest	0	0	0	-5		

we get it for the third element of $\underline{P}_2$. So the third element 1 is the pivot element. We round it. Now we get 2nd iterative table as follows: Third and second row are unchanged and the new first row is the subtraction of 2 times of third row from the first row. Our 2nd table is not optimal. Similarly we find the third iterative table as follows:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-3 $\underline{P}_1$	-5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min. ratio	
$\underline{P}_1$	-3	2	1	0	1/3	0	-2/3		$r_1^{//} = r_1^{/}/3$
$\underline{P}_4$	0	2	0	0	-1/3	1	2/3		$r_2^{//} = r_2^{/} - r_1^{//}$
$\underline{P}_2$	-5	6	0	1	0	0	1		$r_3^{//} = r_3^{/}$
$z_j - c_j$		-36	0	0	-1	0	-3		

Our third table is optimal because its last row does not contain any positive number. Therefore, the optimal solution is $x_1 = 2$, $x_2 = 6$ and $z_{min} = -36$.

<u>Example (4.8)</u>: Find the optimum solution of the following linear programming problem: **[JU-91, NU-00]**

$$\begin{aligned}
&\text{Minimize} && x_2 - 3x_3 + 2x_5 \\
&\text{Subject to} && x_1 + 3x_2 - x_3 + 2x_5 = 7 \\
& && -2x_2 + 4x_3 + x_4 = 12 \\
& && -4x_2 + 3x_3 + 8x_5 + x_6 = 10 \\
& && x_j \geq 0, \quad j = 1, 2, ..., 6
\end{aligned}$$

Solution: Our initial basis consists of $\underline{P}_1$, $\underline{P}_4$, $\underline{P}_6$ and the corresponding solution is $\underline{X}_0 = (x_1, x_4, x_6) = (7, 12, 10)$

Since $c_1 = c_4 = c_6 = 0$, the corresponding value of the objective function, z_0 equals zero. $\underline{P}_3$ is selected to go into the basis, because

$$\max_j (z_j - c_j) = z_3 - c_3 = 3 > 0$$

θ is the minimum of x_{i0}/x_{i3} for $x_{i3} > 0$, that is, min.(12/4, 10/3) = 12/4 = θ_0 = 3 and hence $\underline{P}_4$ is eliminated. We transform the tableau and obtain a new solution.

$X_0^{/} = (x_1, x_2, x_6) = (10, 3, 1)$

and the value of the objective function is –9. In the second step, since $\max_j (z_j^{/} - c_j) = z_2^{/} - c_2 = \frac{1}{2} > 0$ and $\theta_0 = 10/(5/2) = 4$, $\underline{P}_2$ is introduced into the basis and $\underline{P}_1$ is eliminated. We transform the second-step values of tableau and obtain the solution

$X_0^{//} = (x_2, x_3, x_6) = (4, 5, 11)$

Basis	$\underline{C}_B^t$	$c_j \rightarrow$ / $\underline{P}_o$	0 $\underline{P}_1$	1 $\underline{P}_2$	-3 $\underline{P}_3$	0 $\underline{P}_4$	2 $\underline{P}_5$	0 $\underline{P}_6$	Min Ratio θ
$\underline{P}_1$	0	7	1	3	-1	0	2	0	
$\underline{P}_4$	0	12	0	-2	4 Pivot	1	0	0	$12/4 = \theta_0$
$\underline{P}_6$	0	10	0	-4	3	0	8	1	10/3
$z_j - c_j$		0	0	-1	3 Largest	0	-2	0	
$\underline{P}_1$	0	10	1	5/2 Pivot	0	1/4	2	0	10/(5/2)=4
$\underline{P}_3$	-3	3	0	-1/2	1	1/4	0	0	
$\underline{P}_6$	0	1	0	-5/2	0	-3/4	8	1	
$z_j - c_j$		-9	0	1/2 Largest	0	-3/4	-2	0	
$\underline{P}_2$	1	4	2/5	1	0	1/10	4/5	0	
$\underline{P}_3$	-3	5	1/5	0	1	3/10	2/5	0	
$\underline{P}_6$	0	11	1	0	0	-1/2	10	1	
$z_j - c_j$		-11	-1/5	0	0	-4/5	-12/5	0	

with a value of the objective function equal to –11. Since $\max\left(z_j^{//} - c_j\right) \not> 0$, this solution is the minimum feasible solution.

<u>Example (4.9)</u>: Solve the following linear programming problem by simplex method: Maximize $z = 3x_1 + 5x_2$ **[CU-92]**

Subject to $3x_1 + 2x_2 \leq 18$

$x_1 \leq 4$

$x_2 \leq 6$

$x_1, x_2 \geq 0$

Solution: Introducing slack variables $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$ to first, second and third constraints respectively, we get

$$\begin{aligned} &\text{Maximize } z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5 \\ &\text{Subject to } 3x_1 + 2x_2 + x_3 + 0x_4 + 0x_5 = 18 \\ &\qquad x_1 + 0x_2 + 0x_3 + x_4 + 0x_5 = 4 \\ &\qquad 0x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 6 \\ &\qquad x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Now taking the initial simplex table and then taking necessary iterations, we get the following tables.

Basis	C_B^t	c_j / P_o	3 P_1	5 P_2	0 P_3	0 P_4	0 P_5	Min Ratio θ
P_3	0	18	3	2	1	0	0	18/2=9
P_4	0	4	1	0	0	1	0	
P_5	0	6	0	1 Pivot	0	0	1	$6/1=6=\theta_0$
$z_j - c_j$		0	-3	-5 Smallest	0	0	0	
P_3	0	6	3 Pivot	0	1	0	-2	$6/3=2=\theta_0$
P_4	0	4	1	0	0	1	0	4/1 = 4
P_2	5	6	0	1	0	0	1	
$z_j - c_j$		30	-3 Smallest	0	0	0	5	
P_1	3	2	1	0	1/3	0	-2/3	
P_4	0	2	0	0	-1/3	1	2/3	
P_2	5	6	0	1	0	0	1	
$z_j - c_j$		36	0	0	1	0	3	

Since in the last table the optimality conditions $z_j - c_j \geq 0$ for all j are satisfied, it is the optimal table. The optimal solution is $x_1 = 2$, $x_2 = 6$ and $z_{max} = 36$. In the optimal table, none of $z_j - c_j = 0$ for non-basic variables, so the solution is unique.

Alternative method (converting the problem as minimization type): Introducing slack variables $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$ to first, second and third constraints of the given problem respectively, we get,

$$\text{Maximize } z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5$$
$$\text{Subject to } 3x_1 + 2x_2 + x_3 + 0x_4 + 0x_5 = 18$$
$$x_1 + 0x_2 + 0x_3 + x_4 + 0x_5 = 4$$
$$0x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 6$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Converting the problem as minimization type, we get,

$$-\text{Minimize } (-z) = -3x_1 - 5x_2 + 0x_3 + 0x_4 + 0x_5$$
$$\text{Subject to } 3x_1 + 2x_2 + x_3 + 0x_4 + 0x_5 = 18$$
$$x_1 + 0x_2 + 0x_3 + x_4 + 0x_5 = 4$$
$$0x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 6$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Now taking the initial simplex table and then taking necessary iterations, we get the following tables.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	-5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_3$	0	18	3	2	1	0	0	18/2=9
$\underline{P}_4$	0	4	1	0	0	1	0	
$\underline{P}_5$	0	6	0	1 Pivot	0	0	1	$6/1=6=\theta_0$
$z_j - c_j$		0	3	5 Largest	0	0	0	
$\underline{P}_3$	0	6	3 Pivot	0	1	0	-2	$6/3=2=\theta_0$
$\underline{P}_4$	0	4	1	0	0	1	0	4/1 = 4
$\underline{P}_2$	-5	6	0	1	0	0	1	
$z_j - c_j$		-30	3 Largest	0	0	0	-5	
$\underline{P}_1$	-3	2	1	0	1/3	0	-2/3	
$\underline{P}_4$	0	2	0	0	-1/3	1	2/3	
$\underline{P}_2$	-5	6	0	1	0	0	1	
$z_j - c_j$		-36	0	0	-1	0	-3	

Since in the last table the optimality conditions $z_j - c_j \leq 0$ for all j are satisfied, it is the optimal table. The optimal solution is $x_1 = 2$, $x_2 = 6$ and $z_{max} = -z_{min} = -(-36)$. In the optimal table, none of $z_j - c_j = 0$ for non-basic variables, so the solution is unique.

<u>Example (4.10)</u>: Solve the following linear programming problem by simplex method. Minimize $z = 0x_1 - 2x_2 - x_3$ **[JU-90]**

$$\text{Subject to } x_1 + x_2 - 2x_3 \le 7$$
$$-3x_1 + x_2 + 2x_3 \le 3$$
$$x_1, x_2, x_3 \ge 0$$

Solution: Introducing slack variables $x_4 \ge 0$ and $x_5 \ge 0$, we get

$$\text{Minimize } z = 0x_1 - 2x_2 - x_3 + 0x_4 + 0x_5$$
$$\text{Subject to } x_1 + x_2 - 2x_3 + x_4 + 0x_5 = 7$$
$$-3x_1 + x_2 + 2x_3 + 0x_4 + x_5 = 3$$
$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

Now making the initial simplex table and taking necessary iterations, we get the following tables.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	0 $\underline{P}_1$	-2 $\underline{P}_2$	-1 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_4$	0	7	1	1	-2	1	0	7/1=7
$\underline{P}_5$	0	3	-3	1	2	0	1	3/1=3=θ_0
$z_j - c_j$		0	0	2	1	0	0	
$\underline{P}_4$	0	4	4	0	-4	1	-1	4/4=1=θ_0
$\underline{P}_2$	-2	3	-3	1	2	0	1	
$z_j - c_j$		-6	6	0	-3	0	-2	
$\underline{P}_1$	0	1	1	0	-1	1/4	-1/4	
$\underline{P}_2$	-2	6	0	1	-1	3/4	1/4	
$z_j - c_j$		-12	0	0	3	-3/2	-1/2	

Since only $z_3 - c_3 = 3 > 0$ but all components of the vector $\underline{P}_3$ in the final table are negative, the problem has an unbounded solution.

<u>Example (4.11)</u>: Solve the following linear programming problem by simplex method. Minimize $z = -5x_1 - 2x_2$ [JU-89]

$$\text{Subject to } 6x_1 + 10x_2 \le 30$$
$$10x_1 + 4x_2 \le 20$$
$$x_1, x_2 \ge 0$$

Solution: Introducing slack variables $x_3 \ge 0$ and $x_4 \ge 0$, we get

Minimize $z = -5x_1 - 2x_2 + 0x_3 + 0x_4$

Subject to $6x_1 + 10x_2 + x_3 + 0x_4 = 30$

$10x_1 + 4x_2 + 0x_3 + x_4 = 20$

$x_1, x_2, x_3, x_4 \geq 0$

Making initial simplex table and taking necessary iterations, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-5 $\underline{P}_1$	-2 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ
$\underline{P}_3$	0	30	6	10	1	0	30/6 = 5
$\underline{P}_4$	0	20	10	4	0	1	$20/10=3=\theta_0$
$z_j - c_j$		0	5	2	0	0	
$\underline{P}_3$	0	18	0	38/5	1	-3/5	$45/19=\theta_0$
$\underline{P}_1$	-5	2	1	2/5	0	1/10	10/2=5
$z_j - c_j$		-10	0	0	0	-1/2	

Since all $z_j - c_j \leq 0$, the optimality conditions are satisfied. And the optimal solution is $x_1 = 2$, $x_2 = 0$ and $z_{min} = -10$.

Since $z_2 - c_2 = 0$ but $\underline{P}_2$ is non-basic vector, the problem has many optimum solution. Considering $\underline{P}_2$ as entering vector in the second table, we get the following optimal table.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-5 $\underline{P}_1$	-2 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ
$\underline{P}_2$	-2	45/19	0	1	5/38	-3/38	
$\underline{P}_1$	-5	20/19	1	0	-2/19	5/38	
$z_j - c_j$		-10	0	0	0	-1/2	

This is the optimal table as $z_j - c_j \leq 0$. Another solution is $x_1 = 20/19$, $x_2 = 45/19$ and $z_{min} = -10$.

Let the optimum solution from 2nd table be $\underline{X}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and the optimum solution from 3rd table be $\underline{X}_2 = \begin{pmatrix} 20/19 \\ 45/19 \end{pmatrix}$. So, the linear combination $\underline{X}^* = \lambda\underline{X}_1 + (1 - \lambda)\underline{X}_2$, $0 \le \lambda \le 1$ gives infinite number of optimum solutions.

4.8 <u>The artificial basis technique</u>: (ছদ্ম বেসিস পদ্ধতি) Up to this point, we have always assumed that the given linear-programming (LP) problem was feasible and contained a unit matrix that could be used for the initial basis. Although a correct formulation of a problem will usually guarantee that the problem will be feasible, many problems do not contain a unit matrix. For such problems, the method of the artificial basis is a satisfactory way to start the simplex process. This procedure also determines whether or not the problem has any feasible solutions. There are two methods available to solve such problems:

(i) The "big M-method" or "M-technique" or the "method of penalties" due to A. Charnes.
(ii) The "two phase" method due to Dantzig, Orden and Wolfe.

4.8.1 <u>Artificial variable</u>: (ছদ্ম চলক) When the linear programming (LP) problem is in the standard form but there are not so many linearly independent vectors to form a standard basis, then we have to add extra non-negative variable(s) to the left hand side of the constraint(s) to increase the number of independent vectors up to necessary level. These extra non-negative variables are called artificial variables and the corresponding vectors are called artificial vectors. And also the basis is artificial basis.

An artificil variable plays a vital role to solve the LP problems. Usually, artificial variables are added in the '$\ge$' or '=' type constraints to form a unit sub-matrix of coefficient matrix. In the optimum table, if all artificial variables are not at zero level, then the problem has no solution; i.e., artificial variables are the indicators to have solution or not of an LP problem. **[NUH-01]**

Example (4.12): Find the initial basis of the following linear programming (LP) problem: Minimize $2x_1 + x_2$

Subject to
$$3x_1 + x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \geq 2$$
and $x_j \geq 0;\ j = 1, 2, \ldots, 5$

Solution: Firstly, we subtract surplus variables $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$ form the left hand side of each constraint respectively to convert it into standard form as follows:

$$\begin{aligned}
\text{Minimize}\quad & 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \\
\text{Subject to}\quad & 3x_1 + x_2 - x_3 + 0x_4 + 0x_5 = 3 \\
& 4x_1 + 3x_2 + 0x_3 - x_4 + 0x_5 = 6 \\
& x_1 + 2x_2 + 0x_3 + 0x_4 - x_5 = 2 \\
\text{and}\quad & x_j \geq 0;\ j = 1, 2, \ldots, 5
\end{aligned}$$

The problem is in the standard form but there is no independent vector to form a standard basis, so we shall have to add artificial variables to the left hand sides of each constraint. Adding artificial variables $x_6 \geq 0$, $x_7 \geq 0$, $x_8 \geq 0$ to the left hand side of each constraint respectively and adding Mx_6, Mx_7, Mx_8 to the objective function, we get

$$\begin{aligned}
\text{Minimize}\quad & 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 + Mx_6 + Mx_7 + Mx_8 \\
\text{Subject to}\quad & 3x_1 + x_2 - x_3 + 0x_4 + 0x_5 + x_6 + 0x_7 + 0x_8 = 3 \\
& 4x_1 + 3x_2 + 0x_3 - x_4 + 0x_5 + 0x_6 + x_7 + 0x_8 = 6 \\
& x_1 + 2x_2 + 0x_3 + 0x_4 - x_5 + 0x_6 + 0x_7 + x_8 = 2 \\
\text{and}\quad & x_j \geq 0;\qquad j = 1, 2, \ldots, 8
\end{aligned}$$

Now, we have found three independent vectors $\underline{P}_6 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{P}_7 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\underline{P}_8 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. So the initial artificial basis is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4.8.2 <u>The Big M-method</u>:

The standard general linear programming problem is to

$$\begin{aligned}
\text{Minimize}\quad & z = c_1x_1 + c_2x_2 + \ \dots\ + c_nx_n \\
\text{Subject to}\quad & a_{11}x_1 + a_{12}x_2 + \ \dots\ + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ \dots\ + a_{2n}x_n = b_2 \\
& \vdots \\
& a_{i1}x_1 + a_{i2}x_2 + \ \dots\ + a_{in}x_n = b_i \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \ \dots\ + a_{mn}x_n = b_m \\
\text{and}\quad & x_j \geq 0\ ; \quad j = 1, 2, \dots, n.
\end{aligned}$$

For the method of the artificial basis we augment the above system as follows:

$$\begin{aligned}
\text{Minimize}\ & c_1x_1 + c_2x_2 + \ \dots\ + c_nx_n + Mx_{n+1} + Mx_{n+2} + \ \dots\ + Mx_{n+m} \\
\text{Subject to}\ & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{n+1} = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + x_{n+2} = b_2 \\
& \vdots \\
& a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n + x_{n+i} = b_i \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + x_{n+m} = b_m \\
\text{and}\ & x_j \geq 0\ ; \quad j = 1, 2, \dots, n, n+1, \dots, n+m
\end{aligned}$$

The quantity M is taken to be an unspecified large positive number. The vectors $\underline{P}_{n+1}$, $\underline{P}_{n+2}$, ..., $\underline{P}_{n+m}$ form a basis (an artificial basis) for the augmented system. Therefore, for the augmented problem, the first feasible solution is

$$\underline{X}_0 = (x_{n+1,0}, x_{n+2,0}, \dots, x_{n+m,0}) = (b_1, b_2, \dots, b_m) \geq \underline{0}$$

$$\therefore \quad x_{n+1,0}\, \underline{P}_{n+1} + x_{n+2,0}\, \underline{P}_{n+2} + \ \dots\ + x_{n+m,0}\, \underline{P}_{n+m} = \underline{P}_0 \quad \text{---}\ (1)$$

$$Mx_{n+1,0} + Mx_{n+2,0} + \dots + Mx_{n+m,0} = z_0 \quad \text{---}\ (2)$$

And also,

$$x_{1j}\, \underline{P}_{n+1} + x_{2j}\, \underline{P}_{n+2} + \dots + x_{mj}\, \underline{P}_{n+m} = \underline{P}_j \quad \text{---}\ (3)$$

$$Mx_{1j} + Mx_{2j} + ... + Mx_{mj} = z_j \qquad --- \quad (4)$$

Multiplying (3) by θ and then subtracting from (1) we have,

$$(x_{n+1,0}–\theta x_{1j})\underline{P}_{n+1}+(x_{n+2,0}–\theta x_{2j})\underline{P}_{n+2}+...+(x_{n+m,0}–\theta x_{mj})\underline{P}_{n+m}+\theta\underline{P}_j = \underline{P}_0 \qquad --- \quad (5)$$

Multiplying (4) by θ and then subtracting from (2) and after then adding θc_j on both sides, we have

$$(x_{n+1,0} - \theta\, x_{1j})M + (x_{n+2,0} - \theta\, x_{2j})M + ... + (x_{n+m,0} - \theta\, x_{mj})M + \theta c_j = z_0 - \theta\,(z_j - c_j) \qquad --- \quad (6)$$

where, $z_j = M\sum_{i=1}^{m} x_{ij}$. So, $(z_j - c_j) = M\sum_{i=1}^{m} x_{ij} - c_j$

For the first solution each $z_j - c_j$ will then have a M coefficient which are independent of each other. We next set up the associated computational procedure as the given table. For each j, the row free from M component and the row with M component of $z_j - c_j$ have been placed in the (m +1)st and (m +2)nd rows, respectively of that column.

We treat this table exactly like the original simplex table except that the vector introduced into the basis is associated with the largest positive element in the (m+2)nd row. For the first iteration, the vector corresponding to $\max_j \sum_{i=1}^{m} x_{ij}$ is introduced into the basis.

We continue to select a vector to be introduced into the basis, using the element in the (m + 2)nd row as criterion, until either (a) all artificial vectors are eliminated from the basis or (b) no positive (m + 2)nd element exists. The first alternative implies that all the elements in the (m+2)nd row equal to zero and that the corresponding basis is a feasible basis for the original problem.

Sl.	Basis	$\underline{C}_B^t$	$\underline{P}_0$	c_1	c_2	... c_n	M ... M ... M
				$\underline{P}_1$	$\underline{P}_2$	... $\underline{P}_n$	$\underline{P}_{n+1}\ldots\underline{P}_{n+i}\ldots\underline{P}_{n+m}$
1	$\underline{P}_{n+1}$	M	b_1	a_{11}	a_{12}	... a_{1n}	1 ... 0 ... 0
2	$\underline{P}_{n+2}$	M	b_2	a_{21}	a_{22}	... a_{2n}	0 ... 0 ... 0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
i	$\underline{P}_{n+i}$	M	b_i	a_{i1}	a_{i2}	... a_{in}	0 ... 1 ... 0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
m	$\underline{P}_{n+m}$	M	b_m	a_{m1}	a_{m2}	... a_{mn}	0 ... 0 ... 1
m+1	$z_j - c_j$		0	$-c_1$	$-c_2$	... $-c_m$	0 ... 0 ... 0
m+2			$\sum b_i$	$\sum a_{i1}$	$\sum a_{i2}$	... $\sum a_{in}$	0 ... 0 ... 0

We than apply the regular simplex algorithm to determine the minimum feasible solution. In the second alternative, if no positive element exists in (m+2)nd row but at least one artificial vector is in the basis till, i.e, the artificial part of the corresponding value of the objective is greater than zero, the original problem is not feasible.

Example (4.13): Solve the following LP problem using big M-method. Minimize $2x_1 + x_2$ **[JU-97]**

Subject to
$$3x_1 + x_2 - x_3 = 3$$
$$4x_1 + 3x_2 - x_4 = 6$$
$$x_1 + 2x_2 - x_5 = 2$$
and $x_j \geq 0;\ \ j = 1, 2, \ldots, 5$

Solution: Since the LP problem does not contain the initial basis (3 independent coefficient vectors because it contains 3 constraint equations) we need an artificial basis. For finding an artificial basis we add artificial variables x_6, x_7, x_8 to 1st, 2nd, 3rd constraints respectively and add the artificial variables with coefficients big M to the objective function. Then the problem becomes as follows:

Minimize $2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 + Mx_6 + Mx_7 + Mx_8$

Subject to
$$3x_1 + x_2 - x_3 + x_6 = 3$$
$$4x_1 + 3x_2 - x_4 + x_7 = 6$$
$$x_1 + 2x_2 - x_5 + x_8 = 2$$

and $\quad x_j \geq 0; \quad j = 1, 2, \ldots, 8$

Using the above problem we find the following initial tableau.

Sl.	Basis	C_B^t	P_o	2	1	0	0	0	M	M	M	Ratio
				P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	θ
1	P_6	M	3	3	1	-1	0	0	1	0	0	$3/3=\theta_o$
2	P_7	M	6	4	3	0	-1	0	0	1	0	6/4
3	P_8	M	2	1	2	0	0	-1	0	0	1	2/1
3+1	$z_j - c_j$		0	-2	-1	0	0	0	0	0	0	
3+2			11	8 Greatest	6	-1	-1	-1	0	0	0	Coef. of M

Since not all $z_j - c_j \leq 0$ in the 3+2nd row, the table is not optimal. We find the pivot taking the greatest element of 3+2nd row as base and then find the following iterative table as in simplex method.

Sl.	Basis	C_B^t	P_o	2	1	0	0	0	M	M	M	Ratio
				P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	θ
1	P_1	2	1	1	1/3	-1/3	0	0		0	0	1/(1/3)
2	P_7	M	2	0	5/3	4/3	-1	0		1	0	2/(5/3)
3	P_8	M	1	0	5/3	1/3	0	-1		0	1	$1/(5/3)= \theta_o$
3+1	$z_j - c_j$		2	0	-1/3	-2/3	0	0		0	0	
3+2			3	0	10/3 Greatest	5/3	-1	-1		0	0	

Since not all $z_j - c_j \leq 0$ in the 3+2nd row, the table is not optimal. Taking as iterations as before, we get

Sl.	Basis	C_B^t	P_o	2	1	0	0	0	M	M	M	Ratio
				P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	θ
1	P_1	2	4/5	1	0	-2/5	0	1/5		0		
2	P_7	M	1	0	0	1	-1	1		1		$1/1 = \theta_o$
3	P_2	1	3/5	0	1	1/5	0	-3/5		0		(3/5)/(1/5)
3+1	$z_j - c_j$		11/5	0	0	-3/5	0	-1/5		0		
3+2			1	0	0	1 Greatest	-1	1		0		

Since not all $z_j - c_j \leq 0$ in the 3+2nd row, the table is not optimal. Taking as iterations as before, we get

Sl.	Basis	$\underline{C}_B^t$	$\underline{P}_o$	2	1	0	0	0	M	M	M	Ratio
				$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	θ
1	$\underline{P}_1$	2	6/5	1	0	0	-2/5	3/5				(6/5)/(3/5)
2	$\underline{P}_3$	0	1	0	0	1	-1	1				$1/1 = \theta_o$
3	$\underline{P}_2$	1	2/5	0	1	0	1/5	-4/5				
3+1	$z_j - c_j$		14/5	0	0	0	-3/5	2/5 Greatest				
3+2			0	0	0	0	0	0				

Though all $z_j - c_j = 0$ in the 3+2nd row but not all $z_j - c_j \leq 0$ in the 3+1st row, the table is not optimal. Now we find the pivot taking the greatest element of 3+1st row as base and then find the following iterative table.

Sl.	Basis	$\underline{C}_B^t$	$\underline{P}_o$	2	1	0	0	0	M	M	M	Ratio
				$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	θ
1	$\underline{P}_1$	2	3/5	1	0	-3/5	1/5	0				
2	$\underline{P}_5$	0	1	0	0	1	-1	1				
3	$\underline{P}_2$	1	6/5	0	1	4/5	-3/5	0				
3+1	$z_j - c_j$		12/5	0	0	-2/5	-1/5	0				

Since all $z_j - c_j \leq 0$ the table is optimal. The above tableau gives us the extreme point (3/5, 6/5, 0, 0, 1). So, the solution of the problem is $x_1 = 3/5$, $x_2 = 6/5$, $x_3 = 0$, $x_4 = 0$, $x_5 = 1$ and the minimum value of the objective function is 12/5.

Example (4.14): Solve the following LP problem.

$$\begin{aligned} \text{Maximize } z &= 2x_1 + x_2 + 3x_3 \\ \text{Subject to } \quad & x_1 + x_2 + 2x_3 \leq 5 \\ & 2x_1 + 3x_2 + 4x_3 = 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: Introducing slack variable $x_4 \geq 0$ to 1st constraint, we get the standard form as follows:

Maximize $z = 2x_1 + x_2 + 3x_3 + 0x_4$

Subject to $x_1 + x_2 + 2x_3 + x_4 = 5$

$2x_1 + 3x_2 + 4x_3 + 0x_4 = 12$

$x_1, x_2, x_3, x_4 \geq 0$

Since this standard form does not contain the basis, we introduce artificial variable $x_5 \geq 0$ to 2nd constraint to find the basis and add this variable with the objective function with the coefficient –M to apply the big M-method. Then we get

Maximize $z = 2x_1 + x_2 + 3x_3 + 0x_4 - Mx_5$

Subject to $x_1 + x_2 + 2x_3 + x_4 + 0x_5 = 5$

$2x_1 + 3x_2 + 4x_3 + 0x_4 + x_5 = 12$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

We find the following initial simplex table from the problem:

Sl.	Basis	$\underline{C}_B^t$	c_j	2	1	3	0	-M	Ratio
			$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
1	$\underline{P}_4$	0	5	1	1	2	1	0	$5/2 = \theta_o$
2	$\underline{P}_5$	-M	12	2	3	4	0	1	$12/4 = 3$
2+1	$z_j - c_j$		0	-2	-1	-3	0	0	
2+2			-12	-2	-3	-4 Smallest	0	0	Coef. of M

Since not all $z_j - c_j \geq 0$ in the 2+2nd row, the table is not optimal. We find the pivot taking the smallest element of 2+2nd row as base and then find the second iterative table as in simplex method. In the second iterative table not all $z_j - c_j \geq 0$ in the 2+2nd row, the table is not optimal. Taking the same operations as before, we get the third iterative table.

Sl.	Basis	$\underline{C}_B^t$	c_j	2	1	3	0	-M	Ratio
			$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
1	$\underline{P}_3$	3	5/2	1/2	1/2	1	1/2	0	5
2	$\underline{P}_5$	-M	2	0	1	0	-2	1	$2/1=2 = \theta_o$
2+1	$z_j - c_j$		15/2	-1/2	1/2	0	3/2	0	
2+2			-2	0	-1 Smallest	0	2	0	Coef. of M

Sl.	Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2 $\underline{P}_1$	1 $\underline{P}_2$	3 $\underline{P}_3$	0 $\underline{P}_4$	-M $\underline{P}_5$	Ratio θ
1	$\underline{P}_3$	3	3/2	1/2	0	1	3/2	-	$3 = \theta_o$
2	$\underline{P}_2$	1	2	0	1	0	-2	-	
2+1	$z_j - c_j$		13/2	-1/2 Smallest	0	0	5/2	-	
2+2			0	0	0	0	0	-	Coef. of M

Though all $z_j - c_j = 0$ in the 2+2nd row but not all $z_j - c_j \geq 0$ in the 2+1st row, the table is not optimal. Now we find the pivot taking the smallest element of 2+1st row as base and then find the following iterative table.

Sl.	Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2 $\underline{P}_1$	1 $\underline{P}_2$	3 $\underline{P}_3$	0 $\underline{P}_4$	-M $\underline{P}_5$	Ratio θ
1	$\underline{P}_1$	2	3	1	0	2	3	-	
2	$\underline{P}_2$	1	2	0	1	0	-2	-	
2+1	$z_j - c_j$		8	0	0	1	4	-	

Since all $z_j - c_j \geq 0$ the table is optimal. The above tableau gives us the extreme point (3, 2, 0). So, the solution of the problem is $x_1=3$, $x_2 =2$, $x_3 =0$ and the maximum value of the objective function is 8.

Alternative method (converting into minimization type problem): Introducing slack variable $x_4 \geq 0$ to 1st constraint of the given problem and converting it as minimization problem, we get the standard form as follows:

$$\begin{aligned}
&-\text{Minimize } -z = -2x_1 - x_2 - 3x_3 + 0x_4 \\
&\text{Subject to} \quad x_1 + x_2 + 2x_3 + x_4 = 5 \\
&\qquad 2x_1 + 3x_2 + 4x_3 + 0x_4 = 12 \\
&\qquad x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

Since this standard form does not contain the basis, we introduce artificial variable $x_5 \geq 0$ to 2nd constraint to find the basis and add this variable with the objective function with the coefficient M to apply the big M-method. Then we get

$$-\text{Maximize } -z = -2x_1 - x_2 - 3x_3 + 0x_4 + Mx_5$$
$$\text{Subject to} \quad x_1 + x_2 + 2x_3 + x_4 + 0x_5 = 5$$
$$2x_1 + 3x_2 + 4x_3 + 0x_4 + x_5 = 12$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

We find the following initial simplex table from the problem:

Sl.	Basis	C_B^t	c_j	-2	-1	-3	0	M	Ratio
			$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
1	$\underline{P}_4$	0	5	1	1	2	1	0	$5/2 = \theta_o$
2	$\underline{P}_5$	M	12	2	3	4	0	1	$12/4 = 3$
2+1	$z_j - c_j$		0	2	1	3	0	0	
2+2			12	2	3	4 Largest	0	0	Coef. of M

Since not all $z_j - c_j \leq 0$ in the 2+2nd row, the table is not optimal. We find the pivot taking the largest element of 2+2nd row as base and then find the following iterative table as in simplex method.

Sl.	Basis	C_B^t	c_j	-2	-1	-3	0	M	Ratio
			$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
1	$\underline{P}_3$	-3	5/2	1/2	1/2	1	1/2	0	5
2	$\underline{P}_5$	M	2	0	1	0	-2	1	$2/1=2 = \theta_o$
2+1	$z_j - c_j$		-15/2	1/2	-1/2	0	-3/2	0	
2+2			2	0	1 Largest	0	-2	0	Coef. of M

Sl.	Basis	C_B^t	c_j	-2	-1	-3	0	M	Ratio
			$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
1	$\underline{P}_3$	-3	3/2	1/2	0	1	3/2	-	$3 = \theta_o$
2	$\underline{P}_2$	-1	2	0	1	0	-2	-	
2+1	$z_j - c_j$		-13/2	1/2 Largest	0	0	-5/2	-	
2+2			0	0	0	0	0	-	Coef. of M

In the second iterative table not all $z_j - c_j \leq 0$ in the 2+2nd row, the table is not optimal. Taking the same operations as before, we get the third iterative table.

In the third iterative table, though all $z_j - c_j = 0$ in the 2+2nd row but not all $z_j - c_j \leq 0$ in the 2+1st row, the table is not optimal. Now we find the pivot taking the largest element of 2+1st row as base and then find the following iterative table.

Sl.	Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-2 $\underline{P}_1$	-1 $\underline{P}_2$	-3 $\underline{P}_3$	0 $\underline{P}_4$	M $\underline{P}_5$	Ratio θ
1	$\underline{P}_1$	-2	3	1	0	2	3	-	
2	$\underline{P}_2$	-1	2	0	1	0	-2	-	
2+1	$z_j - c_j$		-8	0	0	-1	-4	-	

Since all $z_j - c_j \leq 0$ the table is optimal. The above tableau gives us the extreme point (3, 2, 0). So, the solution of the problem is $x_1=3$, $x_2 =2$, $x_3 =0$ and the maximum value of the objective function is 8.

<u>Example (4.15)</u>: Solve the LP problem by simplex method

$$\begin{aligned}
&\text{Minimize } z = -2x_1 + x_2 - 5x_3 \\
&\text{Subject to} \quad x_1 + 2x_2 + 2x_3 \leq 2 \\
&\qquad 5x_1 + 6x_2 + 8x_3 = 24 \\
&\qquad -2x_1 + 3x_2 + 2x_3 \geq 24 \\
&\qquad x_1, x_2, x_3 \geq 0
\end{aligned}$$

Solution: Introducing slack variable $x_4 \geq 0$ to 1st constraint, surplus variable $x_5 \geq 0$ to 3rd constraint, artificial variables x_6, $x_7 \geq 0$ to 2nd and 3rd constraints respectively, we get the following LP problem.

$$\begin{aligned}
&\text{Minimize } z = -2x_1 + x_2 - 5x_3 + 0x_4 + 0x_5 + Mx_6 + Mx_7 \\
&\text{Subject to} \quad x_1 + 2x_2 + 2x_3 + x_4 = 2 \\
&\qquad 5x_1 + 6x_2 + 8x_3 + x_6 = 24 \\
&\qquad -2x_1 + 3x_2 + 2x_3 - x_5 + x_7 = 24 \\
&\qquad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
\end{aligned}$$

Taking the initial simplex table and doing necessary iterations we get the following tables.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	$\underline{P}_1$ (-2)	$\underline{P}_2$ (1)	$\underline{P}_3$ (-5)	$\underline{P}_4$ (0)	$\underline{P}_5$ (0)	$\underline{P}_6$ (M)	$\underline{P}_7$ (M)	Min Ratio θ
$\underline{P}_4$	0	2	1	2	2	1	0	0	0	$1 = \theta_0$
$\underline{P}_6$	M	24	5	6	8	0	0	1	0	3
$\underline{P}_7$	M	24	-2	3	2	0	-1	0	1	12
$z_j - c_j$		0	2	-1	5	0	0	0	0	
		48	3	9	10 Greatest	0	-1	0	0	Coef. of M
$\underline{P}_3$	-5	1	1/2	1	1	1/2	0	0	0	
$\underline{P}_6$	M	16	1	-2	0	-4	0	1	0	
$\underline{P}_7$	M	22	-3	1	0	-1	-1	0	1	
$z_j - c_j$		-5	-1/2	-6	0	-5/2	0	0	0	
		38	-2	-1	0	-5	-1	0	0	Coef. of M

In the second table, all $z_j - c_j \leq 0$ in the last row but artificial variables are present in the basis at positive level. Hence the problem has no feasible solution.

Example (4.16): Using the big M-method solves the following LP problem. Maximize $z = 4x_1 + 5x_2 + 2x_3$ **[NU-02]**

$$\begin{aligned} \text{Subject to}\quad & -6x_1 + x_2 - x_3 \leq 5 \\ & -2x_1 + 2x_2 - 3x_3 \geq 3 \\ & 2x_2 - 4x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: First we convert the objective function into minimization type. Introducing slack variable $x_4 \geq 0$ to 1st constraint, surplus variable $x_5 \geq 0$ to 2nd constraint, artificial variables $x_6 \geq 0$, $x_7 \geq 0$ to 2nd and 3rd constraints respectively and adding artificial variables with coefficients M to the objective function, we get

$$-\text{Minimize } -z = -4x_1 - 5x_2 - 2x_3 + 0x_4 + 0x_5 + Mx_6 + Mx_7$$

$$\begin{aligned} \text{Subject to}\quad & -6x_1 + x_2 - x_3 + x_4 = 5 \\ & -2x_1 + 2x_2 - 3x_3 - x_5 + x_6 = 3 \\ & 2x_2 - 4x_3 + x_7 = 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

Making the initial simplex table and taking necessary action, we get the following iterative tables.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-4	-5	-2	0	0	M	M	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	
$\underline{P}_4$	0	5	-6	1	-1	1	0	0	0	5/1=5
$\underline{P}_6$	M	3	-2	2	-3	0	-1	1	0	3/2
$\underline{P}_7$	M	1	0	2	-4	0	0	0	1	1/2= θ_0
$z_j - c_j$		0	4	5	2	0	0	0	0	
		4	-2	4	-7	0	-1	0	0	Coef. of M
				Greatest						
$\underline{P}_4$	0	9/2	-6	0	1	1	0	0	-1/2	9/2
$\underline{P}_6$	M	2	-2	0	1	0	-1	1	-1	2 = θ_0
$\underline{P}_2$	-5	1/2	0	1	-2	0	0	0	1/2	
$z_j - c_j$		-5/2	4	0	12	0	0	0	-5/2	
		2	-2	0	1	0	-1	0	-2	Coef. of M
					Greatest					
$\underline{P}_4$	0	5/2	-4	0	0	1	1	-1	1/2	
$\underline{P}_3$	-2	2	-2	0	1	0	-1	1	-1	
$\underline{P}_2$	-5	9/2	-4	1	0	0	-2	2	-3/2	
$z_j - c_j$		-53/2	28	0	0	0	12	-12	6	Free of M
			Greatest							
		0	0	0	0	0	0	-1	-1	Coef. of M

In the last table, all coefficients of M of $z_j - c_j$ are less or equal to zero and all parts free of M of $z_j - c_j$ contains positive number. Here, $z_1 - c_1 = 28$ is the greatest positive number but the vector $\underline{P}_1$ does not contain any positive number. Hence the problem has an unbounded solution.

4.8.3 <u>The two-phase method</u>: (দুই ধাপ পদ্ধতি) According to G.B. Dantzig, A. Orden and P. Wolf, the two phase method is another one to solve those linear programming (LP) problems whose need one or more artificial variables to get an initial basis matrix. The two-phase method avoids the involvement of arbitrary large constant M. This method has two parts. One is phase-I and the other is phase-II.

Phase-I: We consider a new objective function w $= \sum_i a_i$ with artificial variables a_i only. In mathematical point of view all artificial variables must be at zero level at optimum stage. Hence the minimum value of w should be equal to zero. To get the minimum value of w, we consider an auxiliary linear programming problem:

Minimize w $= \sum_i a_i$, subject to the constraints of the original problem. In this phase, we solve this auxiliary linear programming problem by simplex method. At the end of phase-I, the following three cases may arise:

<u>Case-1</u>: If the minimum value of $w \geq 0$, and at least one artificial variable appears in the basis at a positive level, then the given problem has no feasible solution and the procedure terminates.

<u>Case-2</u>: If the minimum value of w = 0, and no artificial variable appears in the basis, then a basic feasible solution to the given problem is obtained. GOTO phase-II.

<u>Case-3</u>: If the minimum value of w = 0 and one or more artificial variables appear in the basis at zero level, then a feasible solution to the original problem is obtained. GOTO phase-II, but take care of these artificial variables so that they never become positive during phase-II computations. If necessary, select one artificial variable as out going variable neglecting minimum ratio criterion for selection of outgoing variable.

Phase-II: Removing all the columns corresponding to the artificial variables from the final table of phase-I and taking the original objective function in lieu of artificial objective function, we construct a new simplex table. The simplex method is then applied to arrive at the optimum solution.

Remarks: 1. In phase-I, the iterations are stopped as soon as the value of the new (artificial) objective function becomes zero

because this is its minimum value. There is no need to continue till the optimality is reached if this value becomes zero earlier than that.
2. The new (artificial) objective function is always of minimization type regardless of whether the original problem is of maximization or minimization type.

<u>Example (4.17)</u>: Solve the following linear programming problem by two-phase method. Minimize $z = \frac{15}{2}x_1 - 3x_2$ **[DU-84]**

$$\text{Subject to } 3x_1 - x_2 - x_3 \geq 3$$
$$x_1 - x_2 + x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

Solution: Introducing surplus variables $x_4 \geq 0$ and $x_5 \geq 0$ to 1^{st} and 2^{nd} constraints respectively, we get the following standard form:

$$\text{Minimize } z = \frac{15}{2}x_1 - 3x_2 + 0x_3 + 0x_4 + 0x_5$$
$$\text{Subject to } 3x_1 - x_2 - x_3 - x_4 + 0x_5 = 3$$
$$x_1 - x_2 + x_3 + 0x_4 - x_5 = 2$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The above standard form does not contain the basis matrix, hence we need to introduce artificial variables $x_6 \geq 0$ and $x_7 \geq 0$ to first and second constraints respectively.

$$\text{Minimize } z = \frac{15}{2}x_1 - 3x_2 + 0x_3 + 0x_4 + 0x_5$$
$$\text{Subject to } 3x_1 - x_2 - x_3 - x_4 + 0x_5 + x_6 + 0x_7 = 3$$
$$x_1 - x_2 + x_3 + 0x_4 - x_5 + 0x_6 + x_7 = 2$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

Phase-I: We consider the auxiliary linear programming problem as follows: Minimize $w = x_6 + x_7$

$$\text{Subject to } 3x_1 - x_2 - x_3 - x_4 + 0x_5 + x_6 + 0x_7 = 3$$
$$x_1 - x_2 + x_3 + 0x_4 - x_5 + 0x_6 + x_7 = 2$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

The simplex table of the auxiliary problem is as follows:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	0	0	0	0	0	1	1	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	
$\underline{P}_6$	1	3	3	-1	-1	-1	0	1	0	$3/3=1=\theta_0$
$\underline{P}_7$	1	2	1	-1	1	0	-1	0	1	2/1 = 2
$w_j - c_j$		5	4	-2	0	-1	-1	0	0	
$\underline{P}_1$	0	1	1	-1/3	-1/3	-1/3	0		0	
$\underline{P}_7$	1	1	0	-2/3	4/3	1/3	-1		1	$1/(4/3)= \theta_0$
$w_j - c_j$		1	0	-2/3	4/3	1/3	-1		0	
$\underline{P}_1$	0	5/4	1	-1/2	0	-1/4	-1/4			
$\underline{P}_3$	0	3/4	0	-1/2	1	1/4	-3/4			
$w_j - c_j$		0	0	0	0	0	0			

Since all $w_j - c_j \leq 0$ and minimum of w = 0 and all artificial variables leave the basis. So the given problem has a basic feasible solution.

Phase-II: Removing all the columns corresponding to the artificial variables from the final table of phase-I and taking the original objective function in lieu of artificial objective function, we construct the following simplex table.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	15/2	-3	0	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	
$\underline{P}_1$	15/2	5/4	1	-1/2	0	-1/4	-1/4	
$\underline{P}_3$	0	3/4	0	-1/2	1	1/4	-3/4	
$z_j - c_j$		75/8	0	-3/4	0	-15/8	-15/8	

Since all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimal basic feasible solution is $x_1 = 5/4$, $x_2 = 0$, $x_3 = 3/4$ and $z_{min} = 75/8$.

<u>Example (4.18)</u>: Solve the following linear programming problem by two-phase method.

$$\text{Minimize } z = x_1 + x_2$$
$$\text{Subject to } 2x_1 + x_2 \geq 4$$

$$x_1 + 7x_2 \geq 7$$
$$x_1, x_2 \geq 0$$

Solution: Introducing surplus variables $x_3 \geq 0$ and $x_4 \geq 0$ to 1st and 2nd constraints respectively, we get the following standard form:

Minimize $z = x_1 + x_2 + 0x_3 + 0x_4$

Subject to $2x_1 + x_2 - x_3 + 0x_4 = 4$

$x_1 + 7x_2 + 0x_3 - x_4 = 7$

$x_1, x_2, x_3, x_4 \geq 0$

The above standard form does not contain the basis matrix, hence we need to introduce artificial variables $x_5 \geq 0$ and $x_6 \geq 0$ to first and second constraints respectively.

Minimize $z = x_1 + x_2 + 0x_3 + 0x_4$

Subject to $2x_1 + x_2 - x_3 + 0x_4 + x_5 + 0x_6 = 4$

$x_1 + 7x_2 + 0x_3 - x_4 + 0x_5 + x_6 = 7$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Phase-I: We consider the auxiliary linear programming problem as follows: Minimize $w = x_5 + x_6$

Subject to $2x_1 + x_2 - x_3 + 0x_4 + x_5 + 0x_6 = 4$

$x_1 + 7x_2 + 0x_3 - x_4 + 0x_5 + x_6 = 7$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

The simplex table of the auxiliary problem is as follows:

Basis	$\underline{C}_B^t$	c_j	0	0	0	0	1	1	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	θ
$\underline{P}_5$	1	4	2	1	-1	0	1	0	3/1=4
$\underline{P}_6$	1	7	1	7	0	-1	0	1	$7/7=1=\theta_0$
$w_j - c_j$		11	3	8	-1	-1	0	0	
$\underline{P}_5$	1	3	13/7	0	-1	1/7	1	-1/7	$21/13=\theta_0$
$\underline{P}_2$	0	1	1/7	1	0	-1/7	0	1/7	7
$w_j - c_j$		3	13/7	0	-1	1/7	0	-8/7	
$\underline{P}_1$	0	21/13	1	0	-7/13	1/13	7/13	-1/13	
$\underline{P}_2$	0	10/13	0	1	1/13	-2/13	-1/13	2/13	
$w_j - c_j$		0	0	0	0	0	-1	-1	

Since all $w_j - c_j \leq 0$ and minimum of $w = 0$ and all artificial variables leave the basis. So the given problem has a basic feasible solution.

Phase-II: Removing all the columns corresponding to the artificial variables from the final table of phase-I and taking the original objective function in lieu of artificial objective function, we construct the following simplex table.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	1	1	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	
$\underline{P}_1$	1	21/13	1	0	-7/13	1/13	
$\underline{P}_2$	1	10/13	0	1	1/13	-2/13	
$z_j - c_j$		31/13	0	0	-6/13	-1/13	

Since all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimal basic feasible solution is $x_1 = 21/13$, $x_2 = 10/13$ and $z_{min} = 31/13$.

<u>Example (4.19)</u>: Solve the following linear programming problem by two-phase method. Maximize $z = 5x_1 + 8x_2$

$$\begin{aligned} \text{Subject to } 3x_1 + 2x_2 &\geq 3 \\ x_1 + 4x_2 &\geq 4 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution: Introducing surplus variables $x_3 \geq 0$, $x_4 \geq 0$ and slack variables $x_5 \geq 0$ to first, second and third constraints respectively, we get the following standard form:

$$\begin{aligned} &\text{Maximize } z = 5x_1 + 8x_2 + 0x_3 + 0x_4 + 0x_5 \\ &\text{Subject to } 3x_1 + 2x_2 - x_3 + 0x_4 + 0x_5 = 3 \\ &\qquad x_1 + 4x_2 + 0x_3 - x_4 + 0x_5 = 4 \\ &\qquad x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 5 \\ &\qquad x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

The above standard form does not contain the basis matrix, hence we need to introduce artificial variables $x_6 \geq 0$ and $x_7 \geq 0$ to first and second constraints respectively.

Maximize $z = 5x_1 + 8x_2 + 0x_3 + 0x_4 + 0x_5$

Subject to $3x_1 + 2x_2 - x_3 + 0x_4 + 0x_5 + x_6 + 0x_7 = 3$

$x_1 + 4x_2 + 0x_3 - x_4 + 0x_5 + 0x_6 + x_7 = 4$

$x_1 + x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 + 0x_7 = 5$

$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$

Phase-I: We consider the auxiliary linear programming problem as follows: Minimize $w = x_6 + x_7$

Subject to $3x_1 + 2x_2 - x_3 + 0x_4 + 0x_5 + x_6 + 0x_7 = 3$

$x_1 + 4x_2 + 0x_3 - x_4 + 0x_5 + 0x_6 + x_7 = 4$

$x_1 + x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 + 0x_7 = 5$

$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$

The simplex table of the auxiliary problem is as follows:

Basis	$\underline{C}_B^t$	c_j	0	0	0	0	0	1	1	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	θ
$\underline{P}_6$	1	3	3	2	-1	0	0	1	0	3/2
$\underline{P}_7$	1	4	1	4	0	-1	0	0	1	$4/4=1=\theta_0$
$\underline{P}_5$	0	5	1	1	0	0	1	0	0	5/1=5
$w_j - c_j$		7	4	6	-1	-1	0	0	0	
$\underline{P}_6$	1	1	5/2	0	-1	1/2	0	1	-1/2	$2/5 = \theta_0$
$\underline{P}_2$	0	1	1/4	1	0	-1/4	0	0	1/4	4
$\underline{P}_5$	0	4	3/4	0	0	1/4	1	0	-1/4	16/3
$w_j - c_j$		1	5/2	0	-1	1/2	0	0	3/2	
$\underline{P}_1$	0	2/5	1	0	-2/5	1/5	0	2/5	-1/5	
$\underline{P}_2$	0	9/10	0	1	1/10	-3/10	0	-1/10	3/10	
$\underline{P}_5$	0	37/10	0	0	3/10	1/10	1	-3/10	-1/10	
$w_j - c_j$		0	0	0	0	0	0	-1	-1	

Since all $w_j - c_j \leq 0$ and minimum of $w = 0$ and all artificial variables leave the basis. So the given problem has a basic feasible solution.

Phase-II: Removing all the columns corresponding to the artificial variables from the final table of phase-I and taking the original objective function in lieu of artificial objective function, we construct the following simplex table.

Basis	C_B^t	c_j / $\underline{P}_o$	5 $\underline{P}_1$	8 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_1$	5	2/5	1	0	-2/5	1/5	0	$2 = \theta_0$
$\underline{P}_2$	8	9/10	0	1	1/10	-3/10	0	
$\underline{P}_5$	0	37/10	0	0	3/10	1/10	1	37
$z_j - c_j$		46/5	0	0	-6/5	-7/5 Smallest	0	
$\underline{P}_4$	0	2	5	0	-2	1	0	
$\underline{P}_2$	8	3/2	3/2	1	-1/2	0	0	
$\underline{P}_5$	0	7/2	-1/2	0	1/2	0	1	$7 = \theta_0$
$z_j - c_j$		12	7	0	-4 Smallest	0	0	
$\underline{P}_4$	0	16	3	0	0	1	4	
$\underline{P}_2$	8	5	1	1	0	0	1	
$\underline{P}_3$	0	7	-1	0	1	0	2	
$z_j - c_j$		40	3	0	0	0	8	

Since all $z_j - c_j \geq 0$, the optimality conditions are satisfied. Hence the optimal basic feasible solution is $x_1 = 0$, $x_2 = 5$ and $z_{max} = 40$.

Alternative method (converting into minimization type problem): Introducing surplus variables $x_3 \geq 0$, $x_4 \geq 0$ and slack variables $x_5 \geq 0$ to first, second and third constraints respectively, we get the following standard form:

$$\begin{aligned} &\text{Maximize } z = 5x_1 + 8x_2 + 0x_3 + 0x_4 + 0x_5 \\ &\text{Subject to } 3x_1 + 2x_2 - x_3 + 0x_4 + 0x_5 = 3 \\ &\qquad x_1 + 4x_2 + 0x_3 - x_4 + 0x_5 = 4 \\ &\qquad x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 5 \\ &\qquad x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

The above standard form does not contain the basis matrix, hence we need to introduce artificial variables $x_6 \geq 0$ and $x_7 \geq 0$ to first and second constraints respectively.

$$\begin{aligned} &\text{Maximize } z = 5x_1 + 8x_2 + 0x_3 + 0x_4 + 0x_5 \\ &\text{Subject to } 3x_1 + 2x_2 - x_3 + 0x_4 + 0x_5 + x_6 + 0x_7 = 3 \\ &\qquad x_1 + 4x_2 + 0x_3 - x_4 + 0x_5 + 0x_6 + x_7 = 4 \\ &\qquad x_1 + x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 + 0x_7 = 5 \\ &\qquad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

Converting the problem into minimization type, we get

$$-\text{Minimize } (-z) = -5x_1 - 8x_2 + 0x_3 + 0x_4 + 0x_5$$

$$\text{Subject to } 3x_1 + 2x_2 - x_3 + 0x_4 + 0x_5 + x_6 + 0x_7 = 3$$

$$x_1 + 4x_2 + 0x_3 - x_4 + 0x_5 + 0x_6 + x_7 = 4$$

$$x_1 + x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 + 0x_7 = 5$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

Phase-I: We consider the auxiliary linear programming problem as follows: Minimize $w = x_6 + x_7$

$$\text{Subject to } 3x_1 + 2x_2 - x_3 + 0x_4 + 0x_5 + x_6 + 0x_7 = 3$$

$$x_1 + 4x_2 + 0x_3 - x_4 + 0x_5 + 0x_6 + x_7 = 4$$

$$x_1 + x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 + 0x_7 = 5$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

The simplex table of the auxiliary problem is as follows:

Basis	$\underline{C}_B^t$	c_j	0	0	0	0	0	1	1	Min Ratio θ
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	
$\underline{P}_6$	1	3	3	2	-1	0	0	1	0	3/2
$\underline{P}_7$	1	4	1	4	0	-1	0	0	1	$4/4=1=\theta_0$
$\underline{P}_5$	0	5	1	1	0	0	1	0	0	5/1=5
$w_j - c_j$		7	4	6	-1	-1	0	0	0	
$\underline{P}_6$	1	1	5/2	0	-1	1/2	0	1	-1/2	$2/5 = \theta_0$
$\underline{P}_2$	0	1	1/4	1	0	-1/4	0	0	1/4	4
$\underline{P}_5$	0	4	3/4	0	0	1/4	1	0	-1/4	16/3
$w_j - c_j$		1	5/2	0	-1	1/2	0	0	3/2	
$\underline{P}_1$	0	2/5	1	0	-2/5	1/5	0	2/5	-1/5	
$\underline{P}_2$	0	9/10	0	1	1/10	-3/10	0	-1/10	3/10	
$\underline{P}_5$	0	37/10	0	0	3/10	1/10	1	-3/10	-1/10	
$w_j - c_j$		0	0	0	0	0	0	-1	-1	

Since all $w_j - c_j \leq 0$ and minimum of $w = 0$ and all artificial variables leave the basis. So the given problem has a basic feasible solution.

Phase-II: Removing all the columns corresponding to the artificial variables from the final table of phase-I and taking the original objective function in the minimization type problem in lieu of artificial objective function, we construct the following simplex table.

Basis	C_B^t	c_j / $\underline{P}_o$	-5 $\underline{P}_1$	-8 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_1$	-5	2/5	1	0	-2/5	1/5	0	$2 = \theta_0$
$\underline{P}_2$	-8	9/10	0	1	1/10	-3/10	0	
$\underline{P}_5$	0	37/10	0	0	3/10	1/10	1	37
$z_j - c_j$		-46/5	0	0	6/5	7/5 Largest	0	
$\underline{P}_4$	0	2	5	0	-2	1	0	
$\underline{P}_2$	-8	3/2	3/2	1	-1/2	0	0	
$\underline{P}_5$	0	7/2	-1/2	0	1/2	0	1	$7 = \theta_0$
$z_j - c_j$		-12	-7	0	4 Largest	0	0	
$\underline{P}_4$	0	16	3	0	0	1	4	
$\underline{P}_2$	-8	5	1	1	0	0	1	
$\underline{P}_3$	0	7	-1	0	1	0	2	
$z_j - c_j$		-40	-3	0	0	0	-8	

Since all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimal basic feasible solution is $x_1 = 0$, $x_2 = 5$ and $z_{max} = -(-40) = 40$.

<u>Example (4.20)</u>: Solve the following linear programming problem by two-phase method. Minimize $z = x_1 + x_2$

$$\text{Subject to } 3x_1 + 2x_2 \geq 30$$
$$-2x_1 - 3x_2 \leq -30$$
$$x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

Solution: Introducing surplus variables $x_3 \geq 0$, $x_4 \geq 0$ and slack variables $x_5 \geq 0$ to first, second and third constraints respectively, we get the following standard form:

$$\text{Minimize } z = x_1 + x_2 + 0x_3 + 0x_4 + 0x_5$$
$$\text{Subject to } 3x_1 + 2x_2 - x_3 + 0x_4 + 0x_5 = 30$$
$$2x_1 + 3x_2 + 0x_3 - x_4 + 0x_5 = 30$$
$$x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 5$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The above standard form does not contain the basis matrix, hence we need to introduce artificial variables $x_6 \geq 0$ and $x_7 \geq 0$ to first and second constraints respectively.

$$\begin{aligned} &\text{Maximize } z = x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \\ &\text{Subject to } 3x_1 + 2x_2 - x_3 + 0x_4 + 0x_5 + x_6 + 0x_7 = 30 \\ &\qquad 2x_1 + 3x_2 + 0x_3 - x_4 + 0x_5 + 0x_6 + x_7 = 30 \\ &\qquad x_1 + x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 + 0x_7 = 5 \\ &\qquad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

Phase-I: We consider the auxiliary linear programming problem as follows: Minimize $w = x_6 + x_7$

$$\begin{aligned} &\text{Subject to } 3x_1 + 2x_2 - x_3 + 0x_4 + 0x_5 + x_6 + 0x_7 = 30 \\ &\qquad 2x_1 + 3x_2 + 0x_3 - x_4 + 0x_5 + 0x_6 + x_7 = 30 \\ &\qquad x_1 + x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 + 0x_7 = 5 \\ &\qquad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

The simplex table of the auxiliary problem is as follows:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	0 $\underline{P}_1$	0 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	1 $\underline{P}_6$	1 $\underline{P}_7$	Min Ratio θ
$\underline{P}_6$	1	30	3	2	-1	0	0	1	0	30/3=10
$\underline{P}_7$	1	30	2	3	0	-1	0	0	1	30/2=15
$\underline{P}_5$	0	5	1	1	0	0	1	0	0	5/1=5=θ_0
$w_j - c_j$		60	5	5	-1	-1	0	0	0	
$\underline{P}_6$	1	15	0	-1	-1	0	-3	1	0	
$\underline{P}_7$	1	20	0	1	0	-1	-2	0	1	
$\underline{P}_1$	0	5	1	1	0	0	1	0	0	
$w_j - c_j$		35	0	0	-1	-1	-5	0	0	

Since all $w_j - c_j \leq 0$ and minimum of $w = 35 \neq 0$ and some artificial variables do not leave the basis, the given problem has no solution.

4.9 Some done examples:

Example (4.21): A furniture company makes tables and chairs. Each table takes 5 hours of carpentry and 10 hours in painting and varnishing shop. Each chair requires 20 hours in carpentry and 15 hours in painting and varnishing. During the current production period, 400 hours of carpentry and 450 hours of painting and varnishing time are available. Each table sold yields a profit of \$45 and each chair yields a profit of \$80. Using simplex method determine the number of tables and chairs to be made to maximize the profit.

Solution: Let x_1 be the number of tables and x_2 be the number of chairs. So, the total profit $45x_1 + 80x_2$ which is the objective function. The objective function $z = 45x_1+80x_2$ is to be maximized. The required carpentry hours are $5x_1 + 20x_2$. Since 400 carpentry hours are available, $4x_1 + 3x_2 \leq 240$.

Similarly, for painting and varnishing, we have $10x_1 + 15x_2 \leq 450$.

The non-negativity conditions $x_1, x_2 \geq 0$

So, the linear programming (LP) form of the given problem is

$$\begin{aligned} &\text{Maximize } z = 45x_1 + 80\,x_2 \\ &\text{Subject to } \quad 5x_1 + 20x_2 \leq 400 \\ &\qquad\qquad\quad 10x_1 + 15x_2 \leq 450 \\ &\qquad\qquad\qquad x_1, x_2 \geq 0 \end{aligned}$$

Introducing slack variables x_3, x_4 and converting it as minimization type, we can rewrite the problem as follows:

$$\begin{aligned} &\text{–Minimize } –z = – 45x_1 – 80x_2 \\ &\text{Subject to } \quad x_1 + 4x_2 + x_3 \qquad = 80 \\ &\qquad\qquad\quad 2x_1 + 3x_2 \qquad + x_4 = 90 \\ &\qquad\qquad\quad x_j, \geq 0;\ j = 1, 2, 3, 4 \end{aligned}$$

Sl.	Basis	C_B^t	$\underline{P}_o$	-45	-80	0	0	Ratio
				$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	θ
1	$\underline{P}_3$	0	80	1	4	1	0	80/4 = 20 =θ_o
2	$\underline{P}_4$	0	90	2	3	0	1	90/3 = 30
3	$z_j – c_j$		0	45	80	0	0	
1	$\underline{P}_2$	-80	20	1/4	1	1/4	0	20/(1/4) = 80
2	$\underline{P}_4$	0	30	5/4	0	-3/4	1	30/(5/4)=24=θ_o
3	$z_j – c_j$		-1600	25	0	-20	0	
1	$\underline{P}_2$	-80	14	0	1	2/5	-1/5	
2	$\underline{P}_1$	-45	24	1	0	-3/5	4/5	
3	$z_j – c_j$		-2200	0	0	-5	-20	

The above tableau gives the extreme point (24, 14), i.e., $x_1 = 24$, $x_2 = 14$. So, the company will earn maximum profit \$2200 if 24 tables and 14 chairs are made.

Mathematica code to solve the above problem is as follows:

ConstrainedMax[45x$_1$ + 80x$_2$,{5x$_1$ + 20x$_2$ ≤ 400,10x$_1$ + 15x$_2$ ≤ 450}, {x$_1$, x$_2$}]

Example (4.22): Solve the following LP problem:

$$\begin{array}{ll} \text{Minimize} & x_1 - x_2 + x_3 \\ \text{Subject to} & x_1 \quad - x_4 \quad - 2x_6 = 5 \\ & x_2 + 2x_4 - 3x_5 + x_6 = 3 \\ & x_3 + 2x_4 - 5x_5 + 6x_6 = 5 \\ & x_i \geq 0 \; ; \; i = 1, 2, ..., 6 \end{array}$$

Solution: From the given problem, we get the following tableau:

Sl	Basis	$\underline{C}_B$	$\underline{P}_o$	1	-1	1	0	0	0	Ratio
				$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	θ
1	$\underline{P}_1$	1	5	1	0	0	-1	0	-2	
2	$\underline{P}_2$	-1	3	0	1	0	2	-3	1	3/1=3
3	$\underline{P}_3$	1	5	0	0	1	2	-5	6	5/6=θ_o
4	$z_j - c_j$		7	0	0	0	-1	-2	3	
1	$\underline{P}_1$	1	20/3	1	0	1/3	-1/3	-5/3	0	
2	$\underline{P}_2$	-1	13/6	0	1	-1/6	5/3	-13/6	0	
3	$\underline{P}_6$	0	5/6	0	0	1/6	1/3	-5/6	1	
4	$z_j - c_j$		9/2	0	0	-1/2	-2	1/2	0	

Since in the second step we find a positive value 1/2 for $z_j - c_j$ but there is no positive number above 1/2. So, we can say that the feasible region is unbounded, i.e., it has an unbounded solution.

Example (4.23): Solve the following LP problem by simplex method. Minimize $z = x_1 - x_2 + x_3 + x_4 + x_5 - x_6$ **[NHU-06]**

$$\begin{array}{ll} \text{Subject to} & x_1 \quad + x_4 \quad + 6x_6 = 9 \\ & 3x_1 + x_2 - 4x_3 \quad + 2x_6 = 2 \\ & x_1 \quad + 2x_3 \quad + x_5 + 2x_6 = 6 \\ & x_j \geq 0, \; j = 1,2, ..., 6 \end{array}$$

Solution: Making initial simplex table and taking necessary iterations, we get the following tables

Basis	C_B^t	c_j / $\underline{P}_o$	1	-1	1	1	1	-1	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_4$	1	9	1	0	0	1	0	6	9/6
$\underline{P}_2$	-1	2	3	1	-4	0	0	2	2/2=1=θ_0
$\underline{P}_5$	1	6	1	0	2	0	1	2	6/2=3
$z_j - c_j$		13	-2	0	5	0	0	7	
$\underline{P}_4$	1	3	-8	-3	12	1	0	0	3/12=θ_0
$\underline{P}_6$	-1	1	3/2	1/2	-2	0	0	1	
$\underline{P}_5$	1	4	-2	-1	6	0	1	0	4/6
$z_j - c_j$		6	-25/2	-7/2	19	0	0	0	
$\underline{P}_3$	1	1/4	-2/3	-1/4	1	1/12	0	0	
$\underline{P}_6$	-1	3/2	1/6	0	0	1/6	0	1	
$\underline{P}_5$	1	5/2	2	1/2	0	-1/2	1	0	5 = θ_0
$z_j - c_j$		5/4	1/6	5/4	0	-19/12	0	0	
$\underline{P}_3$	1	3/2	1/3	0	1	-1/6	1/2	0	
$\underline{P}_6$	-1	3/2	1/6	0	0	1/6	0	1	
$\underline{P}_2$	1	5	4	1	0	-1	2	0	
$z_j - c_j$		-5	-29/6	0	0	-1/3	-5/2	0	

Since all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimal basic feasible solution is $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 5, 3/2, 0, 0, 3/2)$ and $z_{min} = -5$.

Example (4.24): Solve the following LP problem.

$$\text{Maximize } z = x_1 + x_2 + x_3$$
$$\text{Subject to } 2x_1 + x_2 + 2x_3 \leq 3$$
$$4x_1 + 2x_2 + x_3 \leq 2$$
$$x_1, x_2, x_3 \geq 0$$

Solution: Since the given LP problem contains more than two variables the problem can be solved by simplex method only. To solve the problem by simplex method, we convert the problem into minimization type and introduce slack variables $x_4 \geq 0$ and $x_5 \geq 0$ to 1st and 2nd constraints respectively.

$$– \text{Minimize } –z = –x_1 – x_2 – x_3$$
$$\text{Subject to } 2x_1 + x_2 + 2x_3 + x_4 \qquad = 3$$
$$4x_1 + 2x_2 + x_3 \qquad + x_5 = 2$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	C_B^t	c_j / P_o	-1 P_1	-1 P_2	-1 P_3	0 P_4	0 P_5	Min Ratio θ
P_4	0	3	2	1	2	1	0	$3/2=\theta_0$
P_5	0	2	4	2	1	0	1	2/1=2
$z_j – c_j$		0	1	1	1	0	0	
P_3	-1	3/2	1	1/2	1	1/2	0	3
P_5	0	1/2	3	3/2	0	-1/2	1	$1/3 =\theta_0$
$z_j – c_j$		-3/2	0	1/2	0	-1/2	0	
P_3	-1	4/3	0	0	1	2/3	-1/3	
P_2	-1	1/3	2	1	0	-1/3	2/3	
$z_j – c_j$		-5/3	-1	0	0	-1/3	-1/3	

Since all $z_j – c_j \leq 0$, the optimality conditions are satisfied. Hence the optimal basic feasible solution is $(x_1, x_2, x_3) = (0, 1/3, 4/3)$ and $z_{max} = – z_{min} = – (–5/3) = 5/3$.

Example (4.25): Solve the following LP problem by simplex method. Maximize $z = 3x_1 + 6x_2 + 2x_3$

$$\text{Subject to } 3x_1 + 4x_2 + x_3 \leq 2$$
$$x_1 + 3x_2 + 2x_3 \leq 1$$
$$x_1, x_2, x_3 \geq 0$$

Solution: To solve the problem by simplex method, we convert the problem into minimization type and introduce slack variables $x_4 \geq 0$ and $x_5 \geq 0$ to 1st and 2nd constraints respectively. Then we get

$$– \text{Minimize } –z = –3x_1 – 6x_2 – 2x_3$$
$$\text{Subject to } 3x_1 + 4x_2 + x_3 + x_4 \qquad = 2$$
$$x_1 + 3x_2 + 2x_3 \qquad + x_5 = 1$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	C_B^t	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	-6 $\underline{P}_2$	-2 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_4$	0	2	3	4	1	1	0	2/4
$\underline{P}_5$	0	1	1	3	2	0	1	$1/3=\theta_0$
$z_j - c_j$		0	3	6	2	0	0	
$\underline{P}_4$	0	2/3	5/3	0	-5/3	1	-4/3	$2/5=\theta_0$
$\underline{P}_2$	-6	1/3	1/3	1	2/3	0	1/3	1
$z_j - c_j$		-2	1	0	-2	0	-2	
$\underline{P}_1$	-3	2/5	1	0	-1	3/5	-4/5	
$\underline{P}_2$	-6	1/5	0	1	1	-1/5	3/5	
$z_j - c_j$		-12/5	0	0	-1	-3/5	-6/5	

Since all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimal basic feasible solution is $(x_1, x_2, x_3) = (2/5, 1/5, 0)$ and $z_{max} = -z_{min} = -(-12/5) = 12/5$.

<u>Example (4.26)</u>: Solve the following LP problem by simplex method. Minimize $z = 2x_1 + x_2$

$$\text{Subject to } 3x_1 + x_2 \leq 3$$
$$4x_1 + 3x_2 \leq 6$$
$$x_1 + 2x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Solution: Introducing slack variables $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$ to 1st, 2nd and 3rd constraints respectively, we get

$$\text{Minimize } z = 2x_1 + x_2$$
$$\text{Subject to } 3x_1 + x_2 + x_3 = 3$$
$$4x_1 + 3x_2 + x_4 = 6$$
$$x_1 + 2x_2 + x_5 = 2$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following table

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2 $\underline{P}_1$	1 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_3$	0	3	3	1	1	0	0	
$\underline{P}_4$	0	6	4	3	0	1	0	
$\underline{P}_5$	0	2	1	2	0	0	1	
$z_j - c_j$		0	-2	-1	0	0	0	

Since all $z_j - c_j \leq 0$ in the initial simplex table, the optimality conditions are satisfied. So the initial simplex table is optimal and the optimal solution is $(x_1, x_2, x_3) = (0, 0, 0)$ and $z_{min} = 0$.

<u>Example (4.27)</u>: Solve the following LP problem by simplex method.

Maximize $z = 2x_1 + 3x_2$ [JU-93]

Subject to
$$-x_1 + 2x_2 \leq 4$$
$$x_1 + x_2 \leq 6$$
$$x_1 + 3x_2 \leq 9$$
$$x_1, x_2 \geq 0$$

Solution: To solve the problem by simplex method, we convert the problem into minimization type and introduce slack variables $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$ to 1st, 2nd and 3rd constraints respectively. Then we get, – Minimize $-z = -2x_1 - 3x_2$

Subject to
$$-x_1 + 2x_2 + x_3 = 4$$
$$x_1 + x_2 + x_4 = 6$$
$$x_1 + 3x_2 + x_5 = 9$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-2 $\underline{P}_1$	-3 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_3$	0	4	-1	2	1	0	0	4/2=2 Min.
$\underline{P}_4$	0	6	1	1	0	1	0	6/1=6
$\underline{P}_5$	0	9	1	3	0	0	1	9/3=3
$z_j - c_j$		0	2	3	0	0	0	

Basis	C_B^t	c_j / $\underline{P}_o$	-2 $\underline{P}_1$	-3 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_2$	-3	2	-1/2	1	1/2	0	0	
$\underline{P}_4$	0	4	3/2	0	-1/2	1	0	8/3
$\underline{P}_5$	0	3	5/2	0	-3/2	0	1	6/5 Min.
$z_j - c_j$		-6	7/2	0	-3/2	0	0	
$\underline{P}_2$	-3	13/5	0	1	1/5	0	1/5	13
$\underline{P}_4$	0	11/5	0	0	2/5	1	-3/5	11/5 Min.
$\underline{P}_1$	-2	6/5	1	0	-3/5	0	2/5	
$z_j - c_j$		-51/5	0	0	3/5	0	-7/5	
$\underline{P}_2$	-3	3/2	0	1	0	-1/2	1/2	
$\underline{P}_3$	0	11/2	0	0	1	5/2	-3/2	
$\underline{P}_1$	-2	9/2	1	0	0	3/2	-1/2	
$z_j - c_j$		-27/2	0	0	0	-1	-1/2	

Since all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimal basic feasible solution is $(x_1, x_2) = (9/2, 3/2)$ and $z_{max} = -z_{min} = -(-27/2) = 27/2$.

Example (4.28): Solve the following LP problem by simplex method. Maximize $z = 3x_1 + 2x_2$

Subject to $2x_1 - x_2 \geq -2$

$x_1 + 2x_2 \leq 8$

$x_1, x_2 \geq 0$

Solution: Multiplying 1st constraint by –1, we get the given problem as follows: Maximize $z = 3x_1 + 2x_2$

Subject to $-2x_1 + x_2 \leq 2$

$x_1 + 2x_2 \leq 8$

$x_1, x_2 \geq 0$

Converting the problem into minimization type and introducing slack variables $x_3 \geq 0$ and $x_4 \geq 0$ to 1st and 2nd constraints respectively, then we get, – Minimize $-z = -3x_1 - 2x_2$

Subject to $-2x_1 + x_2 + x_3 = 2$

$x_1 + 2x_2 + x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	-2 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ
$\underline{P}_3$	0	2	-2	1	1	0	
$\underline{P}_4$	0	8	1	2	0	1	8/1 Min.
$z_j - c_j$		0	3 Greatest	2	0	0	
$\underline{P}_3$	0	18	0	5	1	2	
$\underline{P}_1$	-3	8	1	2	0	1	
$z_j - c_j$		-24	0	-4	0	-3	

Since all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimal solution is $(x_1,x_2) = (8,0)$ and $z_{max} = -z_{min} = -(-24) = 24$.

Another method (Keeping maximization type problem):

Multiplying 1st constraint of the given problem by –1, we get the problem as follows: Maximize $z = 3x_1 + 2x_2$

Subject to $-2x_1 + x_2 \leq 2$

$x_1 + 2x_2 \leq 8$

$x_1, x_2 \geq 0$

Introducing slack variables $x_3 \geq 0$ and $x_4 \geq 0$ to 1st and 2nd constraints respectively, we get, Maximize $z = 3x_1 + 2x_2$

Subject to $-2x_1 + x_2 + x_3 \qquad = 2$

$x_1 + 2x_2 \qquad + x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

Making initial simplex table and taking necessary iterations, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	3 $\underline{P}_1$	2 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ
$\underline{P}_3$	0	2	-2	1	1	0	
$\underline{P}_4$	0	8	1	2	0	1	8/1 Min.
$z_j - c_j$		0	-3 Smallest	-2	0	0	
$\underline{P}_3$	0	18	0	5	1	2	
$\underline{P}_1$	3	8	1	2	0	1	
$z_j - c_j$		24	0	4	0	3	

Since in the last table all $z_j - c_j \geq 0$, the optimality conditions are satisfied. Hence the optimal solution is $(x_1, x_2) = (8, 0)$ and the maximum value of the objective function, $z_{max} = 24$.

Another method (Keeping maximization type problem and using $c_j - z_j$ in lieu of $z_j - c_j$ in simplex table):

Multiplying 1st constraint of the given problem by –1, we get the problem as follows: Maximize $z = 3x_1 + 2x_2$

$$\text{Subject to } -2x_1 + x_2 \leq 2$$
$$x_1 + 2x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

Introducing slack variables $x_3 \geq 0$ and $x_4 \geq 0$ to 1st and 2nd constraints respectively, we get, Maximize $z = 3x_1 + 2x_2$

$$\text{Subject to } -2x_1 + x_2 + x_3 \quad = 2$$
$$x_1 + 2x_2 \quad + x_4 = 8$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Making initial simplex table and taking necessary iterations, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	3 $\underline{P}_1$	2 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ
$\underline{P}_3$	0	2	-2	1	1	0	
$\underline{P}_4$	0	8	1	2	0	1	8/1 Min.
$c_j - z_j$		0	3 Greatest	2	0	0	
$\underline{P}_3$	0	18	0	5	1	2	
$\underline{P}_1$	3	8	1	2	0	1	
$c_j - z_j$		24	0	-4	0	-3	

Since in the last table all $c_j - z_j \leq 0$, the optimality conditions are satisfied. Hence the optimal solution is $x_1 = 8$ (basic variable), $x_2 = 0$ (non-basic variable) and the maximum value of the objective function, $z_{max} = 24$.

<u>Example (4.29)</u>: Solve the following LP problem by simplex method. Maximize $z = 6x_1 + 5.5x_2 + 9x_3 + 8x_4$ **[NU-02]**

$$\text{Subject to } \quad x_1 + x_2 + x_3 + x_4 \leq 15$$
$$7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120$$
$$3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solution: To solve the problem by simplex method, we convert the problem into minimization type and introduce slack variables $x_5 \geq 0$, $x_6 \geq 0$ and $x_7 \geq 0$ to 1st, 2nd and 3rd constraints respectively. Then we get, $\quad$ –Minimize $-z = -6x_1 - 11/2x_2 - 9x_3 - 8x_4$

$$\begin{aligned} \text{Subject to } x_1 + x_2 + x_3 + x_4 + x_5 &= 15 \\ 7x_1 + 5x_2 + 3x_3 + 2x_4 + x_6 &= 120 \\ 3x_1 + 5x_2 + 10x_3 + 15x_4 + x_7 &= 100 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0 \end{aligned}$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-6	-11/2	-9	-8	0	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	
$\underline{P}_5$	0	15	1	1	1	1	1	0	0	15
$\underline{P}_6$	0	120	7	5	3	2	0	1	0	40
$\underline{P}_7$	0	100	3	5	10	15	0	0	1	10 Min.
$z_j - c_j$		0	6	11/2	9	8	0	0	0	
$\underline{P}_5$	0	5	7/10	1/2	0	-1/2	1	0	-1/10	50/7 Min.
$\underline{P}_6$	0	90	61/10	7/2	0	-5/2	0	1	-3/10	900/61
$\underline{P}_3$	-9	10	3/10	1/2	1	3/2	0	0	1/10	100/3
$z_j - c_j$		-90	33/10	1	0	-11/2	0	0	-9/10	
$\underline{P}_1$	-6	50/7	1	5/7	0	-5/7	10/7	0	1/7	
$\underline{P}_6$	0	325/7	0	-6/7	0	13/7	-61/7	1	4/7	
$\underline{P}_3$	-9	55/7	0	2/7	1	12/7	-3/7	0	1/7	
$z_j - c_j$		-795/7	0	-19/14	0	-22/7	-33/7	0	–3/7	

Since in the 3rd table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. From the 3rd table the optimal solution is $x_1 = 50/7$ (basic), $x_2 = 0$ (non-basic), $x_3 = 55/7$ (basic), $x_4 = 0$ (non-basic), $x_5 = 0$ (non-basic), $x_6 = 325/7$ (basic), $x_7 = 0$ (non-basic) and the maximum value of the given objective function, $z_{max} = -z_{min} = -(-795/7) = 795/7$. Here, x_1, x_2, x_3 and x_4 are decision variables. Hence the optimal basic feasible solution is $(x_1, x_2, x_3, x_4) = (50/7, 0, 55/7, 0)$ and $z_{max} = 795/7$.

Example (4.30): Solve the following LP problem by simplex method. Maximize $z = 3x_1 + 2x_2 + 5x_3$ **[NU-03,07]**

Subject to $x_1 + 2x_2 + x_3 \leq 430$

$x_1 + 4x_2 \leq 420$

$3x_1 + 2x_3 \leq 460$

$x_1, x_2, x_3 \geq 0$

Solution: To solve the problem by simplex method, we convert the problem into minimization type and introduce slack variables $x_4 \geq 0$, $x_5 \geq 0$ and $x_6 \geq 0$ to 1st, 2nd and 3rd constraints respectively. Then we get, $-$Minimize $-z = -3x_1 - 2x_2 - 5x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to $x_1 + 2x_2 + x_3 + x_4 + 0x_5 + 0x_6 = 430$

$x_1 + 4x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 = 420$

$3x_1 + 0x_2 + 2x_3 + 0x_4 + 0x_5 + x_6 = 460$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	-2 $\underline{P}_2$	-5 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	0 $\underline{P}_6$	Min Ratio θ
$\underline{P}_4$	0	430	1	2	1	1	0	0	430
$\underline{P}_5$	0	420	1	4	0	0	1	0	
$\underline{P}_6$	0	460	3	0	2	0	0	1	230 Min.
$z_j - c_j$		0	3	2	5	0	0	0	
$\underline{P}_4$	0	200	-1/2	2	0	1	0	-1/2	100 Min.
$\underline{P}_5$	0	420	1	4	0	0	1	0	105
$\underline{P}_3$	-5	230	3/2	0	1	0	0	1/2	
$z_j - c_j$		-1150	-9/2	2	0	0	0	-5/2	
$\underline{P}_2$	-2	100	-1/4	1	0	1/2	0	-1/4	
$\underline{P}_5$	0	20	2	0	0	-2	1	1	
$\underline{P}_3$	-5	230	3/2	0	1	0	0	1/2	
$z_j - c_j$		-1350	-4	0	0	-1	0	-2	

Since in the 3rd table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimum solution is $(x_1, x_2, x_3) = (0, 100, 230)$ and $z_{max} = -z_{min} = -(-1350) = 1350$.

Example (4.31): Solve the following LP problem by simplex method. Maximize $z = 107x_1 + x_2 + 2x_3$ **[JU-94]**

Subject to $14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5$$

$$3x_1 - x_2 - x_3 \leq 0$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solution: To solve the problem by simplex method, we convert the problem into minimization type, divide 1st constraint by 3 and introduce slack variables $x_5 \geq 0$, $x_6 \geq 0$ to 2nd and 3rd constraints respectively. Then we get,

$$-\text{Minimize} - z = -107x_1 - x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\text{Subject to } \frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 + 0x_5 + 0x_6 = \frac{7}{3}$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 + 0x_4 + x_5 + 0x_6 = 5$$

$$3x_1 - x_2 - x_3 + 0x_4 + 0x_5 + x_6 = 0$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-107 $\underline{P}_1$	-1 $\underline{P}_2$	-2 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	0 $\underline{P}_6$	Min Ratio θ
$\underline{P}_4$	0	7/3	14/3	1/3	-2	1	0	0	1/2
$\underline{P}_5$	0	5	16	1/2	-6	0	1	0	5/16
$\underline{P}_6$	0	0	3	-1	-1	0	0	1	0 Min.
$z_j - c_j$		0	107 Greatest	1	2	0	0	0	
$\underline{P}_4$	0	7/3	0	17/9	-4/9	1	0	-14/9	
$\underline{P}_5$	0	5	0	35/6	-2/9	0	1	-16/3	
$\underline{P}_1$	-107	0	1	-1/3	-1/3	0	0	1/3	
$z_j - c_j$		0	0	110/3	113/3 Greatest	0	0	107/3	

Greatest positive $z_3 - c_3 = 113/3$, but all elements of 3rd column are negative. It indicates that the given problem has an **unbounded solution**.

Example (4.32): Solve the following LP problem by simplex method. Maximize $z = 4x_1 + 10x_2$ **[JU-89]**

Subject to $2x_1 + x_2 \leq 50$

$2x_1 + 5x_2 \leq 100$

$2x_1 + 3x_2 \leq 90$

$x_1, x_2 \geq 0$

Solution: To solve the problem by simplex method, we convert the problem into minimization type and introduce slack variables $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$ to 1st, 2nd and 3rd constraints respectively. Then we get, $-\text{Minimize } -z = -4x_1 - 10x_2 + 0x_3 + 0x_4 + 0x_5$

Subject to $2x_1 + x_2 + x_3 + 0x_4 + 0x_5 = 50$

$2x_1 + 5x_2 + 0x_3 + x_4 + 0x_5 = 100$

$2x_1 + 3x_2 + 0x_3 + 0x_4 + x_5 = 90$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	C_B^t	c_j / $\underline{P}_o$	-4	-10	0	0	0	Min Ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_3$	0	50	2	1	1	0	0	50
$\underline{P}_4$	0	100	2	5	0	1	0	20 Min.
$\underline{P}_5$	0	90	2	3	0	0	1	30
$z_j - c_j$		0	4	10	0	0	0	
$\underline{P}_3$	0	30	8/5	0	1	-1/5	0	75/4 Min.
$\underline{P}_2$	-10	20	2/5	1	0	1/5	0	50
$\underline{P}_5$	0	30	4/5	0	0	3/5	1	75/2
$z_j - c_j$		-200	0	0	0	-10	0	

Since in the 2nd table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimum solution is $(x_1, x_2) = (0, 20)$ and the maximum value of the objective function, $z_{max} = -z_{min} = -(-200) = 200$. Also $z_1 - c_1 = 0$ for the non-basic variable x_1, it indicates that the problem has an **alternative solution**. If we enter the coefficient vector $\underline{P}_1$ associated with the variable x_1 into the basis, then we get the following table.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-4 $\underline{P}_1$	-10 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_1$	-4	75/4	1	0	5/8	-1/8	0	
$\underline{P}_2$	-10	25/2	0	1	-1/4	1/4	0	
$\underline{P}_5$	0	15	0	0	-1/2	-1/2	1	
$z_j - c_j$		-200	0	0	0	-2	0	

This is the optimal table as $z_j - c_j \le 0$. Another solution is $(x_1, x_2) = (75/4, 25/2)$ and $z_{max} = 200$.

Let the optimum solution from 2nd table be $\underline{X}_1 = \begin{pmatrix} 0 \\ 20 \end{pmatrix}$ and the optimum solution from 3rd table be $\underline{X}_2 = \begin{pmatrix} 75/4 \\ 25/2 \end{pmatrix}$. So, the linear combination $\underline{X}^* = \lambda\underline{X}_1 + (1 - \lambda)\underline{X}_2$, $0 \le \lambda \le 1$ gives infinite number of optimum solutions with $z_{max} = 200$.

<u>Example (4.33)</u>: Solve the following LP problem by simplex method. Minimize $z = -x_1 - 2x_2$ **[DU-88]**

$$\text{Subject to } -x_1 + 2x_2 \le 8$$
$$x_1 + 2x_2 \le 12$$
$$x_1 - x_2 \le 3$$
$$x_1, x_2 \ge 0$$

Solution: To solve the problem by simplex method, we introduce slack variables $x_3 \ge 0$, $x_4 \ge 0$ and $x_5 \ge 0$ to 1st, 2nd and 3rd constraints respectively. Then we get,

$$\text{Minimize } z = -x_1 - 2x_2$$
$$\text{Subject to } -x_1 + 2x_2 + x_3 = 8$$
$$x_1 + 2x_2 + x_4 = 12$$
$$x_1 - x_2 + x_5 = 3$$
$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

Making initial simplex table and taking necessary iterations, we get the following table.

Basis	$\underline{C}_B^t$	c_j	-1	-2	0	0	0	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_3$	0	8	-1	2	1	0	0	8/2=4 Min.
$\underline{P}_4$	0	12	1	2	0	1	0	12/2=6
$\underline{P}_5$	0	3	1	-2	0	0	1	
$z_j - c_j$		0	1	2	0	0	0	
$\underline{P}_2$	-2	4	-1/2	1	1/2	0	0	
$\underline{P}_4$	0	4	2	0	-1	1	0	4/2=2Min
$\underline{P}_5$	0	7	1/2	0	1/2	0	0	7/(1/2)=14
$z_j - c_j$		-8	2	0	-1	0	0	
$\underline{P}_2$	-2	5	0	1	1/4	1/4	0	20
$\underline{P}_1$	-1	2	1	0	-1/2	1/2	0	
$\underline{P}_5$	0	6	0	0	3/4	-1/4	1	8 Min.
$z_j - c_j$		-12	0	0	0	-1	-1	

Since in the 3rd table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimum solution is $(x_1, x_2) = (2, 5)$ and the minimum value of the objective function, $z_{min} = -12$. Also $z_3 - c_3 = 0$ for the non-basic variable x_3, it indicates that the problem has an **alternative solution**. If we enter the coefficient vector $\underline{P}_3$ associated with the variable x_3 into the basis, then we get the following table.

Basis	$\underline{C}_B^t$	c_j	-1	-2	0	0	0	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_2$	-2	3	0	1	0	1/3	-1/3	
$\underline{P}_1$	-1	6	1	0	0	1/3	2/3	
$\underline{P}_3$	0	8	0	0	1	-1/3	4/3	
$z_j - c_j$		-12	0	0	0	-1	0	

This is the optimal table as $z_j - c_j \leq 0$. Another solution is $(x_1, x_2) = (6, 3)$ and $z_{min} = -12$. Let $\underline{X}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\underline{X}_2 = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ be first and second optimum solutions. The linear combination $\underline{X}^* = \lambda \underline{X}_1 + (1-\lambda)\underline{X}_2$, for different values of λ in the interval [0, 1] gives infinite number of optimum solutions with $z_{min} = -12$.

<u>Example (4.34)</u>: Solve the following LP problem.

$$\text{Maximize } z = 2x_1 + 3x_2 \qquad \textbf{[JU-95]}$$

$$\text{Subject to } -x_1 + 2x_2 \le 4$$
$$x_1 + x_2 \le 6$$
$$x_1 + 3x_2 \le 9$$

x_1, x_2 are **unrestricted in sign**.

Solution: <u>Graphical solution</u>: Drawing the constraints in the graph paper we find the unbounded solution space, as the variables are unrestricted. We find two vertices of the unbounded solution space, one is A(9/2, 3/2) and another is B(6/5, 13/5). The extreme point A(9/2, 3/2) gives the maximum value of the objective function and the maximum value is z_{maz} = 27/2.

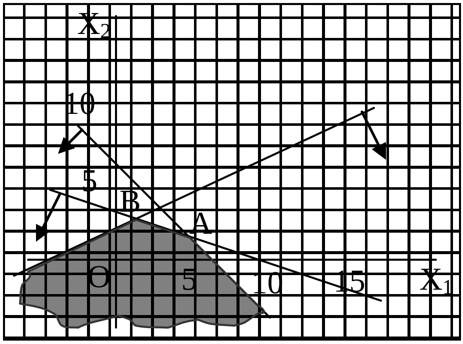

Figure 4.2

So, the optimum solution of the given problem is $x_1 = 9/2$, $x_2 = 3/2$ and $z_{max} = 27/2$.

<u>Solution by simplex method</u>: To make the given problem standard, we consider $x_1 = x_1^{/} - x_1^{//}$, $x_2 = x_2^{/} - x_2^{//}$ where $x_1^{/}, x_1^{//}, x_2^{/}, x_2^{//} \ge 0$ and introduce slack variables $x_3 \ge 0$, $x_4 \ge 0$ and $x_5 \ge 0$ to 1st, 2nd and 3rd constraints respectively. Then we get,

$$\text{Maximize } z = 2(x_1^{/} - x_1^{//}) + 3(x_2^{/} - x_2^{//}) + 0x_3 + 0x_4 + 0x_5$$

$$\text{Subject to } -(x_1^{/} - x_1^{//}) + 2(x_2^{/} - x_2^{//}) + x_3 + 0x_4 + 0x_5 = 4$$
$$x_1^{/} - x_1^{//} + (x_2^{/} - x_2^{//}) + 0x_3 + x_4 + 0x_5 = 6$$
$$x_1^{/} - x_1^{//} + 3(x_2^{/} - x_2^{//}) + 0x_3 + 0x_4 + x_5 = 9$$
$$x_1^{/}, x_1^{//}, x_2^{/}, x_2^{//}, x_3, x_4, x_5 \ge 0$$

To solve the problem by simplex method, we convert the problem into minimization type and then we get,

$$-\text{Minimize } -z = -2x_1' + 2x_1'' - 3x_2' + 3x_2'' + 0x_3 + 0x_4 + 0x_5$$

$$\text{Subject to } -x_1' + x_1'' + 2x_2' - 2x_2'' + x_3 + 0x_4 + 0x_5 = 4$$

$$x_1' - x_1'' + x_2' - x_2'' + 0x_3 + x_4 + 0x_5 = 6$$

$$x_1' - x_1'' + 3x_2' - 3x_2'' + 0x_3 + 0x_4 + x_5 = 9$$

$$x_1', x_1'', x_2', x_2'', x_3, x_4, x_5 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-2 $\underline{P}_1$	2 $\underline{P}_2$	-3 $\underline{P}_3$	3 $\underline{P}_4$	0 $\underline{P}_5$	0 $\underline{P}_6$	0 $\underline{P}_7$	Min Ratio θ
$\underline{P}_5$	0	4	-1	1	2	-2	1	0	0	4/2=2Min
$\underline{P}_6$	0	6	1	-1	1	-1	0	1	0	6/1=6
$\underline{P}_7$	0	9	1	-1	3	-3	0	0	1	9/3=3
$z_j - c_j$		0	2	-2	3	-3	0	0	0	
$\underline{P}_3$	-3	2	-1/2	1/2	1	-1	1/2	0	0	
$\underline{P}_6$	0	4	3/2	-3/2	0	0	-1/2	1	0	8/3
$\underline{P}_7$	0	3	5/2	-5/2	0	0	-3/2	0	1	6/5 Min.
$z_j - c_j$		-6	7/2	-7/2	0	0	-3/2	0	0	
$\underline{P}_3$	-3	13/5	0	0	1	-1	1/5	0	1/5	13
$\underline{P}_6$	0	11/5	0	0	0	0	2/5	1	-3/5	11/5 Min.
$\underline{P}_1$	-2	6/5	1	-1	0	0	-3/5	0	2/5	
$z_j - c_j$		-51/5	0	0	0	0	3/5	0	-7/5	
$\underline{P}_3$	-3	3/2	0	0	1	-1	0	-1/2	1/2	
$\underline{P}_5$	0	11/2	0	0	0	0	1	5/2	-3/2	
$\underline{P}_1$	-2	9/2	1	-1	0	0	0	3/2	-1/2	
$z_j - c_j$		-27/2	0	0	0	0	0	-3/2	-1/2	

Since in the 4th table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. This table indicates us $x_1' = 9/2$, $x_1'' = 0$, $x_2' = 3/2$, $x_2'' = 0$, $x_3 = 11/2$, $x_4 = 0$, $x_5 = 0$ and $z_{max} = 27/2$. So, $x_1 = x_1' - x_1'' = 9/2 - 0 = 9/2$

and $x_2 = x_2' - x_2'' = 3/2 - 0 = 3/2$. Hence the required optimum solution is $(x_1, x_2) = (9/2, 3/2)$ and $z_{max} = 27/2$.

Example (4.35): Solve the following LP problem.

Minimize $z = 3x_1 + 2x_2$ **[DU-92]**

Subject to $x_1 + x_2 \geq 1$

$x_1 + x_2 \leq 7$

$x_1 + 2x_2 \geq 3$

$x_1 \geq 0,\ x_2 \leq 3$

Solution: Graphical solution: Drawing the constraints in the graph paper we find the feasible solution space ABCDA shown shaded. The coordinates of the vertices are A(3, 0), B(3, 4), C(0, 7) and D(0, 3/2). The extreme point D(0, 3/2) gives the minimum value of the objective function, $z_{min} = 3$. So, the optimum solution of the given problem is $x_1 = 0$, $x_2 = 3/2$ and $z_{min} = 3$.

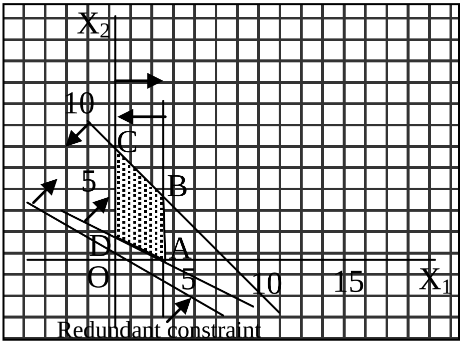

Figure 4.3

Note: The constraint, which has no effect on the solution space is called redundant constraint. Here, the constraint $x_1 + x_2 \geq 1$ is a redundant constraint because it has no effect on the feasible solution space.

Solution by simplex method: To make the variable restricted, we consider $x_2 = 3 - x_2'$ where $x_2' \geq 0$ Then the given problem becomes,

Minimize $z = 3x_1 + 6 - 2x_2'$

Subject to $x_1 - x_2' \geq -2$

$x_1 - x_2' \leq 4$

$x_1 - 2x_2' \geq -3$

$x_1,\ x_2' \geq 0$

To solve the problem by simplex method, we multiply 1st and 3rd constraints by –1 and then we introduce slack variables $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$ to 1st, 2nd and 3rd constraints respectively.

$$\text{Minimize } z = 3x_1 - 2x_2' + 0x_3 + 0x_4 + 0x_5 + 6$$

$$\text{Subject to } \begin{aligned} -x_1 + x_2' + x_3 + 0x_4 + 0x_5 &= 2 \\ x_1 - x_2' + 0x_3 + x_4 + 0x_5 &= 4 \\ -x_1 + 2x_2' + 0x_3 + 0x_4 + x_5 &= 3 \\ x_1, x_2', x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	C_B^t	c_j / P_o	3 P_1	-2 P_2	0 P_3	0 P_4	0 P_5	Min Ratio θ
P_3	0	2	-1	1	1	0	0	2/1=2
P_4	0	4	1	-1	0	1	0	
P_5	0	3	-1	2	0	0	1	3/2 Min.
$z_j - c_j$		0	-3	2	0	0	0	
P_3	0	1/2	2	0	1	0	-1/2	
P_4	0	11/2	1/2	0	0	1	1/2	
P_2	-2	3/2	-1/2	1	0	0	1/2	
$z_j - c_j$		-3	-2	0	0	0	-1	

Since in the 2nd table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. This table indicates us $x_1 = 0, x_2' = 3/2$, $x_3 = 1/2$, $x_4 = 11/2$, $x_5 = 0$ and $z_{min} = -3 + 6 = 3$. So, $x_2 = 3 - x_2' = 3 - 3/2 = 3/2$. Hence the required optimum solution is $(x_1, x_2) = (0, 3/2)$ and $z_{min} = 3$.

Example (4.36): Solve the following LP problem by simplex method. Minimize $z = -6x_1 - 5x_2$ **[DU-85]**

$$\text{Subject to } \begin{aligned} 4x_1 + x_2 &\leq 8 \\ 2x_1 - 3x_2 &\leq 0 \\ 3x_1 - x_2 &\leq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution: To solve the problem by simplex method, we introduce slack variables $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$ to 1st, 2nd and 3rd constraints respectively. Then we get,

$$\text{Minimize } z = -6x_1 - 5x_2 + 0x_3 + 0x_4 + 0x_5$$

$$\begin{aligned} \text{Subject to } 4x_1 + x_2 + x_3 \quad &= 8 \\ 2x_1 - 3x_2 + x_4 \quad &= 0 \\ 3x_1 - x_2 + x_5 &= 3 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

First we detach the coefficients of x_j's and then we make the following initial simplex table.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-6 $\underline{P}_1$	-5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_3$	0	8	4	1	1	0	0	8/1 = 8
$\underline{P}_4$	0	0	2	-3	0	1	0	
$\underline{P}_5$	0	0	3	-1	0	0	1	
$z_j - c_j$		0	6	5	0	0	0	

By simple observation, we see that if we select most positive $z_1 - c_1 = 6$ then we reach a tie of minimum ratio 0 that does not decrease the value of objective function in the next table but if we select $z_2 - c_2 = 5$, i.e., $\underline{P}_2$ as entering vector then we get a minimum ratio 8 that decrease the value of the objective function in the next table. Thus we select $\underline{P}_2$ as entering vector to solve quickly.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-6 $\underline{P}_1$	-5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_2$	-5	8	4	1	1	0	0	
$\underline{P}_4$	0	24	14	0	3	1	0	
$\underline{P}_5$	0	8	7	0	1	0	1	
$z_j - c_j$		- 40	-14	0	-5	0	0	

Since in the 2ndd table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimum solution is $(x_1, x_2) = (0, 8)$ and the minimum value of the objective function, $z_{min} = -40$.

Example (4.37): Solve the following LP problem by simplex method.

Minimize $z = -3x_1 - 9x_2$

Subject to $x_1 + 4x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$

Solution: To solve the problem by simplex method, we introduce slack variables $x_3 \ge 0$ and $x_4 \ge 0$ to 1st and 2nd constraints respectively. Then we get,

Minimize $z = -3x_1 - 9x_2$

Subject to $x_1 + 4x_2 + x_3 = 8$

$x_1 + 2x_2 + x_4 = 4$

$x_1, x_2, x_3, x_4 \ge 0$

Making initial simplex table and taking necessary iterations, we get the following tables.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	-9 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ
$\underline{P}_3$	0	8	1	4	1	0	8/4=2 Min. arbitrarily
$\underline{P}_4$	0	4	1	2	0	1	4/2=2
$z_j - c_j$		0	3	9	0	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	-9 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ
$\underline{P}_2$	-9	2	1/4	1	1/4	0	
$\underline{P}_4$	0	0	1/2	0	-1/2	1	0/(1/2)=0
$z_j - c_j$		-18	3/4	0	-9/4	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	-9 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ
$\underline{P}_2$	-9	2	0	1	1/2	-1/2	
$\underline{P}_1$	-3	0	1	0	-1	2	
$z_j - c_j$		-18	0	0	-3/2	-3/2	

Since in the 3rd table all $z_j - c_j \le 0$, the optimality conditions are satisfied. Hence the optimum solution is $(x_1, x_2) = (0, 2)$ and the minimum value of the objective function, $z_{min} = -18$.

Charnes perturbation method: When the minimum ratio is not unique, we use Charnes perturbation method to solve quickly or avoid cycling. In the Charnes perturbation method, firstly we divide each element in the tied rows by the positive coefficients of the key-column in that row; secondly compare the resulting ratios, column by column, first in the identity and then in the body, from left to right; thirdly and finally the row which first contains the smallest algebraic ratio tells us to decide the pivot element. The first table of the above example is

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	-9 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ	
$\underline{P}_3$	0	8	1	4	1	0	8/4=2	1/4
$\underline{P}_4$	0	4	1	2	0	1	4/2=2	0/2=0 Min.
$z_j - c_j$		0	3	9	0	0		

Using Charnes perturbation method, we decide 2 is the pivot element and taking necessary iteration, we get the following table.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	-9 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ
$\underline{P}_3$	0	0	-1	0	1	-2	
$\underline{P}_2$	-9	2	1/2	1	0	1/2	
$z_j - c_j$		-18	-3/2	0	0	-9/2	

Since in the last table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimum solution is $(x_1, x_2) = (0, 2)$ and the minimum value of the objective function, $z_{min} = -18$.

Example (4.38): (Beale's example of cycling) Solve LP problem

$$\text{Minimize } z = -\frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4$$

$$\text{Subject to } \tfrac{1}{4}x_1 - 60x_2 - \tfrac{1}{25}x_3 + 9x_4 + x_5 = 0$$

$$\tfrac{1}{2}x_1 - 90x_2 - \tfrac{1}{50}x_3 + 3x_4 + x_6 = 0$$

$$0x_1 + 0x_2 + x_3 + x_7 = 1$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables.

Basis	C_B^t	c_j	-3/4	150	-1/50	6	0	0	0	Min Ratio
		P_o	P_1	P_2	P_3	P_4	P_5	P_6	P_7	θ
P_5	0	0	1/4	-60	-1/25	9	1	0	0	0 min.
P6	0	0	1/2	-90	-1/50	3	0	1	0	0
P_7	0	1	0	0	1	0	0	0	1	
$z_j - c_j$		0	3/4	-150	1/50	-6	0	0	0	
P_1	-3/4	0	1	-240	-4/25	36	4	0	0	
P_6	0	0	0	30	3/50	-15	-2	1	0	0/30=0 min
P_7	0	1	0	0	1	0	0	0	1	
$z_j - c_j$		0	0	30	7/50	-33	-3	0	0	
P_1	-3/4	0	1	0	8/25	-84	-12	8	0	0 min.
P_2	150	0	0	1	1/500	-1/2	-1/15	1/30	0	0
P_7	0	1	0	0	1	0	0	0	1	1
$z_j - c_j$		0	0	0	2/25	-18	-1	-1	0	
P_3	-1/50	0	25/8	0	1	-525/2	-75/2	25	0	
P_2	150	0	-1/160	1	0	1/40	1/120	-1/60	0	0 min.
P_7	0	1	-25/8	0	0	525/2	75/2	-25	1	2/525
$z_j - c_j$		0	-1/4	0	0	3	2	-3	0	
P_3	-1/50	0	-125/2	10500	1	0	50	-150	0	0 min.
P_4	6	0	-1/4	40	0	1	1/3	-2/3	0	0
P_7	0	1	125/2	-10500	0	0	-50	150	1	
$z_j - c_j$		0	1/2	-120	0	0	1	-1	0	
P_5	0	0	-5/4	210	1/50	0	1	-3	0	
P_4	6	0	1/6	-30	-1/150	1	0	1/3	0	0/(1/3) =0
P_7	0	1	0	0	1	0	0	0	1	
$z_j - c_j$		0	7/4	-330	-1/50	0	0	2	0	
P_5	0	0	1/4	-60	-1/25	9	1	0	0	
P_6	0	0	1/2	-90	-1/50	3	0	1	0	
P_7	0	1	0	0	1	0	0	0	1	
$z_j - c_j$		0	3/4	-150	1/50	-6	0	0	0	

The last tableau and the initial tableau are same. It indicates that it has a cycling and we never get the optimum solution though the problem has a finite solution. To get the optimal solution, we solve the problem as follows:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	Min Ratio θ	
		c_j	-3/4	150	-1/50	6	0	0	0		
$\underline{P}_5$	0	0	1/4	-60	-1/25	9	1	0	0	0	-60/(1/4)min
$\underline{P}6$	0	0	1/2	-90	-1/50	3	0	1	0	0	-90/(1/2)
$\underline{P}_7$	0	1	0	0	1	0	0	0	1		Go $\underline{P}_2$
$z_j - c_j$		0	3/4	-150	1/50	-6	0	0	0		
$\underline{P}_1$	-3/4	0	1	-240	-4/25	36	4	0	0		
$\underline{P}_6$	0	0	0	30	3/50	-15	-2	1	0	0/30=0 min	
$\underline{P}_7$	0	1	0	0	1	0	0	0	1		
$z_j - c_j$		0	0	30	7/50	-33	-3	0	0		
$\underline{P}_1$	-3/4	0	1	0	8/25	-84	-12	8	0	0	1/(8/25)
$\underline{P}_2$	150	0	0	1	1/500	-1/2	-1/5	1/30	0	0	0 min
$\underline{P}_7$	0	1	0	0	1	0	0	0	1	1	Go $\underline{P}_1$
$z_j - c_j$		0	0	0	2/25	-18	-1	-1	0		
$\underline{P}_1$	-3/4	0	1	-160	0	-4	-4/3	8/3	0		
$\underline{P}_3$	-1/50	0	0	500	1	-250	-100/3	50/3	0		
$\underline{P}_7$	0	1	0	-500	0	250	100/3	-50/3	1	1/250 min	
$z_j - c_j$		0	0	-40	0	2	5/3	-7/3	0		
$\underline{P}_1$	-3/4	2/125	1	-168	0	0	-4/5	12/5	2/125		
$\underline{P}_3$	-1/50	1	0	0	1	0	0	0	1		
$\underline{P}_4$	6	1/250	0	-2	0	1	2/15	-1/15	1/250	(1/250)/(2/15)	
$z_j - c_j$		-1/125	0	-36	0	0	7/5	-11/5	-1/125		
$\underline{P}_1$	-3/4	1/25	1	-180	0	6	0	2	1/25		
$\underline{P}_3$	-1/50	1	0	0	1	0	0	0	1		
$\underline{P}_5$	0	3/100	0	-15	0	15/2	1	-1/2	3/100		
$z_j - c_j$		-1/20	0	-15	0	-21/2	0	-3/2	-1/20		

Since in the 6th table all $z_j - c_j \le 0$, the optimality conditions are satisfied. And this table gives us the optimum solution $x_1 = 1/25$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$, $x_5 = 3/100$, $x_6 = 0$, $x_7 = 0$ and $z_{min} = -1/20$.

Also using Charne's perturbation method we can solve the problem and that gives the same solution.

Example (4.39): A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabean. One acre of corn costs Tk.100 to prepare, requires 7 man-days of work and yields a profit of Tk.30. One acre of wheat costs Tk.120 to prepare, requires 10 man-days of work and yields a profit of Tk.40. One acre of soyabean costs Tk.70 to prepare, requires 8 man-days work and yields a profit of Tk.20. If the farmer has Tk.1,00,000 for preparation and can count an 8,000 man-days of work, how many acres should be allocated to each crop to maximize the profit? [JU-88]

Solution: Mathematical formulation of the problem:

Step-1: The key decision is to determine that how many acres of land should be allocated to each crop.

Step-2: Let x_1, x_2 and x_3 acres of land should be allocated for corn, wheat and soyabean respectively.

Step-3: Feasible alternatives are the sets of the values of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step-4: The objective is to maximize the profit realized from all the three crops, i.e., to maximize $z = 30x_1 + 40x_2 + 20x_3$

Step-5: The constraints (or restrictions) are

$x_1 + x_2 + x_3 \leq 1000$ (Limitation of land)

$100x_1 + 120x_2 + 70x_3 \leq 100000$ (Limitation of preparation cost)

$7x_1 + 10x_2 + 8x_3 \leq 8000$ (Limitation of man-days)

Hence the farmer's problem can be put in the following mathematical form:

$$\text{Maximize } z = 30x_1 + 40x_2 + 20x_3$$
$$x_1 + x_2 + x_3 \leq 1000$$
$$100x_1 + 120x_2 + 70x_3 \leq 100000$$
$$7x_1 + 10x_2 + 8x_3 \leq 8000$$
$$x_1, x_2, x_3 \geq 0$$

Since the LP problem contains more than two variables, the only way to solve the problem is simplex method. To solve the problem by simplex method, we convert it into minimization type and introduce slack variables $x_4, x_5, x_6 \geq 0$ as follows:

$$-\text{Minimize } -z = -30x_1 - 40x_2 - 20x_3 + 0x_4 + 0x_5 + 0x_6$$
$$x_1 + x_2 + x_3 + x_4 = 1000$$
$$100x_1 + 120x_2 + 70x_3 + x_5 = 100000$$
$$7x_1 + 10x_2 + 8x_3 + x_6 = 8000$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables:

Basis	C_B^t	c_j / $\underline{P}_o$	$\underline{P}_1$ (-30)	$\underline{P}_2$ (-40)	$\underline{P}_3$ (-20)	$\underline{P}_4$ (0)	$\underline{P}_5$ (0)	$\underline{P}_6$ (0)	Min Ratio θ
$\underline{P}_4$	0	1000	1	1	1	1	0	0	1000
$\underline{P}_5$	0	100000	100	120	70	0	1	0	10000/12
$\underline{P}_6$	0	8000	7	10	8	0	0	1	800 Min.
$z_j - c_j$		0	30	40	20	0	0	0	
$\underline{P}_4$	0	200	3/10	0	1/5	1	0	-1/10	2000/3
$\underline{P}_5$	0	4000	16	0	-26	0	1	-12	250 Min
$\underline{P}_2$	-40	800	7/10	1	4/5	0	0	1/10	8000/7
$z_j - c_j$		-32000	2	0	-12	0	0	-4	
$\underline{P}_4$	0	125	0	0	11/16	1	-3/160	1/8	
$\underline{P}_1$	-30	250	1	0	-13/8	0	1/16	-3/4	
$\underline{P}_2$	-40	625	0	1	31/16	0	-7/160	5/8	
$z_j - c_j$		-32500	0	0	-35/4	0	-1/8	-5/2	

Since in the 3rd table all $z_j - c_j \leq 0$, it is optimal and the optimal basic feasible solution is $x_1 = 250$, $x_2 = 625$, $x_3 = 0$ and the maximum profit, $z_{max} =$ Tk.32,500. Therefore, to yield maximum profit Tk.32,500 the farmer should grow corn in 250 acres of land and wheat in 625 acres of land.

<u>Example (4.40)</u>: A company sells two products A and B. The company makes profit Tk.8 and Tk.5 per unit of each product respectively. The two products are produced in a common process. The production process has capacity 500 man-days. It takes 2 man-days to produce one unit of A and one man-day per unit of B. The market has been surveyed and it feels that A can be sold 150 units, B of 250 units at most. Form the LP problem and then solve by simplex method, which maximizes the profit.

Solution: Let x_1 and x_2 be the number of product A and B respectively to be produced for maximizing company's total profit satisfying demands and limitations. So, company's total profit is $z = 8x_1 + 5x_2$, limitations are $2x_1 + x_2 \leq 500$, demands are $x_1 \leq 150$,

$x_2 \leq 250$ and the feasibilities are $x_1 \geq 0$, $x_2 \geq 0$. Therefore, the LP form of the given problem is

$$\begin{aligned} \text{Maximize } z &= 8x_1 + 5x_2 \\ \text{Subject to } \quad & 2x_1 + x_2 \leq 500 \\ & x_1 \leq 150 \\ & x_2 \leq 250 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

To solve the LP problem by simplex method, we convert the problem into minimization type and introduce slack variables x_3, x_4, $x_5 \geq 0$. Then we get,

$$\begin{aligned} -\text{Minimize } -z &= -8x_1 - 5x_2 + 0x_3 + 0x_4 + 0x_5 \\ \text{Subject to } \quad & 2x_1 + x_2 + x_3 + 0x_4 + 0x_5 = 500 \\ & x_1 + 0x_2 + 0x_3 + x_4 + 0x_5 = 150 \\ & 0x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 250 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Basis	C_B^t	c_j / $\underline{P}_o$	-8 $\underline{P}_1$	-5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_3$	0	500	2	1	1	0	0	250
$\underline{P}_4$	0	150	1	0	0	1	0	150 Min
$\underline{P}_5$	0	250	0	1	0	0	1	---
$z_j - c_j$		0	8	5	0	0	0	
$\underline{P}_3$	0	200	0	1	1	-2	0	200 Min.
$\underline{P}_1$	-8	150	1	0	0	1	0	---
$\underline{P}_5$	0	250	0	1	0	0	1	250
$z_j - c_j$		-1200	0	5	0	-8	0	
$\underline{P}_2$	-5	200	0	1	1	-2	0	---
$\underline{P}_1$	-8	150	1	0	0	1	0	150
$\underline{P}_5$	0	50	0	0	-1	2	1	25 Min.
$z_j - c_j$		-2200	0	0	-5	2	0	
$\underline{P}_2$	-5	250	0	1	0	0	1	
$\underline{P}_1$	-8	125	1	0	1/2	0	-1/2	
$\underline{P}_4$	0	25	0	0	-1/2	1	1/2	
$z_j - c_j$		-2250	0	0	-4	0	-1	

Since in the 4th table all $z_j - c_j \le 0$, it is optimal and the optimal basic feasible solution is $x_1 = 125$, $x_2 = 250$ and the maximum profit, z_{max}=Tk.2250. Therefore, to earn maximum profit Tk.2250, the producer should produce 125 units of product A and 250 units of product B.

Example (4.41): A company is manufacturing two products A, B and C. The manufacturing times required to make them, the profit and capacity available at each work centre are given by the following table:

Products	Work centres			Profit per unit (in $)
	Matching (in hours)	Fabrication (in hours)	Assembly (in hours)	
A	8	4	2	20
B	2	0	0	6
C	3	3	1	8
Total Capacity	250	150	50	

Company likes to maximize their profit making their products A, B and C. Formulate this linear programming problem and then solve.

Solution: If we consider x_1, x_2 and x_3 be the numbers of products A, B and C respectively to be produced for maximizing the profit. Then company's total profit $z = 20x_1 + 6x_2 + 8x_3$ is to be maximized. And subject to the constraints are $8x_1 + 2x_2 + 3x_3 \le 250$, $4x_1 + 3x_3 \le 150$ and $2x_1 + x_3 \le 50$. Since it is not possible for the manufacturer to produce negative number of the products, it is obvious that $x_1, x_2, x_3 \ge 0$. So, we can summarize the above linguistic linear programming problem as the following mathematical form:

$$\begin{aligned} &\text{Maximize } z = 20x_1 + 6x_2 + 8x_3 \\ &\text{Subject to } 8x_1 + 2x_2 + 3x_3 \le 250 \\ &\qquad 4x_1 + 3x_3 \le 150 \\ &\qquad 2x_1 + x_3 \le 50 \\ &\qquad x_1, x_2, x_3 \ge 0 \end{aligned}$$

Introducing slack variables x_4, x_5, $x_6 \geq 0$ and keeping the problem maximization type we get the following simplex tables.

Basis	C_B^t	c_j / $\underline{P}_o$	20 $\underline{P}_1$	6 $\underline{P}_2$	8 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	0 $\underline{P}_6$	Min Ratio θ	
$\underline{P}_4$	0	250	8	2	3	1	0	0	250/8	
$\underline{P}_5$	0	150	4	3	0	0	1	0	150/4	
$\underline{P}_6$	0	50	2	0	1	0	0	1	50/2 Min	
$z_j - c_j$		0	-20 Smallest	-6	-8	0	0	0		
$\underline{P}_4$	0	50	0	2	-1	1	0	-4	25	
$\underline{P}_5$	0	50	0	3	-2	0	1	-2	50/3 Min	
$\underline{P}_1$	20	25	1	0	1/2	0	0	1/2	--	
$z_j - c_j$		500	0	-6 Smallest	2	0	0	10		
$\underline{P}_4$	0	50/3	0	0	1/3	1	-2/3	-8/3	50	3
$\underline{P}_2$	6	50/3	0	1	-2/3	0	1/3	-2/3	---	$\underline{P}_4/\underline{P}_3$
$\underline{P}_1$	20	25	1	0	1/2	0	0	1/2	50	0Min
$z_j - c_j$		600	0	0	-2 Smallest	0	2	6		
$\underline{P}_4$	0	0	-2/3	0	0	1	-2/3	-3		
$\underline{P}_2$	6	50	4/3	1	0	0	1/3	0		
$\underline{P}_3$	8	50	2	0	1	0	0	1		
$z_j - c_j$		700	4	0	0	0	2	8		

Since the problem is maximization type and in the 4th table all $z_j - c_j \geq 0$, it is optimal and the optimal basic feasible solution is $x_1 = 0$, $x_2 = 50$, $x_3 = 50$ and the maximum profit, z_{max}= \$700. Therefore, to earn maximum profit \$700, the producer should produce 50 units of product B and 50 units of product C.

Mathematica code to solve the problem: **ConstrainedMax[20x₁+6x₂+8x₃, {8x₁+2x₂+3x₃≤250, 4x₁+3x₃≤150, 2x₁+x₃≤50}, {x₁, x₂, x₃}]**

Example (4.42): Maximize $x_1 + 2x_2 + 3x_3 - x_4$

Subject to
$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$
and $x_j \geq 0;\ j = 1, 2, 3, 4.$

Solution: Converting the problem as minimization type, we get

– Minimize $\quad -x_1 - 2x_2 - 3x_3 + x_4$

Subject to

$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$

and $\quad x_j \geq 0;\ j = 1, 2, 3, 4.$

Since in our constraints we get only one basis vector so, we have to take two arbitrary basis vectors. For this we rewrite our problem as follows:

Minimize $\quad -x_1 - 2x_2 - 3x_3 + x_4 + Mx_5 + Mx_6$

Subject to

$$x_1 + 2x_2 + 3x_3 + x_5 = 15$$
$$2x_1 + x_2 + 5x_3 + x_6 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$

and $\quad x_j \geq 0;\ j = 1, 2, . . ., 6$

Sl.	Basis	$\underline{C}_B^t$	$\underline{P}_o$	-1	-2	-3	1	M	M	Ratio
				$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	θ
1	$\underline{P}_5$	M	15	1	2	3	0	1	0	13/3
2	$\underline{P}_6$	M	20	2	1	5	0	0	1	$20/5=\theta_o$
3	$\underline{P}_4$	1	10	1	2	1	1	0	0	10/1
3+1	$z_j - c_j$		10	2	4	4	0	0	0	
3+2			35	3	3	8 Greatest	0	0	0	
1	$\underline{P}_5$	M	3	-1/5	7/5	0	0	0		$3/(7/5)=\theta_o$
2	$\underline{P}_3$	-3	4	2/5	1/5	1	0	0		4/(1/5)
3	$\underline{P}_4$	1	6	3/5	9/5	0	1	0		6/(9/5)
3+1	$z_j - c_j$		-6	2/5	16/5	0	0	0		
3+2			3	-1/5	7/5 Greatest	0	0	0		
1	$\underline{P}_2$	-2	15/7	-1/7	1	0	0			
2	$\underline{P}_3$	-3	25/7	3/7	0	1	0			25/3
3	$\underline{P}_4$	1	15/7	6/7	0	0	1			$15/6=\theta_o$
3+1	$z_j - c_j$		-90/7	6/7 Greatest	0	0	0			
3+2			0	0	0	0	0			
1	$\underline{P}_2$	-2	5/2	0	1	0	1/6			
2	$\underline{P}_3$	-3	5/2	0	0	1	-3/6			
3	$\underline{P}_1$	-1	5/2	1	0	0	7/6			
3+1	$z_j - c_j$		-15	0	0	0	-1			

$\therefore$ (5/2, 5/2, 5/2, 0) is the required extreme point. The minimum value of the new objective function is –15. Hence the maximum value of our given objective function is –(–15) = 15. So, the solution of the problem is $x_1 = 5/2$, $x_2 = 5/2$, $x_3 = 5/2$, $x_4 = 0$ and the maximum value is 15. <u>Mathematica code to solve the problem</u>: **ConstrainedMax[$x_1 + 2x_2 + 3x_3 - x_4$, {$x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$}, {$x_1, x_2, x_3, x_4$}]**

<u>Example (4.43)</u>: Solve the following LP problem by penalty method: Minimize $z = 2x_1 + x_2$

$$\begin{aligned} \text{Subject to } & 3x_1 + x_2 \geq 3 \\ & 4x_1 + 3x_2 \geq 6 \\ & x_1 + 3x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution: Introducing surplus variable x_3 and artificial variable x_4 to first constraint, surplus variable x_5 and artificial variable x_6 to second constraint and slack variable x_7 to third constraint, we get

$$\text{Minimize } z = 2x_1 + x_2 + 0x_3 + Mx_4 + 0x_5 + Mx_6 + 0x_7$$

$$\begin{aligned} \text{Subject to } & 3x_1 + x_2 - x_3 + x_4 = 3 \\ & 4x_1 + 3x_2 - x_5 + x_6 = 6 \\ & x_1 + 3x_2 + x_7 = 3 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

Making initial simplex table and taking necessary iterations we get,

Basis	$\underline{C}_B^t$	c_j	2	1	0	M	0	M	0	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	θ
$\underline{P}_4$	M	3	3	1	-1	1	0	0	0	3/3=1Min
$\underline{P}_6$	M	6	4	3	0	0	-1	1	0	6/4
$\underline{P}_7$	0	3	1	3	0	0	0	0	1	3/1=3
$z_j - c_j$		0	-2	-1	0	0	0	0	0	Free of M
		9	7 Greatest	4	-1	0	-1	0	0	Coef. of M
$\underline{P}_1$	2	1	1	1/3	-1/3		0	0	0	3
$\underline{P}_6$	M	2	0	5/3	4/3		-1	1	0	6/5
$\underline{P}_7$	0	2	0	8/3	1/3		0	0	1	3/4 Min.
$z_j - c_j$		2	0	-1/3	-2/3		0	0	0	Free of M
		2	0	5/3 Greatest	4/3		-1	0	0	Coef. of M

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2	1	0	M	0	M	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	
$\underline{P}_1$	2	3/4	1	0	-3/8		0	0	-1/8	---
$\underline{P}_6$	M	3/4	0	0	9/8		-1	1	-5/8	2/3 Min.
$\underline{P}_2$	1	3/4	0	1	1/8		0	0	3/8	6
$z_j - c_j$		9/4	0	0	-5/8		0	0	1/8	Free of M
		3/4	0	0	9/8 Greatest		-1	0	-5/8	Coef. of M
$\underline{P}_1$	2	1	1	0	0		-1/3		-1/3	
$\underline{P}_3$	0	2/3	0	0	1		-8/9		-5/9	
$\underline{P}_2$	1	2/3	0	1	0		1/9		4/9	
$z_j - c_j$		8/3	0	0	0		-5/8		-2/9	Free of M
		0	0	0	0		0		0	Coef. of M

Since the problem is minimization type and in the 4th table all coefficients of M of $z_j - c_j$ are zero and all constant parts of z_j - c_j are less equal zero hence the table is optimal. The optimal basic feasible solution is $x_1 = 1$, $x_2 = 2/3$ and the minimum value of the objective function is = 8/3.

<u>Mathematica code to solve the problem</u>: **ConstrainedMin[2x_1+ x_2, {3x_1+x_2 $\geq$ 3, 4x_1+3x_3 $\geq$ 6, x_1+3x_3 $\leq$ 3}, {x_1, x_2}]**

<u>Example (4.44)</u>: Solve the following LP problem by big-M method.

$$\begin{array}{ll} \text{Maximize} & 2x_1 - 6x_2 \\ \text{Subject to} & 3x_1 + 2x_2 \leq 6 \\ & x_1 - x_2 \geq -1 \\ & -x_1 - 2x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

Solution: To solve the problem by big-M method first we convert the problem as minimization type then we multiply second constraint by –1 after then introduce slack variables x_3, x_4 to first and second constraints respectively and we introduce surplus variable x_5 and artificial variable x_6 to third constraint. And then we add Mx_6 to the objective function.

$$
\begin{aligned}
-\text{Minimize}\quad & -2x_1 + 6x_2 + 0x_3 + 0x_4 + 0x_5 + Mx_6 \\
\text{Subject to}\quad & 3x_1 + 2x_2 + x_3 = 6 \\
& -x_1 + x_2 + x_4 = 1 \\
& -x_1 - 2x_2 - x_5 + x_6 = 1 \\
& x_j \geq 0;\ j = 1, 2, ..., 6
\end{aligned}
$$

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-2	6	0	0	0	M	Min Ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	θ
$\underline{P}_3$	0	6	3	2	1	0	0	0	
$\underline{P}_4$	0	1	-1	1	0	1	0	0	
$\underline{P}_6$	M	1	-1	-2	0	0	-1	1	
$z_j - c_j$		0	2	-6	0	0	0	0	Free of M
		1	-1	-2	0	0	-1	0	Coef. of M

In the above table, all coefficients of M of $z_j - c_j$ are less equal zero but artificial variable is present in the basis at positive level. Hence the problem has **no feasible solution**.

Example (4.45): Solve the following LP problem by simplex method. Minimize $x_1 + 2x_2$

$$
\begin{aligned}
\text{Subject to}\quad & x_1 - 3x_2 \leq 6 \\
& 2x_1 + 4x_2 \geq 8 \\
& x_1 - 3x_2 \geq -6 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Solution: Multiplying 3rd constraint by –1, we can rewrite the problem as follows: Minimize $x_1 + 2x_2$

$$
\begin{aligned}
\text{Subject to}\quad & x_1 - 3x_2 \leq 6 \\
& 2x_1 + 4x_2 \geq 8 \\
& -x_1 + 3x_2 \leq 6 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Introducing slack variables x_3, x_6 to 1st and 3rd constraints and surplus variable x_4 and artificial variable x_5 to 2nd constraint, we get, Minimize $x_1 + 2x_2 + 0x_3 + 0x_4 + Mx_5 + 0x_6$

$$
\begin{aligned}
\text{Subject to}\quad & x_1 - 3x_2 + x_3 = 6 \\
& 2x_1 + 4x_2 - x_4 + x_5 = 8 \\
& -x_1 + 3x_2 + x_6 = 6
\end{aligned}
$$

$$x_j \geq 0;\ \ j = 1, 2, ..., 6$$

Now, we convert the problem into the simplex table and then we take necessary iterations as follows:

Sl	Basis	$\underline{C}_B$	C_j / $\underline{P}_o$	1 / $\underline{P}_1$	2 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	M / $\underline{P}_5$	0 / $\underline{P}_6$	Min. Ratio θ_o
1	P_3	0	6	1	-3	1	0	0	0	---
2	P_5	M	8	2	4	0	-1	1	0	$8/4 = \theta_o$
3	P_6	0	6	-1	3	0	0	0	1	$6/3 = 2$
4	$z_j - c_j$		0	-1	-2	0	0	0	0	
5			8	2	4	0	0	0	0	
1	P_3	0	12	5/2	0	1	-3/4		0	$12/(5/2)$
2	P_2	2	2	1/2	1	0	-1/4		0	$2/(1/2) = \theta_o$
3	P_6	0	0	-5/2	0	0	3/4		1	
4	$z_j - c_j$		4	0	0	0	-1/2		0	
5			0	0	0	0	0		0	

From the above optimal table, we find the extreme point (0, 2, 12, 0, 0, 0). Therefore, the optimum solution is $x_1 = 0$, $x_2 = 2$ with minimum value of the objective function is 4.

We can bring P_1 vector in the basis with same objective value because in 4^{th} row of P_1 column of 2nd step we get 0 and above this 0 we get two positive elements. Taking 1/2 as pivot we get the following table.

Sl	Basis	$\underline{C}_B$	$\underline{C}_j$ / $\underline{P}_o$	1 / $\underline{P}_1$	2 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	M / $\underline{P}_5$	0 / $\underline{P}_6$	Min. Ratio θ_o
1	P_3	0	2	0	-5	1	1/2		0	
2	P_1	1	4	1	2	0	0		0	
3	P_6	0	10	0	5	0	-1/2		1	
4	$z_j - c_j$		4	0	0	0	-1/2		0	

So, the second optimal solution is $x_1 = 4$, $x_2 = 0$ and $z_{min} = 4$. There are two different optimal solutions. Hence there will exist an infinite number of optimal solutions and that are $\lambda(0,2)+(1-\lambda)(4,0)$ for each value of λ in [0,1].

Example (4.46): Using simplex method, solve the following LP problem. Maximize $x_1 + x_2$

Subject to $x_1 + x_2 \geq 1$

$x_1 - x_2 \leq 1$

$-x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$

Solution: Converting the problem as minimization type and introducing surplus variable x_3 and artificial variable x_6 to 1st constraint, slack variables x_4, x_5 to 2nd and 3rd constraints respectively, we get

– Minimize $-x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 + Mx_6$

Subject to

$$x_1 + x_2 - x_3 + x_6 = 1$$
$$x_1 - x_2 + x_4 = 1$$
$$-x_1 + x_2 + x_5 = 1$$

and $x_j \geq 0$; $j = 1, 2, ..., 6$

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-1	-1	0	0	0	M	Min. ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	θ
$\underline{P}_6$	M	1	1	1	-1	0	0	1	1 Min, it outs M
$\underline{P}_4$	0	1	1	-1	0	1	0	0	1
$\underline{P}_5$	0	1	-1	1	0	0	1	0	---
$z_j - c_j$		0	1	1	0	0	0	0	Free of M
		1	1 Let greatest	1	-1	0	0	0	Coef. of M
$\underline{P}_1$	-1	1	1	1	-1	0	0		---
$\underline{P}_4$	0	0	0	-2	1	1	0		1
$\underline{P}_5$	0	0	0	2	-1	0	1		---
$z_j - c_j$		-1	0	0	1 Greatest	0	0		Free of M
		0	0	0	0	0	0		Coef. of M
$\underline{P}_1$	-1	1	1	-1	0	1	0		
$\underline{P}_3$	0	0	0	-2	1	0	0		
$\underline{P}_5$	0	0	0	0	0	1	1		
$z_j - c_j$		-1	0	2	0	-1	0		

In the 3rd iterative table $z_2 - c_2 > 0$ but there is no positive element in $\underline{P}_2$ to consider a pivot; hence the problem has an **unbounded solution**.

Mathematica code to solve the problem: **ConstrainedMax[$x_1 + x_2$, {$x_1 + x_2 \geq 1$, $x_1 - x_2 \leq 1$, $-x_1 + x_2 \leq 1$}, {x_1, x_2}]**

Example (4.47): Using simplex method, solve the following LP problem.

$$\begin{aligned} \text{Minimize} \quad & x_1 - 2x_2 + 3x_3 \\ \text{Subject to} \quad & -2x_1 + x_2 + 3x_3 = 2 \\ & 2x_1 + 3x_2 + 4x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: Introducing artificial variables x4 and x5 to first and second constraints respectively, we get

$$\begin{aligned} \text{Minimize} \quad & x_1 - 2x_2 + 3x_3 + Mx_4 + Mx_5 \\ \text{Subject to} \quad & -2x_1 + x_2 + 3x_3 + x_4 + 0x_5 = 2 \\ & 2x_1 + 3x_2 + 4x_3 + 0x_4 + x_5 = 1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	1	-2	3	M	M	Min. ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_4$	M	2	-2	1	3	1	0	2/3
$\underline{P}_5$	M	1	2	3	4	0	1	1/4 Min.
$z_j - c_j$		0	-1	2	-3	0	0	Free of M
		3	1	4	7 Greatest	0	0	Coef. of M
$\underline{P}_4$	M	5/4	-5/2	-5/4	0	1	-	
$\underline{P}_3$	3	1/4	1/2	3/4	1	0	-	
$z_j - c_j$		3/4	1/2	17/4	0	0	-	Free of M
		5/4	-5/2	-5/4	0	0	-	Coef. of M

In the 2nd iterative table all coefficients of M of $z_j - c_j$ less or equal zero but artificial vector $\underline{P}_4$ is present in the basis at positive level; hence the problem has **no feasible solution**.

Mathematica code to solve the problem: **ConstrainedMin[$x_1 - 2x_2 + 3x_3$, {$-2x_1 + x_2 + 3x_3 = 2$, $2x_1 + 3x_2 + 4x_3 = 1$ }, {x_1, x_2, x_3}]**

<u>Example (4.48)</u>: Solve the LP problem by big-M method.

Minimize $z = 2x_1 - x_2 + 2x_3$

Subject to $\quad -x_1 + x_2 + x_3 = 4$

$-x_1 + x_2 - x_3 \leq 6$

$x_1 \leq 0$, $x_2 \geq 0$ and x_3 unrestricted.

Solution: Putting $x_1 = -x_1^{/}$, $x_1^{/} \geq 0$ and $x_3 = x_3^{/} - x_3^{//}$, $x_3^{/} \geq 0$, $x_3^{//} \geq 0$ and introducing slack variable $x_4 \geq 0$ to second constraint and artificial variable $x_5 \geq 0$ to first constraint, we get

Minimize $z = -2x_1^{/} - x_2 + 2x_3^{/} - 2x_3^{//} + 0x_4 + Mx_5$

Subject to $\quad x_1^{/} + x_2 + x_3^{/} - x_3^{//} + 0x_4 + x_5 = 4$

$x_1^{/} + x_2 - x_3^{/} + x_3^{//} + x_4 + 0x_5 = 6$

$x_1^{/}, x_2, x_3^{/}, x_3^{//}, x_4, x_5 \geq 0$

Making initial simplex table and taking necessary iterations we get,

Basis	$\underline{C}_B^t$	c_j	-2	-1	2	-2	0	M	Min. ratio
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	θ
$\underline{P}_6$	M	4	1	1	1	-1	0	1	4 Min.
$\underline{P}_5$	0	6	1	1	-1	1	1	0	6
$z_j - c_j$		0	2	1	-2	2	0	0	Free of M
		4	1 Let greatest	1	1	-1	0	0	Coef. of M
$\underline{P}_1$	-2	4	1	1	1	-1	0	-	---
$\underline{P}_5$	0	2	0	0	-2	2	1	-	1
$z_j - c_j$		-8	0	-1	-4	4 Greatest	0	-	Free of M
		0	0	0	0	0	0	-	Coef. of M
$\underline{P}_1$	-2	5	1	1	0	0	1/2	-	
$\underline{P}_4$	-2	1	0	0	-1	1	1/2	-	
$z_j - c_j$		-12	0	-1	0	0	-2	-	

Since in the last table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. The optimal solution is $x_1^{/} = 5$, $x_2 = 0$, $x_3^{/} = 0$, $x_3^{//} = 1$, $x_4 = 0$, $x_5 = 0$ with the minimum value of the objective function -12. Here,

$x_1 = -x_1' = -5$ and $x_3 = x_3' - x_3'' = 0 - 1 = -1$. So, the optimal solution of the given problem is $x_1 = -5$, $x_2 = 0$, $x_3 = -1$ and $z_{min} = -12$.

Example (4.49): Solve the following LP problem by big-M method.

$$\text{Minimize } z = 2x_1 + x_2 - x_3 - x_4$$
$$\text{Subject to } x_1 - x_2 + 2x_3 - x_4 = 2$$
$$2x_1 + x_2 - 3x_3 + x_4 = 6$$
$$x_1 + x_2 + x_3 + x_4 = 7$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Solution: Introducing artificial variables x_5, x_6, x_7 to first, second and third constraints respectively, we get

$$\text{Minimize } z = 2x_1 + x_2 - x_3 - x_4 + Mx_5 + Mx_6 + Mx_7$$
$$\text{Subject to } x_1 - x_2 + 2x_3 - x_4 + x_5 = 2$$
$$2x_1 + x_2 - 3x_3 + x_4 + x_6 = 6$$
$$x_1 + x_2 + x_3 + x_4 + x_7 = 7$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

Making initial simplex table and taking necessary iterations we get,

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2 $\underline{P}_1$	1 $\underline{P}_2$	-1 $\underline{P}_3$	-1 $\underline{P}_4$	M $\underline{P}_5$	M $\underline{P}_6$	M $\underline{P}_7$	Min Ratio θ
$\underline{P}_5$	M	2	1	-1	2	-1	1	0	0	2 Min
$\underline{P}_6$	M	6	2	1	-3	1	0	1	0	3
$\underline{P}_7$	M	7	1	1	1	1	0	0	1	7
$z_j - c_j$		0	-2	-1	2	-1	0	0	0	
		15	4 Greatest	1	0	1	0	0	0	
$\underline{P}_1$	2	2	1	-1	2	-1	-	0	0	
$\underline{P}_6$	M	2	0	3	-7	3	-	1	0	2/3 Min.
$\underline{P}_7$	M	5	0	2	-1	2	-	0	1	5/2
$z_j - c_j$		4	0	-3	5	-1	-	0	0	
		7	0	5	-8	5 Let greatest	-	0	0	
$\underline{P}_1$	2	8/3	1	0	-1/3	0	-	-	0	
$\underline{P}_4$	-1	2/3	0	1	-7/3	1	-	-	0	
$\underline{P}_7$	M	11/3	0	0	11/3	0	-	-	1	1 Min.
$z_j - c_j$		14/3	0	-2	8/3	0	-	-	0	
		11/3	0	0	11/3 Greatest	0	-	-	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2	1	-1	-1	M	M	M	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	
$\underline{P}_1$	2	3	1	0	0	0	-	-	-	
$\underline{P}_4$	-1	3	0	1	0	1	-	-	-	
$\underline{P}_3$	-1	1	0	0	1	0	-	-	-	
$z_j - c_j$		2	0	-2	0	0	-	-	-	
		0	0	0	0	0	-	-	-	

Since in the last table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Optimal solution is $x_1=3$, $x_2=0$, $x_3=1$, $x_4=3$ and $z_{min}= 2$.

Example (4.50): Solve the following LP problem by big-M method.

$$\begin{aligned} &\text{Minimize } z = -3x_1 + x_2 + 3x_3 - x_4 \\ &\text{Subject to } x_1 + 2x_2 - x_3 + x_4 = 0 \\ &\qquad 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\ &\qquad x_1 - x_2 + 2x_3 - x_4 = 6 \\ &\qquad x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Solution: Introducing artificial variables x_5, x_6, x_7 to first, second and third constraints respectively, we get

$$\begin{aligned} &\text{Minimize } z = -3x_1 + x_2 + 3x_3 - x_4 + Mx_5 + Mx_6 + Mx_7 \\ &\text{Subject to } x_1 + 2x_2 - x_3 + x_4 + x_5 = 0 \\ &\qquad 2x_1 - 2x_2 + 3x_3 + 3x_4 + x_6 = 9 \\ &\qquad x_1 - x_2 + 2x_3 - x_4 + x_7 = 6 \\ &\qquad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

Making initial simplex table and taking necessary iterations we get,

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3	1	3	-1	M	M	M	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	
$\underline{P}_5$	M	0	1	2	-1	1	1	0	0	
$\underline{P}_6$	M	9	2	-2	3	3	0	1	0	3 Let min
$\underline{P}_7$	M	6	1	-1	2	-1	0	0	1	3
$z_j - c_j$		0	3	-1	-3	1	0	0	0	
		15	4	-1	4 Let greatest	3	0	0	0	

Basis	C_B^t	c_j / P_o	P_1 (-3)	P_2 (1)	P_3 (3)	P_4 (-1)	P_5 (M)	P_6 (M)	P_7 (M)	Min Ratio θ
P_5	M	3	5/3	4/3	0	2	1	-	0	9/5 Min
P_3	3	3	2/3	-2/3	1	1	0	-	0	9/2
P_7	M	0	-1/3	1/3	0	-3	0	-	1	
$z_j - c_j$		9	5	-3	0	4	0	-	0	
		3	4/3	5/3 Greatest	0	-1	0	-	0	
P_1	-3	9/5	1	4/5	0	6/5	-	-	0	9/4
P_3	3	9/5	0	-6/5	1	1/5	-	-	0	
P_7	M	3/5	0	3/5	0	-13/5	-	-	1	1 Min.
$z_j - c_j$		0	0	-7	0	-2	-	-	0	
		3/5	0	3/5 Greatest	0	-13/5	-	-	0	
P_1	-3	1	1	0	0	14/3	-	-	-	
P_3	3	3	0	0	1	-5	-	-	-	
P_2	1	1	0	1	0	-13/3	-	-	-	
$z_j - c_j$		7	0	0	0	-97/3	-	-	-	
		0	0	0	0	0	-	-	-	

Since in the last table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. The optimal solution is $x_1=1$, $x_2=1$, $x_3=3$, $x_4=0$ and $z_{min}=7$

<u>Example (4.51)</u>: Solve the following LP problem by big-M method.

Maximize $z = x_4 - x_5$

Subject to

$$2x_2 - x_3 - x_4 + x_5 \geq 0$$
$$-2x_1 + 2x_3 - x_4 + x_5 \geq 0$$
$$x_1 - x_2 + 2x_3 - x_4 \geq 0$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Solution: We can rewrite the problem as follows:

–Minimize $z = 0x_1 + 0x_2 + 0x_3 - x_4 + x_5 + 0x_6 + 0x_7 + 0x_8 + Mx_9$

Subject to

$$0x_1 - 2x_2 + x_3 + x_4 - x_5 + x_6 = 0$$
$$2x_1 + 0x_2 - 2x_3 + x_4 - x_5 + x_7 = 0$$
$$-x_1 + x_2 - 2x_3 + x_4 + 0x_5 + x_8 = 0$$
$$x_1 + x_2 + x_3 + 0x_4 + 0x_5 + x_9 = 1$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \geq 0$$

Making initial simplex table and taking necessary iterations we get,

Basis	C_B^t	c_j / $\underline{P}_o$	0	0	0	-1	1	0	0	0	M	Min Ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	$\underline{P}_9$	θ
$\underline{P}_6$	0	0	0	-2	1	1	-1	1	0	0	0	
$\underline{P}_7$	0	0	2	0	-2	1	-1	0	1	0	0	0/2=0 Min
$\underline{P}_8$	0	0	-1	2	0	1	-1	0	0	1	0	
$\underline{P}_9$	M	1	1	1	1	0	0	0	0	0	1	1/1=1
$z_j - c_j$		0	0	0	0	1	-1	0	0	0	0	
		1	1 Let greatest	1	1	0	0	0	0	0	0	
$\underline{P}_6$	0	0	0	-2	1	1	-1	1	0	0	0	0/1=0 Min
$\underline{P}_1$	0	0	1	0	-1	1/2	-1/2	0	1/2	0	0	
$\underline{P}_8$	0	0	0	2	-1	3/2	-3/2	0	1/2	1	0	
$\underline{P}_9$	M	1	0	1	2	-1/2	1/2	0	-1/2	0	1	1/2
$z_j - c_j$		0	0	0	0	1	-1	0	0	0	0	
		1	0	1	2 Greatest	-1/2	1/2	0	-1/2	0	0	
$\underline{P}_3$	0	0	0	-2	1	1	-1	1	0	0	0	
$\underline{P}_1$	0	0	1	-2	0	3/2	-3/2	1	1/2	0	0	
$\underline{P}_8$	0	0	0	0	0	5/2	-5/2	1	1/2	1	0	
$\underline{P}_9$	M	1	0	5	0	-5/2	5/2	-2	-1/2	0	1	1/5 Min
$z_j - c_j$		0	0	0	0	-1	1	0	0	0	0	
		1	0	5 Greatest	0	-5/2	5/2	-2	-1/2	0	0	
$\underline{P}_3$	0	2/5	0	0	1	0	0	1/5	-1/5	0	-	
$\underline{P}_1$	0	2/5	1	0	0	1/2	-1/2	1/5	3/10	0	-	4/5
$\underline{P}_8$	0	0	0	0	0	5/2	-5/2	1	1/2	1	-	0 Min.
$\underline{P}_9$	0	1/5	0	1	0	-1/2	1/2	-2/5	-1/10	0	-	
$z_j - c_j$		0	0	0	0	1 Greatest	-1	0	0	0	-	
		0	0	0	0	0	0	0	0	0	-	
$\underline{P}_3$	0	2/5	0	0	1	0	0	1/5	-1/5	0	-	
$\underline{P}_1$	0	2/5	1	0	0	0	0	0	2/10	-1/5	-	
$\underline{P}_4$	-1	0	0	0	0	1	-1	2/5	1/5	2/5	-	
$\underline{P}_2$	0	1/5	0	1	0	0	0	-1/5	0	1/5	-	
$z_j - c_j$		0	0	0	0	0	0	-2/5	-1/5	-2/5	-	

Since in the last table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. The optimal solution is $x_1 = 2/5$, $x_2 = 1/5$, $x_3 = 2/5$, $x_4 = 0$, $x_5 = 0$ and $z_{max} = 0$.

<u>Example (4.52)</u>: Solve the following LP problem by big-M method.

$$\begin{aligned}
&\text{Maximize } z = 3x_1 - x_2\\
&\text{Subject to } 2x_1 + x_2 \geq 2\\
&\qquad x_1 + 3x_2 \leq 3\\
&\qquad x_2 \leq 4\\
&\qquad x_1, x_2 \geq 0
\end{aligned}$$

Solution: Converting the problem as minimization type and introducing surplus variable $x_3 \geq 0$ to first constraint, slack variables x_4, $x_5 \geq 0$ to second and third constraints respectively and artificial variable $x_6 \geq 0$ to first constraint, we get

$$\begin{aligned}
&-\text{Minimize } -z = -3x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 + Mx_6\\
&\text{Subject to } 2x_1 + x_2 - x_3 + x_6 = 2\\
&\qquad x_1 + 3x_2 + x_4 = 3\\
&\qquad x_2 + x_5 = 4\\
&\qquad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}$$

Making initial simplex table and taking necessary iterations we get,

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	1 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	M $\underline{P}_6$	Min Ratio θ
$\underline{P}_6$	M	2	2	1	-1	0	0	1	2/2=1 Min
$\underline{P}_4$	0	3	1	3	0	1	0	0	3/1=3
$\underline{P}_5$	0	4	0	1	0	0	1	0	
$z_j - c_j$		0	3	-1	0	0	0	0	
		2	2 Greatest	1	-1	0	0	0	
$\underline{P}_1$	-3	1	1	1/2	-1/2	0	0	-	
$\underline{P}_4$	0	2	0	5/2	1/2	1	0	-	4 Min
$\underline{P}_5$	0	4	0	1	0	0	1	-	
$z_j - c_j$		-3	0	-5/2	3/2 Greatest	0	0	-	
		0	0	0	0	0	0	-	
$\underline{P}_1$	-3	3	1	3	0	1	0	-	
$\underline{P}_3$	0	4	0	5	1	2	0	-	
$\underline{P}_5$	0	4	0	1	0	0	1	-	
$z_j - c_j$		-9	0	-10	0	-3	0	-	

Since in the last table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Optimal solution is $x_1 = 3$, $x_2 = 0$ and $z_{max} = -(-9) = 9$.

Example (4.53): Solve the following LP problem by big-M method.

Minimize $z = 3x_1 - x_2$

Subject to $2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_2 \leq 4$

$x_1, x_2 \geq 0$

Solution: Introducing surplus variable $x_3 \geq 0$ to first constraint, slack variables x_4, $x_5 \geq 0$ to second and third constraints respectively and artificial variable $x_6 \geq 0$ to first constraint, we get

$$\text{Minimize } z = 3x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 + Mx_6$$

$$\begin{aligned} \text{Subject to } 2x_1 + x_2 - x_3 \quad\quad + x_6 &= 2 \\ x_1 + 3x_2 \quad + x_4 \quad\quad &= 3 \\ x_2 \quad\quad + x_5 \quad &= 4 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

Making initial simplex table and taking necessary iterations we get,

Basis	C_B^t	c_j / $\underline{P}_o$	3	-1	0	0	0	M	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_6$	M	2	2	1	-1	0	0	1	2/2=1Min
$\underline{P}_4$	0	3	1	3	0	1	0	0	3/1=3
$\underline{P}_5$	0	4	0	1	0	0	1	0	
$z_j - c_j$		0	-3	1	0	0	0	0	
		2	2 Greatest	1	-1	0	0	0	
$\underline{P}_1$	3	1	1	1/2	-1/2	0	0	-	2
$\underline{P}_4$	0	2	0	5/2	1/2	1	0	-	4/5 Min
$\underline{P}_5$	0	4	0	1	0	0	1	-	4
$z_j - c_j$		-3	0	5/2 Greatest	3/2	0	0	-	
		0	0	0	0	0	0	-	
$\underline{P}_1$	3	3/5	1	0	-3/5	-1/5	0	-	
$\underline{P}_2$	-1	4/5	0	1	1/5	2/5	0	-	
$\underline{P}_5$	0	16/5	0	0	-1/5	-2/5	1	-	
$z_j - c_j$		1	0	0	-2	-1	0	-	

Since in the last table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Optimal solution is $x_1 = 3/5$, $x_2 = 4/5$ and $z_{min} = 1$.

<u>Example (4.54)</u>: An animal feed company must produce 200 kg of a mixture consisting of food A and B. The cost of A per kg is \$3 and cost of B per kg is \$5. No more than 80 kg of A can be used and at least 60 kg of B must be used. Find the minimum cost of the mixture.

Solution: Mathematical formulation of the problem:

<u>Step-1</u>: The key decision is to determine that how much food of A and B should be used to make 200 kg mixture with minimum cost.

<u>Step-2</u>: Let x_1 and x_2 kg of food of A and B should be used to make the mixture respectively.

<u>Step-3</u>: Feasible alternatives are the sets of the values of x_1 and x_2 satisfying $x_1 \geq 0$ and $x_2 \geq 0$.

<u>Step-4</u>: The objective is to minimize the cost for making 200 kg mixture, i.e., to minimize $z = 3x_1 + 5x_2$

<u>Step-5</u>: The constraints (or restrictions) are $x_1 + x_2 = 200$, $x_1 \leq 80$, $x_2 \geq 60$.

Hence the company's problem can be put in the following mathematical form: Minimize $z = 3x_1 + 5x_2$

$$\text{Subject to } x_1 + x_2 = 200$$
$$x_1 \leq 80$$
$$x_2 \geq 60$$
$$x_1, x_2 \geq 0$$

To solve the problem by simplex method, we introduce slack variable $x_3 \geq 0$ to second constraint, surplus variable $x_4 \geq 0$ to third constraint and artificial variables x_5, $x_6 \geq 0$ to first and third constraints respectively.

$$\text{Minimize } z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + Mx_5 + Mx_6$$
$$x_1 + x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 = 200$$
$$x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 + 0x_6 = 80$$
$$0x_1 + x_2 + 0x_3 - x_4 + 0x_5 + x_6 = 60$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables:

Basis	C_B^t	c_j / $\underline{P}_o$	3 $\underline{P}_1$	5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	M $\underline{P}_5$	M $\underline{P}_6$	Min Ratio θ
$\underline{P}_5$	M	200	1	1	0	0	1	0	200
$\underline{P}_3$	0	80	1	0	1	0	0	0	---
$\underline{P}_6$	M	60	0	1	0	-1	0	1	60 Min.
$z_j - c_j$		0	-3	-5	0	0	0	0	
		260	1	2 Greatest	0	-1	0	0	
$\underline{P}_5$	M	140	1	0	0	1	1	-	140
$\underline{P}_3$	0	80	1	0	1	0	0	-	80 Min.
$\underline{P}_2$	5	60	0	1	0	-1	0	-	---
$z_j - c_j$		300	-3	0	0	-5	0	-	
		140	1 Let greatest	0	0	1	0	-	
$\underline{P}_5$	M	60	0	0	-1	1	1	-	60 Min
$\underline{P}_1$	3	80	1	0	1	0	0	-	---
$\underline{P}_2$	5	60	0	1	0	-1	0	-	---
$z_j - c_j$		540	0	0	3	-5	0	-	
		60	0	0	-1	1 Greatest	0	-	
$\underline{P}_4$	0	60	0	0	-1	1	-	-	
$\underline{P}_1$	3	80	1	0	1	0	-	-	
$\underline{P}_2$	5	120	0	1	-1	0	-	-	
$z_j - c_j$		840	0	0	-2	0	-	-	

Since in the 4th table all $z_j - c_j \leq 0$, it is optimal and the optimal basic feasible solution is $x_1 = 80$, $x_2 = 120$ and $z_{min} = 840$. Therefore, to produce 200 kg mixture with minimum cost \$840, the producer should use 80 kg of food A and 120 kg of food B.

Example (4.55): Solve the following LP problem by two-phase method.

Minimize $z = 3x_1 - x_2$

Subject to $2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_2 \leq 4$

$x_1, x_2 \geq 0$

Solution: Introducing surplus variable $x_3 \geq 0$ to first constraint and slack variables $x_4 \geq 0$, $x_5 \geq 0$ to second and third constraints respectively, we get the following standard form:

$$\begin{aligned}
\text{Minimize} \quad & z = 3x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 \\
\text{Subject to} \quad & 2x_1 + x_2 - x_3 + 0x_4 + 0x_5 = 2 \\
& x_1 + 3x_2 + 0x_3 + x_4 + 0x_5 = 3 \\
& 0x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 4 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}$$

Phase-I: Since the coefficient matrix of constraints of the above standard LP problem does not contain the 3×3 identity matrix, we introduce an artificial variable $x_6 \geq 0$ to the first constraint. And we consider the auxiliary LP problem as follows:

$$\begin{aligned}
\text{Minimize} \quad & w = x_6 \\
\text{Subject to} \quad & 2x_1 + x_2 - x_3 + 0x_4 + 0x_5 + x_6 = 2 \\
& x_1 + 3x_2 + 0x_3 + x_4 + 0x_5 + 0x_6 = 3 \\
& 0x_1 + x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 = 4 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}$$

The simplex table of the auxiliary problem is as follows:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	0	0	0	0	0	1	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_6$	1	2	2	1	-1	0	0	1	2/2=1Min
$\underline{P}_4$	0	3	1	3	0	1	0	0	3/1=3
$\underline{P}_5$	0	4	0	1	0	0	1	0	---
$w_j - c_j$		2	2 Greatest	1	-1	0	0	0	
$\underline{P}_1$	0	1	1	1/2	-1/2	0	0	-	
$\underline{P}_4$	0	2	0	5/2	1/2	1	0	-	
$\underline{P}_5$	0	4	0	1	0	0	1	-	
$w_j - c_j$		0	0	0	0	0	0	-	

Since all $w_j - c_j \leq 0$, minimum of w = 0 and the artificial variable leaves the basis, the given problem has a basic feasible solution.

Phase-II: Removing the column corresponding to the artificial variable from the final table of phase-I and taking the original objective function in lieu of artificial objective function, we construct the following simplex table.

Basis	C_B^t	c_j / $\underline{P}_o$	3 $\underline{P}_1$	-1 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_1$	3	1	1	1/2	-1/2	0	0	2
$\underline{P}_4$	0	2	0	5/2	1/2	1	0	4/5 Min
$\underline{P}_5$	0	4	0	1	0	0	1	4
$z_j - c_j$		3	0	5/2 Greatest	-3/2	0	0	
$\underline{P}_1$	3	3/5	1	0	-3/5	-1/5	0	
$\underline{P}_2$	-1	4/5	0	1	1/5	2/5	0	
$\underline{P}_5$	0	16/5	0	0	-1/5	-2/5	1	
$z_j - c_j$		1	0	0	-2	-1	0	

Since all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimal basic feasible solution is $x_1 = 3/5$, $x_2 = 4/5$ and $z_{min} = 1$.

Example (4.56): Using two-phase method show that the following LP problem has an unbounded solution.

$$\text{Minimize } z = -5x_1 - x_2 + 2x_3 - x_4$$
$$\text{Subject to } x_1 + 5x_2 - 8x_3 + 3x_4 = 6$$
$$3x_1 - x_2 + x_3 + x_4 = 2$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Solution: Our given LP problem is in standard form. But its coefficient matrix of constraints does not contain a 2×2 identity matrix. To get a 2×2 unit basis matrix we add artificial variables $x_5 \geq 0$ and $x_6 \geq 0$ to first and second constraints respectively and get,

$$\text{Minimize } z = -5x_1 - x_2 + 2x_3 - x_4$$
$$\text{Subject to } x_1 + 5x_2 - 8x_3 + 3x_4 + x_5 = 6$$
$$3x_1 - x_2 + x_3 + x_4 + x_6 = 2$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Phase-I: We consider the auxiliary LP problem as follows:

$$\text{Minimize } w = x_5 + x_6$$
$$\text{Subject to } x_1 + 5x_2 - 8x_3 + 3x_4 + x_5 + 0x_6 = 6$$
$$3x_1 - x_2 + x_3 + x_4 + 0x_5 + x_6 = 2$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

The simplex table of the auxiliary problem is as follows:

Basis	$\underline{C}_B^t$	c_j	0	0	0	0	1	1	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	θ
$\underline{P}_5$	1	6	1	5	-8	3	1	0	6/1= 6
$\underline{P}_6$	1	2	3	-1	1	1	0	1	2/3 Min.
$w_j - c_j$		8	4 Let greatest	4	-7	4	0	0	
$\underline{P}_5$	1	16/3	0	16/3	-25/3	8/3	1	-	1 Min.
$\underline{P}_1$	0	2/3	1	-1/3	1/3	1/3	0	-	- - -
$w_j - c_j$		16/3	0	16/3 Greatest	-25/3	8/3	0	-	
$\underline{P}_2$	0	1	0	1	-25/16	1/2	-	-	
$\underline{P}_1$	0	1	1	0	-3/16	1/2	-	-	
$w_j - c_j$		0	0	0	0	0	-	-	

Since all $w_j - c_j \leq 0$, minimum of w = 0 and the artificial variable leaves the basis, the given problem has an optimum solution.

Phase-II: Removing the column corresponding to the artificial variables from the final table of phase-I and taking the original objective function (Minimize $z = -5x_1 - x_2 + 2x_3 - x_4$) in lieu of artificial objective function, we construct the following simplex table.

Basis	$\underline{C}_B^t$	c_j	-5	-1	2	-1	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	θ
$\underline{P}_2$	-1	1	0	1	-25/16	1/2	
$\underline{P}_1$	-5	1	1	0	-3/16	1/2	
$z_j - c_j$		-6	0	0	5/2 Greatest	-3	

In the above table $z_3 - c_3 > 0$ with all negative elements in 3rd column indicates that the problem has an unbounded solution.

<u>Example (4.57)</u>: A manufacturing company has two machines P and Q to produce three different products X, Y, Z from a common raw material. Different data are given in the table below. Determine the minimum cost of operation and the corresponding operation hours of each machine to fulfill the demand. **[JU-92]**

Machines	Product produced per hour			Machine operating cost per hour in $
	X	Y	Z	
P	2	4	3	9
Q	4	3	2	10
Demand in market	50	24	60	

Solution: Let machine P operates x_1 hours and machine Q operates x_2 hours where $x_1 \geq 0$, $x_2 \geq 0$. So the total operating cost is $9x_1+10x_2$ that is, the objective function is $z = 9x_1 + 10x_2$ which will be minimized. The demand limitations of product X, Y and Z are $2x_1 + 4x_2 \geq 50$, $4x_1 + 3x_2 \geq 24$ and $3x_1 + 2x_2 \geq 60$ respectively. Hence the mathematical formulation of given problem is

$$\begin{aligned} &\text{Minimize } z = 9x_1 + 10x_2 \\ &\text{Subject to } 2x_1 + 4x_2 \geq 50 \\ &\qquad 4x_1 + 3x_2 \geq 24 \\ &\qquad 3x_1 + 2x_2 \geq 60 \\ &\qquad x_1, x_2 \geq 0 \end{aligned}$$

Introducing non-negative surplus variables x_3, x_4 and x_5 to first, second and third constraints respectively, we get following standard form.

$$\begin{aligned} &\text{Minimize } z = 9x_1 + 10x_2 + 0x_3 + 0x_4 + 0x_5 \\ &\text{Subject to } 2x_1 + 4x_2 - x_3 + 0x_4 + 0x_5 = 50 \\ &\qquad 4x_1 + 3x_2 + 0x_3 - x_4 + 0x_5 = 24 \\ &\qquad 3x_1 + 2x_2 + 0x_3 + 0x_4 - x_5 = 60 \\ &\qquad x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Phase-I: Introducing artificial variables x_6, x_7, x_8 and artificial objective function, we get,

$$\begin{aligned} &\text{Minimize } w = x_6 + x_7 + x_8 \\ &\text{Subject to } 2x_1 + 4x_2 - x_3 + 0x_4 + 0x_5 + x_6 + 0x_7 + 0x_8 = 50 \\ &\qquad 4x_1 + 3x_2 + 0x_3 - x_4 + 0x_5 + 0x_6 + x_7 + 0x_8 = 24 \\ &\qquad 3x_1 + 2x_2 + 0x_3 + 0x_4 - x_5 + 0x_6 + 0x_7 + x_8 = 60 \\ &\qquad x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0 \end{aligned}$$

Basis	C_B^t	c_j / $\underline{P}_o$	0 $\underline{P}_1$	0 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	1 $\underline{P}_6$	1 $\underline{P}_7$	1 $\underline{P}_8$	Min Ratio θ
$\underline{P}_6$	1	50	2	4	-1	0	0	1	0	0	25/2
$\underline{P}_7$	1	24	4	3	0	-1	0	0	1	0	8 Min.
$\underline{P}_8$	1	60	3	2	0	0	-1	0	0	1	30
$w_j - c_j$		134	9	9 Let greatest	-1	-1	-1	0	0	0	
$\underline{P}_6$	1	18	-10/3	0	-1	4/3	0	1	-	0	27/2 Min
$\underline{P}_2$	0	8	4/3	1	0	-1/3	0	0	-	0	---
$\underline{P}_8$	1	44	1/3	0	0	2/3	-1	0	-	1	66
$w_j - c_j$		62	-3	0	-1	2 Greatest	1	0	-	0	
$\underline{P}_4$	0	27/2	-5/2	0	-3/4	1	1	-	-	0	---
$\underline{P}_2$	0	25/2	1/2	1	-1/4	0	0	-	-	0	25
$\underline{P}_8$	1	35	2	0	1/2	0	-1	-	-	1	35/2 Min
$w_j - c_j$		0	2 Greatest	0	1/2	0	-1	-	-	0	
$\underline{P}_4$	0	229/4	0	0	-1/8	1	-1/4	-	-	-	
$\underline{P}_2$	0	15/4	0	1	-3/8	0	1/4	-	-	-	
$\underline{P}_1$	0	35/2	1	0	1/4	0	-1/2	-	-	-	
$w_j - c_j$		0	0	0	0	0	0	-	-	-	

Since all $w_j - c_j \leq 0$, minimum of w = 0 and the artificial variables leave the basis, the given problem has a basic feasible solution.

Phase-II: Removing the columns corresponding to the artificial variables from the final table of phase-I and taking the original objective function (Minimize $z = 9x_1 + 10x_2 + 0x_3 + 0x_4 + 0x_5$) in lieu of artificial objective function, we construct the following simplex table.

Basis	C_B^t	c_j / $\underline{P}_o$	9 $\underline{P}_1$	10 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_4$	0	229/4	0	0	-1/8	1	-1/4	
$\underline{P}_2$	10	15/4	0	1	-3/8	0	1/4	
$\underline{P}_1$	9	35/2	1	0	1/4	0	-1/2	
$z_j - c_j$		195	0	0	-3/2	0	-2	

Since all $z_j - c_j \leq 0$, the optimality conditions are satisfied. The optimal basic feasible solution is $x_1 = 35/2$, $x_2 = 15/4$ and $z_{min} = 195$, that is, to fulfill the market demand machine P will work 35/2 hours, machine Q will work 15/4 hours and the minimum machine operating cost will be \$195.

<u>Example (4.58)</u>: Solve the LP problem by using big M method:

Maximize $z = 3x_1 + 2x_2$ **[NUH-01]**

Subject to $2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$

Solution: Introducing slack variable $x_3 \geq 0$ to 1st constraint and surplus variables $x_4 \geq 0$ & artificial variable $x_5 \geq 0$ to 2nd constraint, we get the standard form as follows:

Maximize $z = 3x_1 + 2x_2 + 0x_3 + 0x_4 - Mx_5$

Subject to $2x_1 + x_2 + x_3 = 2$

$3x_1 + 4x_2 - x_4 + x_5 = 12$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

We find the following initial simplex table from the problem:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	3 $\underline{P}_1$	2 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	-M $\underline{P}_5$	Min Ratio θ
$\underline{P}_3$	0	2	2	1	1	0	0	2 min.
$\underline{P}_5$	-M	12	3	4	0	-1	1	3
$z_j - c_j$		0	-3	-2	0	0	0	
		-12	-3	-4 Smallest	0	1	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	3 $\underline{P}_1$	2 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	-M $\underline{P}_5$	Min Ratio θ
$\underline{P}_2$	2	2	2	1	1	0	0	
$\underline{P}_5$	-M	4	-5	0	-4	-1	1	
$z_j - c_j$		4	1	0	2	0	0	
		-4	5	0	4	1	0	

In the second tableau, all $z_j - c_j \geq 0$, but artificial variable appears in the basis at positive level. So, the problem has no solution.

Example (4.59): Solve the LP problem by using big M method.

Maximize $z = x_1 + 5x_2$ **[NUH-03]**

Subject to $3x_1 + 4x_2 \leq 6$

$x_1 + 3x_2 \geq 2$

$x_1, x_2 \geq 0$

Solution: Introducing slack variable $x_3 \geq 0$ to 1st constraint and surplus variables $x_4 \geq 0$ & artificial variable $x_5 \geq 0$ to 2nd constraint, we get the standard form as follows:

Maximize $z = x_1 + 5x_2 + 0x_3 + 0x_4 - Mx_5$

Subject to $3x_1 + 4x_2 + x_3 = 6$

$x_1 + 3x_2 - x_4 + x_5 = 2$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

We find the following initial simplex table from the problem:

Basis	C_B^t	c_j / $\underline{P}_o$	1 $\underline{P}_1$	5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	-M $\underline{P}_5$	Min Ratio θ
$\underline{P}_3$	0	6	3	4	1	0	0	6/4
$\underline{P}_5$	-M	2	1	3	0	-1	1	2/3 min
$z_j - c_j$		0	-1	-5	0	0	0	
		-2	-1	-3 Smallest	0	1	0	

Basis	C_B^t	c_j / $\underline{P}_o$	1 $\underline{P}_1$	5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	-M $\underline{P}_5$	Min Ratio θ
$\underline{P}_3$	0	10/3	5/3	0	1	4/3		
$\underline{P}_2$	5	2/3	1/3	1	0	-1/3		
$z_j - c_j$		10/3	2/3	0	0	-5/3 Smallest		

Basis	C_B^t	c_j / $\underline{P}_o$	1 $\underline{P}_1$	5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	-M $\underline{P}_5$	Min Ratio θ
$\underline{P}_4$	0	5/2	5/4	0	3/4	1		
$\underline{P}_2$	5	31/24	4/3	1	1/4	0		
$z_j - c_j$		115/24	17/3	0	5/4	0		

The 3rd tableau gives the optimal solution $x_1 = 0$, $x_2 = 31/24$ and $z_{max} = 115/24$.

Example (4.60): Solve the following linear programming problem using the simplex method:

$$\text{Maximize } z = x_1 + 4x_2 + 5x_3 \qquad \textbf{[NUH-04]}$$
$$\text{Subject to } 3x_1 + 3x_3 \leq 22$$
$$x_1 + 2x_2 + 3x_3 \leq 14$$
$$3x_1 + 2x_2 \leq 14$$
$$x_1, x_2, x_3 \geq 0$$

Solution: Introducing slack variables $x_4 \geq 0$, $x_5 \geq 0$, $x_6 \geq 0$ to 1st, 2nd and 3rd constraints, we get

$$\text{Maximize } z = x_1 + 4x_2 + 5x_3 + 0x_4 + 0x_5 + 0x_6$$
$$\text{Subject to } 3x_1 + 3x_3 + x_4 = 22$$
$$x_1 + 2x_2 + 3x_3 + x_5 = 14$$
$$3x_1 + 2x_2 + x_6 = 14$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	C_B^t	c_j / $\underline{P}_o$	1	4	5	0	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_4$	0	22	3	0	3	1	0	0	22/3
$\underline{P}_5$	0	14	1	2	3	0	1	0	14/3
$\underline{P}_6$	0	14	3	2	0	0	0	1	
$z_j - c_j$		0	-1	-4	-5 Smallest	0	0	0	

Basis	C_B^t	c_j / $\underline{P}_o$	1	4	5	0	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_4$	0	8	2	-2	0	1	-1	0	
$\underline{P}_3$	5	14/3	1/3	2/3	1	0	1/3	0	7
$\underline{P}_6$	0	14	3	2	0	0	0	1	7 Let min
$z_j - c_j$		70/3	2/3	-2/3 Smallest	0	0	5/3	0	

Basis	C_B^t	c_j / $\underline{P}_o$	1	4	5	0	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_4$	0	22	5	0	0	1	-1	1	
$\underline{P}_3$	5	0	-2/3	0	1	0	1/3	-1/3	
$\underline{P}_2$	4	7	3/2	1	0	0	0	1/2	
$z_j - c_j$		28	5/3	0	0	0	5/3	2	

The 3rd tableau gives the optimal solution $x_1 = 0$, $x_2 = 7$, $x_3 = 0$ and $z_{max} = 28$.

<u>Example (4.61)</u>: Solve the following LP problem by simplex method. Maximize $z = 4x_1 + 5x_2 + 9x_3 + 11x_4$ **[NUH-05]**

$$\text{Subject to} \quad \begin{aligned} x_1 + x_2 + x_3 + x_4 &\leq 15 \\ 7x_1 + 5x_2 + 3x_3 + 2x_4 &\leq 120 \\ 3x_1 + 5x_2 + 10x_3 + 15x_4 &\leq 100 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Solution: To solve the problem by simplex method, we convert the problem into minimization type and introduce slack variables $x_5 \geq 0$, $x_6 \geq 0$ and $x_7 \geq 0$ to 1st, 2nd and 3rd constraints respectively. Then we get, $-\text{Minimize } -z = -4x_1 - 5x_2 - 9x_3 - 11x_4$

$$\text{Subject to} \quad \begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 \qquad\qquad &= 15 \\ 7x_1 + 5x_2 + 3x_3 + 2x_4 \qquad + x_6 \qquad &= 120 \\ 3x_1 + 5x_2 + 10x_3 + 15x_4 \qquad\qquad + x_7 &= 100 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\geq 0 \end{aligned}$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	C_B^t	c_j / $\underline{P}_o$	-4	-5	-9	-11	0	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	
$\underline{P}_5$	0	15	1	1	1	1	1	0	0	15
$\underline{P}_6$	0	120	7	5	3	2	0	1	0	60
$\underline{P}_7$	0	100	3	5	10	15	0	0	1	20/3*
$z_j - c_j$		0	4	5	9	11	0	0	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-4 $\underline{P}_1$	-5 $\underline{P}_2$	-9 $\underline{P}_3$	-11 $\underline{P}_4$	0 $\underline{P}_5$	0 $\underline{P}_6$	0 $\underline{P}_7$	Min Ratio θ
$\underline{P}_5$	0	25/3	4/5	2/5	1/3	0	1	0	-1/15	125/12*
$\underline{P}_6$	0	320/3	33/5	13/3	5/3	0	0	1	-2/15	1600/99
$\underline{P}_4$	-11	20/3	1/5	1/3	2/3	1	0	0	1/15	100/3
$z_j - c_j$		0	9/5	4/3	5/3	0	0	0	-11/15	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-4 $\underline{P}_1$	-5 $\underline{P}_2$	-9 $\underline{P}_3$	-11 $\underline{P}_4$	0 $\underline{P}_5$	0 $\underline{P}_6$	0 $\underline{P}_7$	Min Ratio
$\underline{P}_1$	-4	125/12	1	5/6	5/12	0	5/4	0	-1/12	25
$\underline{P}_6$	0	455/12	0	-7/6	-13/12	0	-33/4	1	5/12	
$\underline{P}_4$	-11	55/12	0	1/6	7/12	1	-1/4	0	1/12	55/7*
$z_j - c_j$		1105/12	0	-1/6	11/12	0	-9/4	0	-7/12	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-4 $\underline{P}_1$	-5 $\underline{P}_2$	-9 $\underline{P}_3$	-11 $\underline{P}_4$	0 $\underline{P}_5$	0 $\underline{P}_6$	0 $\underline{P}_7$	Min Ratio
$\underline{P}_1$	-4	50/7	1	5/7	0	-5/7	10/7	0	-1/7	
$\underline{P}_6$	0	325/7	0	-6/7	0	13/7	-61/7	1	4/7	
$\underline{P}_3$	-9	55/7	0	2/7	1	12/7	3/7	0	1/7	
$z_j - c_j$		695/7	0	-3/7	0	-11/7	-13/7	0	-5/7	

In the 4th table, all $z_j - c_j \leq 0$, the optimality conditions are satisfied. From the 4th table the optimal solution is $x_1 = 50/7$ (basic), $x_2 = 0$ (non-basic), $x_3 = 55/7$ (basic), $x_4 = 0$ (non-basic), $x_5 = 0$ (non-basic), $x_6 = 325/7$ (basic), $x_7 = 0$ (non-basic) and the maximum value of the given objective function, $z_{max} = -z_{min} = -(-695/7) = 695/7$. Here, x_1, x_2, x_3 and x_4 are decision variables. Hence the optimal basic feasible solution is $(x_1, x_2, x_3, x_4) = (50/7, 0, 55/7, 0)$ and $z_{max} = 695/7$.

Example (4.62): Using artificial variable technique, find the optimal solution of the following program:

Minimize $z = 4x_1 + 8x_2 + 3x_3$ **[NUH-05]**

Subject to $x_1 + x_2 \geq 2$

$2x_1 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0$

Solution: Introducing surplus variables $x_4 \geq 0$, $x_5 \geq 0$ to 1st and 2nd constraints, we get

$$\text{Minimize } z = 4x_1 + 8x_2 + 3x_3$$
$$\text{Subject to } x_1 + x_2 \quad - x_4 \quad = 2$$
$$2x_1 \quad + x_3 \quad - x_5 = 5$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	4	8	3	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	
$\underline{P}_2$	8	2	1	1	0	-1	0	2 min
$\underline{P}_3$	3	5	2	0	1	0	-1	5/2
$z_j - c_j$		31	10 Largest	0	0	-8	-3	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	4	8	3	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	
$\underline{P}_1$	4	2	1	1	0	-1	0	
$\underline{P}_3$	3	1	0	-2	1	2	-1	1/2
$z_j - c_j$		11	0	-10	0	2 Largest	-3	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	4	8	3	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	
$\underline{P}_1$	4	5/2	1	0	1/2	0	-1/2	
$\underline{P}_4$	0	1/2	0	-1	1/2	1	-1/2	
$z_j - c_j$		10	0	-8	-1	0	-2	

The 3rd tableau gives the optimal solution $x_1 = 5/2$, $x_2 = 0$, $x_3 = 0$ and $z_{min} = 10$.

4.10 <u>Exercises</u>:

46. Define simplex. Discuss the simplex method for solving a linear programming problem.
47. Write down the criterions of optimum solution.
48. State and prove fundamental theorem of linear programming.

49. Give the statement and the proof of the mini-max theorem of linear programming.
50. Discuss the simplex algorithm for solving a linear programming problem.
51. Define artificial variables and artificial basis. When do you use an artificial basis technique for solving a linear programming problem?
52. Give a brief discussion about the big M-method.
53. What do you know about the two-phase method?
54. How do you recognize that an LP problem is unbounded and the optimal solution is unique, while using the simplex method?
55. A company sells two products A and B. The company makes profit Tk.40 and Tk.30 per unit of each product respectively. The two products are produced in a common process. The production process has capacity 30,000 man-hours. It takes 3 hours to produce one unit of A and one hour per unit of B. The market has been surveyed and it feels that A can be sold 8,000 units, B of 12,000 units. Subject to above limitations form LP problem, which maximizes the profit and solve this problem by simplex method. [Answer: product A = 6000 units, product B = 12000 units and maximum profit = Tk.600000]
56. A farmer has 20 acres of land. He produces tomato and potato and can sell them all. The price he can obtain is Tk.8 per kg. for tomato and Tk.12 per kg. for potato. The average yield per acre is 2000 kg. of tomato and 1500 kg. of potato. Fertilizer is available at Tk.30 per kg. and the amount required per acre is 100 kg. each for tomato and 50 kg. for potato. Labour required for sowing, cultivation and harvesting per acre is 20 man-days for tomato and 15 man-days for potato. The farmer has 180 man-days of labour are available at Tk.80 per man-day. Formulate a linear program for this problem to maximize the farmer's total profit and then solve it by simplex method.

[Answer: He grows tomato on 12 acres of land and get maximum profit Tk.1,83,600]

57. A company produces AM and AM-FM radios. A plant of the company can be operated 48 hours per week. Production of an AM radio will require 2 hours and production of AM-FM radio will require 3 hours each. An AM radio yields Tk.40 as profit and an AM-FM radio yields Tk.80. The marketing department determined that a maximum of 15 AM and 10 AM-FM radios can be sold in a week. Formulate the problem as linear programming problem and solve it by simplex method. [Answer: 9 AM radios, 10 AM-FM radios and maximum profit = Tk.1160 in a week] [NU-02, 05]
58. A firm manufacturing two types of electrical items A and B can make a profit Tk.170 per unit of A, Tk.250 per unit of B. Both A and B uses motors and transformers. Each unit of A requires 2 motors and 4 transformers; each unit of B requires 3 motors and 2 transformers. The total supply of components per month is 210 motors and 300 transformers. Type B requires a voltage stabilizer, which has a supply restricted to 56 units per month. How many of A and B should firm manufacture to maximize the profit? Formulate the problem as LP problem and solve it by simplex method. [Answer: 60 units of product A, 30 units of product B and max. profit = Tk.17,700]
59. A company is manufacturing two products A and B. The manufacturing times required to make them, the profit and capacity available at each work centre are given by the following table:

Products	Work centres			Profit per unit (in $)
	Matching (in hours)	Fabrication (in hours)	Assembly (in hours)	
A B	1 2	5 4	3 1	80 100
Capacity	720	1800	900	

Company likes to maximize their profit making their products A and B. Formulate this linear programming problem and then solve by simplex method.

60. A company produces two types of cowboy hats. Each hat of first type requires twice as much as labour time as the second type. If all hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of first and second types to 150 and 250 hats. Assuming that the profit per hat is Tk.8 for first type and Tk.5 for second type. Formulate the problem as a linear programming problem to determine the number of hats to be produced of each type so as to maximize the profit and solve it by simplex method. [Answer: 125 units of first type, 250 units of second type and maximum profit = Tk.2250]

61. Solve the following LP problems by simplex method:

(i) Minimize $z = 2x_1 + x_2$

Subject to $3x_1 + x_2 \leq 3$

$4x_1 + 3x_2 \leq 6$

$x_1 + 2x_2 \leq 2$

$x_1, x_2 \geq 0$

[Answer: $x_1 = 0$, $x_2 = 0$, and $z_{min.} = 27/2$]

(ii) Maximize $z = 2x_1 + 3x_2$

Subject to $-x_1 + 2x_2 \leq 4$

$x_1 + x_2 \leq 6$

$x_1 + 2x_2 \leq 9$

$x_1, x_2 \geq 0$

[Answer: $x_1 = 9/2$, $x_2 = 3/2$, and $z = 27/2$]

(iii) Maximize $z = 3x_1 + 6x_2 + 2x_3$

Subject to $3x_1 + 4x_2 + x_3 \leq 2$

$x_1 + 3x_2 + 2x_3 \leq 1$

$x_1, x_2, x_3 \geq 0$

[Answer: (2/5, 1/5, 0) and $z_{max.} = 12/5$]

(iv) Minimize $z = 3x_1 + 2x_2$

Subject to $2x_1 - x_2 \geq -2$

$x_1 + 2x_2 \leq 8$

$x_1, x_2 \geq 0$

[Answer: (8, 0) and $z_{min.} = 24$]

(v) Maximize $z = x_1 + x_2 + x_3$

Subject to $2x_1 + x_2 + 2x_3 \leq 3$

$4x_1 + 2x_2 + x_3 \leq 2$

$x_1, x_2, x_3 \geq 0$

[Answer: (0, 1/3, 4/3) and Maximum value = 5/3]

(vi) Maximize $z = 4x_1 + 10x_2$

Subject to $2x_1 + x_2 \leq 50$

$2x_1 + 5x_2 \leq 100$

$2x_1 + 3x_2 \leq 90$

$x_1, x_2 \geq 0$

(vii) Maximize $z = x_1 + x_2 + x_3$
Subject to $4x_1 + 5x_2 + 3x_3 \leq 15$
$10x_1 + 7x_2 + x_3 \leq 12$
$x_1, x_2, x_3 \geq 0$
[Answer: (0, 0, 5) and
Maximum value = 5]

(viii) Maximize $z = 5x_1 + 3x_2$
Subject to $x_1 + x_2 \leq 2$
$5x_1 + 2x_2 \leq 10$
$3x_1 + 8x_2 \leq 12$
$x_1, x_2 \geq 0$
[Ans: (2, 0) and $z_{max} = 10$]

(ix) Maximize $z = 3x_1 + 4x_2$
Subject to $x_1 - x_2 \leq 1$
$-x_1 + x_2 \leq 2$
$x_1, x_2 \geq 0$
[Ans: Unbounded solution]

(x) Minimize $z = x_1 - 3x_2 + 2x_3$
Subject to $3x_1 - x_2 + 2x_3 \leq 7$
$-2x_2 + 4x_3 \leq 12$
$-4x_1 + 3x_2 + 9x_3 \leq 10$
$x_1, x_2, x_3 \geq 0$
[Answer: (0, 4, 5), $z_{min} = -11$]

62. Solve the following LP problem by simplex method:

Maximize $z = 5x_1 + 3x_2 + 7x_3$
Subject to $x_1 + x_2 + 2x_3 \leq 22$
$3x_1 + 2x_2 + x_3 \leq 26$
$x_1 + x_2 + x_3 \leq 18$
$x_1, x_2, x_3 \geq 0$

What will be the solution if the 1st constraint changes to $x_1 + x_2 + 2x_3 \leq 26$.
[Ans: (i) (6, 0, 8), $z_{max} = 86$ (ii) (26/5, 0, 52/5), $z_{max} = 494/5$]

63. Per gram of food X contains 6 units of vitamin A, 7 units of vitamin B and cost Tk.12. Per gram of food Y contains 8 units of vitamin A, 12 units of vitamin B and cost Tk.20. The daily minimum requirements of vitamin A and B are 100 units and 120 units respectively. Find the minimum cost to fulfill the vitamin requirement. [Ans: Food X = 15 grams, food Y = 5/4 grams and minimum cost = Tk.205]

64. A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the tableau below.

Machine	Time per unit (minutes)			Machine capacity (minutes/day)
	Product 1	Product 2	Product 3	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for products 1, 2 and 3 is \$4, \$3 and \$6 respectively. It is assumed that all the amounts produced are consumed in the market.

[Answer: (0, 86, 470/3) and z_{max} = \$1,198]

65. Solve the following LP problems by big-M simplex method:

(i) Minimize $z = x_1 + x_2$

Subject to $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

[Ans: (21/13,10/13),z_{min}=31/13]

(ii) Maximize $z = x_1 + 5x_2$

Subject to $3x_1 + 4x_2 \leq 6$

$x_1 + 3x_2 \geq 3$

$x_1, x_2 \geq 0$

[Ans: (0, 3/2), z_{max} = 15/2]

(iii) Maximize $z = \frac{1}{7}x_1 + 5x_2$

Subject to $3x_1 + 4x_2 \geq 12$

$x_2 \leq 10$

$x_1, x_2 \geq 0$

[Ans: Unbounded solution]

(iv) Minimize $z = 2x_1 - 6x_2$

Subject to $x_1 - x_2 \geq -1$

$-x_1 - 2x_2 \geq 1$

$x_1, x_2 \geq 0$

[Ans: No solution]

(v) Minimize $z = 2x_1 - x_2 + 2x_3$

Subject to $-x_1 + x_2 + x_3 = 4$

$-x_1 + x_2 - x_3 \leq 6$

$x_1 \leq 0$, $x_2 \geq 0$, x_3 unrestricted

[Ans: (–6, 0, –2), z_{min} = –16]

(vi) Maximize $z = 8x_2$

Subject to $x_1 - x_2 \geq 0$

$2x_1 + 3x_2 \leq 6$

x_1, x_2 are unrestricted

[Ans: (6/5,6/5),z_{max} = 48/5]

(vii) Maximize $z = 4x_1+5x_2 +2x_3$
Subject to $2x_1 + x_2 + x_3 \leq 10$
$x_1 + 3x_2 + x_3 \leq 12$
$x_1 + x_2 + x_3 = 6$
$x_1, x_2, x_3 \geq 0$
[Ans: (3, 3, 0), $z_{max} = 27$]

(viii) Minimize $z = 4x_1+ x_2$
Subject to $3x_1 + x_2 = 3$
$4x_1 + 3x_2 \geq 6$
$x_1 + 2x_2 \leq 3$
$x_1, x_2 \geq 0$
[Ans: (3/5,6/5), $z_{min} = 18/5$]

(ix) Minimize $z = x_1 - 3x_2 - 2x_3$
Subject to $3x_1 - x_2 + 2x_3 \geq 7$
$-2x_1 + 4x_2 \leq 12$
$-4x_1 + 3x_2 + 8x_3 \leq 10$
$x_1, x_2, x_3 \geq 0$
[Ans: (78/25,114/25,11/10) and $z_{min} = -319/25$]

(x) Maximize $z = 6x_1 - 3x_2 + 2x_3$
Subject to $2x_1 + x_2 + x_3 \leq 16$
$3x_1 + 2x_2 + x_3 \leq 18$
$x_2 - 2x_3 \geq 8$
$x_1, x_2, x_3 \geq 0$
[Ans: Infeasible solution]

66. Solve the following LP problems by two-phase method:

(i) Minimize $z = x_1 + x_2$
Subject to $2x_1 + x_2 \geq 4$
$x_1 + 7x_2 \geq 7$
$x_1, x_2 \geq 0$
[Ans: (21/13,10/13),z_{min}=31/13]

(ii) Maximize $z = x_1 + 5x_2$
Subject to $3x_1 + 4x_2 \leq 6$
$x_1 + 3x_2 \geq 3$
$x_1, x_2 \geq 0$
[Ans: (0, 3/2), $z_{max} = 15/2$]

(iii) Maximize $z = \frac{1}{7}x_1+5x_2$
Subject to $3x_1 + 4x_2 \geq 12$
$x_2 \leq 10$
$x_1, x_2 \geq 0$
[Ans: Unbounded solution]

(iv) Minimize $z = 2x_1 - 6x_2$
Subject to $x_1 - x_2 \geq -1$
$-x_1 - 2x_2 \geq 1$
$x_1, x_2 \geq 0$
[Ans: No solution]

(v) Minimize $z = 2x_1 - x_2 + 2x_3$
Subject to $-x_1 + x_2 + x_3 = 4$
$-x_1 + x_2 - x_3 \leq 6$
$x_1 \leq 0$, $x_2 \geq 0$, x_3 unrestricted
[Ans: $(-6, 0, -2)$, $z_{min} = -16$]

(vi) Maximize $z = 8x_2$
Subject to $x_1 - x_2 \geq 0$
$2x_1 + 3x_2 \leq 6$
x_1, x_2 are unrestricted
[Ans: $(6/5, 6/5)$, $z_{max} = 48/5$]

(vii) Maximize $z = 4x_1 + 5x_2 + 2x_3$
Subject to $2x_1 + x_2 + x_3 \leq 10$
$x_1 + 3x_2 + x_3 \leq 12$
$x_1 + x_2 + x_3 = 6$
$x_1, x_2, x_3 \geq 0$
[Ans: $(3, 3, 0)$, $z_{max} = 27$]

(viii) Minimize $z = 4x_1 + x_2$
Subject to $3x_1 + x_2 = 3$
$4x_1 + 3x_2 \geq 6$
$x_1 + 2x_2 \leq 3$
$x_1, x_2 \geq 0$
[Ans: $(3/5, 6/5)$, $z_{min} = 18/5$]

(ix) Minimize $z = x_1 - 3x_2 - 2x_3$
Subject to $3x_1 - x_2 + 2x_3 \geq 7$
$-2x_1 + 4x_2 \leq 12$
$-4x_1 + 3x_2 + 8x_3 \leq 10$
$x_1, x_2, x_3 \geq 0$
[Ans: $(78/25, 114/25, 11/10)$ and $z_{min} = -319/25$]

(x) Maximize $z = 6x_1 - 3x_2 + 2x_3$
Subject to $2x_1 + x_2 + x_3 \leq 16$
$3x_1 + 2x_2 + x_3 \leq 18$
$x_2 - 2x_3 \geq 8$
$x_1, x_2, x_3 \geq 0$
[Ans: Infeasible solution]

67. Solve the following LP problem by two-phase method.

Maximize $z = 10x_1 + 7x_3 + x_4 + 5x_5$
Subject to $x_2 + x_3 + x_5 = 5$
$x_1 + x_2 + x_3 + x_5 = 5$
$2x_1 + 3x_2 + 4x_3 + x_4 + x_5 = 10$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

Is it possible to solve the above problem by general simplex method?
[Ans: $(0, 5/2, 0, 0, 5/2)$, $z_{max} = 35$ and No.]

Chapter
05

Duality in Linear Programming

Highlights:

5.1 Introduction: (ভূমিকা) The term duality implies that every linear programming problem whether of maximization or minimization has associated with another linear programming problem based on the same data. The original problem is called the primal problem while the other is called its dual problem. It is important to note that in general, either problem can be considered as primal and the other as its dual. Thus, the two problems constitute the pair of dual problem.

Primal Problem	**Dual Problem**
Minimize $\boxed{12x_1 + 20x_2}$	Maximize $\boxed{100w_1 + 120w_2}$
Subject to	Subject to
$6x_1 + 8x_2 \geq 100$	$6w_1 + 7w_2 \leq 12$
$7x_1 + 12x_2 \geq 120$	$8w_1 + 12w_2 \leq 20$
$x_1, x_2 \geq 0$	$w_1, w_2 \geq 0$

Example (5.1): (**Primal Problem**) Vitamin A and B are found in two different Foods F_1 and F_2. One unit of Food F_1 contains 6 units of vitamin A and 7 units of vitamin B. One unit of Food F_2 contains 8 units of vitamin A and 12 units of vitamin B. One unit of Food F_1 and F_2 costs Tk.12 and Tk.20 respectively. The minimum daily requirements (for a man) of vitamin A and B are 100 and 120 units respectively. Assuming that anything is excess of daily minimum requirement of vitamin A and B is not harmful. Find out the optimum mixture of Food F_1 and F_2 at minimum cost which meets the daily minimum requirement of vitamin A and B. (This is a consumer's problem.)

Mathematical formulation of the primal problem: Let x_1 units of Food F_1 and x_2 units of Food F_2 are required to get the minimum amount of vitamin A and B; then the mathematical formulation is

$$\text{Minimize } f(\underline{x}) = 12x_1 + 20x_2$$
$$\text{Subject to } 6x_1 + 8x_2 \geq 100$$
$$7x_1 + 12x_2 \geq 120$$
$$x_1, x_2 \geq 0$$

Graphical solution of the primal problem: Drawing the constraints in the graph paper we find the shaded unbounded feasible solution space X_1ABCX_2.

The vertices A(120/7, 0), B(15, 5/4) and C(0,25/2) are basic feasible solution of the given problem. And the value of the objective function at A is 205.71, at B is 205 and at C is 250. Here the minimum value is 205 and attain at B(15, 5/4). So, Min. $f(\underline{x})$ = 205.

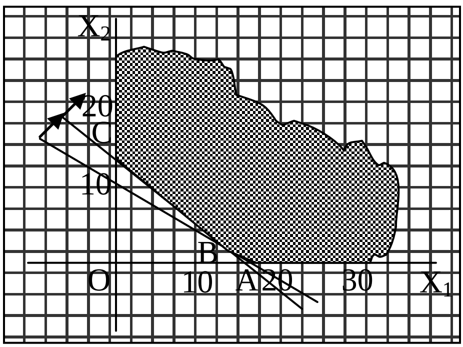

Figure 5.1

(**Dual Problem**) Vitamin A and B are found in two different Foods F_1 and F_2. One unit of Food F_1 contains 6 units of vitamin A and 7 units of vitamin B. One unit of Food F_2 contains 8 units of vitamin A and 12 units of vitamin B. One unit of Food F_1 and F_2 costs Tk.12 and Tk.20 respectively. The minimum daily requirements (for a man) of vitamin A and B are 100 and 120 units respectively. Assuming that anything is excess of daily minimum requirement of vitamin A and B is not harmful. Find out the optimum selling price of vitamin A and B. (This is the seller's problem.)

Mathematical formulation the dual problem: Let w_1 and w_2 be the selling price of vitamin A and B respectively, and then the mathematical formulation is

$$\begin{aligned} &\text{Maximize } g(\underline{w}) = 100w_1 + 120w_2 \\ &\text{Subject to } 6w_1 + 7w_2 \leq 12 \\ &\qquad 8w_1 + 12w_2 \leq 20 \\ &\qquad w_1, w_2 \geq 0 \end{aligned}$$

Graphical solution of the dual problem: Drawing the constraints in the graph paper we find the shaded feasible solution space OABC. The vertices O(0, 0), A(0, 5/3), B(1/4, 3/2) and C(2, 0) are basic feasible solution of the given problem. And the value of the objective function at O is 0, at A is 200, at B is 205 and at C is 200. Here the maximum value is 205 and attain at B(1/4, 3/2). Max. $g(\underline{w}) = 205$.

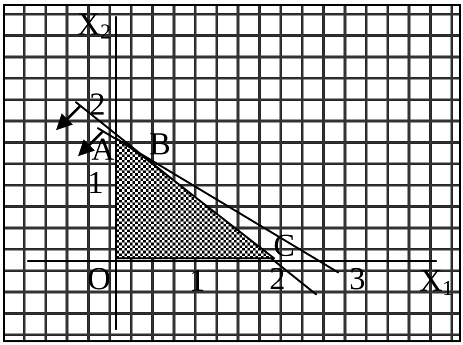

Figure 5.2

<u>Note-1</u>: If either the primal or the dual problem has a finite optimum solution, then the other problem has a finite solution and the value of the objective functions are same, i.e., $\min.f(\underline{x}) = \max.g(\underline{w})$.

If either primal or dual has an unbounded solution then the other has unbounded or no solution.

Note-2: If the primal be Minimize $f(\underline{x}) = \underline{c}\,\underline{x}$

Subject to $\underline{A}\,\underline{x} \geq \underline{b}$

$\underline{x} \geq \underline{0}$

where, $\underline{A} = (a_{ij})_{m\times n}$ is $m\times n$ (coefficient) matrix, $\underline{c} = (c_1, c_2, c_3, ..., c_n)$ is a row (cost) vector, $\underline{x} = (x_1, x_2, x_3, ..., x_n)$ and $\underline{b} = (b_1, b_2, b_3, ..., b_m)$ are column (right hand side) vectors; then the corresponding dual be Maximize $g(\underline{w}) = \underline{b}^T\,\underline{w}$

Subject to $\underline{A}^T\,\underline{w} \leq \underline{c}^T$

$\underline{w} \geq \underline{0}$

$\underline{w} = (w_1, w_2, w_3, ..., w_m)$ is a column vector and $\underline{A}^T$, $\underline{b}^T$, $\underline{c}^T$ are transpose of $\underline{A}$, $\underline{b}$, $\underline{c}$ respectively.

[Remark: If we consider $\underline{w} = (w_1, w_2, w_3, ..., w_m)$ as a row vector then the dual will become Maximize $g(\underline{w}) = \underline{w}\,\underline{b}$

Subject to $\underline{w}\,\underline{A} \leq \underline{c}$

$\underline{w} \geq \underline{0}$ **]**

And if the primal be Maximize $f(\underline{x}) = \underline{c}\,\underline{x}$

Subject to $\underline{A}\,\underline{x} \leq \underline{b}$

$\underline{x} \geq \underline{0}$

then the dual be Minimize $g(\underline{w}) = \underline{b}^T\,\underline{w}$

Subject to $\underline{A}^T\,\underline{w} \geq \underline{c}^T$

$\underline{w} \geq \underline{0}$.

These types of primal-dual problems are called symmetric primal-dual problems.

Example (5.2): Find the dual of the following primal problem:

Maximize $f(\underline{x}) = 5x_1 - 3x_2$

Subject to $4x_1 + 5x_2 \leq 45$

$3x_1 - 7x_2 \leq 15$

$x_1, x_2 \geq 0$

Solution: Here, $\underline{A} = \begin{pmatrix} 4 & 5 \\ 3 & -7 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 45 \\ 15 \end{pmatrix}$, $\underline{c} = (5, -3)$ and so, $\underline{A}^T = \begin{pmatrix} 4 & 3 \\ 5 & -7 \end{pmatrix}$, $\underline{b}^T = (45, 15)$, $\underline{c}^T = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$. Let $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, then the dual problem is

Minimize $g(\underline{w}) = \underline{b}^T \underline{w}$
Subject to $\underline{A}^T \underline{w} \geq \underline{c}^T$
$\underline{w} \geq \underline{0}$

Or, Minimize $g(\underline{w}) = (45, 15)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

Subject to $\begin{pmatrix} 4 & 3 \\ 5 & -7 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \geq \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So, the dual problem becomes, Minimize $g(\underline{w}) = 45w_1 + 15w_2$
Subject to $4w_1 + 3w_2 \geq 5$
$5w_1 - 7w_2 \geq -3$
$w_1, w_2 \geq 0$

Example (5.3): Find the dual of the following primal problem:
Minimize $g(\underline{w}) = 45w_1 + 15w_2$
Subject to $4w_1 + 3w_2 \geq 5$
$5w_1 - 7w_2 \geq -3$
$w_1, w_2 \geq 0$

Solution: Here, $\underline{A} = \begin{pmatrix} 4 & 3 \\ 5 & -7 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, $\underline{c} = (45, 15)$ and so, $\underline{A}^T = \begin{pmatrix} 4 & 5 \\ 3 & -7 \end{pmatrix}$, $\underline{b}^T = (5, -3)$, $\underline{c}^T = \begin{pmatrix} 45 \\ 15 \end{pmatrix}$. Let $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then the dual problem is Maximize $f(\underline{x}) = \underline{b}^T \underline{x}$
Subject to $\underline{A}^T \underline{x} \leq \underline{c}^T$
$\underline{x} \geq \underline{0}$

Or, Maximize $f(\underline{x}) = (5, -3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Subject to $\begin{pmatrix} 4 & 5 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 45 \\ 15 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, the dual problem becomes, Maximize $f(\underline{x}) = 5x_1 - 3x_2$

Subject to $4x_1 + 5x_2 \leq 45$

$3x_1 - 7x_2 \leq 15$

$x_1, x_2 \geq 0$

Note: Example-(5.2) and Example-(5.3) show that the dual of the dual problem is primal.

5.2 <u>Different type of primal-dual problems</u>: (বিভিন্ন ধরনের আদি-দ্বৈত সমস্যা) There are two types of primal-dual problems:

(i) Symmetric primal-dual problems

(ii) Unsymmetric primal-dual problems.

5.2.1 <u>Symmetric primal dual problems</u>: Here, both the primal and the dual are in canonical forms (see §1.6). If the primal be

Minimize $f(\underline{x}) = \underline{c}\ \underline{x}$

Subject to $\underline{A}\ \underline{x} \geq \underline{b}$

$\underline{x} \geq \underline{0}$

where, $\underline{A} = (a_{ij})_{m\times n}$ is $m\times n$ (coefficient) matrix, $\underline{c} = (c_1, c_2, c_3, ..., c_n)$ is a row (cost) vector, $\underline{x} = (x_1, x_2, x_3, ..., x_n)$ is a column vector and $\underline{b} = (b_1, b_2, b_3, ..., b_m)$ is also a column (right hand side) vectors; then the corresponding dual be

Maximize $g(\underline{w}) = \underline{b}^T\ \underline{w}$

Subject to $\underline{A}^T\ \underline{w} \leq \underline{c}^T$

$\underline{w} \geq \underline{0}$

$\underline{w} = (w_1, w_2, w_3, ..., w_m)$ is a column vector and $\underline{A}^T$, $\underline{b}^T$, $\underline{c}^T$ are transpose of $\underline{A}$, $\underline{b}$, $\underline{c}$ respectively.

And if the primal be Maximize $f(\underline{x}) = \underline{c}\ \underline{x}$

Subject to $\underline{A}\ \underline{x} \leq \underline{b}$

$\underline{x} \geq \underline{0}$

then the dual be Minimize $g(\underline{w}) = \underline{b}^T\ \underline{w}$

Subject to $\underline{A}^T\ \underline{w} \geq \underline{c}^T$

$\underline{w} \geq \underline{0}$.

<u>Example (5.4)</u>: Find the dual problem of the following symmetric primal problem:

$$\begin{aligned} &\text{Minimize } 5x_1 + 10x_2 \\ &\text{Subject to } 2x_1 + 3x_2 \geq 5 \\ &\qquad\qquad 4x_1 + 7x_2 \geq 8 \\ &\qquad\qquad x_1, x_2 \geq 0 \end{aligned}$$

Solution: Taking w_1 and w_2 as the dual variable, we get the following dual problem of the given canonical primal problem:

$$\begin{aligned} &\text{Maximize } 5w_1 + 8w_2 \\ &\text{Subject to } 2w_1 + 4w_2 \leq 5 \\ &\qquad\qquad 3w_1 + 7w_2 \leq 10 \\ &\qquad\qquad w_1, w_2 \geq 0 \end{aligned}$$

<u>Example (5.5)</u>: Find the dual problem of the following symmetric primal problem: **[CU-91]**

$$\begin{aligned} &\text{Maximize } 2x_1 + 5x_2 + 4x_3 \\ &\text{Subject to } 5x_1 + 2x_2 + 2x_3 \leq 25 \\ &\qquad\qquad 3x_1 + 8x_2 - 4x_3 \leq 10 \\ &\qquad\qquad x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: Taking w_1 and w_2 as the dual variable, we get the following dual problem of the given symmetric primal problem:

$$\begin{aligned} &\text{Minimize } 25w_1 + 10w_2 \\ &\text{Subject to } 5w_1 + 3w_2 \geq 2 \\ &\qquad\qquad 2w_1 + 8w_2 \geq 5 \\ &\qquad\qquad 2w_1 - 4w_2 \geq 4 \\ &\qquad\qquad w_1, w_2 \geq 0 \end{aligned}$$

5.2.1.1 <u>First algorithm for (symmetric) primal-dual construction</u>: The various steps involved in the construction of a pair of primal-dual linear programming problem are as follows:

<u>Step-1</u>: Using the given data, formulate the basic linear programming problem (maximization or minimization). This is primal problem.

<u>Step-2</u>: Convert the primal problem into the canonical form (see §1.6) using necessary steps.

Step-3: Identify the variables to be used in the dual problem. The number of new variables required in the dual problem equals the number of constraints in the primal.

Step-4: Using the right hand side values of the primal constraints write down the objective function of the dual. If the primal is of maximization (minimization), the dual will be a minimization (maximization) problem.

Step-5: Using the dual variables identified in step-3, write the constraints for the dual problem.

(i) If the primal is a maximization problem, the constraints in the dual must be all of '$\geq$' type. On the other hand, if the primal is a minimization problem, the constraints in the dual must be of '$\leq$' type.

(ii) The row coefficients of the primal constraints become the column coefficients of the dual constraints.

(iii) The coefficients of the primal objective function become the right hand side of the dual constraints set.

(iv) The dual variables are restricted to be non-negative.

Step-6: Making use of step-4 and step-5, write the dual problem. This is the required dual problem of the given LP problem.

[JU-90]

Example (5.6): Find the dual of the following general linear programming problem: **[DU-89]**

Minimize $z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$

subject to
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \geq b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \geq b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \geq b_m \end{array}\right\}$$

$x_j \geq 0;\ j = 1, 2, 3, \ldots, n.$

Solution: Since the given problem with m constraints is in canonical form, taking $w_1, w_2, w_3, ..., w_m$ as dual variables, we get

$$\text{Maximize } u = b_1w_1 + b_2w_2 + \ldots + b_iw_i + \ldots + b_mw_m$$

$$\text{subject to } \left.\begin{array}{l} a_{11}w_1 + a_{21}w_2 + \ldots + a_{i1}w_i + \ldots + a_{m1}w_m \le c_1 \\ a_{12}w_1 + a_{22}w_2 + \ldots + a_{i2}w_i + \ldots + a_{m2}w_m \le c_2 \\ \vdots \\ a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{ij}w_i + \ldots + a_{mj}w_m \le c_j \\ \vdots \\ a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{in}w_i + \ldots + a_{mn}w_m \le c_n \end{array}\right\}$$

$$w_i \ge 0; \ i = 1, 2, 3, ..., m.$$

This is the required dual problem.

<u>Example (5.7)</u>: Reduce the following LP problem into canonical form and then find its dual:

Maximize $z = x_1 + x_2$ **[JU-89]**

Subject to $x_1 + 2x_2 \le 5$

$-3x_1 + x_2 \ge 3$

$x_1 \ge 0$, x_2 is unrestricted in sign.

Solution: Taking $x_2 = x_2^{/} - x_2^{//}$; $x_2^{/}, x_2^{//} \ge 0$, we have

Maximize $z = x_1 + x_2^{/} - x_2^{//}$

Subject to $x_1 + 2(x_2^{/} - x_2^{//}) \le 5$

$-3x_1 + x_2^{/} - x_2^{//} \ge 3$

$x_1, x_2^{/}, x_2^{//} \ge 0$

Multiplying second constraint by –1, we have

Maximize $z = x_1 + x_2^{/} - x_2^{//}$

Subject to $x_1 + 2x_2^{/} - 2x_2^{//} \le 5$

$3x_1 - x_2^{/} + x_2^{//} \le -3$

$x_1, x_2^{/}, x_2^{//} \ge 0$, this is the canonical form.

Since, in the canonical form there are 2 constraints, we consider the following 2 variables w_1, w_2 as dual variables. So, using step-4 and step-5 of the above algorithm of primal-dual constructions, we get the following dual problem:

$$\begin{aligned} &\text{Minimize } u = 5w_1 - 3w_2 \\ &\text{Subject to } \quad w_1 + 3w_2 \geq 1 \\ &\qquad 2w_1 - w_2 \geq 1 \\ &\qquad -2w_1 + w_2 \geq -1 \\ &\qquad w_1, w_2 \geq 0 \end{aligned}$$

Or,

$$\begin{aligned} &\text{Minimize } u = 5w_1 - 3w_2 \\ &\text{Subject to } \quad w_1 + 3w_2 \geq 1 \\ &\qquad 2w_1 - w_2 \geq 1 \\ &\qquad 2w_1 - w_2 \leq 1 \\ &\qquad w_1, w_2 \geq 0 \end{aligned}$$

So,

$$\begin{aligned} &\text{Minimize } u = 5w_1 - 3w_2 \\ &\text{Subject to } \quad w_1 + 3w_2 \geq 1 \\ &\qquad 2w_1 - w_2 = 1 \\ &\qquad w_1, w_2 \geq 0 \end{aligned}$$

This is the required dual problem.

<u>Example (5.8)</u>: Find the dual problem of the following primal problem:

$$\begin{aligned} &\text{Minimize } 3x_1 + 10x_2 \\ &\text{Subject to } 2x_1 + 3x_2 \geq 5 \\ &\qquad 4x_1 + 7x_2 = 8 \\ &\qquad x_1 \geq 0,\ x_2 \geq 2 \end{aligned}$$

Solution: Putting $x_2 = y_2 + 2$; $y_2 \geq 0$, we get

$$\begin{aligned} &\text{Minimize } 3x_1 + 10y_2 + 20 \\ &\text{Subject to } 2x_1 + 3y_2 \geq -1 \\ &\qquad 4x_1 + 7y_2 = -6 \\ &\qquad x_1, y_2 \geq 0 \end{aligned}$$

Or,

$$\begin{aligned} &\text{Minimize } 3x_1 + 10y_2 + 20 \\ &\text{Subject to } 2x_1 + 3y_2 \geq -1 \\ &\qquad 4x_1 + 7y_2 \geq -6 \\ &\qquad 4x_1 + 7y_2 \leq -6 \\ &\qquad x_1, y_2 \geq 0 \end{aligned}$$

Multiplying 3^{rd} constraint by – 1, we get

Or,

$$\begin{aligned} &\text{Minimize } 3x_1 + 10y_2 + 20 \\ &\text{Subject to } 2x_1 + 3y_2 \geq -1 \\ &\qquad 4x_1 + 7y_2 \geq -6 \end{aligned}$$

$$-4x_1 - 7y_2 \geq 6$$
$x_1, y_2 \geq 0$; This is in canonical form.

Since, in the canonical form there are 3 constraints, we consider the following 3 variables w_1, w_2, w_3 as dual variables. So, using step-4 and step-5 of the above algorithm of primal-dual constructions, we get the following dual problem:

$$\begin{aligned} &\text{Maximize } -w_1 - 6w_2 + 6w_3 + 20\\ &\text{Subject to } 2w_1 + 4w_2 - 4w_3 \leq 3\\ &\qquad 3w_1 + 7w_2 - 7w_3 \leq 10\\ &\qquad w_1, w_2, w_3 \geq 0 \end{aligned}$$

Or,
$$\begin{aligned} &\text{Maximize } -w_1 - 6(w_2 - w_3) + 20\\ &\text{Subject to } 2w_1 + 4(w_2 - w_3) \leq 3\\ &\qquad 3w_1 + 7(w_2 - w_3) \leq 10\\ &\qquad w_1, w_2, w_3 \geq 0 \end{aligned}$$

Taking $w_4 = w_2 - w_3$; w_4 is unrestricted, we get

Or,
$$\begin{aligned} &\text{Maximize } -w_1 - 6w_4 + 20\\ &\text{Subject to } 2w_1 + 4w_4 \leq 3\\ &\qquad 3w_1 + 7w_4 \leq 10\\ &\qquad w_1 \geq 0, w_4 \text{ is unrestricted.} \end{aligned}$$

This is the required dual problem.

Example (5.9): Convert the following linear programming problem in canonical form and then find its dual problem:

Maximize $z = x_1 + 4x_2 + 3x_3$ **[DU-92]**

$$\begin{aligned} \text{Subject to } \quad & 2x_1 + 2x_2 - 5x_3 \leq 2\\ & 3x_1 - x_2 + 6x_3 \geq 1\\ & x_1 + x_2 + x_3 = 4\\ & x_1 \geq 3,\ x_2 \leq 0,\ x_3 \text{ unrestricted.} \end{aligned}$$

Solution: We can write the given problem as follows:

$$\left.\begin{aligned} &\text{Maximize } z = x_1 + 4x_2 + 3x_3\\ &\text{Subject to } 2x_1 + 2x_2 - 5x_3 \leq 2\\ &\qquad 3x_1 - x_2 + 6x_3 \geq 1\\ &\qquad x_1 + x_2 + x_3 \geq 4\\ &\qquad x_1 + x_2 + x_3 \leq 4\\ &\quad x_1 \geq 3,\ x_2 \leq 0,\ x_3 \text{ unrestricted.} \end{aligned}\right\} \quad \begin{aligned} &\text{Or, Maximize } z = x_1 + 4x_2 + 3x_3\\ &\text{Subject to } 2x_1 + 2x_2 - 5x_3 \leq 2\\ &\qquad -3x_1 + x_2 - 6x_3 \leq -1\\ &\qquad -x_1 - x_2 - x_3 \leq -4\\ &\qquad x_1 + x_2 + x_3 \leq 4\\ &\quad x_1 \geq 3,\ x_2 \leq 0,\ x_3 \text{ unrestricted.} \end{aligned}$$

Putting $x_1 = x_1' + 3$, $x_2 = -x_2'$, $x_3 = x_3' - x_3''$; $x_1' \geq 0, x_2' \geq 0$, $x_3' \geq 0$, $x_3'' \geq 0$, we get problem as follows:

Maximize $z = x_1' - 4x_2' + 3x_3' - 3x_3'' + 3$

Subject to
$$2x_1' - 2x_2' - 5x_3' + 5x_3'' \leq -4$$
$$-3x_1' - x_2' - 6x_3' + 6x_3'' \leq 8$$
$$-x_1' + x_2' - x_3' + x_3'' \leq -1$$
$$x_1' - x_2' + x_3' - x_3'' \leq 1$$
$$x_1', x_2', x_3', x_3'' \geq 0.$$

This is the canonical form.

Since, in the canonical form there are 4 constraints, we consider the following 4 variables w_1, w_2, w_3, w_4 as dual variables. So, we get the following dual problem:

Minimize $u = -4w_1 + 8w_2 - w_3 + w_4 + 3$

Subject to
$$2w_1 - 3w_2 - w_3 + w_4 \geq 1$$
$$-2w_1 - w_2 + w_3 - w_4 \geq -4$$
$$-5w_1 - 6w_2 - w_3 + w_4 \geq 3$$
$$5w_1 + 6w_2 + w_3 - w_4 \geq -3$$
$$w_1, w_2, w_3, w_4 \geq 0$$

Or, Minimize $u = -4w_1 + 8w_2 - w_3 + w_4 + 3$

Subject to
$$2w_1 - 3w_2 - w_3 + w_4 \geq 1$$
$$-2w_1 - w_2 + w_3 - w_4 \geq -4$$
$$-5w_1 - 6w_2 - w_3 + w_4 \geq 3$$
$$-5w_1 - 6w_2 - w_3 + w_4 \leq 3$$
$$w_1, w_2, w_3, w_4 \geq 0$$

[3rd and 4th constraints together means $-5w_1 - 6w_2 - w_3 + w_4 = 3$]

Or, Minimize $u = -4w_1 + 8w_2 - w_3 + w_4 + 3$

Subject to
$$2w_1 - 3w_2 - w_3 + w_4 \geq 1$$
$$2w_1 + w_2 - w_3 + w_4 \leq 4$$
$$-5w_1 - 6w_2 - w_3 + w_4 = 3$$
$$w_1, w_2, w_3, w_4 \geq 0$$

Taking $w_5 = -w_2$; $w_5 \leq 0$, $w_6 = w_4 - w_3$; w_6 is unrestricted, we get

Or, Minimize $u = -4w_1 - 8w_5 + w_6 + 3$

Subject to $2w_1 + 3w_5 + w_6 \geq 1$

$2w_1 - w_5 + w_6 \leq 4$

$-5w_1 + 6w_5 + w_6 = 3$

$w_1 \geq 0,\ w_5 \leq 0,\ w_6$ unrestricted

This is the required dual problem.

5.2.2 <u>Unsymmetric primal-dual problems</u>: If the primal problem is not in canonical form, then it is called unsymmetric primal-dual problem. If the primal be Minimize $f(\underline{x}) = \underline{c}\ \underline{x}$

Subject to $\underline{A}\ \underline{x} =$ (or, $\leq$ or, $\geq$) $\underline{b}$

$\underline{x} \geq$ (or, $\leq$ or, unrestricted) $\underline{0}$

where, $\underline{A} = (a_{ij})_{m\times n}$ is $m\times n$ (coefficient) matrix, $\underline{c} = (c_1, c_2, c_3, ..., c_n)$ is a row (cost) vector, $\underline{x} = (x_1, x_2, x_3, ..., x_n)$ is a column vector and $\underline{b} = (b_1, b_2, b_3, ..., b_m)$ is also a column (right hand side) vectors; then the corresponding dual be

Maximize $g(\underline{w}) = \underline{b}^T\ \underline{w}$

Subject to $\underline{A}^T\ \underline{w} \leq$ (or, $\geq$ or, $=$) $\underline{c}^T$ [depends on primal's variables]

$\underline{w}$ is unrestricted (or, $\leq \underline{0}$ or, $\geq \underline{0}$) [depends on p's constraints]

$\underline{w} = (w_1, w_2, w_3, ..., w_m)$ is a column vector and $\underline{A}^T$, $\underline{b}^T$, $\underline{c}^T$ are transpose of $\underline{A}$, $\underline{b}$, $\underline{c}$ respectively.

And if the primal be Maximize $f(\underline{x}) = \underline{c}\ \underline{x}$

Subject to $\underline{A}\ \underline{x} =$ (or, $\geq$ or, $\leq$) $\underline{b}$

$\underline{x} \geq$ (or, $\leq$ or, unrestricted) $\underline{0}$

then the dual be

Minimize $g(\underline{w}) = \underline{b}^T\ \underline{w}$

Subject to $\underline{A}^T\ \underline{w} \geq$ (or, $\leq$ or, $=$) $\underline{c}^T$ [depends on p's variables]

$\underline{w}$ is unrestricted (or, $\leq \underline{0}$ or, $\geq$) [depends on p's constraints]

<u>Example (5.10)</u>: Find the dual of the following unsymmetrical primal problem: Minimize $2x_1 + 8x_2$ **[RU-95]**

Subject to $3x_1 + 5x_2 = 16$

$2x_1 + 4x_2 = 12$

x_1, x_2 is unrestricted.

Solution: Here, A = $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$, b = $\begin{pmatrix} 16 \\ 12 \end{pmatrix}$, c = (2, 8) and x = $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

So, $\underline{A}^T = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$, $\underline{b}^T = (16, 12)$, $\underline{c}^T = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$, and taking, $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ as the dual variable, we get,

Maximize $\underline{b}^T\ \underline{w}$

Subject to $\underline{A}^T\ \underline{w} = \underline{c}^T$ [depending on the restriction of p's variables]

$\underline{w}$ is unrestricted [depending on the p's constraints]

Or, Maximize $(16, 12)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

Subject to $\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$

$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is unrestricted

Or, Maximize $16w_1 + 12w_2$

Subject to $3w_1 + 2w_2 = 2$

$5w_1 + 4w_2 = 8$

w_1, w_2 are unrestricted.

This is the required dual problem.

Example (5.11): Find the dual of the following unsymmetrical primal problem:

Minimize $2x_1 + 8x_2$

Subject to $3x_1 + 5x_2 \leq 16$

$2x_1 + 4x_2 = 12$

$x_1 \geq 0$, x_2 is unrestricted.

Solution: Taking w_1, w_2 as dual variable, we get the dual of the given primal as follows:

Maximize $16w_1 + 12w_2$

Subject to $3w_1 + 2w_2 \leq 2$

$5w_1 + 4w_2 = 8$

$w_1 \leq 0$, w_2 is unrestricted.

Example (5.12): Find the dual of the following unsymmetrical primal problem: Maximize $3x_1 - 5x_2$

Subject to $2x_1 + 8x_2 \leq 10$

$7x_1 + 4x_2 = 12$

$x_1 \geq 0,\ x_2 \leq 0$

Solution: Taking w_1, w_2 as dual variable, we get the dual of the given primal as follows:

Minimize $10w_1 + 12w_2$

Subject to $2w_1 + 7w_2 \geq -3$

$8w_1 + 4w_2 \leq 5$

$w_1 \geq 0$, w_2 is unrestricted.

5.2.2.1 Second algorithm for (unsymmetric) primal-dual construction: The various steps involved in the construction of a pair of primal-dual linear programming problem are as follows:

Step-1: Using the given data, formulate the basic linear programming problem (maximization or minimization). This is primal problem.

Step-2: Convert the primal problem into the standard form (see §1.5) using necessary steps.

Step-3: Identify the variables to be used in the dual problem. The number of new variables required in the dual problem equals the number of constraints in the primal.

Step-4: Using the right hand side values of the primal constraints write down the objective function of the dual. If the primal is of maximization (minimization), the dual will be a minimization (maximization) problem.

Step-5: Using the dual variables identified in step-3, write the constraints for the dual problem.

(i) If the primal is a maximization problem, the constraints in the dual must be all of '$\geq$' type. On the other hand, if the primal is a minimization problem, the constraints in the dual must be of '$\leq$' type.

(ii) The row coefficients of the primal constraints become the column coefficients of the dual constraints.
(iii) The coefficients of the primal objective function become the right hand side of the dual constraints set.
(iv) The dual variables are defined to be unrestricted in sign.
Step-6: Making use of step-4 and step-5, write the dual problem. This is the required dual problem of the given LP problem.

Note: To find the dual of any given primal problem, one can use the first algorithm for (symmetric) primal-dual construction.

Example (5.13): Find the dual of the following unsymmetrical primal problem:

$$\begin{aligned} \text{Maximize } & 2x_1 + 5x_2 \\ \text{Subject to } & 3x_1 + 8x_2 = 11 \\ & 5x_1 + 4x_2 = 15 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution: The given primal problem is in standard form. So, taking w_1, w_2 as dual variable and using step-4 & step-5 of the above algorithm, we get the dual of the given primal as follows:

$$\begin{aligned} \text{Minimize } & 11w_1 + 15w_2 \\ \text{Subject to } & 3w_1 + 5w_2 \geq 2 \\ & 8w_1 + 4w_2 \geq 5 \\ & w_1, w_2 \text{ are unrestricted.} \end{aligned}$$

Example (5.14): Find the dual problem of the following unsymmetrical primal problem.

$$\begin{aligned} \text{Minimize } z = \; & x_1 + 4x_2 + 3x_3 \qquad \textbf{[JU-93]} \\ \text{Subject to } \; & 2x_1 + 2x_2 - 5x_3 \leq 2 \\ & 3x_1 - x_2 + 6x_3 \geq 1 \\ & x_1 + x_2 + x_3 = 4 \\ & x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ unrestricted.} \end{aligned}$$

Solution: Step-1: The given problem is a unsymmetrical minimizing primal problem.

Step-2: Putting $x_2 = -x_2'$, $x_3 = x_3' - x_3''$; $x_2' \ge 0$, $x_3' \ge 0$, $x_3'' \ge 0$ and introducing slack variable $s_1 \ge 0$ in 1st constraint and surplus variable $s_2 \ge 0$ in 2nd constraint, we get the problem as follows:

$$\begin{aligned}
\text{Minimize } z = {} & x_1 - 4x_2' + 3x_3' - 3x_3'' + 0.s_1 + 0.s_2 \\
\text{Subject to } & 2x_1 - 2x_2' - 5x_3' + 5x_3'' + s_1 = 2 \\
& 3x_1 + x_2' + 6x_3' - 6x_3'' - s_2 = 1 \\
& x_1 - x_2' + x_3' - x_3'' = 4 \\
& x_1, x_2', x_3', x_3'', s_1, s_2 \ge 0.
\end{aligned}$$

This is the standard form.

Step-3: In the primal problem, there are 3 constraint equations. So, we take 3 dual variables w_1, w_2, w_3.

Step-4: The objective function of the dual is

$$\text{Maximize } u = 2w_1 + w_2 + 4w_3$$

Step-5: The constraints of the dual are as follows:

$$\begin{aligned}
\text{Subject to } & 2w_1 + 3w_2 + w_3 \le 1 \\
& -2w_1 + w_2 - w_3 \le -4 \\
& -5w_1 + 6w_2 + w_3 \le 3 \\
& 5w_1 - 6w_2 - w_3 \le -3 \\
& w_1 + 0w_2 + 0w_3 \le 0 \\
& 0w_1 - w_2 + 0w_3 \le 0 \\
& w_1, w_2, w_3 \text{ are unrestricted in sign.}
\end{aligned}$$

Or,

$$\begin{aligned}
\text{Subject to } & 2w_1 + 3w_2 + w_3 \le 1 \\
& 2w_1 - w_2 + w_3 \ge 4 \\
& -5w_1 + 6w_2 + w_3 = 3 \\
& w_1 \le 0,\ w_2 \ge 0,\ w_3 \text{ is unrestricted in sign.}
\end{aligned}$$

Step-6: So, the required dual problem is as follows:

$$\begin{aligned}
\text{Maximize } u = {} & 2w_1 + w_2 + 4w_3 \\
\text{Subject to } & 2w_1 + 3w_2 + w_3 \le 1 \\
& 2w_1 - w_2 + w_3 \ge 4 \\
& -5w_1 + 6w_2 + w_3 = 3 \\
& w_1 \le 0,\ w_2 \ge 0,\ w_3 \text{ is unrestricted in sign.}
\end{aligned}$$

5.3 <u>Primal-dual tables</u>: (আদি-দ্বৈত তালিকা) Summarizing all the primal dual correspondences for all linear programming problems, we form the following two primal-dual tables to form the dual problem of a given primal problem. **[DU-93]**

Table-1: (If the primal be a maximizing problem)

Criterion	Primal (Maximizing z)	Dual (Minimizing u)
Coefficient matrix	$\underline{A}$	$\underline{A}^T$
Cost vector	$\underline{c}$	$\underline{b}^T$
Right hand side vector	$\underline{b}$	$\underline{c}^T$
Constant term in objective function	If a constant term $\pm K$ with objective function	Then the constant $\pm K$ remains with objective function
Dual variables depending on primal constraints	m constraints	m variables (w_1, w_2, w_3, ..., w_m)
	ith constraint is an equaton	ith variable w_i is unrestricted
	ith constraint is $\leq$ type	ith variable $w_i \geq 0$
	ith constraint is $\geq$ type	ith variable $w_i \leq 0$
	ith constraint is $\approx$ type	ith variable $w_i = 0$
Dual constraints depending on primal variables	n variables (x_1, x_2, x_3, ..., x_n)	n constraints
	jth variable x_j is unrestricted	jth constraint is an equation
	jth variable $x_j \geq 0$ type	jth constraint is $\geq$ type
	jth variable $x_j \leq 0$ type	jth constraint is $\leq$ type
Solutions	Finite optimum solution, z_{max}	Finite optimum solution, $u_{min} = z_{max}$
	Unbounded solution	No solution or unbounded solution
	No solution	No solution or unbounded solution

Example (5.15): Find the dual of the following primal problem:

Maximize $z = 7x_1 + 3x_2 + 8x_3$ **[N.U.M.Sc(Pre.)-03]**

$$\begin{aligned} \text{Subject to } 8x_1 + 2x_2 + x_3 &\geq 7 \\ 3x_1 + 6x_2 + 4x_3 &\geq 4 \\ 4x_1 + x_2 + 5x_3 &\geq 1 \\ x_1 + 5x_2 + 2x_3 &\geq 7 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution: Taking w_1, w_2, w_3, w_4 as dual variables and using the primal-dual table, we get the dual of the given unsymmetric primal problem as follows:

Minimize $u = 7w_1 + 4w_2 + w_3 + 7w_4$

$$\begin{aligned} \text{Subject to } 8w_1 + 3w_2 + 4w_3 + w_4 &\geq 7 \\ 2w_1 + 6w_2 + w_3 + 5w_4 &\geq 3 \\ w_1 + 4w_2 + 5w_3 + 2w_4 &\geq 8 \\ w_1, w_2, w_3, w_4 &\leq 0 \end{aligned}$$

This is the required dual problem.

Example (5.16): Find the dual of the following primal problem:

Maximize $z = 3x_1 + 4x_2 + 5x_3$ **[JU-93]**

$$\begin{aligned} \text{Subject to } 4x_1 + 2x_2 + x_3 &\geq 5 \\ 2x_1 + 5x_2 + 3x_3 &\leq 10 \\ 5x_1 + 2x_2 + 4x_3 &\geq 4 \\ 2x_1 + x_2 + 5x_3 &= 9 \\ x_1, x_2 \geq 0,\ x_3 &\leq 0 \end{aligned}$$

Solution: Taking w_1, w_2, w_3, w_4 as dual variables and using the primal-dual table, we get the dual of the given unsymmetric primal problem as follows:

Minimize $u = 5w_1 + 10w_2 + 4w_3 + 9w_4$

$$\begin{aligned} \text{Subject to } 4w_1 + 2w_2 + 5w_3 + 2w_4 &\geq 3 \\ 2w_1 + 5w_2 + 2w_3 + w_4 &\geq 4 \\ w_1 + 3w_2 + 4w_3 + 5w_4 &\leq 5 \\ w_1, w_3 \leq 0,\ w_2 \geq 0,\ w_4 &\text{ unrestricted.} \end{aligned}$$

This is the required dual problem.

Table-2: (If the primal be a minimizing problem)

Criterion	Primal (Minimizing z)	Dual (Maximizing u)
Coefficient matrix	$\underline{A}$	$\underline{A}^T$
Cost vector	$\underline{c}$	$\underline{b}^T$
Right hand side vector	$\underline{b}$	$\underline{c}^T$
Constant term in objective function	If a constant term $\pm$K with objective function	Then the constant $\pm$K remains with objective function
Dual variables depending on primal constraints	m constraints	m variables (w_1, w_2, w_3, ..., w_m)
	ith constraint is an equaton	ith variable w_i is unrestricted
	ith constraint is $\leq$ type	ith variable $w_i \leq 0$
	ith constraint is $\geq$ type	ith variable $w_i \geq 0$
	ith constraint is $\approx$ type	ith variable $w_i = 0$
Dual constraints depending on primal variables	n variables (x_1, x_2, x_3, ..., x_n)	n constraints
	jth variable x_j is unrestricted	jth constraint is an equation
	jth variable $x_j \geq 0$ type	jth constraint is $\leq$ type
	jth variable $x_j \leq 0$ type	jth constraint is $\geq$ type
Solutions	Finite optimum solution, z_{min}	Finite optimum solution, $u_{max} = z_{min}$
	Unbounded solution	No solution or unbounded solution
	No solution	No solution or unbounded solution

Example (5.17): Find the dual of the following linear programming problem: Minimize $z = 3x_1 + 4x_2 + 2x_3$ **[NUH-04]**

Subject to $3x_1 + x_2 - 2x_3 = 4$

$x_1 - 6x_2 + 3x_3 \leq 1$

$$2x_1 + x_2 - x_3 \geq 2$$
$$x_1, x_2 \geq 0, x_3 \leq 0$$

Solution: Taking w_1, w_2, w_3 as dual variables and using the primal-dual table, we get the dual of the given linear programming problem as follows:

$$\text{Maximize } u = 4w_1 + w_2 + 2w_3$$
$$\text{Subject to } 3w_1 + w_2 + 2w_3 \leq 3$$
$$w_1 - 6w_2 + w_3 \leq 4$$
$$-2w_1 + 3w_2 - w_3 \geq 2$$
$$w_2 \leq 0, w_3 \geq 0, w_1 \text{ unrestricted.}$$

This is the required dual problem.

Example (5.18): Find the dual of the following LP problem:

$$\text{Minimize } z = x_1 + 2x_2 + 3x_3$$ **[NUH-05]**

$$\text{Subject to } x_1 - x_2 + 2x_3 \geq 4$$
$$x_1 + x_2 + 2x_3 \leq 8$$
$$x_2 - x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

Solution: Taking w_1, w_2, w_3 as dual variables and using the primal-dual table, we get the dual of the given LP problem as follows: Maximize $u = 4w_1 + 8w_2 + 2w_3$

$$\text{Subject to } w_1 + w_2 \leq 1$$
$$-w_1 + w_2 + w_3 \leq 2$$
$$2w_1 + 2w_2 - w_3 \leq 3$$
$$w_1, w_3 \geq 0, w_2 \leq 0$$

This is the required dual problem.

Example (5.19): Write the dual of the following LP problem:

$$\text{Minimize } z = 3x_1 + 2x_2 + 7x_3$$
$$\text{Subject to } 9x_1 - x_2 + 2x_3 \geq 5$$
$$3x_1 + x_2 + 5x_3 \leq 25$$
$$2x_1 + 3x_2 - x_3 \approx 8$$
$$x_1, x_2, x_3 \geq 0$$

Solution: Taking w_1, w_2, w_3 as dual variables and using the primal-dual table, we get the dual of the given LP problem as follows: Maximize $u = 5w_1 + 25w_2 + 8w_3$

$$\begin{aligned} \text{Subject to } 9w_1 + 3w_2 + 2w_3 &\le 3 \\ -w_1 + w_2 + 3w_3 &\le 2 \\ 2w_1 + 5w_2 - w_3 &\le 7 \\ w_1 \ge 0,\ w_2 \le 0,\ w_3 &= 0 \end{aligned}$$

As $w_3 = 0$, vanishing w_3, we get the required dual problem as follows: Maximize $u = 5w_1 + 25w_2$

$$\begin{aligned} \text{Subject to } 9w_1 + 3w_2 &\le 3 \\ -w_1 + w_2 &\le 2 \\ 2w_1 + 5w_2 &\le 7 \\ w_1 \ge 0,\ w_2 &\le 0 \end{aligned}$$

5.4 Some theorems on duality: (দ্বৈততার কিছু তত্ত্ব) Some very necessary theorems on duality are discussed below:

Theorem (5.1): If any constraint of the primal be an equation, then the corresponding dual variable will be unrestricted in sign.

Proof: Let the ith constraint of the given general linear programming problem be an equation **[JU-92]**

Minimize $z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$

$$\text{subject to } \left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \ge b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \ge b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n = b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \ge b_m \end{array}\right\}$$

$x_j \ge 0;\ j = 1, 2, 3, \ldots, n.$

We can write the given primal as follows:

Minimize $z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$

$$\text{subject to}\quad \left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \geq b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \geq b_i \\ -a_{i1}x_1 - a_{i2}x_2 - \ldots - a_{ij}x_j - \ldots - a_{in}x_n \geq -b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \geq b_m \end{array}\right\}$$

$x_j \geq 0;\ j = 1, 2, 3, \ldots, n.$

The dual of the above primal can be written as follows:

Maximize $u = b_1w_1 + b_2w_2 + \ldots + b_i(w_i^{/} - w_i^{//}) + \ldots + b_mw_m$

$$\text{subject to}\quad \left.\begin{array}{l} a_{11}w_1 + a_{21}w_2 + \ldots + a_{i1}(w_i^{/} - w_i^{//}) + \ldots + a_{m1}w_m \leq c_1 \\ a_{12}w_1 + a_{22}w_2 + \ldots + a_{i2}(w_i^{/} - w_i^{//}) + \ldots + a_{m2}w_m \leq c_2 \\ \vdots \\ a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{ij}(w_i^{/} - w_i^{//}) + \ldots + a_{mj}w_m \leq c_j \\ \vdots \\ a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{in}(w_i^{/} - w_i^{//}) + \ldots + a_{mn}w_m \leq c_n \end{array}\right\}$$

$w_1, w_2, \ldots, w_{i-1},\ w_i^{/},\ w_i^{//},\ w_{i+1}, \ldots, w_m \geq 0$

Putting $w_i = w_i^{/} - w_i^{//}$ and then w_i is unrestricted in sign, we get

Maximize $u = b_1w_1 + b_2w_2 + \ldots + b_iw_i + \ldots + b_mw_m$

$$\text{subject to}\quad \left.\begin{array}{l} a_{11}w_1 + a_{21}w_2 + \ldots + a_{i1}w_i + \ldots + a_{m1}w_m \leq c_1 \\ a_{12}w_1 + a_{22}w_2 + \ldots + a_{i2}w_i + \ldots + a_{m2}w_m \leq c_2 \\ \vdots \\ a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{ij}w_i + \ldots + a_{mj}w_m \leq c_j \\ \vdots \\ a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{in}w_i + \ldots + a_{mn}w_m \leq c_n \end{array}\right\}$$

$w_1, w_2, ..., w_{i-1}, w_{i+1}, ..., w_m \geq 0$, w_i is unrestricted.
Hence the theorem is proved.

Example (5.20): Find the dual of the following linear programming problem:

Minimize $z = 4x_1 + 5x_2$
Subject to $3x_1 + 2x_2 \geq 20$
$4x_1 + 3x_2 \geq 10$
$x_1 + x_2 = 5$
$x_1, x_2 \geq 0$

Solution: Taking w_1, w_2, w_3 as dual variables and using the primal-dual table, we get the dual of the given LP problem as follows: Maximize $u = 20w_1 + 10w_2 + 5w_3$
Subject to $3w_1 + 4w_2 + w_3 \leq 4$
$2w_1 + 3w_2 + w_3 \leq 5$
$w_1, w_2 \geq 0$, w_3 is unrestricted in sign.

In the primal 3rd constraint is an equation and in the dual 3rd variable is unrestricted in sign.

Example (5.21): Using primal-dual table, write the dual of the following LP problem:

Maximize $z = 5x_1 + 7x_2 - 4x_3$
Subject to $9x_1 + 2x_2 + 2x_3 \leq 25$
$x_1 + 3x_2 + x_3 = 10$
$3x_1 + x_3 \leq 35$
$x_1, x_2, x_3 \geq 0$

Solution: Taking w_1, w_2, w_3 as dual variables and using the primal-dual table, we get the dual of the given LP problem as follows: Minimize $u = 25w_1 + 10w_2 + 35w_3$
Subject to $9w_1 + w_2 + 3w_3 \geq 5$
$2w_1 + 3w_2 \geq 7$
$2w_1 + w_2 + w_3 \geq -4$
$w_1, w_3 \geq 0$, w_2 is unrestricted in sign.

In the primal 2nd constraint is an equation and in the dual 2nd variable is unrestricted in sign.

<u>Theorem (5.2)</u>: If any variable of the primal be unrestricted in sign, then the corresponding dual constraint will be an equation.

Proof: Let the jth variable of the given general linear programming problem be unrestricted in sign.

Minimize $z = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$

subject to
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \geq b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \geq b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \geq b_m \end{array}\right\}$$

$x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \geq 0$, x_j is unrestricted.

Putting $x_j = x_j' - x_j''$, where, $x_j' \geq 0$, $x_j'' \geq 0$ we can write the given primal as follows:

Minimize $z = c_1x_1 + c_2x_2 + \ldots + c_j(x_j' - x_j'') + \ldots + c_nx_n$

subject to
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}(x_j' - x_j'') + \ldots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}(x_j' - x_j'') + \ldots + a_{2n}x_n \geq b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}(x_j' - x_j'') + \ldots + a_{in}x_n \geq b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}(x_j' - x_j'') + \ldots + a_{mn}x_n \geq b_m \end{array}\right\}$$

$x_1, x_2, \ldots, x_{j-1}, x_j', x_j'', x_{j+1}, \ldots, x_n \geq 0$

The dual of the above primal can be written as follows:

Maximize $u = b_1w_1 + b_2w_2 + \ldots + b_iw_i + \ldots + b_mw_m$

$$\text{subject to} \quad \left.\begin{array}{l} a_{11}w_1 + a_{21}w_2 + \ldots + a_{i1}w_i + \ldots + a_{m1}w_m \le c_1 \\ a_{12}w_1 + a_{22}w_2 + \ldots + a_{i2}w_i + \ldots + a_{m2}w_m \le c_2 \\ \vdots \\ a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{ij}w_i + \ldots + a_{mj}w_m \le c_j \\ -a_{1j}w_1 - a_{2j}w_2 - \ldots - a_{ij}w_i - \ldots - a_{mj}w_m \le -c_j \\ \vdots \\ a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{in}w_i + \ldots + a_{mn}w_m \le c_n \end{array}\right\}$$

$w_i \ge 0$; i = 1, 2, 3, ..., n.

We can rewrite the dual as follows:

Maximize $u = b_1w_1 + b_2w_2 + \ldots + b_iw_i + \ldots + b_mw_m$

$$\text{subject to} \quad \left.\begin{array}{l} a_{11}w_1 + a_{21}w_2 + \ldots + a_{i1}w_i + \ldots + a_{m1}w_m \le c_1 \\ a_{12}w_1 + a_{22}w_2 + \ldots + a_{i2}w_i + \ldots + a_{m2}w_m \le c_2 \\ \vdots \\ a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{ij}w_i + \ldots + a_{mj}w_m = c_j \\ \vdots \\ a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{in}w_i + \ldots + a_{mn}w_m \le c_n \end{array}\right\}$$

$w_i \ge 0$; i = 1, 2, 3, ..., m.

Here the jth constraint is an equation, hence the theorem is proved.

Example (5.22): Find the dual of the following linear programming problem:

Minimize $z = 40x_1 + 25x_2$

Subject to $3x_1 + 20x_2 \ge 28$

$4x_1 + 7x_2 \ge 7$

$10x_1 + x_2 \ge 1$

$x_1 \ge 0$, x_2 is unrestricted in sign.

Solution: Taking w_1, w_2, w_3 as dual variables and using the primal-dual table, we get the dual of the given LP problem as follows:

$$\begin{aligned} \text{Maximize } u &= 28w_1 + 7w_2 + w_3 \\ \text{Subject to } & 3w_1 + 4w_2 + 10w_3 \leq 40 \\ & 20w_1 + 7w_2 + w_3 = 25 \\ & w_1, w_2, w_3 \geq 0 \end{aligned}$$

In the primal 2nd variable is unrestricted in sign and in the dual 2nd constraint is equality.

<u>Example (5.23)</u>: Using primal-dual table, write the dual of the following LP problem:

$$\begin{aligned} \text{Maximize } z &= 7x_1 + 7x_2 + 5x_3 \\ \text{Subject to } & 3x_1 + 9x_2 + x_3 \leq 76 \\ & 2x_1 + 6x_2 + x_3 \leq 65 \\ & -4x_1 - x_2 + x_3 \leq 33 \\ & x_2, x_3 \geq 0, x_1 \text{ is unrestricted in sign.} \end{aligned}$$

Solution: Taking w_1, w_2, w_3 as dual variables and using the primal-dual table, we get the dual of the given LP problem as follows: Minimize $u = 76w_1 + 65w_2 + 33w_3$

$$\begin{aligned} \text{Subject to } & 3w_1 + 2w_2 - 4w_3 = 7 \\ & 9w_1 + 6w_2 - w_3 \geq 7 \\ & w_1 + w_2 + w_3 \geq 5 \\ & w_1, w_2, w_3 \geq 0 \end{aligned}$$

In the primal, the 1st variable is unrestricted in sign and in the dual 1st constraint is equality.

<u>Theorem (5.3)</u>: The dual of the dual is the primal itself.

Proof: Let us consider symmetric primal problem as follows:

$$\left.\begin{aligned} \text{Minimize } & \underline{c}\,\underline{x} \\ \text{Subject to } & \underline{A}\,\underline{x} \geq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}\right\} \quad \ldots \quad (1)$$

The dual of the above primal problem is

$$\left.\begin{aligned} \text{Maximize } & \underline{b}^T\underline{w} \\ \text{Subject to } & \underline{A}^T\underline{w} \leq \underline{c}^T \\ & \underline{w} \geq \underline{0} \end{aligned}\right\} \quad \ldots \quad (2)$$

The dual problem (2) can be written as follows:

$$\left.\begin{aligned} &\text{Minimize } -\underline{b}^T\underline{w} \\ &\text{Subject to } -\underline{A}^T\underline{w} \ge -\underline{c}^T \\ &\qquad \underline{w} \ge \underline{0} \end{aligned}\right\} \quad \ldots \quad (3)$$

The dual problem (3) now looks like the primal problem of type (1) and hence considering it as a primal, we find the dual of it as follows:

$$\left.\begin{aligned} &\text{Maximize } (-\underline{c}^T)^T\underline{y} \\ &\text{Subject to } (-\underline{A}^T)^T\underline{y} \le (-\underline{b}^T)^T \\ &\qquad \underline{y} \ge \underline{0} \end{aligned}\right\} \quad \ldots \quad (4)$$

We can write (4) as follows:

$$\left.\begin{aligned} &\text{Maximize } -\underline{c}\,\underline{y} \\ &\text{Subject to } -\underline{A}\,\underline{y} \le -\underline{b} \\ &\qquad \underline{y} \ge \underline{0} \end{aligned}\right\} \quad \ldots \quad (5)$$

Replacing y by x and converting as minimization problem, we get

$$\left.\begin{aligned} &\text{Minimize } \underline{c}\,\underline{x} \\ &\text{Subject to } \underline{A}\,\underline{x} \ge \underline{b} \\ &\qquad \underline{x} \ge \underline{0} \end{aligned}\right\} \quad \ldots \quad (6)$$

Here (6) is exactly the original problem (1) and thus proves that the dual of the dual is the primal itself.

Example (5.24): With an unsymmetrical example, show that the dual of the dual is primal itself. **[CU-90]**

Solution: To prove the dual of the dual is primal itself, we consider the following unsymmetrical primal.

$$\begin{aligned} &\text{Minimize } 50x_1 + 25x_2 \\ &\text{Subject to } 3x_1 + 20x_2 = 23 \\ &\qquad 24x_1 + 7x_2 = 31 \\ &\qquad 11x_1 + 3x_2 = 15 \\ &\qquad x_1, x_2 \ge 0 \end{aligned}$$

Solution: The given primal is in standard form, so taking w_1, w_2, w_3 as dual variables, we get the dual of the given LP problem as follows: Maximize $23w_1 + 31w_2 + 15w_3$

$$\begin{aligned} &\text{Subject to } 3w_1 + 24w_2 + 11w_3 \le 50 \\ &\qquad 20w_1 + 7w_2 + 3w_3 \le 25 \\ &\qquad w_1, w_2, w_3 \text{ are unrestricted.} \end{aligned}$$

Taking $w_1 = w_1' - w_1''$, $w_2 = w_2' - w_2''$, $w_3 = w_3' - w_3''$; $w_1', w_1'', w_2', w_2'', w_3', w_3'' \geq 0$ and introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$ to first and second constraint of the dual respectively and converting it as minimizing problem, we get

Minimize $-23(w_1' - w_1'') - 31(w_2' - w_2'') - 15(w_3' - w_3'')$

Subject to $3(w_1' - w_1'') + 24(w_2' - w_2'') + 11(w_3' - w_3'') + s_1 = 50$

$20(w_1' - w_1'') + 7(w_2' - w_2'') + 3(w_3' - w_3'') + s_2 = 25$

$w_1', w_1'', w_2', w_2'', w_3', w_3'', s_1, s_2 \geq 0$

Multiplying both constraints by – 1, we get

Minimize $-23(w_1' - w_1'') - 31(w_2' - w_2'') - 15(w_3' - w_3'')$

Subject to $-3(w_1' - w_1'') - 24(w_2' - w_2'') - 11(w_3' - w_3'') - s_1 = -50$

$-20(w_1' - w_1'') - 7(w_2' - w_2'') - 3(w_3' - w_3'') - s_2 = -25$

$w_1', w_1'', w_2', w_2'', w_3', w_3'', s_1, s_2 \geq 0$

This dual format looks like the primal. Now taking y_1 and y_2 as dual variables, we get the dual of the dual as follows:

Maximize $-50y_1 - 25y_2$

Subject to $-3y_1 - 20y_2 \leq -23$

$3y_1 + 20y_2 \leq 23$

$-24y_1 - 7y_2 \leq -31$

$24y_1 + 7y_2 \leq 31$

$-11y_1 - 3y_2 \leq -15$

$11y_1 + 3y_2 \leq 15$

$-y_1 \leq 0$

$-y_2 \leq 0$

Or, Minimize $50y_1 + 25y_2$

Subject to $3y_1 + 20y_2 \leq 23$

$24y_1 + 7y_2 \leq 31$

$11y_1 + 3y_2 \leq 15$

$y_1, y_2 \geq 0$

Replacing y_1 by x_1 and y_2 by x_2, we get

Minimize $50x_1 + 25x_2$
Subject to $3x_1 + 20x_2 \leq 23$
$24x_1 + 7x_2 \leq 31$
$11x_1 + 3x_2 \leq 15$
$x_1, x_2 \geq 0$, which is the primal.

That is, dual of the dual is primal itself.

<u>Example (5.25)</u>: With an example, show that the dual of the dual is primal itself.

Solution: To prove the dual of the dual is primal itself, we consider the following primal problem.

Maximize $7x_1 + 13x_2$
Subject to $13x_1 + 22x_2 = 50$
$12x_1 + 5x_2 \geq 13$
$10x_1 + 9x_2 \leq 51$
$x_1, x_2 \geq 0$

Solution: Taking w_1, w_2, w_3 as dual variables and using the primal-dual table, we get the dual of the given primal problem as follows: Minimize $50w_1 + 13w_2 + 51w_3$
Subject to $13w_1 + 12w_2 + 10w_3 \geq 7$
$22w_1 + 5w_2 + 9w_3 \geq 13$
$w_3 \geq 0$, $w_2 \leq 0$, w_1 is unrestricted

Again taking y_1, y_2 as dual variables and using the primal-dual table, we get the dual of the dual problem as follows:

Maximize $7y_1 + 13y_2$
Subject to $13y_1 + 22y_2 = 50$
$12y_1 + 5y_2 \geq 13$
$10y_1 + 9y_2 \leq 51$
$y_1, y_2 \geq 0$

Replacing y_1 by x_1 and y_2 by x_2, we get

Maximize $7x_1 + 13x_2$
Subject to $13x_1 + 22x_2 = 50$
$12x_1 + 5x_2 \geq 13$
$10x_1 + 9x_2 \leq 51$

$x_1, x_2 \geq 0$, this is the primal.
That is, dual of the dual is primal itself.

Theorem (5.4): (**Weak duality theorem**) The value of the objective function of minimization primal problem for any feasible solution is always greater equal to that of its dual. **[NUH-05]**

Proof: We are going to prove this theorem for symmetric primal-dual problem. Let the primal problem be

Minimize $f(\underline{x}) = \underline{c}\,\underline{x}$
Subject to $\underline{A}\,\underline{x} \geq \underline{b}$
$\underline{x} \geq \underline{0}$

then the respected dual problem is

Maximize $g(\underline{w}) = \underline{b}^T\,\underline{w}$
Subject to $\underline{A}^T\,\underline{w} \leq \underline{c}^T$
$\underline{w} \geq \underline{0}$

Again let $\underline{x}^o$ and $\underline{w}^o$ be any feasible solutions of the primal and the dual respectively. Then we have to prove that

$\underline{c}\,\underline{x}^o \geq \underline{b}^T\underline{w}^o$

Since $\underline{x}^o$ is a feasible solution for the primal problem,

$\underline{A}\,\underline{x}^o \geq \underline{b}$ ... (1)

Again since $\underline{w}^o$ is a feasible solution for the dual problem,

$\underline{A}^T\,\underline{w}^o \leq \underline{c}^T$

Or, $(\underline{A}^T\,\underline{w}^o)^T \leq (\underline{c}^T)^T$ [Taking transpose on both sides]

Or, $(\underline{w}^o)^T\underline{A} \leq \underline{c}$ ... (2)

Multiplying both sides of (2) by $\underline{x}^o$, we get

$(\underline{w}^o)^T\underline{A}\,\underline{x}^o \leq \underline{c}\,\underline{x}^o$

Or, $(\underline{w}^o)^T\underline{b} \leq \underline{c}\,\underline{x}^o$ [Using (1)]

Or, $\underline{b}^T\underline{w}^o \leq \underline{c}\,\underline{x}^o$ [$(\underline{w}^o)^T\underline{b}$ is a scalar and so $(\underline{w}^o)^T\underline{b} = \underline{b}^T\underline{w}^o$]

$\therefore$ $\underline{c}\,\underline{x}^o \geq \underline{b}^T\underline{w}^o$

Hence the theorem is proved.

Corollary -1: The value of the objective function of maximization primal problem for any feasible solution is always less equal to that of its dual.

<u>Example (5.26)</u>: Arbitrarily taking two feasible solutions of a primal and respected dual show the weak duality theorem.

Solution: Let the primal be

Minimize $7x_1 + 13x_2$

Subject to $13x_1 + 22x_2 = 35$

$12x_1 + 5x_2 \leq 20$

$10x_1 + 9x_2 \geq 51$

$x_1, x_2 \geq 0$

Then the respected dual be

Maximize $35w_1 + 20w_2 + 51w_3$

Subject to $13w_1 + 12w_2 + 10w_3 \leq 7$

$22w_1 + 5w_2 + 9w_3 \leq 13$

$w_3 \geq 0$, $w_2 \leq 0$, w_1 is unrestricted

We arbitrarily choose a feasible solution $\underline{x}^o = (1, 1)$ of the primal and another feasible solution $\underline{w}^o = (-2, -1, 1)$ of the dual.

Then the value of the objective function of the primal for $\underline{x}^o$ is

$7\times1 + 13\times1 = 20$

And the value of the objective function of the dual for $\underline{w}^o$ is

$35\times(-2) + 20\times(-1) + 51\times1 = -39$

Since the value of the objective function of the primal is greater than that of the dual, the weak duality theorem.

<u>Example (5.27)</u>: Arbitrarily taking two feasible solutions of a primal and respected dual show the corollary-1.

Solution: Let the primal be

Maximize $7x_1 + 13x_2$

Subject to $13x_1 + 22x_2 = 35$

$12x_1 + 5x_2 \geq 13$

$10x_1 + 9x_2 \leq 51$

$x_1, x_2 \geq 0$

Then the respected dual be

Minimize $35w_1 + 13w_2 + 51w_3$

Subject to $13w_1 + 12w_2 + 10w_3 \geq 7$

$22w_1 + 5w_2 + 9w_3 \geq 13$

$w_3 \geq 0$, $w_2 \leq 0$, w_1 is unrestricted

We arbitrarily choose a feasible solution $\underline{x}^o = (1, 1)$ of the primal and another feasible solution $\underline{w}^o = (0, -1, 2)$ of the dual.
Then the value of the objective function of the primal for $\underline{x}^o$ is

$$7\times 1 + 13\times 1 = 20$$

And the value of the objective function of the dual for $\underline{w}^o$ is

$$35\times 0 + 13\times(-1) + 51\times 2 = 89$$

So, it holds the corollary-1.

Theorem (5.5): If $\underline{x}^*$ and $\underline{w}^*$ be any two feasible solutions of the primal

$$\begin{aligned} &\text{Minimize } f(\underline{x}) = \underline{c}\,\underline{x} \\ &\text{Subject to } \underline{A}\,\underline{x} \geq \underline{b} \\ &\qquad\qquad \underline{x} \geq \underline{0} \end{aligned}$$

and the corresponding dual

$$\begin{aligned} &\text{Maximize } g(\underline{w}) = \underline{b}^T\,\underline{w} \\ &\text{Subject to } \underline{A}^T\,\underline{w} \leq \underline{c}^T \\ &\qquad\qquad \underline{w} \geq \underline{0} \end{aligned}$$

respectively and $\underline{c}\,\underline{x}^* = \underline{b}^T\underline{w}^*$. Then $\underline{x}^*$ and $\underline{w}^*$ are the optimum solutions of the primal and dual respectively.

Proof: Let $\underline{x}^o$ and $\underline{w}^o$ be any feasible solutions of the primal and the dual respectively, then by weak duality theorem, we have

$$\underline{c}\,\underline{x}^o \geq \underline{b}^T\underline{w}^o \qquad \ldots \qquad (1)$$

Or, $\underline{c}\,\underline{x}^o \geq \underline{b}^T\underline{w}^*$ [As $\underline{w}$* is a feasible solution of the dual]

Or, $\underline{c}\,\underline{x}^o \geq \underline{c}\,\underline{x}^*$ [Given that $\underline{c}\,\underline{x}^* = \underline{b}^T\underline{w}^*$]

This implies that

$$\text{Minimum } \underline{c}\,\underline{x} = \underline{c}\,\underline{x}^*$$

Therefore, $\underline{x}^*$ is the optimum solution of the primal problem.
Again from (1), we have

$$\underline{c}\,\underline{x}^o \geq \underline{b}^T\underline{w}^o$$

Or, $\underline{c}\,\underline{x}^* \geq \underline{b}^T\underline{w}^o$ [As $\underline{x}$* is a feasible solution of the primal]

Or, $\underline{b}^T\underline{w}^* \geq \underline{b}^T\underline{w}^o$ [Given that $\underline{c}\,\underline{x}^* = \underline{b}^T\underline{w}^*$]

This implies that

$$\text{Maximum } \underline{b}^T\,\underline{w} = \underline{b}^T\underline{w}^*$$

Therefore, $\underline{w}^*$ is the optimum solution of the dual problem.
Hence the theorem is proved.

<u>**Theorem (5.6)**</u>**:** (**Main or basic duality theorem**) If either the primal or the dual problem has a finite optimum solution, then the other problem has a finite solution and the value of the objective functions are same, i.e., Min. f(x) = Max. g(w).

If either primal or dual has an unbounded solution then the other has no solution. **[NUH-04,07, JU-94, DU-95]**

Proof: We are going to prove this theorem for unsymmetric primal-dual problem. Let the primal be: Minimize f(X) = c X

Subject to A X = b

$X \geq 0$

where, $A = (a_{ij})_{m\times n}$ is $m\times n$ (coefficient) matrix, $c = (c_1, c_2, c_3, ..., c_n)$ is a row (cost) vector, $X = (x_1, x_2, x_3, ..., x_n)$ is a column vector and $b = (b_1, b_2, b_3, ..., b_m)$ is also a column (right hand side) vectors; then the corresponding dual be: Maximize g(w) = w b

Subject to $w A \leq c$

where, $w = (w_1, w_2, w_3, ..., w_m)$ is a row vector.

We first assume that the primal is feasible and that a minimum feasible solution has been obtained by the simplex procedure. For discussion purposes, let the first m vectors $P_1, P_2, ..., P_m$ are in the final basis. Let B equal the $m\times m$ matrix $(P_1\ P_2\ ...\ P_m)$. The final computational tableau contains the vectors of the original system $P_1, P_2, ..., P_m, P_{m+1}, ..., P_n$ in terms of the final basis vetors; i.e., for each vector P_j the final tableau contains the vector X_j such that $P_j = BX_j$. Let the $m\times n$ matrix

$$\overline{X} = (X_1\ X_2\ ...\ X_m\ X_{m+1}\ ...\ X_n)$$

be the matrix of coefficients contained in the final simplex tableau. Since we assumed that the final basis contained the first m vectors, we have

$$\overline{X} = \begin{pmatrix} 1 & 0 & ... & 0 & a_{1,m+1} & ... & a_{1n} \\ 0 & 1 & ... & 0 & a_{2,m+1} & ... & a_{2n} \\ \vdots & \vdots & ... & \vdots & \vdots & ... & \vdots \\ 0 & 0 & ... & 1 & a_{m,m+1} & ... & a_{mn} \end{pmatrix}$$

The minimum solution vector is given by $X^o = B^{-1}b$. For this final solution, we then have the following relationships:

$$B^{-1}A = \overline{X} \qquad \therefore A = B\overline{X} \qquad \ldots \qquad (1)$$

$$B^{-1}b = X^o \qquad \therefore b = B X^o \qquad \ldots \qquad (2)$$

$$\text{Min. } f(X) = c^o X^o \qquad \ldots \qquad (3)$$

$$Z = c^o \overline{X} - c \leq 0 \qquad \ldots \qquad (4)$$

Where 0 is a null n – dimensional vector, $c^o = (c_1, c_2, \ldots, c_m)$ is a row vector and so

$$Z = (c^o X_1 - c_1, c^o X_2 - c_2, \ldots, c^o X_n - c_n)$$

$$= (z_1 - c_1, z_2 - c_2, \ldots, z_n - c_n)$$

is a row vector whose elements are non-positive, as they are the $z_j - c_j$ elements corresponding to an optimal solution.

Let $w^o = (w_1^0, w_2^0, \ldots, w_m^0)$ be defined by

$$w^o = c^o B^{-1} \qquad \ldots \qquad (5)$$

Then, by (1), (4) and (5), we have

$$w^o A - c = c^o B^{-1} A - c = c^o \overline{X} - c \leq 0$$

Or, $\quad w^o A \leq c$

The vector w^o is a solution to the dual problem, since it satisfies the dual constraints $w A \leq c$. For this solution the corresponding value of the dual objective function

$$g(w) = w\, b$$

is given by $w^o b$, or from (2) and (3), we have

$$w^o b = c^o B^{-1} b = c^o X^o = \text{Min. } f(X) \qquad \ldots \qquad (6)$$

Hence, for the solution w^o, the value of the dual objective function is equal to the minimum value of the primal objective function.

Now we need only to show that w^o is also an optimum solution for the dual problem.

For any $n \times 1$ vetor X which satisfies $AX = b$ and $X \geq 0$ and any $1 \times m$ vector w satisfies $wA \leq c$, we have

$$wAX = w\, b = g(w) \qquad \ldots \qquad (7)$$

and $\quad wAX \leq c\, X = f(X) \qquad \ldots \qquad (8)$

By (7) and (8), we obtain the important relationship

$$g(w) \leq f(X) \qquad \ldots \qquad (9)$$

for all feasible w and X. The extreme values of the primal and dual are related by

$$\text{Max. } g(w) \leq \text{Min. } f(X) \qquad \ldots \qquad (10)$$

For the dual solution vector w^o, we have from (6)

$$g(w^o) = w^o b = \text{Min. } f(X)$$

and hence, for the optimum solution to the primal problem X^o, and the dual solution w^o, (10) becomes

$$g(w^o) = f(X^o)$$

i.e., for the solution $w^o = c^o B^{-1}$, the dual objective function takes on its maximum value. Therefore, for w^o and X^o we have the corresponding values of the objective functions related by

$$\text{Max. } g(w) = \text{Min. } f(X) \qquad \ldots \qquad (11)$$

The above results have been shown to hold whenever the primal has a finite optimum solution, then the dual also has a finite optimum solution and the value of the objective functions are same.

In a similar fashion, we can show that, when the dual problem has a finite optimum solution, the primal is feasible and (11) holds. To do this, transform the dual problem

$$\text{Max. } g(w) = \text{Max. } wb$$
$$\text{Subject to } wA \leq c$$

to the primal format and show that its dual (which will be feasible by the above theorem) is just the original primal. We have

$$\text{Max. } wb = -\text{Min. } (-wb)$$

Or, $\quad -\text{Max. } wb = \text{Min. } (-wb)$

$$\text{Subject to } wA + w_s I = c$$
$$w_s \geq 0$$

where the w_s is a set of non-negative slack variables. We next transform $w = w_{sk} - w_{sr}$, where w_{sk} and w_{sr} are sets of non-negative variables. Our problem is now

$$\text{Min.} \qquad (-w_{sk} + w_{sr})b + 0\, w_s \quad \ldots \qquad (12a)$$
$$\text{Subject to} \qquad (w_{sk} - w_{sr})A + w_s I = c \ \ldots \qquad (12b)$$
$$w_{sk}, w_{sr}, w_s \geq 0$$

and is in the standard format.

The duality theorem holds for this problem. To determine its dual in the proper format, it is convenient to multiply the constraint equation (12b) by – 1 to obtain

$$-w_{sk}A + w_{sr}A - w_s I = -c \qquad \ldots \qquad (12c)$$

The dual to (12a) and (12c) is then

Max. (– c X) = – Min. c X

Subject to (– A A – I) X $\leq$ (– b b 0)

which is equivalent to

– Max. (– c X) = Min. c X

Subject to

$$\begin{cases} -AX \leq -b \\ AX \leq b \\ -IX \leq 0 \end{cases} \text{ or, } \begin{cases} AX \geq b \\ AX \leq b \\ X \geq 0 \end{cases} \text{ or, } \begin{cases} AX = b \\ X \geq 0 \end{cases}$$

So, the dual problem is Minimize c X
Subject to AX = b
X $\geq$ 0

which is the original primal problem.

This completes the proof of the first part of the theorem.

To prove the second part, we note that, if the primal is unbounded, then we have by (9)

$$g(w) \leq -\infty$$

Any solution to the dual inequalities wA $\leq$ c must have a corresponding value for the dual objective function g(w) = wb which is a lower bound for the primal objective function. Since this contradicts the assumption of unboundedness, we must conclude that there are no solutions to the dual problem and hence that the dual inequalities are inconsistent. A similar argument will show that, when the dual has an unbounded solution, the primal has no solutions.

Note: Due to Dantzig, in a slightly different form, the basic duality theorem can be restated as follows: If feasible solutions to both the primal and dual systems exist, there exists an optimum solution to both system and min. f(x) = max. g(w).

<u>Example (5.28)</u>: Discuss the duality theorem with an example.

Solution: We consider the primal problem:

$$\begin{aligned}
\text{Minimize} \quad & x_2 - 3x_3 + 2x_5 \\
\text{Subject to} \quad & x_1 + 3x_2 - x_3 + 2x_5 = 7 \\
& -2x_2 + 4x_3 + x_4 = 12 \\
& -4x_2 + 3x_3 + 8x_5 + x_6 = 10 \\
\text{and} \quad & x_j \geq 0;\ j = 1, 2, ..., 6
\end{aligned}$$

Here, we have, $A = \begin{pmatrix} 1 & 3 & -1 & 0 & 2 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 0 & -4 & 3 & 0 & 8 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix}$,

$c = (0, 1, -3, 0, 2, 0)$

The dual problem:

$$\begin{aligned}
\text{Maximize} \quad & 7w_1 + 12w_2 + 10w_3 \\
\text{Subject to} \quad & w_1 \leq 0 \\
& 3w_1 - 2w_2 - 4w_3 \leq 1 \\
& -w_1 + 4w_2 + 3w_3 \leq -3 \\
& w_2 \leq 0 \\
& 2w_1 + 8w_3 \leq 2 \\
& w_3 \leq 0
\end{aligned}$$

Solving the primal problem by simplex method, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	0 / $\underline{P}_1$	1 / $\underline{P}_2$	-3 / $\underline{P}_3$	0 / $\underline{P}_4$	2 / $\underline{P}_5$	0 / $\underline{P}_6$	Ratio θ
$\underline{P}_1$	0	7	1	3	-1	0	2	0	
$\underline{P}_4$	0	12	0	-2	4	1	0	0	12/4=3min
$\underline{P}_6$	0	10	0	-4	3	0	8	1	10/3
$z_j - c_j$		0	0	-1	3 Largest	0	-2	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	0 / $\underline{P}_1$	1 / $\underline{P}_2$	-3 / $\underline{P}_3$	0 / $\underline{P}_4$	2 / $\underline{P}_5$	0 / $\underline{P}_6$	Ratio θ
$\underline{P}_1$	0	10	1	5/2	0	1/4	2	0	4 min
$\underline{P}_3$	-3	3	0	-1/2	1	1/4	0	0	
$\underline{P}_6$	0	1	0	-5/2	0	-3/4	8	1	
$z_j - c_j$		-9	0	1/2 Largest	0	-3/4	-2	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	0 / $\underline{P}_1$	1 / $\underline{P}_2$	-3 / $\underline{P}_3$	0 / $\underline{P}_4$	2 / $\underline{P}_5$	0 / $\underline{P}_6$	Ratio θ
$\underline{P}_2$	1	4	2/5	1	0	1/10	4/5	0	
$\underline{P}_3$	-3	5	1/5	0	1	3/10	2/5	0	
$\underline{P}_6$	0	11	1	0	0	-1/2	10	1	
$z_j - c_j$		-11	-1/5	0	0	-4/5	-12/5	0	

The final basis corresponding to an optimal solution to the primal problem is given by the vectors $\underline{P}_2$, $\underline{P}_3$, $\underline{P}_6$, that is,

$$B = (\underline{P}_2 \;\; \underline{P}_3 \;\; \underline{P}_6) = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 4 & 0 \\ -4 & 3 & 1 \end{pmatrix}.$$

In our example, we see that A does contain a unit matrix whose columns correspond to $\underline{P}_1$, $\underline{P}_4$, $\underline{P}_6$. Hence, in our final tableau, which is the third step of the example, the columns which correspond to $\underline{P}_1$, $\underline{P}_4$, $\underline{P}_6$ have been transformed to the inverse of the final basis B. These columns form the set of vectors $\underline{X}_1$, $\underline{X}_4$, $\underline{X}_6$ and

$$\text{hence, } B^{-1} = (\underline{X}_1 \;\; \underline{X}_4 \;\; \underline{X}_6) = \begin{pmatrix} 2/5 & 1/10 & 0 \\ 1/5 & 3/10 & 0 \\ 1 & -1/2 & 1 \end{pmatrix}$$

The corresponding optimum solution X^o is

$$X^o = B^{-1}b$$

$$\text{Or,} \quad \begin{pmatrix} x_2^0 \\ x_3^0 \\ x_6^0 \end{pmatrix} = \begin{pmatrix} 2/5 & 1/10 & 0 \\ 1/5 & 3/10 & 0 \\ 1 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 11 \end{pmatrix}$$

and $\quad c^o = (c_2, c_3, c_6) = (1, -3, 0)$.

Then the minimum value of the objective function is

$$c^o X^o = (1, -3, 0) \begin{pmatrix} 4 \\ 5 \\ 11 \end{pmatrix} = -11$$

From the final simplex tableau, we have

$$\overline{X} = \begin{pmatrix} 2/5 & 1 & 0 & 1/10 & 4/5 & 0 \\ 1/5 & 0 & 1 & 3/10 & 2/5 & 0 \\ 1 & 0 & 0 & -1/2 & 10 & 1 \end{pmatrix}$$

The vector Z for the optimum solution is given by

$Z = c^o\overline{X} - c$

$$= (1, -3, 0) \begin{pmatrix} 2/5 & 1 & 0 & 1/10 & 4/5 & 0 \\ 1/5 & 0 & 1 & 3/10 & 2/5 & 0 \\ 1 & 0 & 0 & -1/2 & 10 & 1 \end{pmatrix} - (0, 1, -3, 0, 2, 0)$$

$= (-1/5, 0, 0, -4/5, -12/5, 0) \leq \underline{0}$

The elements of Z are just those elements contained in the (m+1)st row of the final tableau, i.e., the $z_j - c_j$ elements.

As we have shown in the proof of the duality theorem, the optimum solution w^o to the dual is given by

$$w^o = c^oB^{-1} = (1, -3, 0) \begin{pmatrix} 2/5 & 1/10 & 0 \\ 1/5 & 3/10 & 0 \\ 1 & -1/2 & 1 \end{pmatrix} = (-1/5, -4/5, 0)$$

We cheek this solution by substituting it in the dual constraints, and we have $w^oA \leq c$

$$\text{Or,} \quad (-1/5, -4/5, 0) \begin{pmatrix} 1 & 3 & -1 & 0 & 2 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 0 & -4 & 3 & 0 & 8 & 1 \end{pmatrix} \leq (0, 1, -3, 0, 2, 0)$$

Or, $(-1/5, 1, -5, -4/5, -2/5, 0) \leq (0, 1, -3, 0, 2, 0)$

We have as the value of the dual objective function

$$w^ob = (-1/5, -4/5, 0) \begin{pmatrix} 7 \\ 12 \\ 10 \end{pmatrix} = -11$$

Note: The values of the variables for the optimum solution to the dual do not have to be obtained by multiplying c^oB^{-1} if the matrix A contains a unit matrix. We have

$$w^o = c^oB^{-1} = c^o(\underline{X}_1\ \ \underline{X}_4\ \ \underline{X}_6)$$

Or, $\quad (w_1^0, w_2^0,\ w_3^0) = (c^o\underline{X}_1\ \ c^o\underline{X}_4\ \ c^o\underline{X}_6)$

By definition of z_j we have $c^o\underline{X}_1 = z_1$, $c^o\underline{X}_4 = z_4$ and $c^o\underline{X}_6 = z_6$. In the (m+1)st row of the final tableau, we have for each $\underline{X}_j$, the corresponding $z_j - c_j$ element. For j = 1, 4, 6, we note that the corresponding $c_j = 0$, and hence the elements in the (m+1)st row corresponding to j = 1, 4, 6 are equal to the corresponding values of the dual variables, that is, $w_1^0 = z_1$, $w_2^0 = z_4$ and $w_3^0 = z_6$. If a vector which formed the unit matrix had a $c_j \neq 0$, then the value of this c_j would have to be added back to the corresponding $z_j - c_j$ in the final tableau in order to obtain the correct value for the w_i^0. We note that w_i^0 is equal to the z_j which has, for its corresponding unit vector in the initial simplex tableau, the vector whose unit element is in the position i. In our example, $w_2^0 = z_4$, since $\underline{P}_4$ is a unit vector with its unit element in position i = 2.

<u>Example (5.29)</u>: Examine the duality theorem holds for symmetric primal-dual problem.

Solution: Consider the symmetric general primal problem:

Minimize $f(X) = c\,X$ ... (1)

Subject to $\quad AX \geq b$ ... (2)

and $\quad X \geq 0$ ... (3)

The corresponding dual problem is

Maximize $g(w) = w\,b$ ... (4)

Subject to $\quad w\,A \leq c$ ... (5)

and $\quad w \geq 0$ ... (6)

We next show that the duality theorem also applies symmetric primal-dual problems. Let the m non-negative elements of the column vector $Y = (y_1, y_2, ..., y_m)$ be the surplus variables which transform the primal constraints into a set of equations. The equivalent linear programming problem in terms of partitioned matrices is

$$\text{Minimize } f(X,Y) = (c \mid \underline{0}) \left(\frac{X}{Y} \right) \quad \ldots \quad (7)$$

$$\text{Subject to} \quad (A \mid -I) \left(\frac{X}{Y} \right) = b \quad \ldots \quad (8)$$

$$\text{and} \quad X \geq 0,\ Y \geq 0 \quad \ldots \quad (9)$$

Here, $\underline{0} = (0, 0, ..., 0)$ is an m-component row vector and I is an $m \times m$ identity matrix. The dual of this transformed primal is to find a vector w which

$$\text{Maximize } g(w) = w\,b \quad \ldots \quad (10)$$

$$\text{Subject to} \quad w\,(A \mid -I) \leq (c \mid \underline{0}) \quad \ldots \quad (11)$$

We see that (11) decomposes to (5) and (6), respectively, that is,

$$w\,A \leq c \quad \text{and}$$

$$-\,w\,I \leq \underline{0}$$

The last expression is equivalent to $w \geq \underline{0}$.

That is, the dual of the transformed unsymmetric primal is

$$\text{Maximize } g(w) = w\,b$$

$$\text{Subject to} \quad w\,A \leq c$$

$$w \geq 0$$

which is the dual of given symmetric primal.

The original problems are now given as unsymmetric primal-dual problem, and so the duality theorem holds, as it is already proved.

Another proof of duality theorem: We are going to prove this theorem for symmetric primal-dual problem. Let the primal be

$$\text{Minimize } f(\underline{x}) = \underline{c}\,\underline{x}$$

$$\text{Subject to } \underline{A}\,\underline{x} \geq \underline{b}$$

$$\underline{x} \geq \underline{0}$$

where, $\underline{A} = (a_{ij})_{m \times n}$ is $m \times n$ (coefficient) matrix, $\underline{c} = (c_1, c_2, c_3, ..., c_n)$ is a row (cost) vector, $\underline{x} = (x_1, x_2, x_3, ..., x_n)$ is a column vector and $\underline{b} = (b_1, b_2, b_3, ..., b_m)$ is also a column (right hand side) vectors; then the corresponding dual be

Maximize $g(\underline{w}) = \underline{b}^T \underline{w}$
Subject to $\underline{A}^T \underline{w} \le \underline{c}^T$
$\underline{w} \ge \underline{0}$

$\underline{w} = (w_1, w_2, w_3, ..., w_m)$ is a column vector and $\underline{A}^T$, $\underline{b}^T$, $\underline{c}^T$ are transpose of $\underline{A}$, $\underline{b}$, $\underline{c}$ respectively.

Let us convert the constraints of the primal in the following form.

$$\underline{A}\,\underline{x} - \underline{I}_m\underline{x}_s = \underline{b},\ \underline{x} \ge \underline{0},\ \underline{x}_s \ge \underline{0} \qquad ... \qquad (1)$$

where $\underline{x}_s$ is a column vector of non-negative surplus variables and $\underline{I}_m$ is an $m \times m$ unit matrix and

$\underline{c}$ becomes $\underline{c}_s = (c_1, c_2, c_3, ..., c_n, 0, 0, ..., 0_{n+m})$

Suppose we solve the primal problem by simplex method and get a finite solution $\underline{X}_o = \underline{B}^{-1}\underline{b}$ where $\underline{B}$ be $(m\times m)$ matrix formed by taking the vectors of initial table whose corresponding final vectors are in the basis. Let in the final table, we get first m vectors in the basis. The final computational tableau contains the vectors of the original system $\underline{P}_1, \underline{P}_2, \underline{P}_3, ...\ \underline{P}_n\ \ \underline{P}_{n+1}\ ...\ \underline{P}_{n+m}$ in terms of the final basis vetors; i.e., for each vector $\underline{P}_j$ the final tableau contains the vector $\underline{X}_j$ such that $\underline{P}_j = \underline{B}\,\underline{X}_j \qquad ... \qquad (2)$

i.e., $(\underline{A} \mid - \underline{I}_m) = \underline{B}\,\underline{X} \qquad ... \qquad (3)$

where, $\underline{X} = (\underline{X}_1\ \ \underline{X}_2\ \ ...\ \ \underline{X}_n\ \ \underline{X}_{n+1}\ \ ...\ \underline{X}_{n+m})$ and $(\underline{A} \mid - \underline{I}_m)$ is a partitioned matrix.

Optimum value of the primal objective function, Min. $f(\underline{x}) = \underline{c}_o\underline{X}_o$ and $\underline{Z} \le \underline{0}$ or, $\underline{z} - \underline{c}_s \le \underline{0}$ or, $\underline{c}_o\underline{X} - \underline{c}_s \le \underline{0}$ or, $\underline{c}_o\underline{X} \le \underline{c}_s \quad ... \quad (4)$

Let $\quad \underline{w}_0^T = \underline{c}_o\underline{B}^{-1}$ [i.e., $\underline{w}_o = (\underline{c}_o\underline{B}^{-1})^T = (\underline{B}^{-1})^T \underline{c}_0^T$]

Or, $\quad \underline{w}_0^T(\underline{A} \mid - \underline{I}_m) = \underline{c}_o\underline{B}^{-1}(\underline{A} \mid - \underline{I}_m)$ [Multiplying by $(\underline{A} \mid - \underline{I}_m)$]

Or, $\quad \underline{w}_0^T(\underline{A} \mid - \underline{I}_m) = \underline{c}_o\underline{X}$ [Using (3)]

Or, $\quad (\underline{w}_0^T\underline{A} \mid - \underline{w}_0^T\underline{I}_m) \le \underline{c}_s$ [Using (4)]

Or, $\quad (\underline{w}_0^T\underline{A} \mid - \underline{w}_0^T) \le (c_1, c_2, c_3, ..., c_n, 0, 0, ..., 0_{n+m})$

Or, $\quad \underline{w}_0^T\underline{A} \le \underline{c}$ and $-\underline{w}_0^T \le \underline{0}$ [By separating]

Or, $\quad \underline{w}_0^T\underline{A} \le \underline{c}$ and $\underline{w}_0^T \ge \underline{0}$

Or, $\quad \underline{A}^T\underline{w}_o \le \underline{c}^T$ and $\underline{w}_o \ge \underline{0}$ [Taking transpose on both sides]

which shows that $\underline{w}_o$ is a feasible solution to the dual problem.
For the feasible solution $\underline{w}_o$, the value of the dual objective function is

$g(\underline{w}_o) = \underline{b}^T\, \underline{w}_o$ [This is a scalar]

$= \underline{w}_0^T\, \underline{b}$ [$\because$ $\underline{b}^T\, \underline{w}_o = \underline{w}_0^T\, \underline{b}$ as transpose of a scalar is itself]

$= \underline{c}_o\underline{B}^{-1}\underline{b}$

$= \underline{c}_o\underline{X}_o = \text{Min. } f(\underline{x})$ ... (5)

Now, we have to prove that $\underline{w}_o = (\underline{c}_o\underline{B}^{-1})^T$ is the optimum solution of the dual problem.
Let $\underline{w}$ be any solution of the dual problem, then from the dual constraint, we get $\underline{A}^T\, \underline{w} \le \underline{c}^T$

Or, $\underline{w}^T\underline{A} \le \underline{c}$ [Taking transpose on both sides]

Or, $\underline{w}^T\underline{A}\, \underline{x} \le \underline{c}\, \underline{x}$ [Multiplying by $\underline{x}$]

Or, $\underline{w}^T\underline{A}\, \underline{x} \le f(\underline{x})$ ... (6)

And from primal constraint, we have $\underline{A}\, \underline{x} \ge \underline{b}$

Or, $\underline{w}^T\underline{A}\, \underline{x} \ge \underline{w}^T\underline{b}$

Or, $\underline{w}^T\underline{A}\, \underline{x} \ge \underline{b}^T\underline{w}$ [$\because \underline{w}^T\underline{b} = \underline{b}^T\underline{w}$]

Or, $\underline{w}^T\underline{A}\, \underline{x} \ge g(\underline{w})$ ... (7)

By (6) and (7) we obtain the important relationship

$g(\underline{w}) \le f(\underline{x})$ ... (8) for all feasible $\underline{w}$ and $\underline{x}$.

The extreme values of the primal and the dual are related by

$\text{Max. } g(\underline{w}) \le \text{Min. } f(\underline{x})$ ... (9)

For $\underline{w}_o$ and $\underline{X}_o$ we have the corresponding values of the objective functions related by

$\text{Max. } g(\underline{w}) = \text{Min. } f(\underline{x})$ ... (10)

Hence, $\underline{w}_o = (\underline{c}_o\underline{B}^{-1})^T$ is the optimum solution of the dual problem.
The above results have been shown to hold whenever the primal has a finite optimum solution; the dual also has a finite optimum solution.
Similarly, we can prove that if the dual has a finite optimum solution then the primal has also a finite optimum solution. Also we know that the dual of the dual is the primal.
Now we are going to prove the last part of the theorem.

Let if the primal has an unbounded optimum solution, then from (8), we get $\quad g(w) \le -\infty$

But we know that the solution of the dual is the lower bound of the solution set of the primal. Since this contradicts the assumption of unboundedness, we must conclude that there are no solutions to the dual problem. Similarly, we can prove the converse.

Example (5.30): Solve the following LP problem and find its dual and hence solve the dual problem.

$$\begin{aligned} &\text{Minimize } -x_1 - x_2 - x_3 \\ &\text{Subject to } 2x_1 + x_2 + 2x_3 \le 3 \\ &\qquad\qquad 4x_1 + 2x_2 + x_3 \le 2 \\ &\qquad\qquad x_1, x_2, x_3 \ge 0 \end{aligned}$$

Solution: Introducing slack variables $x_4 \ge 0$ and $x_5 \ge 0$ to first and second constraints respectively, we can write the given LP problem as follows:

$$\begin{aligned} &\text{Minimize } -x_1 - x_2 - x_3 \\ &\text{Subject to } 2x_1 + x_2 + 2x_3 + x_4 \qquad = 3 \\ &\qquad\qquad 4x_1 + 2x_2 + x_3 \qquad + x_5 = 2 \\ &\qquad\qquad x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

Solving the primal problem by simplex method, we get

Basis	$\underline{C}_B^t$	c_j	-1	-1	-1	0	0	Ratio
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_4$	0	3	2	1	2	1	0	3/2 min.
$\underline{P}_5$	0	2	4	2	1	0	1	2/1=2
$z_j - c_j$		0	1	1	1	0	0	

Let greatest

Basis	$\underline{C}_B^t$	c_j	-1	-1	-1	0	0	Ratio
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_3$	-1	3/2	1	1/2	0	1/2	0	3
$\underline{P}_5$	0	1/2	3	3/2	1	-1/2	1	1/3 min.
$z_j - c_j$		-3/2	0	1/2	0	-1/2	0	

greatest

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-1 / $\underline{P}_1$	-1 / $\underline{P}_2$	-1 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Ratio θ
$\underline{P}_3$	-1	4/3	1	0	0	2/3	-1/3	
$\underline{P}_2$	-1	1/3	2	1	1	-1/3	2/3	
$z_j - c_j$		-5/3	-1	0	0	-1/3	-1/3	

Therefore, the optimum solution of the primal is (0, 1/3, 4/3) and the minimum value of the primal objective function is –5/3.

The dual of the given problem is as follows:

$$\begin{aligned} &\text{Maximize } 3w_1 + 2w_2 \\ &\text{Subject to } 2w_1 + 4w_2 \le -1 \\ &\qquad w_1 + 2w_2 \le -1 \\ &\qquad 2w_1 + w_2 \le -1 \\ &\qquad w_1, w_2 \le 0 \end{aligned}$$

From the third table, we find

$$c^o = (-1, -1),\ b = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

Therefore, the optimum solution to the dual is

$$w^o = (c^oB^{-1})^T = \left((-1,-1)\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \right)^T = (-1/3, -1/3)^T = \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix}$$

i.e., $w_1 = -1/3,\ w_2 = -1/3$

And the minimum value of the dual objective function is

$$b^Tw^0 = (3, 2)\begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix} = -5/3$$

Theorem (5.7): (Complementary slackness theorem) For optimal feasible solutions of the primal and the dual problems whenever inequality occurs in the kth relation of either system (the corresponding slack variable is positive), then the kth variable of its dual vanishes; if the kth variable is positive in either system, the kth relation if its dual is equality (the corresponding slack variable is zero). **[JU-91, RU-97]**

Proof: We prove this theorem for symmetric primal-dual problems. Let the primal be

Minimize $f(x) = c_1x_1 + c_2x_2 + \ldots + c_jx_j + \ldots + c_nx_n$ ... (1)

subject to
$$\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \geq b_2 \\
\vdots \\
a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \geq b_i \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \geq b_m
\end{array}\right\} \quad \ldots \quad (2)$$

$x_j \geq 0$, j = 1, 2, ..., n

The corresponding dual is as follows:

Maximize $g(w) = b_1w_1 + b_2w_2 + \ldots + b_iw_i + \ldots + b_mw_m$ ... (3)

subject to
$$\left.\begin{array}{l}
a_{11}w_1 + a_{21}w_2 + \ldots + a_{i1}w_i + \ldots + a_{m1}w_m \leq c_1 \\
a_{12}w_1 + a_{22}w_2 + \ldots + a_{i2}w_i + \ldots + a_{m2}w_m \leq c_2 \\
\vdots \\
a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{ij}w_i + \ldots + a_{mj}w_m \leq c_j \\
\vdots \\
a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{in}w_i + \ldots + a_{mn}w_m \leq c_n
\end{array}\right\} \quad \ldots \quad (4)$$

$w_i \geq 0$; i = 1, 2, 3, ..., m.

From (2), we get

$$\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n - x_{n+1} = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n - x_{n+2} = b_2 \\
\vdots \\
a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n - x_{n+i} = b_i \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n - x_{n+m} = b_m
\end{array}\right\} \quad \ldots \quad (2')$$

$x_j \geq 0$, j = 1, 2, ..., n, n+1, ..., n+m

From (4), we get

$$\left.\begin{array}{ll}
a_{11}w_1 + a_{21}w_2 + \ldots + a_{i1}w_i + \ldots + a_{m1}w_m + w_{m+1} & = c_1 \\
a_{12}w_1 + a_{22}w_2 + \ldots + a_{i2}w_i + \ldots + a_{m2}w_m \quad + w_{m+2} & = c_2 \\
\vdots & \\
a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{ij}w_i + \ldots + a_{mj}w_m \quad + w_{m+j} & = c_j \\
\vdots & \\
a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{in}w_i + \ldots + a_{mn}w_m \quad + w_{m+n} & = c_n
\end{array}\right\} \ldots \ (4^/)$$

$w_i \geq 0$; i = 1, 2, 3, ..., m, m+1, ..., m+n.

We next multiply the ith equation of ($2^/$) by the corresponding dual variable w_i (i = 1, 2, ..., m), add the resulting set of equations, and subtract this sum from (1) to obtain

$$(c_1 - \sum_{i=1}^{m} a_{i1}w_i)x_1 + (c_2 - \sum_{i=1}^{m} a_{i2}w_i)x_2 + \ldots + (c_n - \sum_{i=1}^{m} a_{in}w_i)x_n +$$

$$w_1x_{n+1} + w_2x_{n+2} + \ldots + w_mx_{n+m} = f(x) - \sum_{i=1}^{m} w_ib_i$$

Noting that $w_{m+j} = c_j - \sum_{i=1}^{m} a_{ij}w_i$ and $g(w) = \sum_{i=1}^{m} w_ib_i$, we have

$$w_{m+1}x_1 + w_{m+2}x_2 + \ldots + w_{m+n}x_n + w_1x_{n+1} + \ldots + w_mx_{n+m} = f(x) - g(w) \quad \ldots \ (5)$$

From the duality theorem, we have for an optimum solution $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)$ to the primal problem and $w^0 = (w_1^0, w_2^0, \ldots, w_m^0)$ an optimum solution to the dual problem that $f(x^0) - g(w^0) = 0$. Thus, for these optimum solutions and corresponding slack variables $x_{n+i}^0 \geq 0$ and $w_{m+j}^0 \geq 0$, (5) becomes

$$w_{m+1}^0\, x_1^0 + w_{m+2}^0\, x_2^0 + \ldots + w_{m+n}^0\, x_n^0 + w_1^0\, x_{n+1}^0 + w_2^0\, x_{n+2}^0 + \ldots + w_m^0\, x_{n+m}^0 = 0 \quad \ldots \ (6)$$

We note that the terms $w_{m+j}^0\, x_j^0$ are the product of the jth slack variable of the dual and the jth variable of the primal; while the terms $w_i^0 \, x_{n+i}^0$ are the product of the ith variable of the dual and the ith slack variable of the primal. Since all variables are restricted to be non-negative, all the product terms of (6) are non-negative and, as the sum of these terms must be equal to zero, they individually must be equal to zero. Thus

$$w_{m+j}^0 \, x_j^0 = 0 \text{ for all j} \qquad \dots \qquad (7)$$

and $$w_i^0 \, x_{n+i}^0 = 0 \text{ for all i.} \qquad \dots \qquad (8)$$

If $w_{m+k}^0 > 0$, we must have $x_k^0 = 0$; if $x_{n+k}^0 > 0$, then $w_k^0 = 0$, which establishes the first part of the theorem.

If $x_k^0 > 0$, we must have $w_{m+k}^0 = 0$; if $w_k^0 > 0$, then $x_{n+k}^0 = 0$, which completes the theorem.

Another proof of complementary slackness theorem: Here we shall prove this theorem for unsymmetrical primal-dual problems. Let the primal be

$$\text{Minimize } f(x) = c_1x_1 + c_2x_2 + \dots + c_jx_j + \dots + c_nx_n \qquad \dots \qquad (a)$$

$$\text{subject to } \left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1j}x_j + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2j}x_j + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{ij}x_j + \dots + a_{in}x_n = b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mj}x_j + \dots + a_{mn}x_n = b_m \end{array}\right\} \qquad \dots \qquad (b)$$

$$x_j \geq 0,\ j = 1, 2, \dots, n$$

The corresponding dual is as follows:

$$\text{Maximize } g(w) = b_1w_1 + b_2w_2 + \dots + b_iw_i + \dots + b_mw_m \qquad \dots \qquad (c)$$

$$\text{subject to} \left.\begin{array}{l} a_{11}w_1 + a_{21}w_2 + ... + a_{i1}w_i + ... + a_{m1}w_m \le c_1 \\ a_{12}w_1 + a_{22}w_2 + ... + a_{i2}w_i + ... + a_{m2}w_m \le c_2 \\ \vdots \\ a_{1j}w_1 + a_{2j}w_2 + ... + a_{ij}w_i + ... + a_{mj}w_m \le c_j \\ \vdots \\ a_{1n}w_1 + a_{2n}w_2 + ... + a_{in}w_i + ... + a_{mn}w_m \le c_n \end{array}\right\} \quad ... \quad (d)$$

w_i (i = 1, 2, 3, ..., m) unrestricted.

From (d), we get

$$\left.\begin{array}{l} a_{11}w_1 + a_{21}w_2 + ... + a_{i1}w_i + ... + a_{m1}w_m + w_{m+1} = c_1 \\ a_{12}w_1 + a_{22}w_2 + ... + a_{i2}w_i + ... + a_{m2}w_m + w_{m+2} = c_2 \\ \vdots \\ a_{1j}w_1 + a_{2j}w_2 + ... + a_{ij}w_i + ... + a_{mj}w_m + w_{m+j} = c_j \\ \vdots \\ a_{1n}w_1 + a_{2n}w_2 + ... + a_{in}w_i + ... + a_{mn}w_m + w_{m+n} = c_n \end{array}\right\} \quad ... \quad (d^{/})$$

$w_{m+j} \ge 0$; j = 1, 2, ..., n.

We next multiply the ith equation of (b) by the corresponding dual variable w_i (i = 1, 2, ..., m), add the resulting set of equations, and subtract this sum from (a) to obtain

$$(c_1 - \sum_{i=1}^{m} a_{i1}w_i)x_1 + (c_2 - \sum_{i=1}^{m} a_{i2}w_i)x_2 + ... + (c_n - \sum_{i=1}^{m} a_{in}w_i)x_n = f(x) - \sum_{i=1}^{m} w_i b_i$$

Noting that $w_{m+j} = c_j - \sum_{i=1}^{m} a_{ij}w_i$ and $g(w) = \sum_{i=1}^{m} w_i b_i$, we have

$$w_{m+1}x_1 + w_{m+2}x_2 + ... + w_{m+n}x_n = f(x) - g(w) \quad ... \quad (e)$$

From the duality theorem, we have for an optimum solution $x^0 = (x_1^0, x_2^0, ..., x_n^0)$ to the primal problem and $w^0 = (w_1^0, w_2^0, ..., w_m^0)$

an optimum solution to the dual problem that f(x^0) – g(w^0) = 0. Thus, for these optimum solutions and corresponding slack variables $w^0_{m+j} \geq 0$, (e) becomes

$$w^0_{m+1}\, x^0_1 + w^0_{m+2}\, x^0_2 + ... + w^0_{m+n}\, x^0_n = 0 \qquad ... \qquad (f)$$

Since w^0_{m+j} and x^0_j are restricted to be non-negative, hence all terms of (f) are individually must be equal to zero. Thus

$$w^0_{m+j}\, x^0_j = 0 \text{ for all j} \qquad ... \qquad (g)$$

If $w^0_{m+k} > 0$, we must have $x^0_k = 0$; and also we know that the primal is the dual of its dual. Hence the first part of the theorem is proved.

If $x^0_k > 0$, we must have $w^0_{m+k} = 0$; which proves the second part of the theorem. These complete the theorem.

Example (5.31): Solve the following LP problem and find its dual and hence solve the dual problem and discuss the complementary slackness theorem. Minimize $2x_1 + x_2$

Subject to $3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \geq 2$

$x_1, x_2 \geq 0$

Solution: Since the LP problem does not contain the initial basis (3 independent coefficient vectors because it contains 3 constraint equations) we need an artificial basis. For finding an artificial basis we add artificial variables x_6, x_7, x_8 to 1st, 2nd, 3rd constraints respectively and add the artificial variables with coefficients big M to the objective function. Then the problem becomes as follows:

Minimize $2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 + Mx_6 + Mx_7 + Mx_8$

Subject to $3x_1 + x_2 - x_3 + x_6 = 3$

$4x_1 + 3x_2 - x_4 + x_7 = 6$

$x_1 + 2x_2 - x_5 + x_8 = 2$

and $x_j \geq 0$; $j = 1, 2, . . ., 8$

Using the above problem we find the following initial tableau.

Sl.	Basis	C_B^t	$\underline{P}_o$	2	1	0	0	0	M	M	M	Ratio
				$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	θ
1	$\underline{P}_6$	M	3	3	1	-1	0	0	1	0	0	$3/3=\theta_o$
2	$\underline{P}_7$	M	6	4	3	0	-1	0	0	1	0	6/4
3	$\underline{P}_8$	M	2	1	2	0	0	-1	0	0	1	2/1
3+1	$z_j - c_j$		0	-2	-1	0	0	0	0	0	0	
3+2			11	8 Greatest	6	-1	-1	-1	0	0	0	Coef. of M

Since not all $z_j - c_j \leq 0$ in the 3+2nd row, the table is not optimal. We find the pivot taking the greatest element of 3+2nd row as base and then find the following iterative table as in simplex method.

Sl.	Basis	C_B^t	$\underline{P}_o$	2	1	0	0	0	M	M	M	Ratio
				$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	θ
1	$\underline{P}_1$	2	1	1	1/3	-1/3	0	0		0	0	1/(1/3)
2	$\underline{P}_7$	M	2	0	5/3	4/3	-1	0		1	0	2/(5/3)
3	$\underline{P}_8$	M	1	0	5/3	1/3	0	-1		0	1	$1/(5/3)= \theta_o$
3+1	$z_j - c_j$		2	0	-1/3	-2/3	0	0		0	0	
3+2			3	0	10/3 Greatest	5/3	-1	-1		0	0	

Since not all $z_j - c_j \leq 0$ in the 3+2nd row, the table is not optimal. Taking as iterations as before, we get

Sl.	Basis	C_B^t	$\underline{P}_o$	2	1	0	0	0	M	M	M	Ratio
				$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	θ
1	$\underline{P}_1$	2	4/5	1	0	-2/5	0	1/5		0		
2	$\underline{P}_7$	M	1	0	0	1	-1	1		1		$1/1 = \theta_o$
3	$\underline{P}_2$	1	3/5	0	1	1/5	0	-3/5		0		(3/5)/(1/5)
3+1	$z_j - c_j$		11/5	0	0	-3/5	0	-1/5		0		
3+2			1	0	0	1 Greatest	-1	1		0		

Since not all $z_j - c_j \leq 0$ in the 3+2nd row, the table is not optimal. Taking as iterations as before, we get

Sl.	Basis	$\underline{C}_B^t$	$\underline{P}_o$	2	1	0	0	0	M	M	M	Ratio
				$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	θ
1	$\underline{P}_1$	2	6/5	1	0	0	-2/5	3/5				(6/5)/(3/5)
2	$\underline{P}_3$	0	1	0	0	1	-1	1				$1/1 = \theta_o$
3	$\underline{P}_2$	1	2/5	0	1	0	1/5	-4/5				
3+1	$z_j - c_j$		14/5	0	0	0	-3/5	2/5 Greatest				
3+2			0	0	0	0	0	0				

Though all $z_j - c_j = 0$ in the 3+2nd row but not all $z_j - c_j \leq 0$ in the 3+1st row, the table is not optimal. Now we find the pivot taking the greatest element of 3+1st row as base and here $\underline{P}_6 = -\underline{P}_3$, $\underline{P}_7 = -\underline{P}_4$, $\underline{P}_8 = -\underline{P}_5$, and then find the following iterative table.

Sl.	Basis	$\underline{C}_B^t$	$\underline{P}_o$	2	1	0	0	0	M	M	M	Ratio
				$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	θ
1	$\underline{P}_1$	2	3/5	1	0	-3/5	1/5	0	3/5	-1/5	0	
2	$\underline{P}_5$	0	1	0	0	1	-1	1	-1	1	-1	
3	$\underline{P}_2$	1	6/5	0	1	4/5	-3/5	0	-4/5	3/5	0	
3+1	$z_j - c_j$		12/5	0	0	-2/5	-1/5	0	2/5	1/5	0	

Since all $z_j - c_j \leq 0$ the table is optimal. The above tableau gives us the extreme point (3/5, 6/5, 0, 0, 1). So, the solution of the problem is $x_1 = 3/5$, $x_2 = 6/5$, $x_3 = 0$, $x_4 = 0$, $x_5 = 1$ and the minimum value of the objective function is 12/5.

The dual of the given problem is as follows:

$$\text{Maximize } 3w_1 + 6w_2 + 2w_3$$
$$\text{Subject to } 3w_1 + 4w_2 + w_3 \leq 2$$
$$w_1 + 3w_2 + 2w_3 \leq 1$$
$$w_1, w_2, w_3 \geq 0$$

From the third table, we find

$$c^o = (2, 0, 1) \text{ and } B^{-1} = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -1 & 1 & -1 \\ -4/5 & 3/5 & 0 \end{pmatrix}$$

Therefore, the optimum solution to the dual is

$$w^o = c^oB^{-1} = (2, 0, 1)\begin{pmatrix} 3/5 & -1/5 & 0 \\ -1 & 1 & -1 \\ -4/5 & 3/5 & 0 \end{pmatrix} = (2/5, 1/5, 0)$$

i.e., $w_1 = 2/5$, $w_2 = 1/5$, $w_3 = 0$

And the minimum value of the dual objective function is

$3(2/5) + 6(1/5) + 2(0) = 12/5$

Here for optimum solutions,

1st variable of primal, $x_1 = 3/5$ (positive) and hence, 1st constraint of dual $3(2/5) + 4(1/5) + 0 = 2$ is equality.

2nd variable of primal, $x_2 = 6/5$ (positive) and hence, 2nd constraint of dual $2/5 + 3(1/5) + 2(0) = 1$ is equality.

1st variable of dual, $w_1 = 2/5$ (positive) and hence, 1st constraint of prial $3(3/5) + 6/5 = 3$ is equality.

2nd variable of dual, $w_2 = 1/5$ (positive) and hence, 2nd constraint of prial $4(3/5) + 3(6/5) = 6$ is equality.

That is, if the kth variable is positive in either system then the kth relation of its dual is equality.

3rd constraint $3/5 + 2(6/5) = 3 \geq 2$ is inequality and hence, 3rd variable of dual $w_3 = 0$.

That is, if the kth constraint of either system is inequality then the kth variable of its dual vanishes.

5.5 <u>Complementary slackness conditions</u>: (কমতি-পরিপূরক শর্তসমূহ)

From complementary slackness theorem, we have

$w^0_{m+j}\ x^0_j = 0$ for all j = 1, 2, ..., n ... (1)

and $w^0_i\ x^0_{n+i} = 0$ for all i = 1, 2, ..., m ... (2)

Generally, these two equations are known as the complementary slackness conditions. In other words, the complementary slackness conditions are as follows:

(i) If the kth variable of the primal x^0_k is positive, then the corresponding kth dual constraint will be an equation at the optimum stage, i.e., if $x^0_k > 0$ then $w^0_{m+k} = 0$.

(ii) If the kth variable of the dual w_k^0 is positive, then the corresponding kth primal constraint will be an equation at the optimum stage, i.e., if $w_k^0 > 0$ then $x_{n+k}^0 = 0$.

(iii) If the kth constraint of the primal is a strict inequality at the optimum stage then the corresponding kth dual variable w_k^0 must be zero, i.e., if $x_{n+k}^0 > 0$ then $w_k^0 = 0$.

(iv) If the kth constraint of the dual is a strict inequality at the optimum stage then the corresponding kth primal variable x_k^0 must be zero, i.e., if $w_{m+k}^0 > 0$ then $x_k^0 = 0$.

(v) If the kth slack or surplus variable x_{n+k}^0 of the primal problem appears in the optimum solution at positive level, then the corresponding dual optimum solution contains kth dual legitimate variable w_k^0 at zero level and vice versa, i.e., if $x_{n+k}^0 > 0$ then $w_k^0 = 0$ and if $w_{m+k}^0 > 0$ then $x_k^0 = 0$.

(vi) If the kth legitimate variable x_k^0 of the primal problem appears in the optimum solution at positive level, then the corresponding dual optimum solution contains kth dual slack or surplus variable w_{m+k}^0 at zero level and vice versa, i.e., if $x_k^0 > 0$ then $w_{m+k}^0 = 0$ and if $w_k^0 > 0$ then $x_{n+k}^0 = 0$.

Example (5.32): Solve the dual of the following LP problem and then using complementary slackness conditions solve the given primal. Maximize $z = x_1 + 2x_2 + 3x_3 + 4x_4$ **[JU-88]**

Subject to $x_1 + 2x_2 + 2x_3 + 3x_4 \leq 20$

$2x_1 + x_2 + 3x_3 + 2x_4 \leq 20$

$x_1, x_2, x_3, x_4 \geq 0$

Solution: Introducing slack variables $x_5 \geq 0$ and $x_6 \geq 0$ to the 1st and 2nd constraints respectively, we get the standard form of the primal as follows:

Maximize $z = x_1 + 2x_2 + 3x_3 + 4x_4 + 0x_5 + 0x_6$

$$\begin{aligned}\text{Subject to}\quad x_1 + 2x_2 + 2x_3 + 3x_4 + x_5 \quad &= 20\\ 2x_1 + x_2 + 3x_3 + 2x_4 \quad + x_6 &= 20\\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0\end{aligned}$$

Taking w_1 and w_2 as dual variables, we get the dual of the given primal as follows: Minimize $u = 20w_1 + 20w_2$

$$\begin{aligned}\text{Subject to}\quad w_1 + 2w_2 &\geq 1\\ 2w_1 + w_2 &\geq 2\\ 2w_1 + 3w_2 &\geq 3\\ 3w_1 + 2w_2 &\geq 4\\ w_1, w_2 &\geq 0\end{aligned}$$

Drawing the dual constraints in the graph paper, we find the shaded unbounded feasible solution space ABC. The vertices A(3/2, 0), B(6/5, 1/5) and C(0, 2) are basic feasible solution of the dual problem. And the value of the objective function of dual at A is 30, at B is 28 and at C is 40. Here the minimum value is 28 and attain at B(6/5, 1/5). u_{min} = 28.

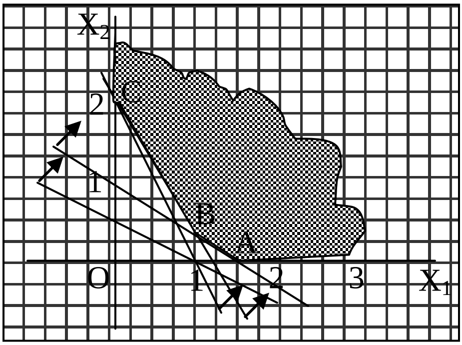

Figure 5.3

Therefore, the optimum solution of the dual is $w_1 = 6/5$, $w_2 = 1/5$ and $u_{min} = 28$.

Complementary slackness conditions imply that at the optimum stage $w_1x_5 = 0$ ... (1)

$w_2x_6 = 0$ ... (2)

and if the kth constraint of the dual is a strict inequality then the corresponding kth primal variable x_k must be zero. ... (3)

As $w_1 = 6/5 > 0$, (1) implies $x_5 = 0$.

As $w_2 = 1/5 > 0$, (2) implies $x_6 = 0$.

As the 1st constraint $w_1 + 2w_2 = 6/5 + 2(1/5) = 8/5 \geq 1$ is an inequality, (3) implies $x_1 = 0$.

As the 2nd constraint $2w_1 + w_2 = 2(6/5) + 1/5 = 13/5 \geq 2$ is an inequality, (3) implies $x_2 = 0$.
As the 3rd constraint $2w_1 + 3w_2 = 2(6/5) + 3(1/5) = 15/5 = 3$ is an equality, (3) implies $x_3 \geq 0$.
As the 4th constraint $3w_1 + 2w_2 = 3(6/5) + 2(1/5) = 20/5 = 4$ is an equality, (3) implies $x_4 \geq 0$.
Putting $x_1 = 0$, $x_2 = 0$, $x_5 = 0$ and $x_6 = 0$ in the constraints of the standard primal problem, we have

$$2x_3 + 3x_4 = 20$$
$$3x_3 + 2x_4 = 20$$

Solving these system, we get $x_3 = 4$, $x_4 = 4$ and the maximum value of the primal objective function is $3(4) + 4(4) = 28$.
Thus, the optimal solution of the primal is $x_1 = 0$, $x_2 = 0$, $x_3 = 4$, $x_4 = 4$ and $z_{max} = 28$.

5.6 <u>Economic interpretation of primal-dual problems</u>: (আদি-দ্বৈত সমস্যার অর্থনৈতিক ব্যাখ্যা) Let us briefly investigate the economic interpretation of the activity-analysis problem and its dual. The primal problem can be written so as to

$$\text{Maximize } cx$$
$$\text{Subject to } Ax \leq b$$
$$x \geq 0$$

where the a_{ij} represent the number of units of resource i required to produce one unit of commodity j, the b_i represent the maximum number of units of resource i available, and the c_j represent the value (profit) per unit of commodity j produced. The corresponding dual problem is to

$$\text{Minimize } wb$$
$$\text{Subject to } wA \geq c$$
$$w \geq 0.$$

Whereas the physical interpretation of the primal is straightforward, the corresponding interpretation of the dual is not so evident. The primal problem is to

$$\text{Maximize } \sum_{j=1}^{n}\left(\frac{value}{output\ \ j}\right)(output\ \ j) = \text{(value)}$$

$$\text{Subject to } \sum_{j=1}^{n}\left(\frac{input\ \ i}{output\ \ j}\right)(output\ \ j) \le \text{(input i)};\ \ i = 1, 2, ..., m$$

$$\text{(output j)} \ge 0;\ \ j = 1, 2, ..., n$$

And the dual is to

$$\text{Minimize } \sum_{i=1}^{m}(input\ \ i)\ w_i = (?)$$

$$\text{Subject to } \sum_{i=1}^{m} w_i\left(\frac{input\ \ i}{output\ \ j}\right) \ge \left(\frac{value}{output\ \ j}\right);\ \ j = 1, 2, ..., n$$

$$w_i \ge 0;\ \ i = 1, 2, ..., m.$$

We see that the dual constraints will be consistent if the w_is are in units of value per unit of input i. The dual problem would then be to

$$\text{Minimize } \sum_{i=1}^{m}(input\ \ i)\left(\frac{value}{input\ \ i}\right) = \text{(value)}$$

$$\text{Subject to } \sum_{i=1}^{m}\left(\frac{value}{input\ \ i}\right)\left(\frac{input\ \ i}{output\ \ j}\right) \ge \left(\frac{value}{output\ \ j}\right);\ \ j = 1, 2, ..., n$$

$$\left(\frac{value}{input\ \ i}\right) \ge 0;\ \ i = 1, 2, ..., m.$$

Verbal descriptions of the primal and the dual problems can then be stated as follows:

<u>The primal problem</u>: With a given unit of value of each output (c_j) and a given upper limit for the availability of each input (b_i) how much of each output (x_j) should be produced in order to maximize the value of the total output?

<u>The dual problem</u>: With a given availability of each input (b_i) and a given lower limit of unit value for each output (c_j) what unit

values should be assigned to each input (w_i) in order to minimize the value of the total input?

Note: The variable w_is are referred to by various names, e.g., accounting prices, accounting fictitious, marginal prices, or **shadow prices** etc.

5.7 <u>Shadow price</u>: (ছায়া মূল্য) The rate of change to the optimum value of the objective function with respect to the resource is known as shadow price of that resource. In the linear programming problem, the optimum value of the objective function is $z_{optimum}$ and availability of the resources is $b = (b_i)$, so the shadow price is

$$\frac{\partial(z_{optimum})}{\partial b} = \frac{\partial(c^0 B^{-1} b)}{\partial b} \qquad [\because z_{optimum} = c^o B^{-1} b]$$
$$= c^o B^{-1}$$
$$= w^o \qquad [\because w^o = c^o B^{-1}]$$

where w^o is the optimum solution of the dual. That is, unit change of resource b_i makes the change of the optimum value of z by w_i, and it is valid so long as the optimal basis remains unchanged for any change of b_i. Some times, determination of shadow prices are more important than the solution of the LP problem, because a business man can decide whether certain changes in his model increase the profit or decrease the loss. **[JU-93]**

<u>Example (5.33)</u>: A farmer has 1,000 acres of land on which he can grow corn, wheat or soybean. One acre of corn costs Tk.100 to prepare requires 7 man-days of work and yields a profit of Tk.30. One acre of wheat costs Tk.120 to prepare requires 10 man-days of work and yields a profit of Tk.40. One acre of soybean costs Tk.70 to prepare requires 8 man-days work and yields a profit of Tk.20. If the farmer has Tk.1, 00,000 for preparation and can count 8,000 man-days of work, how many acres should be allocated to each crop to maximize the profit? And also find the shadow prices for lands, preparation costs, man-days and interpret then economically.

Solution: (**Formulation of the primal**) Mathematical formulation of the problem:

Step-1: The key decision is to determine that how many acres of land should be allocated to each crop.

Step-2: Let x_1, x_2 and x_3 acres of land should be allocated for corn, wheat and soyabean respectively.

Step-3: Feasible alternatives are the sets of the values of x_1, x_2 and x_3 satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step-4: The objective is to maximize the profit realized from all the three crops, i.e., to maximize $z = 30x_1 + 40x_2 + 20x_3$

Step-5: The constraints (or restrictions) are

$x_1 + x_2 + x_3 \leq 1000$ (Limitation of land)

$100x_1 + 120x_2 + 70x_3 \leq 100000$ (Limitation of preparation cost)

$7x_1 + 10x_2 + 8x_3 \leq 8000$ (Limitation of man-days)

Hence the farmer's problem can be put in the following mathematical form:

$$
\begin{aligned}
\text{Maximize } z = 30x_1 + 40x_2 + 20x_3 & \\
x_1 + x_2 + x_3 &\leq 1000 \\
100x_1 + 120x_2 + 70x_3 &\leq 100000 \\
7x_1 + 10x_2 + 8x_3 &\leq 8000 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

(**Solution of the primal**) Since the LP problem contains more than two variables, the only way to solve the problem is simplex method. To solve the problem by simplex method, we convert it into minimization type and introduce slack variables $x_4, x_5, x_6 \geq 0$ as follows:

$$
\begin{aligned}
\text{–Minimize } -z = -30x_1 - 40x_2 - 20x_3 + 0x_4 + 0x_5 + 0x_6 & \\
x_1 + x_2 + x_3 + x_4 &= 1000 \\
100x_1 + 120x_2 + 70x_3 + x_5 &= 100000 \\
7x_1 + 10x_2 + 8x_3 + x_6 &= 8000 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Making initial simplex table and taking necessary iterations, we get the following tables:

Basis	C_B^t	c_j / $\underline{P}_o$	-30 $\underline{P}_1$	-40 $\underline{P}_2$	-20 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	0 $\underline{P}_6$	Min Ratio θ
$\underline{P}_4$	0	1000	1	1	1	1	0	0	1000
$\underline{P}_5$	0	100000	100	120	70	0	1	0	10000/12
$\underline{P}_6$	0	8000	7	10	8	0	0	1	800 Min.
$z_j - c_j$		0	30	40	20	0	0	0	
$\underline{P}_4$	0	200	3/10	0	1/5	1	0	-1/10	2000/3
$\underline{P}_5$	0	4000	16	0	-26	0	1	-12	250 Min
$\underline{P}_2$	-40	800	7/10	1	4/5	0	0	1/10	8000/7
$z_j - c_j$		-32000	2	0	-12	0	0	-4	
$\underline{P}_4$	0	125	0	0	11/16	1	-3/160	1/8	
$\underline{P}_1$	-30	250	1	0	-13/8	0	1/16	-3/4	
$\underline{P}_2$	-40	625	0	1	31/16	0	-7/160	5/8	
$z_j - c_j$		-32500	0	0	-35/4	0	-5/4	-5/2	

Since in the 3rd table all $z_j - c_j \leq 0$, it is optimal and the optimal basic feasible solution is $x_1 = 250$, $x_2 = 625$, $x_3 = 0$ and the maximum profit, z_{max} = Tk.32,500. Therefore, to yield maximum profit Tk.32,500 the farmer should grow corn in 250 acres of land and wheat in 625 acres of land.

Determination of shadow prices: Taking w_1, w_2 and w_3 as dual variables, we get the dual of the given primal as follows:

$$\text{Minimize } u = 1000w_1 + 100000w_2 + 8000w_3$$
$$\text{Subject to } \quad w_1 + 100w_2 + 7w_3 \geq 30$$
$$w_1 + 120w_2 + 10w_3 \geq 40$$
$$w_1 + 70w_2 + 8w_3 \geq 20$$
$$w_1, w_2, w_3 \geq 0$$

From the third table, we find

$$c^o = (0, 30, 40),\ b = \begin{pmatrix} 1000 \\ 100000 \\ 8000 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} 1 & -3/160 & 1/8 \\ 0 & 1/16 & -3/4 \\ 0 & -7/160 & 5/8 \end{pmatrix}$$

Therefore, the optimum solution to the dual is

$$w^o = c^oB^{-1} = (0, 30, 40)\begin{pmatrix} 1 & -3/160 & 1/8 \\ 0 & 1/16 & -3/4 \\ 0 & -7/160 & 5/8 \end{pmatrix} = (0, 1/8, 5/2)$$

i.e., $w_1 = 0$, $w_2 = 1/8$, $w_3 = 5/2$

$$\text{and } u_{min} = (0, 1/8, 5/2)\begin{pmatrix} 1000 \\ 100000 \\ 8000 \end{pmatrix} = 32500.$$

We know that the optimum values of the dual variables are respected shadow prices. So, $w_1 = 0$ is the shadow price of land, $w_2 = 1/8$ is the shadow price of preparing cost and $w_3 = 5/2$ is the shadow price of man-day.

Economic interpretation: The shadow price 0 for land means the optimum value of the primal objective function does not change with the change of the land availability. Similarly, shadow prices 1/8 and 5/2 say that the value of the primal objective function changes by 1/8 and 5/2 with unit change of preparing cost and man-day respectively.

Example (5.34): A company sells two products A and B. The company makes profit Tk.8 and Tk.5 from per unit of each product respectively. The two products are produced in a common process. The production process has capacity 500 man-days. It takes 2 man-days to produce one unit of A and one man-day per unit of B. The market has been surveyed and it feels that A can be sold 150 units, B of 250 units at most. Form the LP problem and then solve by simplex method, which maximizes the profit. And also find the shadow prices for man-day, demand for A, demand for B and interpret then economically.

Solution: Let x_1 and x_2 be the number of product A and B respectively to be produced for maximizing company's total profit satisfying demands and limitations. So, company's total profit is $z = 8x_1 + 5x_2$, limitations are $2x_1 + x_2 \leq 500$, demands are $x_1 \leq 150$,

$x_2 \leq 250$ and the feasibilities are $x_1 \geq 0$, $x_2 \geq 0$. Therefore, the LP form of the given problem is

$$\begin{aligned} \text{Maximize } z &= 8x_1 + 5x_2 \\ \text{Subject to } & 2x_1 + x_2 \leq 500 \\ & x_1 \leq 150 \\ & x_2 \leq 250 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

To solve the LP problem by simplex method, we convert the problem into minimization type and introduce slack variables x_3, x_4, $x_5 \geq 0$. Then we get,

$$\begin{aligned} \text{Maximize } z &= 8x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5 \\ \text{Subject to } & 2x_1 + x_2 + x_3 + 0x_4 + 0x_5 = 500 \\ & x_1 + 0x_2 + 0x_3 + x_4 + 0x_5 = 150 \\ & 0x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 250 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Basis	$\underline{C}_B^t$	c_j	8	5	0	0	0	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_3$	0	500	2	1	1	0	0	250
$\underline{P}_4$	0	150	1	0	0	1	0	150 Min
$\underline{P}_5$	0	250	0	1	0	0	1	---
$z_j - c_j$		0	-8	-5	0	0	0	
$\underline{P}_3$	0	200	0	1	1	-2	0	200 Min.
$\underline{P}_1$	8	150	1	0	0	1	0	---
$\underline{P}_5$	0	250	0	1	0	0	1	250
$z_j - c_j$		1200	0	-5	0	8	0	
$\underline{P}_2$	5	200	0	1	1	-2	0	---
$\underline{P}_1$	8	150	1	0	0	1	0	150
$\underline{P}_5$	0	50	0	0	-1	2	1	25 Min.
$z_j - c_j$		2200	0	0	5	-2	0	
$\underline{P}_2$	5	250	0	1	0	0	1	
$\underline{P}_1$	8	125	1	0	1/2	0	-1/2	
$\underline{P}_4$	0	25	0	0	-1/2	1	1/2	
$z_j - c_j$		2250	0	0	4	0	1	

Since in the 4th table all $z_j - c_j \geq 0$, it is optimal and the optimal basic feasible solution is $x_1 = 125$, $x_2 = 250$ and the maximum profit, z_{max}=Tk.2250. Therefore, to earn maximum profit Tk.2250, the producer should produce 125 units of product A and 250 units of product B.

Determination of shadow prices: Taking w_1, w_2 and w_3 as dual variables, we get the dual of the given primal as follows:

$$\begin{aligned} &\text{Minimize } u = 500w_1 + 150w_2 + 250w_3 \\ &\text{Subject to} \quad 2w_1 + w_2 \geq 8 \\ &\qquad\qquad\quad w_1 + w_3 \geq 5 \\ &\qquad\qquad\quad w_1, w_2 \geq 0 \end{aligned}$$

We know that for primal optimal solution, optimal w_i is equal to $(z_j - c_j) + c_j$ which has, for its corresponding unit vector in the initial simplex tableau, the vector whose unit element is in position i. So, from the optimal table, we find

$$w_1 = 4,\ w_2 = 0,\ w_3 = 1 \text{ and } u_{min} = 2250.$$

We know that the optimum values of the dual variables are respected shadow prices. So, $w_1 = 4$ is the shadow price of man-day, $w_2 = 0$ is the shadow price of demand for A and $w_3 = 1$ is the shadow price of demand for B.

Economic interpretation: The shadow price 4 for man-day means the optimum value of the primal objective function changes by 4 with unit change of the man-day availability. Similarly, shadow prices 0 and 1 say that the value of the primal objective function changes by 0 and 1 with unit change of demand for A and demand for B respectively.

<u>Example (5.35)</u>: Maximize $z = 10x_1 + 6x_2 + 4x_3$

$$\begin{aligned} \text{Subject to } & x_1 + x_2 + x_3 \leq 100 \text{ (Administration)} \\ & 10x_1 + 4x_2 + 5x_3 \leq 600 \text{ (Labour)} \\ & 2x_1 + 2x_2 + 6x_3 \leq 300 \text{ (Material)} \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

where x_1, x_2, x_3 are the number of products 1, 2 and 3 respectively.

(a) Find the optimal product mix by simplex method.

(b) Determine the shadow prices of all the resources.
(c) What are the ranges of the availability of the resources so that the current solution is still optimal? **[DU-92]**
Solution: (a) Given that

$$\begin{aligned} \text{Maximize } z &= 10x_1 + 6x_2 + 4x_3 \\ \text{Subject to } x_1 + x_2 + x_3 &\le 100 \text{ (Administration)} \\ 10x_1 + 4x_2 + 5x_3 &\le 600 \text{ (Labour)} \\ 2x_1 + 2x_2 + 6x_3 &\le 300 \text{ (Material)} \\ x_1, x_2, x_3 &\ge 0, \end{aligned}$$

where x_1, x_2, x_3 are the number of products 1, 2 and 3 respectively. We first transform the given problem to its standard form where x_4, x_5 and x_6 are slack variables.

$$\begin{aligned} \text{Maximize } z &= 10x_1 + 6x_2 + 4x_3 \\ \text{Subject to } x_1 + x_2 + x_3 + x_4 &= 100 \\ 10x_1 + 4x_2 + 5x_3 + x_5 &= 600 \\ 2x_1 + 2x_2 + 6x_3 + x_6 &= 300 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\ge 0 \end{aligned}$$

The simplex iterations are shown in the following tableau.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	10 / $\underline{P}_1$	6 / $\underline{P}_2$	4 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	0 / $\underline{P}_6$	Ratio θ
$\underline{P}_4$	0	100	1	1	1	1	0	0	100
$\underline{P}_5$	0	600	10	4	5	0	1	0	60 *
$\underline{P}_6$	0	300	2	2	6	0	0	1	150
$z_j - c_j$		0	-10 Smallest	-6	-4	0	0	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	10 / $\underline{P}_1$	6 / $\underline{P}_2$	4 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	0 / $\underline{P}_6$	Ratio θ
$\underline{P}_4$	0	40	0	3/5	1/2	1	-1/10	0	200/3*
$\underline{P}_1$	10	60	1	2/5	1/2	0	1/10	0	150
$\underline{P}_6$	0	180	0	6/5	5	0	-1/5	1	150
$z_j - c_j$		600	0	-2 Smallest	1	0	1	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	10 / $\underline{P}_1$	6 / $\underline{P}_2$	4 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	0 / $\underline{P}_6$	Ratio θ
$\underline{P}_2$	6	200/3	0	1	5/6	5/3	-1/6	0	
$\underline{P}_1$	10	100/3	1	0	1/6	-2/3	1/6	0	
$\underline{P}_6$	0	100	0	0	4	-2	0	1	
$z_j - c_j$		2200/3	0	0	8/3	10/3	2/3	0	

The optimal production mix is to make 100/3 units of product 1, 200/3 units of product 2 and 0 unit of product 3. This optimal program yields the maximum revenue of 2200/3.

(b) Here, $c^o = (6, 10, 0)$, $b = \begin{pmatrix} 100 \\ 600 \\ 300 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 5/3 & -1/6 & 0 \\ -2/3 & 1/6 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

So, $(w_1, w_2, w_3) = c^oB^{-1} = (6, 10, 0) \begin{pmatrix} 5/3 & -1/6 & 0 \\ -2/3 & 1/6 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

$= (10/3, 2/3, 0)$

i.e., $w_1 = 10/3,\ w_2 = 2/3,\ w_3 = 0$

Therefore, 10/3, 2/3 and 0 are shadow prices of the resources administration, labour and material respectively.

(c) Let $b^o = \begin{pmatrix} b_1 \\ 600 \\ 300 \end{pmatrix}$.

To keep the 3rd simplex table optimal $x^* = B^{-1}b^o \geq 0$

i.e., $\begin{pmatrix} 5/3 & -1/6 & 0 \\ -2/3 & 1/6 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ 600 \\ 300 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Or, $(5/3)b_1 - 100 \geq 0 \quad \Rightarrow b_1 \geq 60$

$(-2/3)b_1 + 100 \geq 0 \quad \Rightarrow b_1 \leq 150$

$$-2b_1 + 300 \geq 0 \qquad \Rightarrow b_1 \leq 150$$

So, the range is $60 \leq b_1$ (administration) ≤ 150

Again let $b^o = \begin{pmatrix} 100 \\ b_2 \\ 300 \end{pmatrix}$.

To keep the 3rd simplex table optimal $x^* = B^{-1}b^o \geq 0$

i.e., $$\begin{pmatrix} 5/3 & -1/6 & 0 \\ -2/3 & 1/6 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 100 \\ b_2 \\ 300 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Or, $(-1/6)b_2 - 500/3 \geq 0 \quad \Rightarrow b_2 \leq 1000$

$(1/6)b_2 - 200/3 \geq 0 \quad \Rightarrow b_2 \geq 400$

So, the range is $400 \leq b_2$ (labour) ≤ 1000

Again let $b^o = \begin{pmatrix} 100 \\ 600 \\ b_3 \end{pmatrix}$.

To keep the 3rd simplex table optimal $x^* = B^{-1}b^o \geq 0$

i.e., $$\begin{pmatrix} 5/3 & -1/6 & 0 \\ -2/3 & 1/6 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 100 \\ 600 \\ b_3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Or, $b_3 - 200 \geq 0 \quad \Rightarrow b_3 \geq 200$

So, the range is $200 \leq b_3$ (material)

5.8 <u>Dual simplex method</u>: (দ্বৈত সিমপে- ক্স পদ্ধতি) The method for solving LP problem has been developed with the help of the properties of its dual problem is known as the dual simplex method. This method is used to solve problems which start dual feasible, i.e., whose primal is optimal but infeasible. In this method we try to move from infeasible optimality to the feasible optimality. Using this method, we can solve so many maximization as well as minimization problems without using artificial variables. The only disadvantage of this method is that, all types of problems cannot be solved by using this method. The method is applicable to solve those problems where at the initial stage $z_j - c_j \geq 0$ (for maximization problems) and at least one initial solution is negative.

5.9 <u>Dual simplex algorithm</u>: (দ্বৈত সিমপে-ক্স এ্যালগরিদম) This algorithm consists of the following steps:

Step-1: Convert the problem into maximization problem if it is initially in the minimization form.

Step-2: Convert '$\leq$' type constraints, if any into '$\geq$' type, multiplying by – 1.

Step-3: Adding slack variables convert the inequality constraints into equalities and obtain the initial basic solution. After then form the dual simplex tableau with above information.

Step-4: Compute $z_j - c_j$ for every column.

(i) If all $z_j - c_j \geq 0$ and all $b_i \geq 0$, the solution found above is the optimum solution.

(ii) If all $z_j - c_j \geq 0$ and at least one b_i is negative, then GOTO step-5.

(iii) If any $z_j - c_j$ is negative, the method fails.

Step-5: Select the row that contains the most negative b_i. This is the pivot row or key row. The corresponding basic variable leaves the current basis.

Step-6: Look at the elements of the pivot row.

(i) If all elements of pivot row are non-negative, the problem has no feasible solution.

(ii) If at least one element of pivot row is negative, calculate the ratio of the corresponding $z_j - c_j$ to these elements. Select the largest ratio and the corresponding column is the pivot column or key column and the associated variable is the entering variable.

Step-7: The common element of the pivot row and the pivot column is the pivot element. Mark up the pivot element by a circle.

Step-8: As in the regular simplex method, convert the pivot element to unity and all other elements of pivot column to zero in the next table to get an improved solution.
Step-9: Repeat Step-4 to Step-8 until an optimum solution is attained or indicate to have no feasible solution.

<u>Example (5.36)</u>: If possible, solve the following LP problem by dual simplex method. Minimize $z = x_1 + 5x_2$

$$\text{Subject to } 3x_1 + 4x_2 \leq 6$$
$$x_1 + 3x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

Solution: Converting the problem as maximization problem and all constraints as $\leq$ type, we get Maximize $Z = -x_1 - 5x_2$

$$\text{Subject to } 3x_1 + 4x_2 \leq 6$$
$$-x_1 - 3x_2 \leq -3$$
$$x_1, x_2 \geq 0$$

Adding slack variables $x_3 \geq 0$ and $x_4 \geq 0$ to 1st and 2nd constraint respectively, we get,

$$\text{Maximize } Z = -x_1 - 5x_2 + 0x_3 + 0x_4$$
$$\text{Subject to } 3x_1 + 4x_2 + x_3 + 0x_4 = 6$$
$$-x_1 - 3x_2 + 0x_3 + x_4 = -3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Here initial basic solution is $x_3 = 6$, $x_4 = -3$, which is infeasible. Now we form the initial dual simplex tableau and consecutive iterative tableau as follow:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-1 / $\underline{P}_1$	-5 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Negative of $\underline{P}_o$
$\underline{P}_3$	0	6	3	4	1	0	
$\underline{P}_4$	0	-3	-1	-3	0	1	Pivot row
$z_j - c_j$		0	1	5	0	0	
$\dfrac{z_j - c_j}{y_{2j}}$; $y_{2j} < 0$			-1 Pivot column	-5/3			

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-1 / $\underline{P}_1$	-5 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Negative of $\underline{P}_o$
$\underline{P}_3$	0	-3	0	-5	1	3	Pivot row
$\underline{P}_1$	-1	3	1	3	0	-1	
$z_j - c_j$		-3	0	2	0	1	
$\dfrac{z_j - c_j}{y_{1j}}$; $y_{1j} < 0$				-2/5 Pivot column			

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-1 / $\underline{P}_1$	-5 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Negative of $\underline{P}_o$
$\underline{P}_2$	-5	3/5	0	1	-1/5	-3/5	
$\underline{P}_1$	-1	6/5	1	0	3/5	4/5	
$z_j - c_j$		-21/5	0	0	2/5	11/5	

Since all $z_j - c_j \geq 0$ and all $b_i \geq 0$, we reached at the optimum stage. And the optimum solution is $x_1 = 6/5$, $x_2 = 3/5$ and hence $z_{min} = -Z_{max} = -(-21/5) = 21/5$.

Example (5.37): Solve the following LP problem by dual simplex method.

Minimize $z = 2x_1 + 2x_2 + 4x_3$ **[CU-94]**

Subject to $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

Solution: Converting the problem as maximization problem and all constraints as $\leq$ type, we get

Maximize $Z = -2x_1 - 2x_2 - 4x_3$

Subject to $-2x_1 - 3x_2 - 5x_3 \leq -2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

Adding slack variables $x_4 \geq 0$, $x_5 \geq 0$ and $x_6 \geq 0$ to 1st, 2nd and 3rd constraint respectively, we get,

$$\begin{aligned} &\text{Maximize } Z = -2x_1 - 2x_2 - 4x_3 \\ &\text{Subject to } -2x_1 - 3x_2 - 5x_3 + x_4 = -2 \\ &\quad 3x_1 + x_2 + 7x_3 + x_5 = 3 \\ &\quad x_1 + 4x_2 + 6x_3 + x_6 = 5 \\ &\quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Here initial basic solution is $x_4 = -2$, $x_5 = 3$, $x_6 = 5$ which is infeasible.

Now we form the initial dual simplex tableau and consecutive iterative tableau as follow:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-2 / $\underline{P}$	-2 / $\underline{P}_2$	-4 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	0 / $\underline{P}_6$	Negative of $\underline{P}_o$
$\underline{P}_4$	0	-2	-2	-3	-5	1	0	0	-2 P. row
$\underline{P}_5$	0	3	3	1	7	0	1	0	
$\underline{P}_6$	0	5	1	4	6	0	0	1	
$z_j - c_j$		0	2	2	4	0	0	0	
$\frac{z_j - c_j}{y_{1j}}$; $y_{1j} < 0$			-1	-2/3 Largest P.column	-4/5				

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-2 / $\underline{P}$	-2 / $\underline{P}_2$	-4 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	0 / $\underline{P}_6$	Negative of $\underline{P}_o$
$\underline{P}_2$	-2	2/3	2/3	1	5/3	-1/3	0	0	
$\underline{P}_5$	0	7/3	7/3	0	16/3	1/3	1	0	
$\underline{P}_6$	0	7/3	-5/3	0	-2/3	4/3	0	1	
$z_j - c_j$		-4/3	2/3	0	2/3	2/3	0	0	

Since all $z_j - c_j \geq 0$ and all $b_i \geq 0$, we reached at the optimum stage. And the optimum solution is $x_1 = 0$, $x_2 = 2/3$, $x_3 = 0$ and hence $z_{min} = -Z_{max} = -(-4/3) = 4/3$.

5.10 <u>Some done examples</u>:

<u>Example (5.38)</u>: Write the dual of the following symmetrical primal problem.

$$\begin{aligned} \text{Maximize } & 3x_1 + 9x_2 + 4x_3 \\ \text{Subject to } & 2x_1 + 2x_2 + 7x_3 \le 14 \\ & 3x_1 + 6x_2 - 3x_3 \le 12 \\ & 5x_1 - 8x_2 + 4x_3 \le 10 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Solution: Taking w_1, w_2 and w_3 as the dual variable, we get the following dual problem of the given symmetric primal problem:

$$\begin{aligned} \text{Minimize } & 14w_1 + 12w_2 + 10w_3 \\ \text{Subject to } & 2w_1 + 3w_2 + 5w_3 \ge 3 \\ & 2w_1 + 6w_2 - 8w_3 \ge 9 \\ & 7w_1 - 3w_2 + 4w_3 \ge 4 \\ & w_1, w_2, w_3 \ge 0 \end{aligned}$$

Example (5.39): Reduce the following LP problem into canonical form and then find its dual:

$$\begin{aligned} \text{Maximize } \quad & z = 3x_1 + x_2 \\ \text{Subject to } \quad & 5x_1 + 2x_2 \le 5 \\ & -3x_1 + x_2 \ge 3 \\ & x_1 \ge 0,\ x_2 \text{ is unrestricted in sign.} \end{aligned}$$

Solution: Taking $x_2 = x_2' - x_2''$; $x_2', x_2'' \ge 0$, we have

$$\begin{aligned} \text{Maximize } \quad & z = 3x_1 + x_2' - x_2'' \\ \text{Subject to } \quad & 5x_1 + 2(x_2' - x_2'') \le 5 \\ & -3x_1 + x_2' - x_2'' \ge 3 \\ & x_1, x_2', x_2'' \ge 0 \end{aligned}$$

Multiplying second constraint by –1, we have

$$\begin{aligned} \text{Maximize } \quad & z = 3x_1 + x_2' - x_2'' \\ \text{Subject to } \quad & 5x_1 + 2x_2' - 2x_2'' \le 5 \\ & 3x_1 - x_2' + x_2'' \le -3 \\ & x_1, x_2', x_2'' \ge 0 \end{aligned}$$

This is the canonical form.

Since, in the canonical form there are 2 constraints, we consider the following 2 variables w_1, w_2 as dual variables. So, using step-4 and step-5 of the primal-dual constructions algorithm, we get the following dual problem:

$$\begin{aligned} &\text{Minimize } u = 5w_1 - 3w_2 \\ &\text{Subject to } 5w_1 + 3w_2 \geq 3 \\ &\quad 2w_1 - w_2 \geq 1 \\ &\quad -2w_1 + w_2 \geq -1 \\ &\quad w_1, w_2 \geq 0 \end{aligned}$$

Or,

$$\begin{aligned} &\text{Minimize } u = 5w_1 - 3w_2 \\ &\text{Subject to } 5w_1 + 3w_2 \geq 3 \\ &\quad 2w_1 - w_2 \geq 1 \\ &\quad 2w_1 - w_2 \leq 1 \\ &\quad w_1, w_2 \geq 0 \end{aligned}$$

So,

$$\begin{aligned} &\text{Minimize } u = 5w_1 - 3w_2 \\ &\text{Subject to } 5w_1 + 3w_2 \geq 3 \\ &\quad 2w_1 - w_2 = 1 \\ &\quad w_1, w_2 \geq 0 \end{aligned}$$

This is the required dual problem.

Example (5.40): Using primal-dual table, find the dual problem of the following linear programming problem.

$$\begin{aligned} &\text{Minimize } 5x_1 + 10x_2 \\ &\text{Subject to } 3x_1 + 3x_2 \geq 7 \\ &\quad 7x_1 + 7x_2 = 10 \\ &\quad x_1 \geq 0, x_2 \geq 2 \end{aligned}$$

Solution: Putting $x_2 = y_2 + 2$; $y_2 \geq 0$, we get

$$\begin{aligned} &\text{Minimize } 5x_1 + 10y_2 + 20 \\ &\text{Subject to } 3x_1 + 3y_2 \geq 1 \\ &\quad 7x_1 + 7y_2 = -4 \\ &\quad x_1, y_2 \geq 0 \end{aligned}$$

Taking w_1, w_2 as dual variables and using primal-dual table, we get the dual of the above LP problem as follows:

$$\begin{aligned} &\text{Maximize } w_1 - 4w_2 + 20 \\ &\text{Subject to } 3w_1 + 7w_2 \leq 5 \\ &\quad 3w_1 + 7w_2 \leq 10 \\ &\quad w_1 \geq 0, w_2 \text{ is unrestricted.} \end{aligned}$$

This is the required dual problem.

<u>Example (5.41)</u>: Using matrices, find the dual problem of the following linear programming problem.

Minimize $3x_1 + 4x_2 + x_3$ **[RU-92]**

Subject to $2x_1 - 3x_2 + x_3 = 5$

$x_1 + 7x_2 - 5x_3 = 6$

$4x_1 + 3x_2 - 3x_3 = 9$

$x_1, x_2, x_3 \geq 0$

Solution: We can write the given LP problem as follows:

Minimize CX

Subject to AX = b

$X \geq 0$

where, C = (3, 4, 1), $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 7 & -5 \\ 4 & 3 & -3 \end{pmatrix}$, $b = \begin{pmatrix} 5 \\ 6 \\ 9 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

The dual of the given problem is

Maximize Wb

Subject to $WA \leq C$

W is unrestricted

Or, Maximize $(w_1, w_2, w_3)\begin{pmatrix} 5 \\ 6 \\ 9 \end{pmatrix}$

Subject to $(w_1, w_2, w_3)\begin{pmatrix} 2 & -3 & 1 \\ 1 & 7 & -5 \\ 4 & 3 & -3 \end{pmatrix} \leq (3, 4, 1)$

w_1, w_2, w_3 are unrestricted.

Or, Maximize $5w_1 + 6w_2 + 9w_3$

Subject to $2w_1 + w_2 + 4w_3 \leq 3$

$-3w_1 + 7w_2 + 3w_3 \leq 4$

$w_1 - 5w_2 - 3w_3 \leq 1$

w_1, w_2, w_3 are unrestricted.

This is the required dual problem.

<u>Example (5.42)</u>: Using primal-dual table, find the dual problem of the following linear programming problem.

Minimize $3x_1 + 4x_2 + x_3$

Subject to $2x_1 - 3x_2 + x_3 \geq 5$

$x_1 + 7x_2 - 5x_3 \geq 6$

$4x_1 + 3x_2 - 3x_3 = 9$

$x_1, x_2, x_3 \geq 0$

Solution: Taking w_1, w_2, w_3 as dual variables and using primal-dual table, we get the dual of the above LP problem as follows:

Maximize $5w_1 + 6w_2 + 9w_3$

Subject to $2w_1 + w_2 + 4w_3 \leq 3$

$-3w_1 + 7w_2 + 3w_3 \leq 4$

$w_1 - 5w_2 - 3w_3 \leq 1$

$w_1, w_2 \geq 0$, w_3 are unrestricted.

This is the required dual problem.

<u>Example (5.43)</u>: Write the dual of the following LP problem:

Maximize $z = 5x_1 - 2x_2 + 4x_3$

Subject to $2x_1 - 5x_2 + 12x_3 \geq 10$

$7x_1 + 5x_2 - 2x_3 \leq 30$

$3x_1 + 9x_2 - 2x_3 \approx 15$

$x_1, x_2 \geq 0$, x_3 unrestricted

Solution: Taking w_1, w_2, w_3 as dual variables and using the primal-dual table, we get the dual of the given LP problem as follows: Minimize $u = 10w_1 + 30w_2 + 15w_3$

Subject to $2w_1 + 7w_2 + 3w_3 \geq 5$

$-5w_1 + 5w_2 + 9w_3 \geq -2$

$12w_1 - 2w_2 - 2w_3 = 4$

$w_1 \leq 0$, $w_2 \geq 0$, $w_3 = 0$

As $w_3 = 0$, vanishing w_3, we get the required dual problem as follows: Minimize $u = 10w_1 + 30w_2$

Subject to $2w_1 + 7w_2 \geq 5$

$-5w_1 + 5w_2 \geq -2$

$12w_1 - 2w_2 = 4$

$w_1 \leq 0$, $w_2 \geq 0$

Example (5.44): Using primal-dual table, find the dual of the following linear programming problem.

$$\begin{aligned}
\text{Maximize } z &= x_1 + 4x_2 + 3x_3 \\
\text{Subject to } & 2x_1 + 2x_2 - 5x_3 \leq 2 \\
& 3x_1 - x_2 + 6x_3 \geq 1 \\
& x_1 + x_2 + x_3 = 4 \\
& x_1 \geq 3,\ x_2 \leq 0,\ x_3 \text{ unrestricted.}
\end{aligned}$$

Solution: Putting $x_1 = y_1 + 3$; $y_2 \geq 0$ we get the given primal as follows:

$$\begin{aligned}
\text{Maximize } z &= y_1 + 4x_2 + 3x_3 + 3 \\
\text{Subject to } & 2y_1 + 2x_2 - 5x_3 \leq -4 \\
& 3y_1 - x_2 + 6x_3 \geq -8 \\
& y_1 + x_2 + x_3 = 1 \\
& y_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ unrestricted.}
\end{aligned}$$

Taking w_1, w_2, w_3 as dual variables and using primal-dual table, we find the dual of the given problem as follows:

$$\begin{aligned}
\text{Minimize } u &= -4w_1 - 8w_2 + w_3 + 3 \\
\text{Subject to } & 2w_1 + 3w_2 + w_3 \geq 1 \\
& 2w_1 - w_2 + w_3 \leq 4 \\
& -5w_1 + 6w_2 + w_3 = 3 \\
& w_1 \geq 0,\ w_2 \leq 0,\ w_3 \text{ unrestricted.}
\end{aligned}$$

This is the required dual problem.

Example (5.45): Using primal-dual construction algorithm, find the dual problem of the following unsymmetrical primal problem.

$$\begin{aligned}
\text{Minimize } z &= 9x_1 + 4x_2 + 3x_3 \qquad \textbf{[DU-93]} \\
\text{Subject to } & 4x_1 + 2x_2 - 5x_3 \leq 5 \\
& 2x_1 - x_2 + 6x_3 \geq 3 \\
& 2x_1 + x_2 + x_3 = 7 \\
& x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ unrestricted.}
\end{aligned}$$

Solution: To find the dual of the given problem, we use the following steps of primal-dual construction algorithm.

Step-1: The given problem is an unsymmetrical minimizing primal problem.

Step-2: Putting $x_2 = -x_2'$, $x_3 = x_3' - x_3''$; $x_2' \geq 0$, $x_3' \geq 0$, $x_3'' \geq 0$ and introducing slack variable $s_1 \geq 0$ in 1st constraint and surplus variable $s_2 \geq 0$ in 2nd constraint, we get the problem as follows:

$$\text{Minimize } z = 9x_1 - 4x_2' + 3x_3' - 3x_3'' + 0.s_1 + 0.s_2$$

$$\begin{aligned} \text{Subject to} \quad & 4x_1 - 2x_2' - 5x_3' + 5x_3'' + s_1 = 5 \\ & 2x_1 + x_2' + 6x_3' - 6x_3'' - s_2 = 3 \\ & 2x_1 - x_2' + x_3' - x_3'' = 7 \\ & x_1, x_2', x_3', x_3'', s_1, s_2 \geq 0. \end{aligned}$$

This is the standard form.

Step-3: In the primal problem, there are 3 constraint equations. So, we take 3 dual variables w_1, w_2, w_3.

Step-4: The objective function of the dual is

$$\text{Maximize } u = 5w_1 + 3w_2 + 7w_3$$

Step-5: The constraints of the dual are as follows:

$$\begin{aligned} \text{Subject to} \quad & 4w_1 + 2w_2 + 2w_3 \leq 9 \\ & -2w_1 + w_2 - w_3 \leq -4 \\ & -5w_1 + 6w_2 + w_3 \leq 3 \\ & 5w_1 - 6w_2 - w_3 \leq -3 \\ & w_1 + 0w_2 + 0w_3 \leq 0 \\ & 0w_1 - w_2 + 0w_3 \leq 0 \\ & w_1, w_2, w_3 \text{ are unrestricted in sign.} \end{aligned}$$

Or,

$$\begin{aligned} \text{Subject to} \quad & 4w_1 + 2w_2 + 2w_3 \leq 9 \\ & 2w_1 - w_2 + w_3 \geq 4 \\ & -5w_1 + 6w_2 + w_3 = 3 \\ & w_1 \leq 0 \\ & w_2 \geq 0 \\ & w_1, w_2, w_3 \text{ is unrestricted in sign.} \end{aligned}$$

Or,

$$\begin{aligned} \text{Subject to} \quad & 4w_1 + 2w_2 + 2w_3 \leq 9 \\ & 2w_1 - w_2 + w_3 \geq 4 \\ & -5w_1 + 6w_2 + w_3 = 3 \\ & w_1 \leq 0, w_2 \geq 0, w_3 \text{ is unrestricted in sign.} \end{aligned}$$

Step-6: So, the required dual problem is as follows:

$$\begin{aligned} &\text{Maximize } u = 5w_1 + 3w_2 + 7w_3 \\ &\text{Subject to } 4w_1 + 2w_2 + 2w_3 \le 9 \\ &\qquad 2w_1 - w_2 + w_3 \ge 4 \\ &\qquad -5w_1 + 6w_2 + w_3 = 3 \\ &\qquad w_1 \le 0,\ w_2 \ge 0,\ w_3 \text{ is unrestricted in sign.} \end{aligned}$$

Example (5.46): Using primal-dual construction algorithm, find the dual problem of the following unsymmetrical primal problem.

$$\begin{aligned} &\text{Maximize} && x_1 - x_2 + 3x_3 + 2x_4 \\ &\text{Subject to} && x_1 + x_2 \ge -1 \\ &&& x_1 - 3x_2 - x_3 \le 7 \\ &&& x_1 + x_3 - 3x_4 = -2 \\ &&& x_1, x_4 \ge 0,\ x_2, x_3 \text{ unrestricted.} \end{aligned}$$

Solution: To find the dual of the given problem, we use the following steps of primal-dual construction algorithm.

Step-1: The given problem is an unsymmetrical maximizing primal problem.

Step-2: Putting $x_2 = x_2^{/} - x_2^{//}$, $x_3 = x_3^{/} - x_3^{//}$; $x_2^{/} \ge 0$, $x_2^{//} \ge 0$, $x_3^{/} \ge 0$, $x_3^{//} \ge 0$ and introducing surplus variable $s_1 \ge 0$ in 1st constraint and slack variable $s_2 \ge 0$ in 2nd constraint, we get the problem as follows: Maximize $x_1 - (x_2^{/} - x_2^{//}) + 3(x_3^{/} - x_3^{//}) + 2x_4$

$$\begin{aligned} \text{Subject to} \quad & -x_1 - (x_2^{/} - x_2^{//}) + s_1 = 1 \\ & x_1 - 3(x_2^{/} - x_2^{//}) - (x_3^{/} - x_3^{//}) + s_2 = 7 \\ & -x_1 - (x_3^{/} - x_3^{//}) + 3x_4 = 2 \\ & x_1, x_2^{/}, x_2^{//}, x_3^{/}, x_3^{//}, x_4, s_1, s_2 \ge 0. \end{aligned}$$

This is the standard form.

Step-3: In the primal problem, there are 3 constraint equations. So, we take 3 dual variables w_1, w_2, w_3.

Step-4: The objective function of the dual is

$$\text{Minimize } w_1 + 7w_2 + 2w_3$$

Step-5: The constraints of the dual are as follows:

$$- w_1 + w_2 - w_3 \geq 1$$
$$- w_1 - 3w_2 \geq -1$$
$$w_1 + 3w_2 \geq 1$$
$$- w_2 - w_3 \geq 3$$
$$w_2 + w_3 \geq -3$$
$$3w_3 \geq 2$$
$$w_1 \geq 0$$
$$w_2 \geq 0$$

w_1, w_2, w_3 are unrestricted in sign.

Or,
$$- w_1 + w_2 - w_3 \geq 1$$
$$w_1 + 3w_2 = 1$$
$$w_2 + w_3 = -3$$
$$3w_3 \geq 2$$

$w_1, w_2 \geq 0$, w_3 unrestricted in sign.

Step-6: So, the required dual problem is as follows:

Minimize $w_1 + 7w_2 + 2w_3$

Subject to
$$- w_1 + w_2 - w_3 \geq 1$$
$$w_1 + 3w_2 \qquad = 1$$
$$w_2 + w_3 = -3$$
$$3w_3 \geq 2$$

$w_1, w_2 \geq 0$, w_3 unrestricted in sign.

Example (5.47): Using primal-dual construction algorithm, find the dual problem of the following unsymmetrical primal problem.

Minimize $z = 7x_1 + 2x_2$ **[DU-89]**

Subject to
$$2x_1 - 3x_2 \leq 12$$
$$5x_1 - 2x_2 \geq 2$$
$$2x_1 + 3x_2 \approx 10$$
$$x_1, x_2 \geq 0$$

Solution: To find the dual of the given problem, we use the following steps of primal-dual construction algorithm.

Step-1: The given problem is am unsymmetrical minimizing primal problem.

Step-2: Adding slack variable s_1, $-s_2$, $s_3 - s_4$ in 1st, 2nd and 3rd constraints respectively, where $s_1, s_2, s_3, s_4 \geq 0$, we get the problem as follows:

$$\begin{array}{ll} \text{Minimize } z = & 7x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 \\ \text{Subject to} & 2x_1 - 3x_2 + s_1 = 12 \\ & 5x_1 - 2x_2 - s_2 = 2 \\ & 2x_1 + 3x_2 + s_3 - s_4 = 10 \\ & x_1, x_2, s_1, s_2, s_3, s_4 \geq 0 \end{array}$$

This is the standard form.

Step-3: We consider w_1, w_2, w_3 as dual variable, because there are 3 constraints in the primal problem.

Step-4: The objective function of the dual is

$$\text{Maximize } u = 12w_1 + 2w_2 + 10w_3$$

Step-5: The constraints of the dual are as follows:

$$\begin{array}{ll} \text{Subject to} & 2w_1 + 5w_2 + 2w_3 \leq 7 \\ & -3w_1 - 2w_2 + 3w_3 \leq 2 \\ & w_1 + 0w_2 + 0w_3 \leq 0 \\ & 0w_1 - w_2 + 0w_3 \leq 0 \\ & 0w_1 + 0w_2 + w_3 \leq 0 \\ & 0w_1 + 0w_2 - w_3 \leq 0 \\ & w_1, w_2, w_3 \text{ are unrestricted in sign.} \end{array}$$

$$\begin{array}{ll} \text{Or, Subject to} & 2w_1 + 5w_2 + 2w_3 \leq 7 \\ & -3w_1 - 2w_2 + 3w_3 \leq 2 \\ & w_1 \leq 0 \\ & -w_2 \leq 0 \\ & w_3 \leq 0 \\ & -w_3 \leq 0 \\ & w_1, w_2, w_3 \text{ are unrestricted in sign.} \end{array}$$

$$\begin{array}{ll} \text{Or, Subject to} & 2w_1 + 5w_2 + 2w_3 \leq 7 \\ & -3w_1 - 2w_2 + 3w_3 \leq 2 \\ & w_1 \leq 0,\ w_2 \geq 0,\ w_3 = 0 \end{array}$$

Step-6: So, the dual problem given linear programming problem is as follows: Maximize $u = 12w_1 + 2w_2 + 10w_3$

Subject to $2w_1 + 5w_2 + 2w_3 \leq 7$

$-3w_1 - 2w_2 + 3w_3 \leq 2$

$w_1 \leq 0,\ w_2 \geq 0,\ w_3 = 0$

Or, Maximize $u = 12w_1 + 2w_2$

Subject to $2w_1 + 5w_2 \leq 7$

$-3w_1 - 2w_2 \leq 2$

$w_1 \leq 0,\ w_2 \geq 0$

This is the required dual problem.

Example (5.48): Using primal-dual construction algorithm, find the dual problem of the following unsymmetrical primal problem.

$$\begin{aligned} \text{Maximize } z = \ & 5x_1 + 3x_2 + x_3 \\ \text{Subject to } \ & 5x_1 + 3x_2 - 5x_3 \leq 13 \\ & 3x_1 - 2x_2 + 6x_3 \geq 3 \\ & x_1 + 2x_2 + x_3 \approx 5 \\ & x_1, x_2 \geq 0, x_3 \leq 0 \end{aligned}$$

Solution: To find the dual of the given problem, we use the following steps of primal-dual construction algorithm.

Step-1: The given problem is am unsymmetrical maximizing primal problem.

Step-2: Putting $x_3 = -y_3$ and introducing slack variable s_1, $-s_2$, $s_3 - s_4$ in 1st, 2nd and 3rd constraints respectively, where $y_3, s_1, s_2, s_3, s_4 \geq 0$, we get the problem as follows:

$$\begin{aligned} \text{Maximize } z = \ & 5x_1 + 3x_2 - y_3 + 0s_1 + 0s_2 + 0s_3 + 0s_4 \\ \text{Subject to } \ & 5x_1 + 3x_2 + 5y_3 + s_1 = 13 \\ & 3x_1 - 2x_2 - 6y_3 - s_2 = 3 \\ & x_1 + 2x_2 - y_3 + s_3 - s_4 = 5 \\ & x_1, x_2, y_3, s_1, s_2, s_3, s_4 \geq 0 \end{aligned}$$

This is the standard form.

Step-3: We consider w_1, w_2, w_3 as dual variable, because there are 3 constraints in the primal problem.

Step-4: The objective function of the dual is

$$\text{Minimize } u = 13w_1 + 3w_2 + 5w_3$$

Step-5: The constraints of the dual are as follows:

$$\begin{aligned} \text{Subject to } & 5w_1 + 3w_2 + w_3 \geq 5 \\ & 3w_1 - 2w_2 + 2w_3 \geq 3 \\ & 5w_1 - 6w_2 - w_3 \geq -1 \\ & w_1 + 0w_2 + 0w_3 \geq 0 \\ & 0w_1 - w_2 + 0w_3 \geq 0 \\ & 0w_1 + 0w_2 + w_3 \geq 0 \\ & 0w_1 + 0w_2 - w_3 \geq 0 \\ & w_1, w_2, w_3 \text{ are unrestricted in sign.} \end{aligned}$$

$$\begin{aligned} \text{Or, Subject to } & 5w_1 + 3w_2 + w_3 \geq 5 \\ & 3w_1 - 2w_2 + 2w_3 \geq 3 \\ & 5w_1 - 6w_2 - w_3 \geq -1 \\ & w_1 \geq 0 \\ & -w_2 \geq 0 \\ & w_3 \geq 0 \\ & -w_3 \geq 0 \\ & w_1, w_2, w_3 \text{ are unrestricted in sign.} \end{aligned}$$

$$\begin{aligned} \text{Or, Subject to } & 5w_1 + 3w_2 + w_3 \geq 5 \\ & 3w_1 - 2w_2 + 2w_3 \geq 3 \\ & 5w_1 - 6w_2 - w_3 \geq -1 \\ & w_1 \geq 0, w_2 \leq 0, w_3 = 0 \end{aligned}$$

Step-6: So, the dual problem given linear programming problem is as follows:

$$\begin{aligned} & \text{Minimize } u = 13w_1 + 3w_2 + 5w_3 \\ & \text{Subject to } 5w_1 + 3w_2 + w_3 \geq 5 \\ & \qquad 3w_1 - 2w_2 + 2w_3 \geq 3 \\ & \qquad 5w_1 - 6w_2 - w_3 \geq -1 \\ & \qquad w_1 \geq 0, w_2 \leq 0, w_3 = 0 \end{aligned}$$

Or,

$$\begin{aligned} & \text{Minimize } u = 13w_1 + 3w_2 \\ & \text{Subject to } 5w_1 + 3w_2 \geq 5 \\ & \qquad 3w_1 - 2w_2 \geq 3 \\ & \qquad 5w_1 - 6w_2 \geq -1 \\ & \qquad w_1 \geq 0, w_2 \leq 0 \end{aligned}$$

This is the required dual problem.

<u>Example (5.49)</u>: Solve the following LP problem using the duality theorem.

Maximize $x_1 + x_2 + x_3$

Subject to $2x_1 + x_2 + 2x_3 \leq 3$

$4x_1 + 2x_2 + x_3 \leq 2$

$x_1, x_2, x_3 \geq 0$

Solution: We can write the given LP problem as follows:

Maximize $x_1 + x_2 + x_3$

Subject to $2x_1 + x_2 + 2x_3 + x_4 \quad = 3$

$4x_1 + 2x_2 + x_3 \quad + x_5 = 2$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Solving the primal problem by simplex method, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	1 / $\underline{P}_1$	1 / $\underline{P}_2$	1 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Ratio θ
$\underline{P}_4$	0	3	2	1	2	1	0	3/2 min.
$\underline{P}_5$	0	2	4	2	1	0	1	2/1=2
$z_j - c_j$		0	-1	-1	-1 Let smallest	0	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	1 / $\underline{P}_1$	1 / $\underline{P}_2$	1 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Ratio θ
$\underline{P}_3$	1	3/2	1	1/2	0	1/2	0	3
$\underline{P}_5$	0	1/2	3	3/2	1	-1/2	1	1/3 min.
$z_j - c_j$		3/2	0	-1/2 Smallest	0	1/2	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	1 / $\underline{P}_1$	1 / $\underline{P}_2$	1 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Ratio θ
$\underline{P}_3$	1	4/3	1	0	0	2/3	-1/3	
$\underline{P}_2$	1	1/3	2	1	1	-1/3	2/3	
$z_j - c_j$		5/3	1	0	0	1/3	1/3	

Therefore, the optimum solution of the primal is (0, 1/3, 4/3, 0, 0) and the maximum value of the primal objective function is 5/3.

The dual of the given problem as follows:

Minimize $3w_1 + 2w_2$

Subject to $2w_1 + 4w_2 \geq 1$

$w_1 + 2w_2 \geq 1$

$2w_1 + w_2 \geq 1$

$w_1, w_2 \geq 0$

From the third table, we find

$$c^o = (1, 1),\ b = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

Therefore, the optimum solution to the dual is

$$w^o = c^oB^{-1} = (1, 1) \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} = (1/3, 1/3)$$

i.e., $w_1 = 1/3,\ w_2 = 1/3$

And the minimum value of the dual objective function is

$$w^0b = (1/3, 1/3)\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 5/3$$

Example (5.50): Solve the following LP problem and find its dual and hence solve the dual problem.

Maximize $z = 3x_1 - x_2$

Subject to $2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_2 \leq 4$

$x_1, x_2 \geq 0$

Solution: Introducing surplus variable $x_3 \geq 0$ to first constraint, slack variables x_4, $x_5 \geq 0$ to second and third constraints respectively and artificial variable $x_6 \geq 0$ to first constraint, we get

Maximize $z = 3x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 - Mx_6$

Subject to $2x_1 + x_2 - x_3 + x_6 = 2$

$x_1 + 3x_2 + x_4 = 3$

$x_2 + x_5 = 4$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Making initial simplex table and taking necessary iterations we get,

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	3 / $\underline{P}_1$	-1 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	-M / $\underline{P}_6$	Min Ratio θ
$\underline{P}_6$	-M	2	2	1	-1	0	0	1	2/2=1Min
$\underline{P}_4$	0	3	1	3	0	1	0	0	3/1=3
$\underline{P}_5$	0	4	0	1	0	0	1	0	
$z_j - c_j$		0	-3	1	0	0	0	0	
		2	-2 Smallest	-1	1	0	0	0	
$\underline{P}_1$	3	1	1	1/2	-1/2	0	0	-	
$\underline{P}_4$	0	2	0	5/2	1/2	1	0	-	4 Min
$\underline{P}_5$	0	4	0	1	0	0	1	-	
$z_j - c_j$		3	0	5/2	-3/2 Greatest	0	0	-	
		0	0	0	0	0	0	-	
$\underline{P}_1$	3	3	1	3	0	1	0	0	$\underline{P}_3$ and $\underline{P}_6$ differ with a '–' sign
$\underline{P}_3$	0	4	0	5	1	2	0	-1	
$\underline{P}_5$	0	4	0	1	0	0	1	0	
$z_j - c_j$		9	0	10	0	3	0	-	

Since in the last table all $z_j - c_j \geq 0$, the optimality conditions are satisfied. Optimal solution is $x_1 = 3$, $x_2 = 0$ and $z_{max} = 9$.

Using primal–dual table we find the dual of the given problem is as follows:

$$\text{Minimize } 2w_1 + 3w_2 + 4w_3$$
$$\text{Subject to } 2w_1 + w_2 \geq 3$$
$$w_1 + 3w_2 + w_3 \geq -1$$
$$w_1 \leq 0,\ w_2, w_3 \geq 0$$

From the third table, we find

$$c^o = (3, 0, 0) \text{ and } B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Therefore, the optimum solution to the dual is

$$w^o = c^o B^{-1} = (3, 0, 0) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (0, 3, 0)$$

i.e., $w_1 = 0, w_2 = 3, w_3 = 0$

And the minimum value of the dual objective function is

$$2(0) + 3(3) + 4(0) = 9.$$

Example (5.51): Consider the following LP problem

Minimize $g(x_1, x_2) = -x_2$ **[JU-94]**

Subject to $x_1 - x_2 \geq 1$

$-x_1 + x_2 \geq 2$

$x_1, x_2 \geq 0$

(i) Give the dual of the above problem,

(ii) Sketch the constraints set of both the dual and primal.

Solution: (i) Taking w_1 and w_2 as dual variables, we can write the dual of the given LP problem as follows:

Maximize $f(w_1, w_2) = w_1 + 2w_2$

Subject to $w_1 - w_2 \leq 0$

$-w_1 + w_2 \leq -1$

$w_1, w_2 \geq 0$

(ii) Figure-5.4 represents the graph of the primal constraints, which shows that the primal problem has no feasible solution, because there is no feasible region. And Figure-5.5 represents the graph of the dual constraints, which shows that the dual problem also has no feasible solution, because there is no feasible region.

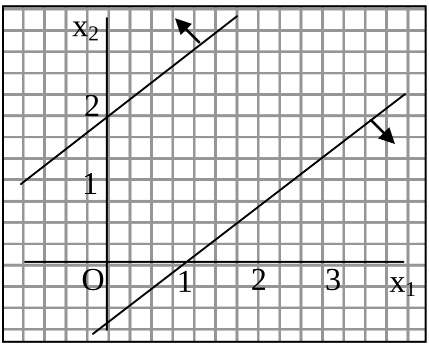

Figure 5.4

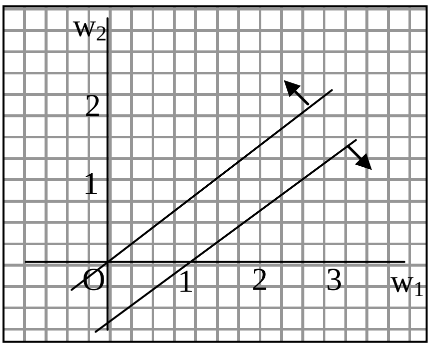

Figure 5.5

Example (5.52): Solve the following LP problem and find its dual and hence solve the dual problem.

$$\begin{aligned} &\text{Maximize } 14x_1 + 4x_2 \\ &\text{Subject to } 2x_1 + x_2 \leq 3 \\ &\qquad x_1 - x_2 \leq 1 \\ &\qquad x_1, x_2 \geq 0 \end{aligned}$$

Solution: We can write the given LP problem as follows:

$$\begin{aligned} &\text{Maximize } 14x_1 + 4x_2 \\ &\text{Subject to } 2x_1 + x_2 + x_3 \qquad = 3 \\ &\qquad x_1 - x_2 \qquad + x_4 = 1 \\ &\qquad x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Solving the primal problem by simplex method, we get

Basis	C_B^t	c_j / P_0	14 / P_1	4 / P_2	0 / P_3	0 / P_4	Ratio θ
P_3	0	3	2	1	1	0	3/2
P_4	0	1	1	-1	0	1	1 min.
$z_j - c_j$		0	-14 Smallest	-4	0	0	

Basis	C_B^t	c_j / P_0	14 / P_1	4 / P_2	0 / P_3	0 / P_4	Ratio θ
P_3	0	1	0	3	1	-2	1/3 min.
P_1	14	1	1	-1	0	1	
$z_j - c_j$		14	0	-18 Smallest	0	14	

Basis	C_B^t	c_j / P_0	14 / P_1	4 / P_2	0 / P_3	0 / P_4	Ratio θ
P_2	4	1/3	0	1	1/3	-2/3	
P_1	14	4/3	1	0	1/3	1/3	
$z_j - c_j$		20	0	0	6	2	

Third tableau gives the optimum solution of the primal is $x_1 = 4/3$, $x_2 = 1/3$ and objective$_{max}$ = 20.

The dual of the given problem as follows:

Minimize $3w_1 + w_2$
Subject to $2w_1 + w_2 \geq 14$
$w_1 - w_2 \geq 4$
$w_1, w_2 \geq 0$

From the third table, we find

$$c^o = (4, 14),\ b = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{pmatrix}$$

Therefore, the optimum solution to the dual is

$$w^o = c^o B^{-1} = (4, 14) \begin{pmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{pmatrix} = (6, 2)$$

i.e., $w_1 = 6, w_2 = 2$

And the minimum value of the dual objective function is

$$w^0 b = (6, 2)\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 20.$$

Note:(Solution of dual using complementary slackness conditions)
Complementary slackness conditions imply that at the optimum stage $w_3x_1 = 0$ ... (1)
$w_4x_2 = 0$ ... (2)
where w_3, w_4 are dual surplus variables.
and if the kth constraint of the primal is a strict inequality then the corresponding kth dual variable w_k must be zero. ... (3)
As $x_1 = 4/3 > 0$, (1) implies $w_3 = 0$.
As $x_2 = 1/3 > 0$, (2) implies $w_4 = 0$.
As the 1st constraint $2x_1 + x_2 = 2(4/3) + 1/3 = 3$ is an equality, (3) implies $w_1 \geq 0$.
As the 1st constraint $x_1 - x_2 = 4/3 - 1/3 = 1$ is an equality, (3) implies $w_2 \geq 0$.
Considering surplus variables $w_3 = 0$, $w_4 = 0$ we have from dual constraints $2w_1 + w_2 = 14$
$w_1 - w_2 = 4$
Solving these system, we get $w_1 = 6$, $w_2 = 2$ and the maximum value of the dual objective function is $3(6) + 2 = 20$.

So, dual solution is $w_1 = 6$, $w_2 = 2$ & $\text{dual}_{min} = 20$.

<u>Example (5.53)</u>: Solve the following LP problem using its dual:

Maximize $z = 5x_1 - 2x_2 + 3x_3$

Subject to $2x_1 + 2x_2 - x_3 \geq 2$ **[NUH-07]**

$3x_1 - 4x_2 \leq 3$

$x_2 + 3x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

Solution: The given problem may be expressed as

Maximize $z = 5x_1 - 2x_2 + 3x_3$

Subject to $-2x_1 - 2x_2 + x_3 \leq -2$

$3x_1 - 4x_2 \leq 3$

$x_2 + 3x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

The associated dual is given by

Minimize $u = -2w_1 + 3w_2 + 5w_3$

Subject to $-2w_1 + 3w_2 \geq 5$

$-2w_1 - 4w_2 + w_3 \geq -2$

$w_1 + 3w_3 \geq 3$

$w_1, w_2, w_3 \geq 0$

The standard form of the dual problem is as follows:

Minimize $u = -2w_1 + 3w_2 + 5w_3 + 0w_4 + 0w_5 + 0w_6 + Mw_7 + Mw_8$

Subject to $-2w_1 + 3w_2 - w_4 + w_7 = 5$

$2w_1 + 4w_2 - w_3 + w_5 = 2$

$w_1 + 3w_3 - w_6 + w_8 = 3$

$w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8 \geq 0$

Solving the dual problem by simplex method, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-2	3	5	0	0	0	M	M	Ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	θ
$\underline{P}_7$	M	5	-2	3	0	-1	0	0	1	0	5/3
$\underline{P}_5$	0	2	2	4	-1	0	1	0	0	0	1/2 min
$\underline{P}_8$	M	3	1	0	3	0	0	-1	0	1	
$z_j - c_j$		0	2	-3	-5	0	0	0	0	0	
		8	-1	3	3	-1	0	-1	0	0	

Let largest

Basis	C_B^t	c_j	-2	3	5	0	0	0	M	M	Ratio
		P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	θ
P_7	M	7/2	-7/2	0	3/4	-1	-3/4	0	1	0	14/3
P_2	3	1/2	1/2	1	-1/4	0	1/4	0	0	0	
P_8	M	3	1	0	3	0	0	-1	0	1	1 min
$z_j - c_j$		3/2	7/2	0	-23/4	0	3/4	0	0	0	
		13/2	-5/2	0	15/4	-1	-3/4	-1	0	0	
					Largest						

Basis	C_B^t	c_j	-2	3	5	0	0	0	M	M	Ratio
		P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	θ
P_7	M	11/4	-15/4	0	0	-1	-3/4	1/4	1	-1/4	11
P_2	3	3/4	7/12	1	0	0	1/4	-1/12	0	1/12	
P_3	5	1	1/3	0	1	0	0	-1/3	0	1/3	
$z_j - c_j$		29/4	65/12	0	0	0	3/4	-23/12	0	23/12	
		11/4	-15/4	0	0	-1	-3/4	1/4	0	-5/4	
								Largest			

Basis	C_B^t	c_j	-2	3	5	0	0	0	M	M	Ratio
		P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	θ
P_6	0	11	-15	0	0	-4	-3	1	4	-1	
P_2	3	5/3	-2/3	1	0	-1/3	0	0	1/3	0	
P_3	5	14/3	-14/3	0	1	-4/3	-1	0	4/3	0	
$z_j - c_j$		85/3	-70/3	0	0	-23/3	-5	0	23/3	0	
		0	0	0	0	0	0	0	0	0	

Since in the 4th tableau all $z_j - c_j \leq 0$ except artificial columns, the optimality conditions are satisfied. And this table gives the optimum solution of the dual problem $w_1 = 0$, $w_2 = 5/3$, $w_3 = 14/3$ and $u_{min} = 85/3$. Also from the 4th tableau, we find the optimal solution of the given LP problem as follows:

$x_1 = z_7 - c_7 + c_7 = 23/3$

$x_2 = -(z_5 - c_5 + c_5) = -(-5) = 5$ [$\because$ 2nd constraint of the dual was multiplied by –1 to convert into standard form]

$x_3 = z_8 - c_8 + c_8 = 0$

and $z_{max} = 85/3$.

<u>Example (5.54)</u>: Solving the primal problem by simplex method, find the solution of its dual problem. **[RU-93]**

$$\begin{aligned} &\text{Maximize } z = 5x_1 + 4x_2 \\ &\text{Subject to } 3x_1 + 4x_2 \le 24 \\ &\qquad 3x_1 + 2x_2 \le 18 \\ &\qquad\qquad x_2 \le 5 \\ &\qquad x_1, x_2 \ge 0 \end{aligned}$$

Solution: We can write the given LP problem as follows:

$$\begin{aligned} &\text{Maximize } z = 5x_1 + 4x_2 \\ &\text{Subject to } 3x_1 + 4x_2 + x_3 = 24 \\ &\qquad 3x_1 + 2x_2 + x_4 = 18 \\ &\qquad x_2 + x_5 = 5 \\ &\qquad x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

Solving the primal problem by simplex method, we get

Basis	C_B^t	c_j	5	4	0	0	0	Ratio
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_3$	0	24	3	4	1	0	0	8
$\underline{P}_4$	0	18	3	2	0	1	0	6 min.
$\underline{P}_5$	0	5	0	1	0	0	1	
$z_j - c_j$		0	-5 Smallest	-4	0	0	0	

Basis	C_B^t	c_j	5	4	0	0	0	Ratio
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_3$	0	6	0	2	1	-1	0	3 min.
$\underline{P}_1$	5	6	1	2/3	0	1/3	0	9
$\underline{P}_5$	0	5	0	1	0	0	1	5
$z_j - c_j$		30	0	-2/3 Smallest	0	5/3	0	

Basis	C_B^t	c_j	5	4	0	0	0	Ratio
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_2$	4	3	0	1	1/2	-1/2	0	
$\underline{P}_1$	5	4	1	0	-1/3	2/3	0	
$\underline{P}_5$	0	2	0	0	-1/2	1/2	1	
$z_j - c_j$		32	0	0	1/3	4/3	0	

Third tableau gives the optimum solution of the primal is $x_1 = 4$, $x_2 = 3$ and $z_{max} = 32$.
The dual of the given problem is as follows:

Minimize $u = 24w_1 + 18w_2 + 5w_3$

Subject to $3w_1 + 3w_2 \geq 5$

$4w_1 + 2w_2 + w_3 \geq 4$

$w_1, w_2, w_3 \geq 0$

Also from the 3rd tableau, we find the optimal solution of the dual problem as follows:

$w_1 = z_3 - c_3 + c_3 = 1/3$

$w_2 = z_4 - c_4 + c_4 = 4/3$

$w_3 = z_5 - c_5 + c_5 = 0$

and $u_{min} = 32$.

Example (5.55): A pension fund manager is considering investing in two shares A and B. It is estimated that (i) share A will earn a dividend of 12% per annum and share B 4% per annum, (ii) growth in the market value in one year of share A will be 10 paisa per Tk.1 invested and in B 40 paisa per Tk.1 invested.
He requires investing the minimum total sum which will give dividend income of at least Tk.600 per annum and growth in one year of at least Tk.1000 on the initial investment. You are required to (a) state the mathematical formulation of the problem, and (b) compute the minimum sum to be invested to meet the manager's objective by using the simplex method on the dual problem.
Solution: The given problem, stated in an appropriate mathematical form is as follows:

Minimize $z = x_1 + x_2$ (total investment)

Subject to $0.12x_1 + 0.04x_2 \geq 600$

$0.10x_1 + 0.40x_2 \geq 1000$

$x_1, x_2 \geq 0$

where x_1 and x_2 denote the number of units of shares A and B respectively.
The dual of the above problem is

Maximize u = 600w1 + 1000w2

Subject to $0.12w_1 + 0.10w_2 \leq 1$

$0.04w_1 + 0.40w_2 \leq 1$

$w_1, w_2 \geq 0$

The standard form of the dual problem is as follows:

Maximize u = 600w1 + 1000w2

Subject to $0.12w_1 + 0.10w_2 + w_3 \quad = 1$

$0.04w_1 + 0.40w_2 \quad + w_4 = 1$

$w_1, w_2, w_3, w_4 \geq 0$

The simplex tableau of the dual problem as follows:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	600 / $\underline{P}_1$	1000 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Ratio θ
$\underline{P}_3$	0	1	0.12	0.10	1	0	10
$\underline{P}_4$	0	1	0.04	0.40	0	1	5/2 min.
$z_j - c_j$		0	-600	-1000 Smallest	0	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	600 / $\underline{P}_1$	1000 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Ratio θ
$\underline{P}_3$	0	0.75	0.11	0	1	-0.25	75/11 min.
$\underline{P}_2$	1000	2.5	0.10	1	0	2.5	25
$z_j - c_j$		2500	-500 Smallest	0	0	2500	

Basis	$\underline{C}_B^t$	c_j $\underline{P}_0$	600 / $\underline{P}_1$	1000 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Ratio θ
$\underline{P}_1$	600	75/11	1	0	100/11	-25/11	
$\underline{P}_2$	1000	20/11	0	1	-10/11	30/11	
$z_j - c_j$		65000/11	0	0	50000/11	15000/11	

Third tableau gives the following optimal basic feasible solution to the dual problem: $w_1 = 75/11$, $w_2 = 20/11$ and u_{max} = Tk.65000/11. Also the optimal solution to the given primal problem as obtained from the third tableau as follows:

$x_1 = 50000/11$, $x_2 = 15000/11$ and z_{min} = Tk.65000/11.

Example (5.56): Solve the dual of the given LP problem and hence using complementary slackness conditions find the solution of the primal. Minimize $z = x_1 + 4x_2 + 3x_4$ **[JU-95]**

$$\text{Subject to } x_1 + 2x_2 - x_3 + x_4 \geq 3$$
$$-2x_1 - x_2 + 4x_3 + x_4 \geq 2$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Solution: Introducing surplus variables $x_5 \geq 0$ and $x_6 \geq 0$, we get the standard form of the primal problem as follows:

$$\text{Minimize } z = x_1 + 4x_2 + 3x_4$$
$$\text{Subject to } x_1 + 2x_2 - x_3 + x_4 - x_5 = 3$$
$$-2x_1 - x_2 + 4x_3 + x_4 - x_6 = 2$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

The dual of the given primal is as follows:

$$\text{Maximize } u = 3w_1 + 2w_2$$
$$\text{Subject to } \quad w_1 - 2w_2 \leq 1$$
$$2w_1 - w_2 \leq 4$$
$$-w_1 + 4w_2 \leq 0$$
$$w_1 + w_2 \leq 3$$
$$w_1, w_2 \geq 0$$

Introducing slack variables $w_3 \geq 0$, $w_4 \geq 0$, $w_5 \geq 0$, $w_6 \geq 0$, we get the standard form of the dual problem as follows:

$$\text{Maximize } u = 3w_1 + 2w_2$$
$$\text{Subject to } \quad w_1 - 2w_2 + w_3 = 1$$
$$2w_1 - w_2 + w_4 = 4$$
$$-w_1 + 4w_2 + w_5 = 0$$
$$w_1 + w_2 + w_6 = 3$$
$$w_1, w_2, w_3, w_4, w_5, w_6 \geq 0$$

Drawing the dual constraints in the graph paper, we find the shaded feasible solution space OAB. The vertices O(0, 0), A(1, 0) and B(2, 1/2) give the value of the dual objective function 0, 3 and 7 respectively. Here the maximum value is 7 and attain at B(2, 1/2). So the optimum solution of the dual is $w_1 = 2$, $w_2 = 1/2$ and $u_{max} = 7$.

Complementary slackness conditions imply that at the optimum stage

$w_1 x_5 = 0$ ... (1)

$w_2 x_6 = 0$ ... (2)

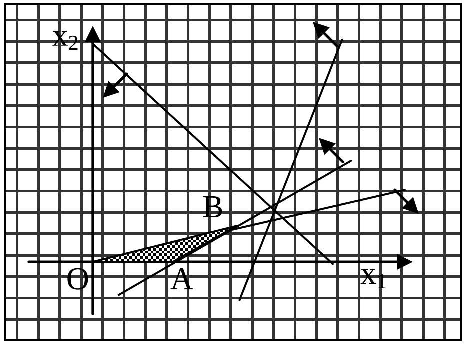

Figure 5.6

and if the kth constraint of the dual is a strict inequality then the corresponding kth primal variable x_k must be zero. ... (3)

As $w_1 = 2 > 0$, (1) implies $x_5 = 0$.

As $w_2 = 1/2 > 0$, (2) implies $x_6 = 0$.

As the 1st constraint $w_1 - 2w_2 = 2 - 2(1/2) = 1$ is an equality, (3) implies $x_1 \geq 0$.

As the 2nd constraint $2w_1 - w_2 = 2(2) - 1/2 = 7/2 \leq 4$ is an inequality, (3) implies $x_2 = 0$.

As the 3rd constraint $-w_1 + 4w_2 = -2 + 4(1/2) = 0$ is an equality, (3) implies $x_3 \geq 0$.

As the 4th constraint $w_1 + w_2 = 2 + 1/2 = 3/2 \leq 3$ is an inequality, (3) implies $x_4 = 0$.

Putting $x_2 = 0$, $x_4 = 0$, $x_5 = 0$ and $x_6 = 0$ in the constraints of the standard primal problem, we have

$$x_1 - x_3 = 3$$
$$-2x_1 + 4x_3 = 2$$

Solving these system, we get $x_3 = 4$, $x_1 = 7$ and the minimum value of the primal objective function is $7 + 4(0) + 3(0) = 7$.

Thus, the optimal solution of the primal is $x_1 = 7$, $x_2 = 0$, $x_3 = 4$, $x_4 = 0$ and $z_{min} = 7$.

<u>Example (5.57)</u>: Solve the given primal problem and hence using complementary slackness conditions find the solution of the dual.

$$\text{Maximize } z = 3x_1 + 2x_2$$
$$\text{Subject to } -x_1 + 2x_2 \leq 4$$
$$3x_1 + 2x_2 \leq 14$$
$$x_1 - x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Solution: Introducing slack variables $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$, we get the standard form of the primal problem as follows:

$$\text{Maximize } z = 3x_1 + 2x_2$$
$$\text{Subject to } -x_1 + 2x_2 + x_3 = 4$$
$$3x_1 + 2x_2 + x_4 = 14$$
$$x_1 - x_2 + x_5 = 3$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The dual of the given primal is as follows:

$$\text{Minimize } u = 4w_1 + 14w_2 + 3w_3$$
$$\text{Subject to } -w_1 + 3w_2 + w_3 \geq 3$$
$$2w_1 + 2w_2 - w_3 \geq 2$$
$$w_1, w_2, w_3 \geq 0$$

Introducing surplus variables $w_4 \geq 0$, $w_5 \geq 0$, we get the standard form of the dual problem as follows:

$$\text{Minimize } u = 4w_1 + 14w_2 + 3w_3$$
$$\text{Subject to } -w_1 + 3w_2 + w_3 - w_4 = 3$$
$$2w_1 + 2w_2 - w_3 - w_5 = 2$$
$$w_1, w_2, w_3, w_4, w_5 \geq 0$$

Drawing the primal constraints in the graph paper, we find the shaded feasible solution space OABCD. The vertices O(0, 0), A(3, 0), B(4, 1), C(5/2, 13/4) and D(0, 2) give the value of the dual objective function 0, 9, 14, 14 and 4 respectively.

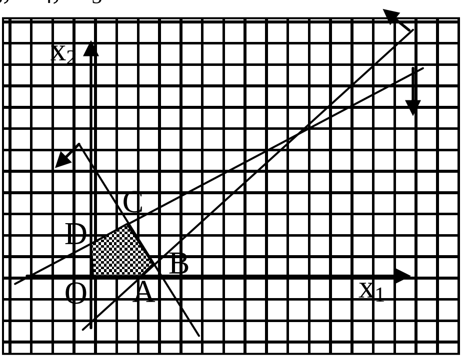

Figure 5.7

Here the maximum value is 14 and attained at B(4, 1) and C(5/2, 13/4). So the optimum solution of the dual is $(x_1, x_2) = (4,1)$ or (5/2, 13/4) and $z_{max} = 14$.

Complementary slackness conditions imply that at the optimum stage

$$x_1w_4 = 0 \quad \ldots \quad (1)$$

$$x_2w_5 = 0 \quad \ldots \quad (2)$$

and if the kth constraint of the primal is a strict inequality then the corresponding kth dual variable w_k must be zero. ... (3)

<u>For primal solution $(x_1, x_2) = (4, 1)$</u>:

As $x_1 = 4 > 0$, (1) implies $w_4 = 0$.

As $x_2 = 1 > 0$, (2) implies $w_5 = 0$.

As the 1st constraint $-x_1 + 2x_2 = -4 + 2(1) = -2 \leq 4$ is an inequality, (3) implies $w_1 = 0$.

As the 2nd constraint $3x_1 + 2x_2 = 3(4) + 2(1) = 14$ is an equality, (3) implies $w_2 \geq 0$.

As the 3rd constraint $x_1 - x_2 = 4 - 1 = 3$ is an equality, (3) implies $w_3 \geq 0$.

Putting $w_1 = 0$, $w_4 = 0$ and $w_5 = 0$ in the constraints of the standard dual problem, we have

$$3w_2 + w_3 = 3$$

$$2w_2 - w_3 = 2$$

Solving these system, we get $w_2 = 1$, $w_3 = 0$ and the minimum value of the dual objective function is $4(0) + 14(1) + 3(0) = 14$.

Thus, the optimal solution of the dual is $w_1 = 0$, $w_2 = 1$, $w_3 = 0$ and $u_{min} = 14$.

<u>For primal solution $(x_1, x_2) = (5/2, 13/4)$</u>:

As $x_1 = 5/2 > 0$, (1) implies $w_4 = 0$.

As $x_2 = 13/4 > 0$, (2) implies $w_5 = 0$.

As the 1st constraint $-x_1 + 2x_2 = -5/2 + 2(13/4) = 4$ is an equality, (3) implies $w_1 \geq 0$.

As the 2nd constraint $3x_1 + 2x_2 = 3(5/2) + 2(13/4) = 14$ is an equality, (3) implies $w_2 \geq 0$.

As the 3rd constraint $x_1 - x_2 = 5/2 - 13/4 = -3/4 \leq 3$ is an inequality, (3) implies $w_3 = 0$.
Putting $w_3 = 0$, $w_4 = 0$ and $w_5 = 0$ in the constraints of the standard dual problem, we have

$$-w_1 + 3w_2 = 3$$
$$2w_1 + 2w_2 = 2$$

Solving these system, we get $w_1 = 0$, $w_2 = 1$ and the minimum value of the dual objective function is $4(0) + 14(1) + 3(0) = 14$.
Thus, for both primal optimal solutions, dual has unique optimal solution $w_1 = 0$, $w_2 = 1$, $w_3 = 0$ and $u_{min} = 14$.

<u>Example (5.58)</u>: Solve the given primal problem and hence using complementary slackness conditions find the solution of the dual.

$$\begin{aligned} \text{Maximize } z = \ & 2x_1 + x_2 + 3x_3 \\ \text{Subject to} \quad & x_1 + x_2 + 2x_3 \leq 5 \\ & 2x_1 + 3x_2 + 4x_3 = 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: Introducing slack variable $x_4 \geq 0$ to 1st constraint, we get the standard form as follows:

$$\begin{aligned} \text{Maximize } z = \ & 2x_1 + x_2 + 3x_3 + 0x_4 \\ \text{Subject to} \quad & x_1 + x_2 + 2x_3 + x_4 = 5 \\ & 2x_1 + 3x_2 + 4x_3 + 0x_4 = 12 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Since this standard form does not contain the basis, we introduce artificial variable $x_5 \geq 0$ to 2nd constraint to find the basis and add this variable with the objective function with the coefficient $-M$ to apply the big M-method. Then we get

$$\begin{aligned} \text{Maximize } z = \ & 2x_1 + x_2 + 3x_3 + 0x_4 - Mx_5 \\ \text{Subject to} \quad & x_1 + x_2 + 2x_3 + x_4 + 0x_5 = 5 \\ & 2x_1 + 3x_2 + 4x_3 + 0x_4 + x_5 = 12 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

We find the following initial simplex table from the problem:

Sl.	Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2 $\underline{P}_1$	1 $\underline{P}_2$	3 $\underline{P}_3$	0 $\underline{P}_4$	-M $\underline{P}_5$	Ratio θ
1	$\underline{P}_4$	0	5	1	1	2	1	0	$5/2 = \theta_o$
2	$\underline{P}_5$	-M	12	2	3	4	0	1	$12/4 = 3$
2+1	$z_j - c_j$		0	-2	-1	-3	0	0	
2+2			-12	-2	-3	-4 Smallest	0	0	Coef. of M

Sl.	Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2 $\underline{P}_1$	1 $\underline{P}_2$	3 $\underline{P}_3$	0 $\underline{P}_4$	-M $\underline{P}_5$	Ratio θ
1	$\underline{P}_3$	3	5/2	1/2	1/2	1	1/2	0	5
2	$\underline{P}_5$	-M	2	0	1	0	-2	1	$2/1=2 = \theta_o$
2+1	$z_j - c_j$		15/2	-1/2	1/2	0	3/2	0	
2+2			-2	0	-1 Smallest	0	2	0	Coef. of M

Sl.	Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2 $\underline{P}_1$	1 $\underline{P}_2$	3 $\underline{P}_3$	0 $\underline{P}_4$	-M $\underline{P}_5$	Ratio θ
1	$\underline{P}_3$	3	3/2	1/2	0	1	3/2	-	$3 = \theta_o$
2	$\underline{P}_2$	1	2	0	1	0	-2	-	
2+1	$z_j - c_j$		13/2	-1/2 Smallest	0	0	5/2	-	
2+2			0	0	0	0	0	-	Coef. of M

Sl.	Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2 $\underline{P}_1$	1 $\underline{P}_2$	3 $\underline{P}_3$	0 $\underline{P}_4$	-M $\underline{P}_5$	Ratio θ
1	$\underline{P}_1$	2	3	1	0	2	3	-	
2	$\underline{P}_2$	1	2	0	1	0	-2	-	
2+1	$z_j - c_j$		8	0	0	1	4	-	

Since all $z_j - c_j \geq 0$ the table is optimal. The above tableau gives us the extreme point (3, 2, 0). So, the optimal solution of the problem is $x_1 = 3$, $x_2 = 2$, $x_3 = 0$ and the maximum value of the objective function is 8.

The dual of the given problem is as follows:

Minimize $u = 5w_1 + 12w_2$

Subject to $w_1 + 2w_2 \geq 2$
$w_1 + 3w_2 \geq 1$
$2w_1 + 4w_2 \geq 3$
$w_1 \geq 0$, w_2 unrestricted.

Complementary slackness conditions imply that at the optimum stage $x_1w_3 = 0$ ... (1)
$x_2w_4 = 0$ ... (2)
$x_3w_5 = 0$ ... (3)

and if the kth constraint of the primal is a strict inequality then the corresponding kth dual variable w_k must be zero. ... (4)

As $x_1 = 3 > 0$, (1) implies $w_3 = 0$ i.e., dual 1st constraint is equality.
As $x_2 = 2>0$, (2) implies $w_4 = 0$ i.e., dual 2nd constraint is equality.
As $x_3 = 0$, (3) implies $w_5 \geq 0$ i.e, dual 3rd constraint may be inequality.

So, we find the dual constraints as follows:

$$w_1 + 2w_2 = 2$$
$$w_1 + 3w_2 = 1$$
$$2w_1 + 4w_2 \geq 3$$

Solving the first two equations, we get $w_1 = 4$, $w_2 = -1$ and this solution satisfies the 3rd inequality. Thus, the optimal solution of the dual is $w_1 = 4$, $w_2 = -1$ and $u_{min} = 5(4) + 12(-1) = 8$.

Example (5.59): A production firm manufactures four types of products using two types of raw materials A and B. Requirements of raw material for different products, availability of raw materials and profit per unit of each product are given below:

Raw material	Requirement of raw materials				Availability of raw materials
	i	ii	iii	iv	
A	1	2	2	3	20
B	2	1	3	2	20
Profit per unit products	1	2	3	4	

Find how many units of different products to be manufactured to maximize the total profit. And also find the shadow prices for raw materials and interpret then economically. [JU-95]

Solution: Formulation of the primal: Let x_1, x_2, x_3 and x_4 units of different products to be manufactured respectively to maximize total profit. Then, we get, the given problem mathematically as follows

$$\text{Maximize } z = x_1 + 2x_2 + 3x_3 + 4x_4$$
$$\text{Subject to } \quad x_1 + 2x_2 + 2x_3 + 3x_4 \leq 20$$
$$2x_1 + x_2 + 3x_3 + 2x_4 \leq 20$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Solution of the primal: Introducing slack variables $x_5 \geq 0$ and $x_6 \geq 0$ to the 1st and 2nd constraints respectively, we get the standard form of the primal as follows:

$$\text{Maximize } z = x_1 + 2x_2 + 3x_3 + 4x_4 + 0x_5 + 0x_6$$
$$\text{Subject to } \quad x_1 + 2x_2 + 2x_3 + 3x_4 + x_5 \quad = 20$$
$$2x_1 + x_2 + 3x_3 + 2x_4 \quad + x_6 = 20$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Solving the primal problem by simplex method, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	1 / $\underline{P}_1$	2 / $\underline{P}_2$	3 / $\underline{P}_3$	4 / $\underline{P}_4$	0 / $\underline{P}_5$	0 / $\underline{P}_6$	Ratio θ
$\underline{P}_5$	0	20	1	2	2	3	1	0	20/3 min.
$\underline{P}_6$	0	20	2	1	3	2	0	1	10
$z_j - c_j$		0	-1	-2	-3	-4 Smallest	0	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	1 / $\underline{P}_1$	2 / $\underline{P}_2$	3 / $\underline{P}_3$	4 / $\underline{P}_4$	0 / $\underline{P}_5$	0 / $\underline{P}_6$	Ratio θ
$\underline{P}_4$	4	20/3	1/3	2/3	2/3	1	1/3	0	10
$\underline{P}_6$	0	20/3	4/3	-1/3	5/3	0	-2/3	1	4 min.
$z_j - c_j$		80/3	1/3	2/3	-1/3 Smallest	0	4/3	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	1 / $\underline{P}_1$	2 / $\underline{P}_2$	3 / $\underline{P}_3$	4 / $\underline{P}_4$	0 / $\underline{P}_5$	0 / $\underline{P}_6$	Ratio θ
$\underline{P}_4$	4	4	-1/5	4/5	0	1	3/5	-2/5	
$\underline{P}_3$	3	4	4/5	-1/5	1	0	-2/5	3/5	
$z_j - c_j$		28	3/5	3/5	0	0	6/5	1/5	

Since all $z_j - c_j \geq 0$ the optimality conditions are satisfied and the optimal solution of the primal is $x_1 = 0$, $x_2 = 0$, $x_3 = 4$, $x_4 = 4$ and $z_{max} = 28$.

Determination of shadow prices: Taking w_1 and w_2 as dual variables, we get the dual of the given primal as follows:

$$\begin{aligned} &\text{Minimize } u = 20w_1 + 20w_2 \\ &\text{Subject to} \quad w_1 + 2w_2 \geq 1 \\ &\qquad\qquad 2w_1 + w_2 \geq 2 \\ &\qquad\qquad 2w_1 + 3w_2 \geq 3 \\ &\qquad\qquad 3w_1 + 2w_2 \geq 4 \\ &\qquad\qquad w_1, w_2 \geq 0 \end{aligned}$$

From the third table, we find

$$c^o = (4, 3),\ b = \begin{pmatrix} 20 \\ 20 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} 3/5 & -2/5 \\ -2/5 & 3/5 \end{pmatrix}$$

Therefore, the optimum solution to the dual is

$$w^o = c^oB^{-1} = (4, 3)\begin{pmatrix} 3/5 & -2/5 \\ -2/5 & 3/5 \end{pmatrix} = (6/5, 1/5)$$

i.e., $w_1 = 6/5$, $w_2 = 1/5$ and $u_{min} = (6/5, 1/5)\begin{pmatrix} 20 \\ 20 \end{pmatrix} = 28.$

We know that the optimum values of the dual variables are respected shadow prices. So, $w_1 = 6/5$ is the shadow price of raw material A and $w_2 = 1/5$ is the shadow price of raw material B.

Economic interpretation: The shadow price of raw material A is 6/5 means the optimum value of the primal objective function will change by 6/5 with unit change of the availability of raw material A up to certain limit. And the shadow price of raw material B is

1/5 means the optimum value of the primal objective function will change by 1/5 with unit change of the availability of raw material B up to certain limit.

<u>Example (5.60)</u>: A company is manufacturing two products A, B and C. The manufacturing times required making them; the profit and capacity available at each work centre are given by the following table:

Products	Work centres			Profit per unit (in $)
	Matching (in hours)	Fabrication (in hours)	Assembly (in hours)	
A	8	4	2	20
B	2	0	0	6
C	3	3	1	8
Total Capacity	250	150	50	

Company likes to maximize their profit making their products A, B and C. Formulate this linear programming problem and then solve. And also find the shadow prices for matching hours, fabrication hours, assembling hours availability and interpret then economically.

Solution: If we consider x_1, x_2 and x_3 be the numbers of products A, B and C respectively to be produced for maximizing the profit. Then company's total profit $z = 20x_1 + 6x_2 + 8x_3$ is to be maximized. And subject to the constraints are $8x_1 + 2x_2 + 3x_3 \leq 250$, $4x_1 + 3x_3 \leq 150$ and $2x_1 + x_3 \leq 50$. Since it is not possible for the manufacturer to produce negative number of the products, it is obvious that $x_1, x_2, x_3 \geq 0$. So, we can summarize the above linguistic linear programming problem as the following mathematical form: Maximize $z = 20x_1 + 6x_2 + 8x_3$

$$\text{Subject to } 8x_1 + 2x_2 + 3x_3 \leq 250$$
$$4x_1 + 3x_3 \leq 150$$
$$2x_1 + x_3 \leq 50$$
$$x_1, x_2, x_3 \geq 0$$

Introducing slack variables x_4, x_5, $x_6 \geq 0$ and keeping the problem maximization type we get the following simplex tables.

Basis	C_B^t	c_j	20	6	8	0	0	0	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	θ
$\underline{P}_4$	0	250	8	2	3	1	0	0	250/8
$\underline{P}_5$	0	150	4	3	0	0	1	0	150/4
$\underline{P}_6$	0	50	2	0	1	0	0	1	50/2 Min
$z_j - c_j$		0	-20 Smallest	-6	-8	0	0	0	
$\underline{P}_4$	0	50	0	2	-1	1	0	-4	25
$\underline{P}_5$	0	50	0	3	-2	0	1	-2	50/3 Min
$\underline{P}_1$	20	25	1	0	1/2	0	0	1/2	--
$z_j - c_j$		500	0	-6 Smallest	2	0	0	10	
$\underline{P}_4$	0	50/3	0	0	1/3	1	-2/3	-8/3	50 | 3
$\underline{P}_2$	6	50/3	0	1	-2/3	0	1/3	-2/3	--- | $\underline{P}_4/\underline{P}_3$
$\underline{P}_1$	20	25	1	0	1/2	0	0	1/2	50 | 0Min
$z_j - c_j$		600	0	0	-2 Smallest	0	2	6	
$\underline{P}_4$	0	0	-2/3	0	0	1	-2/3	-3	
$\underline{P}_2$	6	50	4/3	1	0	0	1/3	0	
$\underline{P}_3$	8	50	2	0	1	0	0	1	
$z_j - c_j$		700	4	0	0	0	2	8	

Since the problem is maximization type and in the 4th table all $z_j - c_j \geq 0$, it is optimal and the optimal basic feasible solution is $x_1 = 0$, $x_2 = 50$, $x_3 = 50$ and the maximum profit, $z_{max} = \$700$. Therefore, to earn maximum profit \$700, the producer should produce 50 units of product B and 50 units of product C.

Determination of shadow prices: We know that for primal optimal solution, optimal w_i is equal to $(z_j - c_j) + c_j$ which has, for its corresponding unit vector in the initial simplex tableau, the vector whose unit element is in position i. So, from the optimal table, we find

$$w_1 = 0,\ w_2 = 2,\ w_3 = 8$$

which are shadow prices for matching hours, fabrication hours, assembling hours availability respectively.

Economic interpretation: The shadow price 0 for matching hour's availability means the optimum value of the primal objective function does not change with the change of the matching hour availability. Similarly, shadow prices 2 and 8 say that the value of the primal objective function changes by 2 and 8 with unit change of fabrication hours and assembling hour's availability respectively.

Example (5.61): A company which manufactures three products A, B, C, requiring two resources labour and material wants to maximize the total profit. Let the linear program of its problem is as follows:

$$\begin{aligned} &\text{Maximize } z = 3x_1 + x_2 + 5x_3 \\ &\text{Subject to } 6x_1 + 3x_2 + 5x_3 \le 45 \text{ (Labour)} \\ &\qquad 3x_1 + 4x_2 + 5x_3 \le 30 \text{ (Material)} \\ &\qquad x_1, x_2, x_3 \ge 0 \end{aligned}$$

(a) Find the optimal solution of the problem by simplex method.
(b) Determine the shadow prices of all the resources.
(c) Suppose an additional 15 units of materials may be obtained at a cost of $10. Is it profitable to do so?
(d) Find the optimal solution when the available material is increased to 60 units.

Solution: (a) Given that

$$\begin{aligned} &\text{Maximize } z = 3x_1 + x_2 + 5x_3 \\ &\text{Subject to } 6x_1 + 3x_2 + 5x_3 \le 45 \text{ (Labour)} \\ &\qquad 3x_1 + 4x_2 + 5x_3 \le 30 \text{ (Material)} \\ &\qquad x_1, x_2, x_3 \ge 0 \end{aligned}$$

We first transform the given problem to its standard form where x_4 and x_5 are slack variables.

$$\begin{aligned} &\text{Maximize } z = 3x_1 + x_2 + 5x_3 \\ &\text{Subject to } 6x_1 + 3x_2 + 5x_3 + x_4 \quad = 45 \\ &\qquad 3x_1 + 4x_2 + 5x_3 \quad + x_5 = 30 \\ &\qquad x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

The simplex iterations are shown in the following tableau.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	3 / $\underline{P}_1$	1 / $\underline{P}_2$	5 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Ratio θ
$\underline{P}_4$	0	45	6	3	5	1	0	9
$\underline{P}_5$	0	30	3	4	5	0	1	6 *
$z_j - c_j$		0	-3	-1	-5 Smallest	0	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	3 / $\underline{P}_1$	1 / $\underline{P}_2$	5 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Ratio θ
$\underline{P}_4$	0	15	3	-1	0	1	-1	
$\underline{P}_3$	5	6	3/5	4/5	1	0	1/5	
$z_j - c_j$		30	0	3	0	0	1	

The optimal problem is to make 6 units of product C, and 0 unit of product A and B. This optimal program yields the maximum revenue of 2200/3.

(b) Here, $c^o = (0, 5)$, $b = \begin{pmatrix} 45 \\ 30 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1/5 \end{pmatrix}$

So, $(w_1, w_2) = c^oB^{-1} = (0, 5) \begin{pmatrix} 1 & -1 \\ 0 & 1/5 \end{pmatrix} = (0, 1)$

i.e., $w_1 = 0, w_2 = 1$.

Therefore, 0 and 1 are shadow prices of the resources labour and material respectively.

(c) If 15 units of materials are added then $b^o = \begin{pmatrix} 45 \\ 45 \end{pmatrix}$

So, $x^* = B^{-1}b^o = \begin{pmatrix} 1 & -1 \\ 0 & 1/5 \end{pmatrix}\begin{pmatrix} 45 \\ 45 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Therefore, the optimal table remains optimal and the value of $z = 5\times9 = 45$. As the additional 15 units' material cost \$10, so, net profit is $45 - 10 = 35$ which is greater than the optimal value 30 (obtained before). So, adding 15 units' material is profitable.

(d) When the material is increased to 60 units then $b^o = \begin{pmatrix} 45 \\ 60 \end{pmatrix}$.

So, $x^* = B^{-1}b^o = \begin{pmatrix} 1 & -1 \\ 0 & 1/5 \end{pmatrix}\begin{pmatrix} 45 \\ 60 \end{pmatrix} = \begin{pmatrix} -15 \\ 12 \end{pmatrix} \ngeq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the optimal table no longer optimal. Taking iterations by dual simplex method with the following simplex table, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	3 / $\underline{P}_1$	1 / $\underline{P}_2$	5 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Negative of $\underline{P}_o$
$\underline{P}_4$	0	-15	3	-1	0	1	-1	Pivot row
$\underline{P}_3$	5	12	3/5	4/5	1	0	1/5	
$z_j - c_j$		60	0	3	0	0	1	
$\frac{z_j - c_j}{y_{1j}}$; $y_{1j}<0$				-3			-1 Pivot column	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	3 / $\underline{P}_1$	1 / $\underline{P}_2$	5 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Negative of $\underline{P}_o$
$\underline{P}_5$	0	15	-3	1	0	-1	-1	
$\underline{P}_3$	5	9	6/5	1	1	1/5	0	
$z_j - c_j$		45	3	4	0	1	0	

So, the optimal solution is 45 when the material is increased to 60 units.

<u>Example (5.62)</u>: A factory manufactures four products, which require three resources, labour, material and administration. The unit profits on these products are Tk.6, Tk.1, Tk.4 and Tk.5 respectively. There are 30 hours of labour, 15 kg. of materials and 5 hours of administration per day. In order to determine the optimal product mix, the following LP model is formulated and solved.

$$\begin{aligned} &\text{Maximize } z = 6x_1 + x_2 + 4x_3 + 5x_4 \\ &\text{Subject to } 2x_1 + x_2 + x_3 + x_4 \le 30 \text{ (Labour)} \\ &\qquad x_1 + 2x_3 + x_4 \le 15 \text{ (Material)} \\ &\qquad x_1 + x_2 + x_3 \le 5 \text{ (Administration)} \\ &\qquad x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

where x_1, x_2, x_3, x_4 are hourly production level of products 1,2,3,4.

(a) Find the optimal solution of the problem by simplex method.

(b) Determine the shadow prices of all the resources.

(c) Find the optimal solution when the available material is increased to 20 units.

(d) If 25 units labour used instead of 30, what will be the solution?

(e) If 15 units labour used instead of 30 then what will be the solution?

Solution: (a) Given that

$$\begin{aligned} &\text{Maximize } z = 6x_1 + x_2 + 4x_3 + 5x_4 \\ &\text{Subject to } 2x_1 + x_2 + x_3 + x_4 \le 30 \text{ (Labour)} \\ &\qquad x_1 + 2x_3 + x_4 \le 15 \text{ (Material)} \\ &\qquad x_1 + x_2 + x_3 \le 5 \text{ (Administration)} \\ &\qquad x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

We first transform the given problem to its standard form where x_5, x_6 and x_7 are slack variables.

$$\begin{aligned} &\text{Maximize } z = 6x_1 + x_2 + 4x_3 + 5x_4 \\ &\text{Subject to } 2x_1 + x_2 + x_3 + x_4 + x_5 = 30 \\ &\qquad x_1 + 2x_3 + x_4 + x_6 = 15 \\ &\qquad x_1 + x_2 + x_3 + x_7 = 5 \\ &\qquad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \ge 0 \end{aligned}$$

The simplex iterations are shown in the following tableau.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	6 $\underline{P}_1$	1 $\underline{P}_2$	4 $\underline{P}_3$	5 $\underline{P}_4$	0 $\underline{P}_5$	0 $\underline{P}_6$	0 $\underline{P}_7$	Ratio θ
$\underline{P}_5$	0	30	2	1	1	1	1	0	0	15
$\underline{P}_6$	0	15	1	0	2	1	0	1	0	15
$\underline{P}_7$	0	5	1	1	1	0	0	0	1	5 min.
$z_j - c_j$		0	-6 Smallest	-1	-4	-5	0	0	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	6 / $\underline{P}_1$	1 / $\underline{P}_2$	4 / $\underline{P}_3$	5 / $\underline{P}_4$	0 / $\underline{P}_5$	0 / $\underline{P}_6$	0 / $\underline{P}_7$	Ratio θ
$\underline{P}_5$	0	20	0	-1	-1	1	1	0	-2	20
$\underline{P}_6$	0	10	0	-1	1	1	0	1	-1	10 min.
$\underline{P}_1$	6	5	1	1	1	0	0	0	1	
$z_j - c_j$		30	0	5	2	-5 Smallest	0	0	6	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	6 / $\underline{P}_1$	1 / $\underline{P}_2$	4 / $\underline{P}_3$	5 / $\underline{P}_4$	0 / $\underline{P}_5$	0 / $\underline{P}_6$	0 / $\underline{P}_7$	Ratio θ
$\underline{P}_5$	0	10	0	0	-2	0	1	-1	-1	
$\underline{P}_4$	5	10	0	-1	1	1	0	1	-1	
$\underline{P}_1$	6	5	1	1	1	0	0	0	1	
$z_j - c_j$		80	0	0	7	0	0	5	1	

The optimal production mix is to make 5 units of product 1, 10 units of product 4 and 0 unit of product 2 and 3. This optimal program yields the maximum revenue of Tk.80.

(b) Here, $c^o = (0, 5, 6)$, $b = \begin{pmatrix} 30 \\ 15 \\ 5 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

So, $(w_1, w_2, w_3) = c^oB^{-1} = (0, 5, 6) \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = (0, 5, 1)$

i.e., $w_1 = 0$, $w_2 = 5$, $w_3 = 1$.

Therefore, 0, 5 and 1 are shadow prices of the resources labour, material and administration respectively.

(c) If 20 units of materials are used instead of 15 units then

$b^o = \begin{pmatrix} 30 \\ 20 \\ 5 \end{pmatrix}$. So, $x^* = B^{-1}b^o = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 30 \\ 20 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ 5 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Therefore, the optimal table remains optimal.

$\therefore$ New optimal solution is $x_1 = 5$, $x_4 = 15$, $x_2 = x_3 = 0$ and the value of $z = 6\times5 + 5\times15 = 105$.

(d) If 25 units of labours are used instead of 30 units then

$$b^o = \begin{pmatrix} 25 \\ 15 \\ 5 \end{pmatrix}. \text{ So, } x^* = B^{-1}b^o = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 25 \\ 15 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, the optimal table remains optimal.

$\therefore$ New optimal solution is $x_1 = 5$, $x_4 = 10$, $x_2 = x_3 = 0$ and the value of $z = 6\times5 + 5\times10 = 80$.

(e) If 15 units of labours are used instead of 30 units then

$$b^o = \begin{pmatrix} 15 \\ 15 \\ 5 \end{pmatrix}. \text{ So, } x^* = B^{-1}b^o = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 15 \\ 15 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \\ 5 \end{pmatrix} \ngeq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \text{ the}$$

optimal table no longer optimal. Taking iterations by dual simplex method with the following simplex table, we get

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	6	1	4	5	0	0	0	Negative of $\underline{P}_o$
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	
$\underline{P}_5$	0	-5	0	0	-2	0	1	-1	-1	Pivot row
$\underline{P}_4$	5	10	0	-1	1	1	0	1	-1	
$\underline{P}_1$	6	5	1	1	1	0	0	0	1	
$z_j - c_j$		80	0	0	7	0	0	5	1	
$\frac{z_j - c_j}{y_{1j}}$;$y_{1j} <0$					-7/2			-5	-1 Pivot column	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	6	1	4	5	0	0	0	Negative of $\underline{P}_o$
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	
$\underline{P}_7$	0	5	0	0	2	0	-1	1	1	
$\underline{P}_4$	5	15	0	-1	3	1	-1	2	0	
$\underline{P}_1$	6	0	1	1	-1	0	1	-1	0	
$z_j - c_j$		75	0	0	5	0	1	4	0	

So, optimal solution is 75 when the labour is decreased to 15 units.

Example (5.63): Solve the following LP problem by dual simplex method.

$$\text{Minimize } z = x_1 + x_2$$
$$\text{Subject to } 2x_1 + x_2 \geq 2$$
$$-x_1 - x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

Solution: Converting the problem as maximization problem and all constraints as $\leq$ type, we get

$$\text{Maximize } Z = -x_1 - x_2$$
$$\text{Subject to } -2x_1 - x_2 \leq -2$$
$$x_1 + x_2 \leq -1$$
$$x_1, x_2 \geq 0$$

Adding slack variables $x_3 \geq 0$ and $x_4 \geq 0$ to 1st and 2nd constraint respectively, we get,

$$\text{Maximize } Z = -x_1 - x_2$$
$$\text{Subject to } -2x_1 - x_2 + x_3 = -2$$
$$x_1 + x_2 + x_4 = -1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Here, initial basic solution is $x_3 = -2$, $x_4 = -1$ which is infeasible. Now we form the initial dual simplex tableau and consecutive iterative tableau as follow:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-1 / $\underline{P}$	-1 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Negative of $\underline{P}_o$
$\underline{P}_3$	0	-2	-2	-1	1	0	-2 P. row
$\underline{P}_4$	0	-1	1	1	0	1	-1
$z_j - c_j$		0	1	1	0	0	
$\dfrac{z_j - c_j}{y_{1j}}$; $y_{1j} < 0$			-1/2 Largest P.column	-1			

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-1 / $\underline{P}$	-1 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Negative of $\underline{P}_o$
$\underline{P}_3$	-1	1	1	1/2	-1/2	0	
$\underline{P}_4$	0	-2	0	1/2	1/2	1	-2 P. row
$z_j - c_j$		-1	0	1/2	1/2	0	

Since all $z_j - c_j \geq 0$ and $b_2 \leq 0$, but all elements in the pivot row are non-negative, the problem has no feasible solution.

<u>Example (5.64)</u>: Solve the following LP problem by dual simplex method. Maximize $z = -3x_1 - 2x_2$

Subject to $x_1 + x_2 \geq 1$, $x_1 + x_2 \leq 7$, $x_1 + 2x_2 \geq 10$, $x_2 \leq 3$

$x_1, x_2 \geq 0$

Solution: Converting all constraints as $\leq$ type, we get

$$\begin{aligned} &\text{Maximize } z = -3x_1 - 2x_2 \\ &\text{Subject to } \quad -x_1 - x_2 \leq -1 \\ &\qquad\qquad x_1 + x_2 \leq 7 \\ &\qquad\qquad -x_1 - 2x_2 \leq -10 \\ &\qquad\qquad x_2 \leq 3 \\ &\qquad\qquad x_1, x_2 \geq 0 \end{aligned}$$

Adding slack variables $x_3 \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$ and $x_6 \geq 0$ to 1st, 2nd, 3rd and 4th constraints respectively, we get,

$$\begin{aligned} &\text{Maximize } z = -3x_1 - 2x_2 \\ &\text{Subject to } \quad -x_1 - x_2 + x_3 = -1 \\ &\qquad\qquad x_1 + x_2 + x_4 = 7 \\ &\qquad\qquad -x_1 - 2x_2 + x_5 = -10 \\ &\qquad\qquad x_2 + x_6 = 3 \\ &\qquad\qquad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Now we form the initial dual simplex tableau and consecutive iterative tableau as follow:

Basis	C_B^t	c_j / P_0	-3 / P	-2 / P_2	0 / P_3	0 / P_4	0 / P_5	0 / P_6	Negative of P_o
P_3	0	-1	-1	-1	1	0	0	0	-2
P_4	0	7	1	1	0	1	0	0	
P_5	0	-10	-1	-2	0	0	1	0	-10 P.row
P_6	0	3	0	1	0	0	0	1	
$z_j - c_j$		0	3	2	0	0	0	0	
$\dfrac{z_j - c_j}{y_{3j}}$; $y_{3j} < 0$			-3	-1 Largest P.column					

Basis	C_B^t	c_j / $\underline{P}_0$	-3 $\underline{P}$	-2 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	0 $\underline{P}_6$	Negative of $\underline{P}_o$
$\underline{P}_3$	0	4	-1/2	0	1	0	-1/2	0	
$\underline{P}_4$	0	2	1/2	0	0	1	1/2	0	
$\underline{P}_2$	-2	5	1/2	1	0	0	-1/2	0	
$\underline{P}_6$	0	-2	-1/2	0	0	0	1/2	1	-2 P.row
$z_j - c_j$		-10	2	0	0	0	1	0	
$\dfrac{z_j - c_j}{y_{3j}}$; $y_{3j} < 0$			-4 P.column						

Basis	C_B^t	c_j / $\underline{P}_0$	-3 $\underline{P}$	-2 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	0 $\underline{P}_6$	Negative of $\underline{P}_o$
$\underline{P}_3$	0	6	0	0	1	0	-1	-1	
$\underline{P}_4$	0	0	0	0	0	1	1	1	
$\underline{P}_2$	-2	3	0	1	0	0	0	1	
$\underline{P}_1$	-3	4	1	0	0	0	-1	-2	
$z_j - c_j$		-18	0	0	0	0	3	4	

Since all $z_j - c_j \geq 0$ and all $b_i \geq 0$, we reached at the optimum stage. And the optimum solution is $x_1 = 4$, $x_2 = 3$ and hence $z_{max} = -18$.

<u>Example (5.65)</u>: Solve the following LP problem by dual simplex method.

$$\text{Minimize } z = 3x_1 + 8x_2$$
$$\text{Subject to } 2x_1 + 5x_2 \leq 30$$
$$x_1 + x_2 \leq 15$$
$$x_1, x_2 \geq 0$$

Solution: Converting the problem as maximization problem and all constraints as $\leq$ type, we get

$$\text{Maximize } Z = -3x_1 - 8x_2$$
$$\text{Subject to } 2x_1 + 5x_2 \leq 30$$
$$x_1 + x_2 \leq 15$$
$$x_1, x_2 \geq 0$$

Adding slack variables $x_3 \geq 0$ and $x_4 \geq 0$ to 1st and 2nd constraint respectively, we get, Maximize $Z = -3x_1 - 8x_2$

Subject to $2x_1 + 5x_2 + x_3 = 30$

$x_1 + x_2 + x_4 = 15$

$x_1, x_2, x_3, x_4 \geq 0$

Here, initial basic solution is $x_3 = 30$, $x_4 = 15$ which is feasible. Now we form the initial dual simplex tableau and consecutive iterative tableau as follow:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	-3 / $\underline{P}$	-8 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Negative of $\underline{P}_o$
$\underline{P}_3$	0	30	2	5	1	0	
$\underline{P}_4$	0	15	1	1	0	1	
$z_j - c_j$		0	3	8	0	0	

Since all $z_j - c_j \geq 0$ and all $b_i \geq 0$, the initial solution is optimal. That is, the solution of the problem is $x_1 = 0$, $x_2 = 0$ and $z_{min} = 0$.

Example (5.66): Solve the following LP problem by dual simplex method. Minimize $z = -3x_1 + 8x_2$

Subject to $3x_1 + 5x_2 \leq 50$

$x_1 + 3x_2 \leq 15$

$x_1, x_2 \geq 0$

Solution: Converting the problem as maximization problem and all constraints as $\leq$ type, we get Maximize $Z = 3x_1 - 8x_2$

Subject to $3x_1 + 5x_2 \leq 50$

$x_1 + 3x_2 \leq 15$

$x_1, x_2 \geq 0$

Adding slack variables $x_3 \geq 0$ and $x_4 \geq 0$ to 1st and 2nd constraint respectively, we get, Maximize $Z = 3x_1 - 8x_2$

Subject to $3x_1 + 5x_2 + x_3 = 50$

$x_1 + 3x_2 + x_4 = 15$

$x_1, x_2, x_3, x_4 \geq 0$

Here, initial basic solution is $x_3 = 30$, $x_4 = 15$ which is feasible. Now we form the initial dual simplex tableau as follow:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	3 / $\underline{P}$	-8 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Negative of $\underline{P}_o$
$\underline{P}_3$	0	50	3	5	1	0	
$\underline{P}_4$	0	15	1	3	0	1	
$z_j - c_j$		0	-3	8	0	0	

Since $z_1 - c_1 \leq 0$, the dual simplex method is not applicable.

Example (5.67): Solve the following LP problem by using dual simplex method.

$$\begin{aligned} &\text{Minimize } z = x_1 + 2x_2 + 3x_3 \\ &\text{Subject to } x_1 - x_2 + x_3 \geq 4 \\ &\qquad x_1 + x_2 + 2x_3 \leq 8 \\ &\qquad x_2 - x_3 \geq 2 \\ &\qquad x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: Converting the problem as maximization problem and all constraints as $\leq$ type, we get

$$\begin{aligned} &\text{Maximize } Z = -x_1 - 2x_2 - 3x_3 \\ &\text{Subject to } -x_1 + x_2 - x_3 \leq -4 \\ &\qquad x_1 + x_2 + 2x_3 \leq 8 \\ &\qquad -x_2 + x_3 \leq -2 \\ &\qquad x_1, x_2, x_3 \geq 0 \end{aligned}$$

Adding slack variables $x_4 \geq 0$, $x_5 \geq 0$ and $x_6 \geq 0$ to 1st, 2nd and 3rd constraint respectively, we get,

$$\begin{aligned} &\text{Maximize } Z = -x_1 - 2x_2 - 3x_3 \\ &\text{Subject to } -x_1 + x_2 - x_3 + x_4 = -4 \\ &\qquad x_1 + x_2 + 2x_3 + x_5 = 8 \\ &\qquad -x_2 + x_3 + x_6 = -2 \\ &\qquad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Here, initial basic solution is $x_4 = -4$, $x_5 = 8$, $x_6 = -2$ which is infeasible.

Now we form the initial dual simplex tableau and consecutive iterative tableau as follow:

Basis	C_B^t	c_j	-1	-2	-3	0	0	0	Negative of $\underline{P}_o$
		$\underline{P}_0$	$\underline{P}$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_4$	0	-4	-1	1	-1	1	0	0	-4 min.
$\underline{P}_5$	0	8	1	1	2	0	1	0	
$\underline{P}_6$	0	-2	0	-1	1	0	0	1	-2
$z_j - c_j$		0	1	2	3	0	0	0	
$\dfrac{z_j - c_j}{y_{1j}}$; $y_{1j} < 0$			-1 Largest P.column		-3				

Basis	C_B^t	c_j	-1	-2	-3	0	0	0	Negative of $\underline{P}_o$
		$\underline{P}_0$	$\underline{P}$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_1$	-1	4	1	-1	1	-1	0	0	
$\underline{P}_5$	0	4	0	2	1	1	1	0	
$\underline{P}_6$	0	-2	0	-1	1	0	0	1	-2 min.
$z_j - c_j$		-4	0	3	2	1	0	0	
$\dfrac{z_j - c_j}{y_{3j}}$; $y_{3j} < 0$				-3 P.column					

Basis	C_B^t	c_j	-1	-2	-3	0	0	0	Negative of $\underline{P}_o$
		$\underline{P}_0$	$\underline{P}$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_1$	-1	6	1	0	0	-1	0	-1	
$\underline{P}_5$	0	0	0	0	3	1	1	2	
$\underline{P}_2$	-2	2	0	1	-1	0	0	-1	
$z_j - c_j$		-10	0	0	5	1	0	3	

Since all $z_j - c_j \geq 0$ and all $b_i \geq 0$, the 3rd tableau gives the optimum solution. That is, the optimum solution of the problem is $x_1 = 6$, $x_2 = 2$, $x_3 = 0$ and $z_{min} = -Z_{max} = -(-10) = 10$.

5.11 Exercises:

1. What do you mean by duality in linear programming problem?
2. Discuss different types of primal-dual problems with examples.
3. Write an algorithm to construct the dual problem of a given primal problem.

4. Form the primal-dual table for a maximizing problem.
5. Create a primal-dual table to find the dual of a minimizing primal problem.
6. Prove that the dual of the dual is the primal itself.
7. Prove that if any constraint of the primal be an equation, then the corresponding dual variable will be unrestricted in sign.
8. State and prove weak duality theorem.
9. Show that the value of the objective function of maximization primal problem for any feasible solution is always less equal to that of its dual.
10. Prove that for optimal feasible solutions of the primal and the dual problems whenever inequality occurs in the kth relation of either system (the corresponding slack variable is positive), then the kth variable of its dual vanishes; if the kth variable is positive in either system, the kth relation if its dual is equality (the corresponding slack variable is zero).
11. From the complementary slackness theorem extract the complementary slackness conditions.
12. Discuss economical interpretation of the primal- dual problems.
13. State and prove maid duality theorem.
14. Explain the duality theory of linear programming problem.
15. Define shadow price. Why shadow price is very important in LP programming?
16. For solving a linear programming problem, write the dual simplex algorithm.
17. Write down the advantages and disadvantages of the dual simplex method.
18. Find dual of the following linear programming problems:

(i) Maximize $5x_1 + 12x_2 + 2x_3$
S/t $8x_1 + 2x_2 + 3x_3 \leq 20$
$9x_1 + 6x_2 - 4x_3 \leq 12$
$x_1, x_2, x_3 \geq 0$

(ii) Minimize $5x_1 + 12x_2 + 2x_3$
S/t $8x_1 + 2x_2 + 3x_3 \geq 20$
$9x_1 + 6x_2 - 4x_3 \geq 12$
$x_1, x_2, x_3 \geq 0$

(iii) Maximize $3x_1 + 2x_2 + 2x_3$
S/t $2x_1 + 7x_2 + 3x_3 \leq 50$
$6x_1 - 7x_2 + 4x_3 \leq 23$
$3x_1 - 4x_2 - 2x_3 \leq 15$
$x_1, x_2, x_3 \geq 0$

(iv) Minimize $x_1 - 3x_2 + 2x_3$
S/t $7x_1 - 4x_2 + 3x_3 \geq 20$
$2x_1 + 4x_2 - 2x_3 \geq -3$
$8x_1 + 3x_2 - 4x_3 \geq 10$
$x_1, x_2, x_3 \geq 0$

(v) Maximize $5x_1 - 2x_2$
S/t $-2x_1 + 7x_2 \leq 8$
$6x_1 + 7x_2 = 20$
$4x_1 - 4x_2 \geq 3$
$x_1, x_2 \geq 0$

(vi) Maximize $3x_1 + 2x_2 + 2x_3 + x_4$
S/t $x_1 + 8x_2 + x_3 + x_4 = 11$
$x_1 - 2x_2 - 4x_3 - 2x_4 \leq 1$
$2x_1 - 4x_2 + 2x_3 + 5x_4 \approx 5$
$x_1, x_2, x_3, x_4 \geq 0$

(vii) Min. $z = 3x_1 + 2x_2$
S/t $x_1 + 7x_2 = 50$
$6x_1 + 7x_2 \leq 70$
$4x_1 - 3x_2 \approx 15$
$5x_1 - 4x_2 \geq 3$
$x_1 \geq 0, x_2 \leq 0$

(viii) Min. $3x_1 + 2x_2 + 2x_3 + x_4 - 3x_5$
S/t $3x_1 + x_2 - 2x_3 + x_4 + 2x_5 = 15$
$5x_1 - 2x_2 + 4x_3 + 3x_5 \leq 25$
$7x_1 - x_2 + 4x_3 + x_4 \geq 5$
$2x_1 + 2x_2 + 2x_3 + 5x_4 - x_5 \approx 20$
$x_1, x_2, x_5 \geq 0, x_4 \leq 0$, x_3 unrestricted

(ix) Min. $z = x_1 + 3x_2$
S/t $x_1 + 2x_2 \leq -6$
$-x_1 + x_2 \geq 7$
$x_1 \geq 0$, x_2 unrestricted

(x) Min. $z = 2x_1 + x_3 + x_4$
S/t $x_1 - x_2 + x_3 + x_4 = 8$
$3x_1 - x_3 - 2x_4 \geq 9$
$x_1, x_2 \geq 0, x_3 \leq 0$, x_4 unrestricted

(xi) Min. $z = x_1 + 3x_2$
S/t $x_1 + 2x_2 \leq -6$
$x_1 - x_2 \leq 2$
$-x_1 + x_2 \geq 7$
$x_1, x_2 \geq 0$

(xii) Max. $z = 3x_1 + 2x_2 + 2x_3$
S/t $x_1 + 2x_2 + 3x_3 \leq 20$
$2x_1 - 3x_2 + 4x_3 \geq 10$
$6x_1 + 4x_2 - 2x_3 = 12$
$x_1 \geq 0, x_2 \leq 0$, x3 unrestricted

(xiii) Max. $z = 3x_1 + 2x_2 + 3x_3$
S/t $2x_1 + 2x_2 - 5x_3 \leq 10$
$4x_1 - x_2 + 6x_3 \geq 2$
$x_1 + x_2 + x_3 = 5$
$x_1 \geq 5, x_2 \leq 0$, x_3 unrestricted

(xiv) Min. $z = 3x_1 + 5x_2 + 2x_3$
S/t $6x_1 + 3x_2 - 5x_3 \leq 13$
$4x_1 - 3x_2 + 6x_3 \geq 3$
$2x_1 + x_2 + x_3 \approx 3$
$x_1 \geq 2, x_2 \leq 0$, x_3 unrestricted

19. Using duality solve the following LP problem.

(i) Min. $z = 10x_1 + 6x_2 + 2x_3$
S/t $-x_1 + x_2 + x_3 \geq 1$
$3x_1 + x_2 - x_3 \geq 2$
$x_1, x_2, x_3 \geq 0$
[Ans: $x_1 = 1/4$, $x_2 = 5/4$, $x_3 = 0$ and $z_{min} = 10$]

(ii) Min. $z = x_1 + x_2$
S/t $2x_1 + x_2 \geq 2$
$-x_1 - x_2 \geq 1$
$x_1, x_2 \geq 0$
[Ans: No feasible solution]

(iii) Min. $z = 3x_1 + 2x_2 + x_3 + 4x_4$
S/t $2x_1 + 4x_2 + 5x_3 + x_4 \geq 10$
$3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$
$5x_1 + 2x_2 + x_3 + 6x_4 \geq 15$
$x_1, x_2, x_3, x_4 \geq 0$
[Ans: $x_1 = 65/23$, $x_2 = 0$, $x_3 = 20/23$, $x_4 = 0$ and $z_{min} = 215/23$]

(iv) Min. $z = 4x_1 + 2x_2$
S/t $3x_1 + x_2 \geq 27$
$x_1 + x_2 \geq 21$
$x_1 + 2x_2 \geq 30$
$x_1, x_2 \geq 0$
[Ans: $x_1 = 3$, $x_2 = 18$ and $z_{min} = 48$]

20. Solving the dual of the given LP problem, find the solution of the given problem:
Maximize $14x_1 + 4x_2$
Subject to $2x_1 + x_2 \leq 3$
$x_1 - x_2 \leq 1$
$x_1, x_2 \geq 0$
[Ans: $x_1 = 4/3$, $x_2 = 1/3$ and Objective$_{max}$ = 20.]

21. Solving the primal find the solution of its dual.
Maximize $z = 5x_1 - 2x_2 + 3x_3$
Subject to $2x_1 + 2x_2 - x_3 \geq 2$
$3x_1 - 4x_2 \leq 3$
$x_2 + 3x_3 \leq 5$
$x_1, x_2, x_3 \geq 0$ [Ans: $w_1 = 0$, $w_2 = 5/3$, $w_3 = 14/3$ and $u_{min} = 85/3$, w_is are dual variables]

22. Solving the dual problem of the given LP problem by simplex method, find the solution of the given problem.
Maximize $z = 5x_1 + 4x_2$
Subject to $3x_1 + 4x_2 \leq 24$
$3x_1 + 2x_2 \leq 18$
$x_2 \leq 5$
$x_1, x_2 \geq 0$

[Ans: $x_1 = 4$, $x_2 = 3$ and $z_{max} = 32$]

23. A unit of a industry uses 60 units of raw material to produce two products P_1 adn P_2. Each unit of P_1 and P_2 requires 2 units and one unit of raw material and yield Tk.60 and Tk.40 as profit respectively. Market condition suggests that P_1 should not be less than 25 units and P_2 should be less than 35 units.
    (i) Find a suitable production plan to maximize the profit.
    (ii) Find the shadow price for raw material.
    (iii) Explain the concept of shadow price in planning and in determination for raw material reserves.
24. A firm makes two products P_1 and P_2. Each product requires production time on each of the two machines:

Machines	Time requirement (hour)		Available time in hours
	P_1	P_2	
M_1	6	4	60
M_2	1	2	22
Profit per unit	Tk.3	Tk.4	

Determine the optimal production mix to maximize the profit and find the shadow prices, the cost per hour of machine M_1 and M_2.

25. Solve the dual of the given primal problem and hence using complementary slackness conditions find the solution of the primal. Maximize $z = 3x_1 + 2x_2$

$$\text{Subject to } -x_1 + 2x_2 \le 4$$
$$3x_1 + 2x_2 \le 14$$
$$x_1 - x_2 \le 3$$
$$x_1, x_2 \ge 0$$

[Ans: $(x_1, x_2) = (4, 1)$ or $(5/2, 13/4)$ and $z_{max} = 14$]

26. Solve the following LP problem by simplex method. Hence find the solution of its dual using complementary slackness conditions:

$$\text{Minimize } z = 3x_1 - x_2$$
$$\text{Subject to } 2x_1 + x_2 \geq 2$$
$$x_1 + 3x_2 \leq 3$$
$$x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

[Ans: dual variables $w_1 = 0$, $w_2 = 3$, $w_3 = 0$ and $u_{max} = 9$]

27. Solve the dual of the following LP problem by simplex method. Hence find the solution of the primal using complementary slackness conditions:

$$\text{Minimize } z = 3x_1 + x_2$$
$$\text{Subject to } x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

[Ans: $x_1 = 0$, $x_2 = 1$ and $z_{min} = -1$]

28. Solve the dual of the following LP problem by simplex method. Hence find the solution of the primal using complementary slackness conditions:

$$\text{Minimize } z = 4x_1 - 5x_2 - 2x_3$$
$$\text{Subject to } 6x_1 + x_2 - x_3 \leq 5$$
$$2x_1 + 2x_2 - 3x_3 \geq 3$$
$$2x_2 - 4x_3 \geq 1$$
$$x_1 \leq 0, x_3 \geq 0, x_2 \text{ unrestricted}$$

[Ans: $x_1 = 0$, $x_2 = 3/2$, $x_3 = 0$ and $z_{max} = -15/2$]

29. Solve the following LP problems by using dual simplex method:

(i) Min. $z = 3x_1 + x_2$

S/t $x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

$x_1, x_2 \geq 0$

[Ans: $x_1 = 0$, $x_2 = 1$ and $z_{min} = -1$]

(ii) Min. $z = 2x_1 + x_2$

S/t $3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

[Ans: $x_1 = 3/5$, $x_2 = 6/5$ and $z_{min} = 12/5$]

(iii) Min. $z = 10x_1 + 6x_2 + 2x_3$
S/t $-x_1 + x_2 + x_3 \geq 1$
$3x_1 + x_2 - x_3 \geq 2$
$x_1, x_2, x_3 \geq 0$
[Ans: $x_1 = 1/4$, $x_2 = 5/4$, $x_3 = 0$ and $z_{min} = 10$]

(iv) Min. $z = x_1 + x_2$
S/t $2x_1 + x_2 \geq 2$
$-x_1 - x_2 \geq 1$
$x_1, x_2 \geq 0$
[Ans: No feasible solution]

(v) Min. $z = 3x_1 + 2x_2 + x_3 + 4x_4$
S/t $2x_1 + 4x_2 + 5x_3 + x_4 \geq 10$
$3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$
$5x_1 + 2x_2 + x_3 + 6x_4 \geq 15$
$x_1, x_2, x_3, x_4 \geq 0$
[Ans: $x_1 = 65/23$, $x_2 = 0$, $x_3 = 20/23$, $x_4 = 0$ and $z_{min} = 215/23$]

(vi) Min. $z = 4x_1 + 2x_2$
S/t $3x_1 + x_2 \geq 27$
$x_1 + x_2 \geq 21$
$x_1 + 2x_2 \geq 30$
$x_1, x_2 \geq 0$
[Ans: $x_1 = 3$, $x_2 = 18$ and $z_{min} = 48$]

(vii) Min. $z = x_1 + 4x_2 + 3x_3$
S/t $x_1 + 2x_2 - x_3 + x_4 \geq 3$
$-2x_1 - x_2 + 4x_3 + x_4 \geq 2$
$x_1, x_2, x_3, x_4 \geq 0$
[Ans: $x_1 = 7$, $x_2 = 0$, $x_3 = 3$, $x_4 = 0$ and $z_{min} = 7$]

(viii) Max. $z = 2x_1 - 3x_2$
S/t $-x_1 + x_2 \geq -2$
$-5x_1 - 4x_2 \geq -46$
$7x_1 + 2x_2 \geq 32$
$x_1, x_2 \geq 0$
[Ans: Dual simplex method is not applicable]

(ix) Max. $z = 4x_1 - 5x_2 - 2x_3$
S/t $6x_1 + x_2 - x_3 \leq 5$
$2x_1 + 2x_2 - 3x_3 \geq 3$
$2x_2 - 4x_3 \geq 1$
$x_1 \leq 0$, $x_3 \geq 0$, x_2 unrestricted
[Ans: $x_1 = 0$, $x_2 = 3/2$, $x_3 = 0$ and $z_{max} = -15/2$]

(x) Max. $z = -4x_1 - 3x_2$
S/t $x_1 + x_2 \geq 1$
$x_1 + x_2 \leq 7$
$x_1 + 2x_2 \geq 10$
$x_2 \leq 3$
$x_1, x_2 \geq 0$
[Ans: $x_1 = 4$, $x_2 = 3$ and $z_{max} = -25$]

Chapter

06

Sensitivity Analysis

Highlights:

6.1 Sensitivity Analysis: (সেনসিটিভিটি বিশে-ষন) Once the optimal solution to a linear programming problem has been attained it may be desirable to study how the current solution changes when the parameters of the problem are changed. The changes in parameters of the problem may be discrete or continuous. The study of the effect of discrete changes in parameters on the optimal solution is called sensitivity analysis or post optimality analysis.

[NU-05, 06,07]

Note: The continuous changes in parameters are called parametric programming.

In other words: Sensitivity analysis refers to the study of the changes in the optimal solution of the variables and optimal value of the objective function (z) due to changes in the input data coefficients.

For a discussion of sensitivity analysis we shall confine ourselves to the following changes in the data and how to handle these changes –

(i) Changes in the objective function coefficients (c).
(ii) Changes in the right hand side constants (b).
(iii) Changes in the constraints or coefficient matrix (A).

6.2 An analytic example: (একটি ব্যাখ্যামূলক উদাহরন) With the help of the following illustrative example (Example-6.1), we shall present the above discussion.

Example (6.1): (A production mix problem) A company plans production on three of their products A, B and C. The unit profits on these products are \$2, \$3 and \$1 respectively and they require two resources labour and material. The company's operations research department formulates the following linear programming model for determining the optimal product mix.

$$\text{Maximize } z = 2x_1 + 3x_2 + x_3$$
$$\text{Subject to } \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \le 1$$
$$\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3}x_3 \le 3$$
$$x_1, x_2, x_3 \ge 0$$

where x_1, x_2, x_3 are the number of products which A, B, C produced.

Discussion: The simplex tableau with addition of slack variables x_4 and x_5 is given below:

Basis	C_B^t	c_j	2	3	1	0	0	Ratio of
		P_0	P_1	P_2	P_3	P_4	P_5	P_o & P_2
P_4	0	1	1/3	1/3	1/3	1	0	3
P_5	0	3	1/3	4/3	7/3	0	1	9/4 min
$z_j - c_j$		0	-2	-3 Smallest	-1	0	0	Table-1

Basis	C_B^t	c_j	2	3	1	0	0	Ratio of
		P_0	P_1	P_2	P_3	P_4	P_5	P_o & P_1
P_4	0	1/4	1/4	0	-1/4	1	-1/4	1 min
P_2	3	9/4	1/4	1	7/4	0	3/4	9
$z_j - c_j$		27/4	-5/4 Smallest	0	17/4	0	9/4	Table-2

Basis	$\underline{C}_B^t$	c_j	2	3	1	0	0	Ratio of $\underline{P}_o$ & $\underline{P}_j$
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	
$\underline{P}_1$	2	1	1	0	-1	4	-1	
$\underline{P}_2$	3	2	0	1	2	-1	1	
$z_j - c_j$		8	0	0	3	5	1	Table-3

Since the given LP problem is already in canonical form, no artificial variables are needed by simplex method, after some iteration we get the optimal table-3. From the optimal tableau, we see that the optimal production mix is to produce 1 unit of product A and 2 units of product B for a total profit of \$8. That is, $x_1 = 1$, $x_2 = 2$, $x_3 = 0$ and $z_{max} = \$8$.

By performing a sensitivity analysis it is possible to obtain valuable information regarding alternative production schedule in the neighbourhood of the optimal solution.

Variation in objective function coefficients c_j:

Case-1: Changing the objective function coefficient of a non-basic variable**:**

In the optimal production mix shown in table-3 product C is not produced because of its low profit of \$1 per unit (c_3). One may be interested in finding the range on the values of c_3 such that the current optimal solution remains optimal. It is clear that when c_3 decreases it has no effect on the present optimal solution.

However when c_3 is increased beyond certain value, product C may become profitable to produce.

When the value of c_3 changes, the value of the relative profit coefficients of the non-basic variable x_3 ($\bar{z}_3$) [where $\bar{z}_j = z_j - c_j$] changes in the optimal tableau. Table-3 is optimal as long as $\bar{z}_3$ is non-negative. That is,

$$\bar{z}_3 = (2, 3)\begin{pmatrix} -1 \\ 2 \end{pmatrix} - c_3 = 4 - c_3 \geq 0$$

$\Rightarrow \quad c_3 \leq 4$

This means that as long as the unit profit of product C is less than or equal to \$4, it is not profitable to produce product C.

Suppose the unit profit on product C is increased to \$6, i.e., $c_3 = 6$

then $\quad \bar{z}_3 = (2, 3)\begin{pmatrix} -1 \\ 2 \end{pmatrix} - c_3 = 4 - 6 = -2$ and the current product is not optimal. The maximum profit can be increased further by producing product C. In other words table-3 is non-optimal, since $\underline{P}_3$ can inter the basis to increase z.

Basis	$\underline{C}_B^t$	c_j	2	3	6	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_3$
$\underline{P}_1$	2	1	1	0	-1	4	-1	
$\underline{P}_2$	3	2	0	1	2	-1	1	1 min
$z_j - c_j$		8	0	0	-2 Smallest	5	1	Table-4

Basis	$\underline{C}_B^t$	c_j	2	3	6	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_j$
$\underline{P}_1$	2	2	1	1/2	0	7/2	-1/2	
$\underline{P}_3$	6	1	0	1/2	1	-1/2	1/2	
$z_j - c_j$		10	0	1	0	4	2	Table-5

By the minimum ratio rule $\underline{P}_2$ leaves the basis. The new optimal solution can be determined as shown in above table-5. The new (optimal) product mix is to produce 2 units of product A and 1 unit of product C with a maximum profit \$10.

Case-2: Changing the objective function coefficients of a basic variable:

Suppose we want to determine the effect of changes on the unit profit of product A (c_1). It is clear that when c_1 decreases below a certain level, it may not be profitable to include product A in the optimal product mix. Even when c_1 increased, it is possible that it may change the optimal product mix at some level. This happens, because product A may become so profitable that the optimal mix may include only product A. Hence there is an upper and a lower limit on the variation of c_1 within the optimal solution given by table-3 is not affected.

To determine the range of c_1 observe that a change in c_1 changes the profit vector of the basic variables ($\underline{C}_B$), where $\underline{C}_B$ = (c_1, c_2). It can verified that the relative profit coefficients of the basic variables namely $\bar{z}_1$ and $\bar{z}_2$ will not be affected and they will still remain at zero value. However, the relative profits of non-basic variables $\bar{z}_3$, $\bar{z}_4$, $\bar{z}_5$ will changes. But as long as $\bar{z}_j$ remain non-negative 3rd table is still optimal. Thus

$$\bar{z}_3 = (c_1, 3)\begin{pmatrix}-1\\2\end{pmatrix} - 1 = 5 - c_1 \geq 0 \quad \Rightarrow \quad c_1 \leq 5$$

$$\bar{z}_4 = (c_1, 3)\begin{pmatrix}4\\-1\end{pmatrix} - 0 = 4c_1 - 3 \geq 0 \quad \Rightarrow \quad c_1 \geq 3/4$$

$$\bar{z}_5 = (c_1, 3)\begin{pmatrix}-1\\1\end{pmatrix} - 0 = 3 - c_1 \geq 0 \quad \Rightarrow \quad c_1 \leq 3$$

Therefore, table-3 will remain optimal as long as $c_1 \in [3/4, 3]$. Off course as c_1 changes, the optimal value of the objective function will change.

For example, when $c_1 = 1$, the optimal solution is given by $x_1 = 1$, $x_2 = 2$, $x_3 = 0$ but the maximum profit, z = \$7.

Note: When the value of c_1 goes beyond the range provided the sensitivity analysis table-3 will no longer be optimal, once again we can apply the simplex method to determine the new optimal solution.

<u>Remark</u>: If $c_1 = 4$, determine the effect on the product mix and hence find the optimal product mix.

Solution: When $c_1 = 4$,

$$\bar{z}_3 = 5 - 4 = 1$$
$$\bar{z}_4 = 4 \times 4 - 3 = 13$$
$$\bar{z}_5 = 3 - 4 = -1$$

As $\bar{z}_5$ becomes negative, the solution given in table-3 no longer remains optimal. Slack variable x_5 enters into the basis variables. This is shown as following.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	4 / $\underline{P}_1$	3 / $\underline{P}_2$	1 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Ratio of $\underline{P}_o$ & $\underline{P}_5$
$\underline{P}_1$	4	1	1	0	-1	4	-1	
$\underline{P}_2$	3	2	0	1	2	-1	1	
$z_j - c_j$		10	0	0	1	13	-1 Smallest	Table-6

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_0$	4 / $\underline{P}_1$	3 / $\underline{P}_2$	1 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Ratio of $\underline{P}_o$ & $\underline{P}_j$
$\underline{P}_1$	4	3	1	1	1	3	0	
$\underline{P}_5$	0	2	0	1	2	-1	1	
$z_j - c_j$		12	0	1	3	12	0	Table-7

Thus the optimal product mix changes to $x_1 = 3$, $x_2 = 0$, $x_3 = 0$, that is, the optimal product mix is to produce 3 units of product A with a total profit of $12.

Case-3: Changing the objective function coefficients of both the basic and non-basic variables:

Suppose, the profit on all three products are changed such that the objective function becomes

$$z = x_1 + 4x_2 + 2x_3$$

The effect on the optimal product mix can be determined by checking whether the $\bar{z}_j$ row in table-3 remains non-negative, that is, $\bar{z}_1 = \bar{z}_2 = 0$

$$\bar{z}_3 = (1, 4)\begin{pmatrix} -1 \\ 2 \end{pmatrix} - 2 = 7 - 2 = 5 \geq 0$$

$$\bar{z}_4 = (1, 4)\begin{pmatrix} 4 \\ -1 \end{pmatrix} - 0 = 0$$

$$\bar{z}_5 = (1, 4)\begin{pmatrix} -1 \\ 1 \end{pmatrix} - 0 = 3 \geq 0$$

Hence the optimal solution does not change. The optimal product mix remains $x_1 = 1$, $x_2 = 2$, $x_3 = 0$ and $z_{max} = \$(1 + 4.2 + 2.0) = \$9$.

Variation in right hand side constants b_i:

Suppose that an additional one unit of labour is made available and the company is interested in determining how this affects the optimal product mix.

In this case, the vector of constants in table-1 change from $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. The basic columns are the columns corresponding to the optimal basic variables x_1 and x_2 in the initial table. Hence the basis matrix corresponding to table-3 is given by

$$B = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 4/3 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}.$$

The values of the new right hand sides constants in table-3 due to the increased labour is given by

$$\bar{x} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \text{ which is positive vector.}$$

Hence table-3 still remains feasible. Since the change in right hand side vector does not change the value of the relative profit factors, i.e., table-3 remains optimal and the new optimal product mix is x

$x_1 = 5$, $x_2 = 1$, $x_3 = 0$ and $z_{max} = \$13$.

Note: Both the optimal solution and the optimal value of the objective function have changed due to a variation in the availability of labour, but the optimal basis has not changed. In other words, it is still optimal to produce only the two products A and B. The only difference lies in the quantities of A and B produced.

Supposed the extra unit of labour can be obtained by allowing overtime which costs an additional \$4 to the company. The company may want to find out whether it is profitable to use overtime labour. This may be found by comparing the increased profit by employing overtime to its added cost. In our example, the increased profit is \$13 – \$8 = \$5 which is more than the cost of overtime \$4. It is therefore profitable to get the additional 1 unit of labour.

The increased profit of \$5 per unit of increased in labour availability is called the <u>shadow price</u> for labour constraints, which is discussed in chapter 5.

The optimal dual solution (w_1^o, w_2^0) is given by

$$
\begin{aligned}
(w_1^o,\ w_2^0) &= \text{optimal simplex multipliers corresponding to table-3}\\
&= C_B B^{-1}\\
&= (2, 3)\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\\
&= (5, 1)
\end{aligned}
$$

Then the optimal dual solution $w_1^o = 5$, $w_2^0 = 1$ represent the shadow prices of labour and material respectively.

Note that the shadow prices reflect the net change in the optimal value of z per unit increase as long as the variation in the constant resources.

Now we have to find the range of shadow price to remain the optimal table unchanged. Let b_1 denote the amount of labour available and b^* denote the new vector of constant in the initial table. Hence

$$b^* = \begin{pmatrix} b_1 \\ 3 \end{pmatrix}$$

For the simplex tableau given by table-3 to be optimal we should have $B^{-1}b^* \geq 0$. Since $B^{-1} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$, we get

$$B^{-1}b^* = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4b_1 - 3 \\ -b_1 + 3 \end{pmatrix}$$

So that $B^{-1}b^*$ is non-negative as long as

$$\begin{aligned} 4b_1 - 3 \geq 0 \quad & \text{or,} \quad b_1 \geq 3/4 \\ -b_1 + 3 \geq 0 \quad & \text{or,} \quad b_1 \leq 3 \end{aligned}$$

This means that x_1 and x_2 will remain in the optimal product mix as long as $3/4 \leq b_1 \leq 3$ and the optimal solution is

$$x_1 = 4b_1 - 3,\ x_2 = -b_1 + 3,\ x_3 = 0$$

and the maximum profit

$$z = 2(4b_1 - 3) + 3(-b_1 + 3) = \$(5b_1 + 3)$$

Let us now consider the case when the labour availability is increased to 4 units, i.e., the constant vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ reduces to $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. The new value of the constants in the 3rd table will be given by

$$\bar{x} = B^{-1}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ -1 \end{pmatrix}$$

This means that table-3 is no longer optimal, since $x_1 = 13$, $x_2 = -1$, $x_3 = 0$ is impossible. In this case table-3 becomes as follows as which should be solved by dual simplex method.

Basis	C_B^t	c_j / $\underline{P}_0$	2 / $\underline{P}_1$	3 / $\underline{P}_2$	1 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Negative of $\underline{P}_o$
$\underline{P}_1$	2	13	1	0	-1	4	-1	
$\underline{P}_2$	3	-1	0	1	2	-1	1	-1 min
$z_j - c_j$		23	0	0	3	5	1	
$\frac{z_j - c_j}{y_{2j}}$; $y_{2j} \le 0$						-5		Table-8

Basis	C_B^t	c_j / $\underline{P}_0$	2 / $\underline{P}_1$	3 / $\underline{P}_2$	1 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Negative of $\underline{P}_o$
$\underline{P}_1$	2	9	1	4	7	0	3	
$\underline{P}_4$	0	1	0	-1	-2	1	-1	
$z_j - c_j$		18	0	5	13	0	6	Table-9

Table-9 is optimal since the right hand side constants are positive. The new availability is increased to 4 units is given by $x_1 = 9$, $x_2 = x_3 = 0$ and the maximum profit is \$18, i.e., the optimal product mix is to produce only the product A of 9 units.

Variation in the constraints or coefficients matrix (a_{ij}):

The constraints matrix or the coefficients matrix (a_{ij}) may be changed by

(i) Adding new variables or activities

(ii) Changing the resources requirements of the existing activities

(iii) Adding new constraints.

Case-1: Adding new variables or activities:

Suppose the company research and development department has come out with a new product D, which requires 1 unit of labour and 1 unit material. The new product has sufficiently market and can be sold at a unit profit of \$3. The company wants to know whether it is economical to manufacture product D.

The new product D in our possible product mix is mathematically equivalent to adding a variable (say x_6) and a column $\begin{pmatrix}1\\1\end{pmatrix}$ in the initial table. The present optimal product mix in table-3 is optimal as long as the relative profit factor of the new product namely

$$\bar{z}_6 \geq 0$$

We see that

$$\bar{z}_6 = (1, 4)\begin{pmatrix}-1\\1\end{pmatrix} - 0 = 3 \geq 0$$

This shows that producing product D will not improve the present value of the maximum profit. In case if $\bar{z}_6 < 0$ then the simplex method will be performed again to get the optimal solution.

Case-2: Variation in the resources requirements of the existing activities:

When the constraint's coefficients of non-basic variable (say C) change, its effect on the optimal solution can be studied by the same steps as given in case-1.

On the other hand, if the constraints coefficients of a basic variable (e.g., A or B) change the basis matrix itself is effected. Under such circumstances it may be better to solve the linear program over again.

Case-3: Adding new constraints:

Consider the addition of an administrative service constraint to the problem where in the product A, B, C requires 1, 2, 1 hour of administrative hour are 10. This amount to adding a new constraint of the form

$$x_1 + 2x_2 + x_3 \leq 10$$

to the original formulation of the problem. Since in table-3 $x_1 = 1$, $x_2 = 2$, $x_3 = 0$ so that $1 + 2\times2 + 0 = 5 \leq 10$ so that the optimal solution satisfies the new constraint. Hence the optimal mix is not altered.

Again, suppose the available administrative hours are only 4 then the new constraint becomes

$$x_1 + 2x_2 + x_3 \le 4$$

and the present optimal solution in table-3 does not satisfy it, i.e., $5 \le 4$.

Now we construct the new tableau doing some row operations as follows:

Basis	$\underline{C}_B^t$	c_j	2	3	1	0	0	0	We try to make
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_1$	2	1	1	0	-1	4	-1	0	$\underline{P}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$
$\underline{P}_2$	3	2	0	1	2	-1	1	0	
$\underline{P}_6$	0	4	1	2	1	0	0	1	
$z_j - c_j$		8	0	0	3	5	1	0	Table-10

Basis	$\underline{C}_B^t$	c_j	2	3	1	0	0	0	We try to make
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_1$	2	1	1	0	-1	4	-1	0	$\underline{P}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$
$\underline{P}_2$	3	2	0	1	2	-1	1	0	
$\underline{P}_6$	0	3	0	2	2	-4	1	1	
$z_j - c_j$		8	0	0	3	5	1	0	Table-11

Basis	$\underline{C}_B^t$	c_j	2	3	1	0	0	0	We use dual
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_1$	2	1	1	0	-1	4	-1	0	simplex method as -1 in $\underline{P}_0$
$\underline{P}_2$	3	2	0	1	2	-1	1	0	
$\underline{P}_6$	0	-1	0	0	-2	-2	-1	1	
$z_j - c_j$		8	0	0	3	5	1	0	
$\frac{z_j - c_j}{y_{2j}}$; $y_{3j}<0$					-3/2	-5/2	-1 Greatest		Table-12

Basis	$\underline{C}_B^t$	c_j	2	3	1	0	0	0	
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_1$	2	2	1	0	1	6	0	-1	
$\underline{P}_2$	3	1	0	1	0	-3	0	1	
$\underline{P}_5$	0	1	0	0	2	2	1	-1	
$z_j - c_j$		7	0	0	1	3	0	1	Table-13

Since table-12 is dual feasible, the dual simplex method is applied to find the new optimal solution (table-13) and the corresponding optimal product mix is to produce 2 units of product A and 1 unit of product B. The maximum profit has been produced from 8 to 7 due to the addition of the new constraint.

Note: Whenever a new constraint is added to a linear program, the old optimal value will always be better or equal to the new optimum value. Thus the addition of a new constraint can not improve the optimal value of any linear programming problem.

6.3 <u>Variation in the objective function coefficients (c)</u>:
(উদ্দেশ্যমূলক আপেক্ষকের সহগগুলোর পরিবর্তন)

Let us consider the following general LP problem:

$$\text{Maximize } z = \underline{c}\,\underline{x}$$
$$\text{Subject to } \underline{A}\,\underline{x} = \underline{b};\ \underline{b} \geq \underline{0}$$
$$\underline{x} \geq \underline{0}$$

Suppose $\underline{x}_B$ be the optimum solution with the optimum basis $\underline{B}$, then $\underline{x}_B = \underline{B}^{-1}\underline{b}$

Let the objective function coefficients vector $\underline{c}$ be replaced by $\underline{c}^{/}$, then the LP problem becomes

$$\text{Maximize } z = \underline{c}^{/}\,\underline{x}$$
$$\text{Subject to } \underline{A}\,\underline{x} = \underline{b};\ \underline{b} \geq \underline{0}$$
$$\underline{x} \geq \underline{0}$$

and we see that there is no change in the constraints.

Let the quantity Δc_j to be added to the jth component of $\underline{c}$ so that

$$c_j' = c_j + \Delta c_j$$

Then two cases may arise.
Case-I: c_j be the coefficient of a non-basic variable.
Case-II: c_j be the coefficient of a basic variable.

Case-I: <u>c_j be the coefficient of a non-basic variable</u>:
At the optimum stage, the relative profit coefficients

$$\bar{z}_j = z_j' - c_j' = z_j - (c_j + \Delta c_j)$$

So, the current optimum solution will be unaltered if

$$z_j' - c_j' \geq 0$$

Or, $\quad z_j - (c_j + \Delta c_j) \geq 0$

Or, $\quad \Delta c_j \leq z_j - c_j$, this shows that Δc_j has no lower bound.

So, the current solution remains optimal with the change of c_j upto $\Delta c_j \leq z_j - c_j$. As $x_j = 0$ (non-basic variable), any change of c_j does not change the value of the objective function.

Remark: For minimizing problem, the current solution remains optimal if $\Delta c_j \geq z_j - c_j$.

Case-II: <u>c_j be the coefficient of a basic variable</u>:
Let $x_k = \underline{x}_{Br}$ be the rth basic variable and the corresponding profit coefficient $c_k = c_{Br}$. Let c_k changes to $c_{Br}' = c_{Br} + \Delta c_{Br}$ and so $\underline{c}_B$ changes to $\underline{c}_B'$. These changes will not affect the relative profit coefficients of the basic variables and they will still remain at zero level. At the optimum stage, the relative profit coefficients of the non-basic variables become

$$\bar{z}_j = z_j' - c_j'$$

$$= \underline{c}_B' \underline{y}_j - c_j$$

$$= \sum_{\substack{i=1 \\ i \neq r}}^{m} c_{Bi} y_{ij} - c_j + (c_{Br} + \Delta c_{Br}) y_{rj}$$

$$= \sum_{i=1}^{m} c_{Bi} y_{ij} - c_j + \Delta c_{Br} y_{rj}$$

$$= z_j - c_j + \Delta c_{Br} y_{rj} \qquad [\because \sum_{\substack{i=1 \\ i \neq r}}^{m} c_{Bi} y_{ij} = z_j - c_j]$$

The current solution remains optimal if

$$z_j - c_j + \Delta c_{Br} y_{rj} \geq 0 \qquad \ldots \qquad (1)$$

If $y_{rj} = 0$ then from (1) $z_j - c_j \geq 0$, optimality is satisfied.

If $y_{rj} > 0$ then from (1), we have $\Delta c_{Br} \geq \dfrac{-(z_j - c_j)}{y_{rj}}$ ... (2)

and if $y_{rj} < 0$ then from (1), we have $\Delta c_{Br} \leq \dfrac{-(z_j - c_j)}{y_{rj}}$ ... (3)

From inequality (2), (3) and $\Delta c_{Br} = \Delta c_k$, we have,

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{rj}}; y_{rj} > 0\right\} \leq \Delta c_k \leq \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{rj}}; y_{rj} < 0\right\} \ldots (4)$$

for all non-basic vectors $\underline{P}_j$.

Condition (4) gives the changing range of the objective function coefficient of rth basic variable x_k remaining the current solution optimal. And the value of the objective function changes by $\Delta c_k x_k$.

Remark: For minimizing problem, the current solution remains optimal if

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{rj}}; y_{rj} < 0\right\} \leq \Delta c_k \leq \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{rj}}; y_{rj} > 0\right\}$$

for all non-basic vectors $\underline{P}_j$.

Note: If the coefficients of the objective function change beyond their limit, we solve the problem by simplex method rearranging the optimal tableau again.

Example (6.2): After solving the following LP problem find the range of c_2 for which the optimal solution does not change.

Maximize $z = 2x_1 + 3x_2 + x_3$ **[JU-93]**

Subject to $\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \leq 1$

$\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3}x_3 \leq 3$

$x_1, x_2, x_3 \geq 0$

Solution: The simplex tableau with addition of slack variables x_4 and x_5 is given below:

Basis	C_B^t	c_j	2	3	1	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_2$
$\underline{P}_4$	0	1	1/3	1/3	1/3	1	0	3
$\underline{P}_5$	0	3	1/3	4/3	7/3	0	1	9/4 min
$z_j - c_j$		0	-2	-3 Smallest	-1	0	0	Table-1

Basis	C_B^t	c_j	2	3	1	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_1$
$\underline{P}_4$	0	1/4	1/4	0	-1/4	1	-1/4	1 min
$\underline{P}_2$	3	9/4	1/4	1	7/4	0	3/4	9
$z_j - c_j$		27/4	-5/4 Smallest	0	17/4	0	9/4	Table-2

Basis	C_B^t	c_j	2	3	1	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_j$
$\underline{P}_1$	2	1	1	0	-1	4	-1	
$\underline{P}_2$	3	2	0	1	2	-1	1	
$z_j - c_j$		8	0	0	3	5	1	Table-3

Tableau-3 gives the optimal solution, $x_1 = 1$, $x_2 = 2$, $x_3 = 0$ and $z_{max} = 8$.
Since c_2 is the coefficient of the 2nd basic variable in the objective function, the changing range Δc_2 of c_2 is given by

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} > 0\right\} \le \Delta c_2 \le \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} < 0\right\}$$

for all non-basic vectors $\underline{P}_j$.

Or, $\quad Max\left\{\frac{-3}{2}, \frac{-1}{1}\right\} \le \Delta c_2 \le Min\left\{\frac{-5}{-1}\right\}$

Or, $\quad -1 \le \Delta c_2 \le 5$

So, the range of c_2 is $3 - 1 \le c_2 \le 3 + 5$ or, $2 \le c_2 \le 8$ for which the optimal solution does not change.

<u>Example (6.3)</u>: Solve the following LP problem by simplex method and discuss the variation of c_j ($j = 1, 2$) for which the optimal solution does not change.

Minimize $z = -3x_1 - 9x_2$

Subject to $x_1 + 4x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$

Solution: To solve the problem by simplex method, we introduce slack variables $x_3 \ge 0$ and $x_4 \ge 0$ to 1st and 2nd constraints respectively. Then we get,

Minimize $z = -3x_1 - 9x_2$

Subject to $x_1 + 4x_2 + x_3 = 8$

$x_1 + 2x_2 + x_4 = 4$

$x_1, x_2, x_3, x_4 \ge 0$

Making initial simplex table and taking necessary iterations, we get the following tables.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3	-9	0	0	Min Ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	θ
$\underline{P}_3$	0	8	1	4	1	0	8/4=2 Min. arbitrarily
$\underline{P}_4$	0	4	1	2	0	1	4/2=2
$z_j - c_j$		0	3	9	0	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	-9 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ
$\underline{P}_2$	-9	2	1/4	1	1/4	0	
$\underline{P}_4$	0	0	1/2	0	-1/2	1	0/(1/2)=0
$z_j - c_j$		-18	3/4	0	-9/4	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	-3 $\underline{P}_1$	-9 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Min Ratio θ
$\underline{P}_2$	-9	2	0	1	1/2	-1/2	
$\underline{P}_1$	-3	0	1	0	-1	2	
$z_j - c_j$		-18	0	0	-3/2	-3/2	

Since in the 3rd table all $z_j - c_j \leq 0$, the optimality conditions are satisfied. Hence the optimum solution is $x_1 = 0$ $x_2 = 2$ and the minimum value of the objective function, $z_{min} = -18$.

Since c_1 is the coefficient of the 2nd basic variable in the objective function, the changing range Δc_1 of c_1 is given by

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} < 0\right\} \leq \Delta c_1 \leq \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} > 0\right\}$$

for non-basic vectors $\underline{P}_3$ and $\underline{P}_4$.

Or, $$Max\left\{\frac{3/2}{-1}\right\} \leq \Delta c_1 \leq Min\left\{\frac{3/2}{2}\right\}$$

Or, $-3/2 \leq \Delta c_1 \leq 3/4$

So, the range of c_1 is $-3 - 3/2 \leq c_1 \leq -3 + 3/4$ or, $-9/2 \leq c_1 \leq -9/4$ for which the optimal solution does not change.

And since c_2 is the coefficient of the 1st basic variable in the objective function, the changing range Δc_2 of c_2 is given by

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{1j}}; y_{1j} < 0\right\} \leq \Delta c_2 \leq \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{1j}}; y_{1j} > 0\right\}$$

for non-basic vectors $\underline{P}_3$ and $\underline{P}_4$.

Or, $\quad Max\left\{\frac{3/2}{-1/2}\right\} \le \Delta c_2 \le Min\left\{\frac{3/2}{1/2}\right\}$

Or, $\quad -3 \le \Delta c_2 \le 3$

So, the range of c_2 is $-9-3 \le c_2 \le -9+3$ or, $-12 \le c_2 \le -6$, for which the optimal solution does not change.

Example (6.4): Solve the following LP problem and discuss the variation of c_j (j = 1, 2, 3) for which the optimal solution does not change.

$$\text{Maximize } z = 2x_1 + x_2 + 3x_3$$
$$\text{Subject to } \quad x_1 + x_2 + 2x_3 \le 5$$
$$2x_1 + 3x_2 + 4x_3 = 12$$
$$x_1, x_2, x_3 \ge 0$$

Solution: Introducing slack variable $x_4 \ge 0$ to 1st constraint and artificial variable $x_5 \ge 0$ to 2nd constraint, we get the problem as follows:

$$\text{Maximize } z = 2x_1 + x_2 + 3x_3 + 0x_4 - Mx_5$$
$$\text{Subject to } \quad x_1 + x_2 + 2x_3 + x_4 + 0x_5 = 5$$
$$2x_1 + 3x_2 + 4x_3 + 0x_4 + x_5 = 12$$
$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

We find the following initial and iterative simplex tables:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2	1	3	0	-M	Ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_4$	0	5	1	1	2	1	0	$5/2 = \theta_o$
$\underline{P}_5$	-M	12	2	3	4	0	1	12/4 = 3
$z_j - c_j$		0	-2	-1	-3	0	0	
		-12	-2	-3	-4 Smallest	0	0	Coef. of M

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	2	1	3	0	-M	Ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_3$	3	5/2	1/2	1/2	1	1/2	0	5
$\underline{P}_5$	-M	2	0	1	0	-2	1	$2/1=2 = \theta_o$
$z_j - c_j$		15/2	-1/2	1/2	0	3/2	0	
		-2	0	-1 Smallest	0	2	0	Coef. of M

Basis	$\underline{C}_B^t$	c_j	2	1	3	0	-M	Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_3$	3	3/2	1/2	0	1	3/2	-	$3 = \theta_o$
$\underline{P}_2$	1	2	0	1	0	-2	-	
$z_j - c_j$		13/2	-1/2 Smallest	0	0	5/2	-	
		0	0	0	0	0	-	Coef. of M

Basis	$\underline{C}_B^t$	c_j	2	1	3	0	-M	Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_1$	2	3	1	0	2	3	-	
$\underline{P}_2$	1	2	0	1	0	-2	-	
$z_j - c_j$		8	0	0	1	4	-	

Since all $z_j - c_j \geq 0$ the table is optimal. The above tableau gives us the solution of the problem: $x_1 = 3$, $x_2 = 2$, $x_3 = 0$ and $z_{max} = 8$.

Variation of c_1: Since c_1 is the coefficient of the 1st basic variable in the objective function, the changing range Δc_1 of c_1 is given by

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{1j}}; y_{1j} > 0\right\} \leq \Delta c_1 \leq \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{1j}}; y_{1j} < 0\right\}$$

for all non-basic vectors $\underline{P}_j$.

Or, $\quad Max\left\{\frac{-1}{2}, \frac{-4}{3}\right\} \leq \Delta c_1$, no upper bound as $y_{1j} \not< 0$.

Or, $\quad -1/2 \leq \Delta c_1$

So, the range of c_1 is $2 - 1/2 \leq c_1$ or, $c_1 \geq 3/2$ for which the optimal solution does not change.

Variation of c_2: Since c_2 is the coefficient of the 2nd basic variable in the objective function, the changing range Δc_2 of c_2 is given by

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} > 0\right\} \leq \Delta c_2 \leq \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} < 0\right\}$$

for all non-basic vectors $\underline{P}_j$.

Or, no lower bound as $y_{2j} \not> 0$ and $\Delta c_2 \le Min\left\{\dfrac{-4}{-2}\right\}$

Or, $\Delta c_2 \le 2$

So, the range of c_2 is $c_2 \le 1 + 2$ or, $c_2 \le 3$ for which the optimal solution does not change.

Variation of c_3: Since c_3 is the coefficient of a non-basic variable in the objective function, the changing range Δc_3 of c_3 is given by

$$\Delta c_3 \le z_3 - c_3 \text{ and } \Delta c_3 \text{ has no lower bound.}$$

Or, $\Delta c_3 \le 1$

So, the range of c_3 is $c_3 \le 3 + 1$ or, $c_3 \le 4$ for which the optimal solution does not change.

6.4 Variation in the right hand side constants (b): (ডান পার্শ্বস্থ ধ্রুবকের পরিবর্তন) Let us consider the following general LP problem:

$$\text{Maximize } z = \underline{c}\,\underline{x}$$
$$\text{Subject to } \underline{A}\,\underline{x} = \underline{b};\ \underline{b} \ge \underline{0}$$
$$\underline{x} \ge \underline{0}$$

Suppose $\underline{x}_B$ be the optimum solution with the optimum basis $\underline{B}$, then $\underline{x}_B = \underline{B}^{-1}\underline{b}$

Now let the kth component b_k of the requirement vector $\underline{b}$ be changed to $b_k + \Delta b_k$ so that the requirement vector becomes $\underline{b}^*$.

Then $\underline{b}^* = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k + \Delta b_k \\ \vdots \\ b_m \end{pmatrix}$; and let $\underline{x}_B = \begin{pmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bk} \\ \vdots \\ x_{Bm} \end{pmatrix}$ and the new basic feasible solution $\underline{x}_B^*$.

$$\therefore \qquad \underline{x}_B^* = \underline{B}^{-1}\underline{b}^* = \underline{B}^{-1}\underline{b} + \sum_{i=1}^{m}\beta_{ik}\Delta b_k = \underline{x}_B + \sum_{i=1}^{m}\beta_{ik}\Delta b_k \,;$$

where β_{ik} (i =1, 2, ..., m) are elements of the kth column $\underline{\beta}_k$ of $\underline{B}^{-1}$.

If the new solution is feasible, then

$$\underline{x}_B^* \geq 0$$

Or, $$\underline{x}_B + \sum_{i=1}^{m}\beta_{ik}\Delta b_k \geq 0$$

Or, $x_{Bi} + \beta_{ik}\Delta b_k \geq 0$; for individual i = 1, 2, ..., m. ... (1)

If $\beta_{ik} > 0$ then from (1), $\Delta b_k \geq -\dfrac{x_{Bi}}{\beta_{ik}}$ ... (2)

And if $\beta_{ik} < 0$ then from (1), $\Delta b_k \leq -\dfrac{x_{Bi}}{\beta_{ik}}$ ... (3)

Combination of (2) and (3) gives the interval of Δb_k so that the optimality remains unaffected as follows:

$$\underset{i}{Max}\left\{\frac{-x_{Bi}}{\beta_{ik}}; \beta_{ik} > 0\right\} \leq \Delta b_k \leq \underset{i}{Min}\left\{\frac{-x_{Bi}}{\beta_{ik}}; \beta_{ik} < 0\right\}$$

If no $\beta_{ik} > 0$, there is no lower bound and if no $\beta_{ik} < 0$, there is no upper bound of displacement of the right hand side constant b_k without affecting the optimality.

<u>Note</u>: The change of right hand side constants does not affect the net evaluation row $z_j - c_j$, so its excessive changes make the problem dual feasible but primal infeasible. Hence, if the right hand side constants change beyond their limit, we solve the problem by dual simplex method rearranging the optimal tableau again.

Example (6.5): After solving the following LP problem find the range of b_1 for which the optimality is unaffected.

Maximize $z = 2x_1 + 3x_2 + x_3$ **[DU-93]**

$$\text{Subject to } \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \le 1$$

$$\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3}x_3 \le 3$$

$$x_1, x_2, x_3 \ge 0$$

Solution: The simplex tableau with addition of slack variables x_4 and x_5 is given below:

Basis	C_B^t	c_j	2	3	1	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_2$
$\underline{P}_4$	0	1	1/3	1/3	1/3	1	0	3
$\underline{P}_5$	0	3	1/3	4/3	7/3	0	1	9/4 min
$z_j - c_j$		0	-2	-3 Smallest	-1	0	0	Table-1

Basis	C_B^t	c_j	2	3	1	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_1$
$\underline{P}_4$	0	1/4	1/4	0	-1/4	1	-1/4	1 min
$\underline{P}_2$	3	9/4	1/4	1	7/4	0	3/4	9
$z_j - c_j$		27/4	-5/4 Smallest	0	17/4	0	9/4	Table-2

Basis	C_B^t	c_j	2	3	1	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_j$
$\underline{P}_1$	2	1	1	0	-1	4	-1	
$\underline{P}_2$	3	2	0	1	2	-1	1	
$z_j - c_j$		8	0	0	3	5	1	Table-3

Tableau-3 gives the optimal solution, $x_1 = 1$, $x_2 = 2$, $x_3 = 0$ and $z_{max} = 8$.

<u>Variation of b_1</u>: Here, $\underline{x}_B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\underline{B}^{-1} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$

We know that the changing range of Δb_1 of b_1 so that the optimality remains unaffected as follows:

$$\underset{i}{Max}\left\{\frac{-x_{Bi}}{\beta_{ik}};\beta_{ik}>0\right\} \le \Delta b_k \le \underset{i}{Min}\left\{\frac{-x_{Bi}}{\beta_{ik}};\beta_{ik}<0\right\}$$

Or, $\quad Max\left\{\frac{-1}{4}\right\} \le \Delta b_1 \le Min\left\{\frac{-2}{-1}\right\}$ $\quad$ [$\because$ k = 1, 1st column of B^{-1} gives β_{ik}]

Or, $\quad -1/4 \le \Delta b_1 \le 2$

So, the displacement range of b_1 is $1 - 1/4 \le b_1 \le 1 + 2$ or, $3/4 \le b_1 \le 3$ for which the optimality remains unaffected.

<u>Example (6.6)</u>: Solve the following LPP:

Maximize $z = 5x_1 + 12x_2 + 4x_3$ **[NUH-05]**

Subject to $x_1 + 2x_2 + x_3 \le 5$

$2x_1 - x_2 + 3x_3 = 2$

$x_1, x_2, x_3 \ge 0$

and describe the effect of changing the requirement vector form $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$, on the optimum solution.

Solution: Introducing slack variable $x_4 \ge 0$ and artificial variable $x_5 \ge 0$ to 1st and 2nd constraints respectively of the given problem, we get

Maximize $z = 5x_1 + 12x_2 + 4x_3 + 0x_4 - Mx_5$

Subject to $x_1 + 2x_2 + x_3 + x_4 + 0x_5 = 5$

$2x_1 - x_2 + 3x_3 + 0x_4 + x_5 = 2$

$x_1, x_2, x_3, x_4, x_5 \ge 0$

The initial simplex tableau and iterative tableau are as given below:

Basis	$\underline{C}_B^t$	c_j	5	12	4	0	-M	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_3$
$\underline{P}_4$	0	5	1	2	1	1	0	5
$\underline{P}_5$	-M	2	2	-1	3	0	1	2/3 min
$z_j - c_j$		0	-5	-12	-4	0	0	Table-1
		-2	-2	1	-3 Smallest	0	0	

Basis	$\underline{C}_B^t$	c_j	5	12	4	0	-M	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_2$
$\underline{P}_4$	0	13/3	1/3	7/3	0	1	-1/3	13/7
$\underline{P}_3$	4	2/3	2/3	-1/3	1	0	1/3	
$z_j - c_j$		8/3	-7/3	-40/3 Smallest	0	0		Table-2

Basis	$\underline{C}_B^t$	c_j	5	12	4	0	-M	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_2$
$\underline{P}_2$	12	13/7	1/7	1	0	3/7	-1/7	13
$\underline{P}_3$	4	9/7	5/7	0	1	1/7	2/7	9/5 min.
$z_j - c_j$		192/9	-3/7 Smallest	0	0	40/7		Table-3

Basis	$\underline{C}_B^t$	c_j	5	12	4	0	-M	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_o$ & $\underline{P}_j$
$\underline{P}_2$	12	8/5	0	1	-1/5	2/5	-1/5	
$\underline{P}_1$	5	9/5	1	0	7/5	1/5	2/5	
$z_j - c_j$		141/5	0	0	3/5	29/5		Table-4

Table-4 gives the optimal solution $x_1 = 9/5$, $x_2 = 8/5$, $x_3 = 0$ and $z_{max} = 141/5$

<u>Second part</u>: Given that $\underline{b} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$, $\underline{B}^{-1} = \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix}$.

So the new values of the current basic variables are given by

$$\underline{x} = \underline{B}^{-1}\underline{b} = \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 12/5 \\ 11/5 \end{pmatrix} \geq \underline{0}.$$

Hence, the current basic solution consisting of x_2, x_1 remains feasible and optimal at the new values $x_1 = 11/5$, $x_2 = 12/5$ and $x_3 = 0$. The new $z_{max} = 5\times\frac{11}{5} + 12\times\frac{12}{5} + 4\times 0 = \frac{199}{5}$

Example (6.7): Solve the following LPP:

Maximize: $Z = 3x_1 + 5x_2$ **[NUH-06]**

Subject to $x_1 + x_2 \leq 1$

$2x_1 + 3x_2 \leq 1$

$x_1, x_2 \geq 0$

and describe the effect of changing the requirement vector form $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$, on the optimum solution.

Solution: Introducing slack variables $x_3 \geq 0$, $x_4 \geq 0$ to 1st and 2nd constraints, we get the initial and iterative simplex tableau as follows:

Basis	$\underline{C}_B^t$	c_j	3	5	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_o$ & $\underline{P}_2$
$\underline{P}_3$	0	1	1	1	1	0	1
$\underline{P}_4$	0	1	2	3	0	1	1/3 min
$z_j - c_j$		0	-3	-5 Smallest	0	0	Table-1

Basis	$\underline{C}_B^t$	c_j	3	5	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_o$ & $\underline{P}_j$
$\underline{P}_3$	0	2/3	1/3	0	1	-1/3	
$\underline{P}_2$	5	1/3	2/3	1	0	1/3	
$z_j - c_j$		5/3	1/3	0	0	4/3	Table-2

Table-2 gives the optimal solution as follows:

$x_1 = 0$, $x_2 = 1/3$ and $z_{max} = 5/3$.

Second part: Given that $\underline{b} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and from Table-3, we get,

$$\underline{x} = \begin{pmatrix} x_3 \\ x_2 \end{pmatrix},\ \underline{B}^{-1} = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}.$$

So the new values of the current basic variables are given by

$$\underline{x} = \underline{B}^{-1}\underline{b} = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \geq \underline{0}.$$

Hence, the current basic solution consisting of x_3, x_2 remains feasible and optimal at the new values $x_1 = 0$, $x_2 = 1$. The new $z_{max} = 3\times0 + 5\times1 = 5$.

Example (6.8): Solve the following LP problem by simplex method and discuss the range of discrete changes in the requirement vector $\underline{b}$ so that the optimality not affected.

Maximize $z = 3x_1 + 2x_2 + 5x_3$ **[JU-91]**

Subject to
$$x_1 + 2x_2 + x_3 \leq 430$$
$$x_1 + 4x_2 \leq 420$$
$$3x_1 + 2x_3 \leq 460$$
$$x_1, x_2, x_3 \geq 0$$

Solution: Introducing slack variables $x_4 \geq 0$, $x_5 \geq 0$ and $x_6 \geq 0$ to 1st, 2nd and 3rd constraints respectively, we get

Maximize $z = 3x_1 + 2x_2 + 5x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to
$$x_1 + 2x_2 + x_3 + x_4 + 0x_5 + 0x_6 = 430$$
$$x_1 + 4x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 = 420$$
$$3x_1 + 0x_2 + 2x_3 + 0x_4 + 0x_5 + x_6 = 460$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	C_B^t	c_j / $\underline{P}_o$	3	2	5	0	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_4$	0	430	1	2	1	1	0	0	430
$\underline{P}_5$	0	420	1	4	0	0	1	0	
$\underline{P}_6$	0	460	3	0	2	0	0	1	230 Min.
$z_j - c_j$		0	-3	-2	-5	0	0	0	Table-1
$\underline{P}_4$	0	200	-1/2	2	0	1	0	-1/2	100 Min.
$\underline{P}_5$	0	420	1	4	0	0	1	0	105
$\underline{P}_3$	5	230	3/2	0	1	0	0	1/2	
$z_j - c_j$		1150	9/2	-2	0	0	0	5/2	Table-2
$\underline{P}_2$	2	100	-1/4	1	0	1/2	0	-1/4	
$\underline{P}_5$	0	20	2	0	0	-2	1	1	
$\underline{P}_3$	5	230	3/2	0	1	0	0	1/2	
$z_j - c_j$		1350	4	0	0	1	0	2	Table-3

Since in the 3rd table all $z_j - c_j \geq 0$, the optimality conditions are satisfied. Hence the optimum solution is $x_1 = 0$, $x_2 = 100$, $x_3 = 230$ and $z_{max} = 1350$.

Discussion of variation of $\underline{b}$:

Here, $\underline{x}_B = \begin{pmatrix} 100 \\ 20 \\ 230 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 430 \\ 420 \\ 460 \end{pmatrix}$ and $\underline{B}^{-1} = \begin{pmatrix} 1/2 & 0 & -1/4 \\ -2 & 1 & 1 \\ 0 & 0 & 1/2 \end{pmatrix}$

Variation of b_1: We know that the variation range of Δb_1 of b_1 so that the optimality remains unaffected as follows:

$$\underset{i}{Max}\left\{\frac{-x_{Bi}}{\beta_{i1}}; \beta_{i1} > 0\right\} \leq \Delta b_1 \leq \underset{i}{Min}\left\{\frac{-x_{Bi}}{\beta_{i1}}; \beta_{i1} < 0\right\}$$

Or, $\quad Max\left\{\dfrac{-100}{1/2}\right\} \leq \Delta b_1 \leq Min\left\{\dfrac{-20}{-2}\right\}$ [$\because$ k = 1, 1st column of B^{-1} gives β_{i1}]

Or, $\quad -200 \le \Delta b_1 \le 10$

So, the displacement range of b_1 is $430 - 200 \le b_1 \le 430 + 10$ or, $230 \le b_1 \le 440$ for which the optimality remains unaffected.

Variation of b_2: We know that the variation range of Δb_2 of b_2 so that the optimality remains unaffected as follows:

$$\underset{i}{Max}\left\{\frac{-x_{Bi}}{\beta_{i2}};\beta_{i2}>0\right\} \le \Delta b_2 \le \underset{i}{Min}\left\{\frac{-x_{Bi}}{\beta_{i1}};\beta_{i1}<0\right\}$$

Or, $\quad Max\left\{\frac{-20}{1}\right\} \le \Delta b_2 \le$ no upper bound [$\because$ k = 2, 2nd column of B^{-1} gives β_{i2}]

Or, $\quad -20 \le \Delta b_2 \le$ no upper bound

So, the displacement range of b_2 is $420 - 20 \le b_2$ or, $b_2 \ge 400$ for which the optimality remains unaffected.

Variation of b_3: We know that the variation range of Δb_3 of b_3 so that the optimality remains unaffected as follows:

$$\underset{i}{Max}\left\{\frac{-x_{Bi}}{\beta_{i3}};\beta_{i3}>0\right\} \le \Delta b_3 \le \underset{i}{Min}\left\{\frac{-x_{Bi}}{\beta_{i3}};\beta_{i3}<0\right\}$$

Or, $Max\left\{\frac{-20}{1},\frac{-230}{1/2}\right\} \le \Delta b_3 \le Min\left\{\frac{-100}{-1/4}\right\}$ [$\because$ k = 3, 3rd column of B^{-1} gives β_{i3}]

Or, $\quad -20 \le \Delta b_3 \le 400$

So, the displacement range of b_3 is $460 - 20 \le b_3 \le 460 + 400$ or, $440 \le b_3 \le 860$ for which the optimality remains unaffected.

6.5 <u>Variation in the coefficients matrix (A)</u>: (সহগ মেট্রিক্সের পরিবর্তন)

The constraints matrix or the coefficients matrix A may be changed by three ways.

(i) Adding new variables or activities.

(ii) Changing the resources requirements of the existing activities.

(ii) Adding new constraints.
The variations are discussed with examples in the following arts.

6.5.1 <u>Adding new variables or activities</u>: Let us consider the following general LP problem:

Maximize $z = \underline{c}\ \underline{x}$
Subject to $\underline{A}\ \underline{x} = \underline{b};\ \underline{b} \geq \underline{0}$
$\underline{x} \geq \underline{0}$

Let a non-negative variable x_{n+1} with a profit c_{n+1} be added in the considered LP problem. It introduces an extra column in the coefficients matrix $\underline{A}$. We compute the net evaluation of the new variable to test the optimality of the current solution.

$$z_{n+1} - c_{n+1} = \underline{c}_B\, \underline{y}_{n+1} - c_{n+1} \text{ where } \underline{y}_{n+1} = \underline{B}^{-1}\underline{a}_{n+1}$$

(i) If $z_{n+1} - c_{n+1} \geq 0$, then the current solution remains optimal.
(ii) If $z_{n+1} - c_{n+1} < 0$, then the current solution is no longer optimal. For this case, introducing $\underline{a}_{n+1}$ into the current optimal table and using simplex method, we shall find the new optimum solution.

Here it is to be noted that addition of any new non-negative variable does not affect the feasibility of the current solution.
Note: For minimizing problem $z_{n+1} - c_{n+1} \leq 0$ indicates the current solution remains optimal.

<u>Example (6.9)</u>: Solve the LP problem by using big M method.

Maximize $z = x_1 + 5x_2$ **[DU-98]**
Subject to $3x_1 + 4x_2 \leq 6$
$x_1 + 3x_2 \geq 2$
$x_1, x_2 \geq 0$

and discuss the effect of adding new non-negative variable x_6 with profit coefficient $c_6 = 1$ and constraints coefficients $a_{16} = 1$, $a_{26} = 1$.
Solution: Introducing slack variable $x_3 \geq 0$ to 1st constraint and surplus variables $x_4 \geq 0$ & artificial variable $x_5 \geq 0$ to 2nd constraint, we get the standard form as follows:

$$\begin{array}{ll} \text{Maximize } z = & x_1 + 5x_2 + 0x_3 + 0x_4 - Mx_5 \\ \text{Subject to} & 3x_1 + 4x_2 + x_3 \qquad\qquad = 6 \\ & x_1 + 3x_2 \qquad - x_4 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

We find the following initial simplex table from the problem:

Basis	$\underline{C}_B^t$	c_j	1	5	0	0	-M	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_3$	0	6	3	4	1	0	0	6/4
$\underline{P}_5$	-M	2	1	3	0	-1	1	2/3 min
$z_j - c_j$		0	-1	-5	0	0	0	Table-1
		-2	-1	-3 Smallest	0	1	0	

Basis	$\underline{C}_B^t$	c_j	1	5	0	0	-M	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_3$	0	10/3	5/3	0	1	4/3	-4/3	
$\underline{P}_2$	5	2/3	1/3	1	0	-1/3	1/3	
$z_j - c_j$		10/3	2/3	0	0	-5/3 Smallest		Table-2

Basis	$\underline{C}_B^t$	c_j	1	5	0	0	-M	Min Ratio
		$\underline{P}_o$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_4$	0	5/2	5/4	0	3/4	1	-1	
$\underline{P}_2$	5	31/24	4/3	1	1/4	0	0	
$z_j - c_j$		115/24	17/3	0	5/4	0		Table-3

The 3rd tableau gives the optimal solution $x_1 = 0$, $x_2 = 31/24$ and $z_{max} = 115/24$.

Second part: Here, $\underline{a}_6 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 3/4 & -1 \\ 1/4 & 0 \end{pmatrix}$, $\underline{c}_B = (0, 5)$, $c_6 = 1$

$$\therefore \qquad \underline{y}_6 = \underline{B}^{-1}\underline{a}_6 = \begin{pmatrix} 3/4 & -1 \\ 1/4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix}$$

And $z_6 - c_6 = \underline{c}_B\, \underline{y}_6 - c_6 = (0, 5)\begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix} - 1 = 5/4 - 1 = 1/4 \geq 0.$

Since $z_6 - c_6 \geq 0$, the current solution is optimal. So, the optimal solution is $x_1 = 0$, $x_2 = 31/24$ and $z_{max} = 115/24$.

6.5.2 <u>Changing the resources requirements of the existing activities</u>: When the constraint's coefficients of non-basic variable change, its effect on the optimal solution can be studied by the same steps as given in §6.5.1.

On the other hand, if the constraints coefficient of a basic variable changes, it affect the basis matrix. Under such circumstances it may be better to solve the linear program over again.

6.5.3 <u>Adding new constraints</u>: Let us consider the following general LP problem:

$$\text{Maximize } z = \underline{c}\,\underline{x}$$
$$\text{Subject to } \underline{A}\,\underline{x} = \underline{b};\ \underline{b} \geq \underline{0}$$
$$\underline{x} \geq \underline{0}$$

After the optimality, let the constraint $\underline{a}_{n+1}\underline{x} \leq b_{m+1}$ is to be added newly, that is,

$$a_{m+1,1}x_1 + a_{m+2,2}x_2 + \ldots + a_{m+1,n}x_n \leq b_{m+1}$$

be the new constraint. If the current optimal solution satisfies the constraint, there is nothing to do. Otherwise, first introducing necessary variables (it may be necessary to introduce artificial variable) make the constraint an equation and then put it at the bottom of the optimal table.

Adding new constraints does not affect the optimality but it may affect the basis.

Then appropriate row operations can make the current basis to the right form. After then, use the dual simplex method to get the new optimal solution. We try to understand this by the following example.

Example (6.10): Solve the following LP problem

$$\text{Maximize } z = 3x_1 + 4x_2 + x_3 + 7x_4$$
$$\text{Subject to } 8x_1 + 3x_2 + 4x_3 + x_4 \le 7$$
$$2x_1 + 6x_2 + x_3 + 5x_4 \le 3$$
$$x_1 + 4x_2 + 5x_3 + 2x_4 \le 8$$
$$x_1, x_2, x_3, x_4 \ge 0$$

Find the optimal solution if the constraint $2x_1 + 3x_2 + x_3 + 5x_4 \le 2$ be added.

Solution: Introducing slack variables $x_5 \ge 0$, $x_6 \ge 0$, $x_7 \ge 0$ to first, second and third constraints respectively, we have

$$\text{Maximize } z = 3x_1 + 4x_2 + x_3 + 7x_4$$
$$\text{Subject to } 8x_1 + 3x_2 + 4x_3 + x_4 + x_5 = 7$$
$$2x_1 + 6x_2 + x_3 + 5x_4 + x_6 = 3$$
$$x_1 + 4x_2 + 5x_3 + 2x_4 + x_7 = 8$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \ge 0$$

Making initial simplex table and taking necessary iterations, we get the following tables.

Basis	$\underline{C}_B^t$	c_j	3	4	1	7	0	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_o$ & $\underline{P}_4$
$\underline{P}_5$	0	7	8	3	4	1	1	0	0	7
$\underline{P}_6$	0	3	2	6	1	5	0	1	0	3/5 min
$\underline{P}_7$	0	8	1	4	5	2	0	0	1	4
$z_j - c_j$		0	-3	-4	-1	-7 Smallest	0	0	0	Table-1

Basis	$\underline{C}_B^t$	c_j	3	4	1	7	0	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_o$ & $\underline{P}_1$
$\underline{P}_5$	0	32/5	38/5	9/5	19/5	0	1	-1/5	0	16/19
$\underline{P}_4$	7	3/5	2/5	6/5	1/5	1	0	1/5	0	3/2
$\underline{P}_7$	0	34/5	1/5	8/5	23/5	0	0	-2/5	1	34
$z_j - c_j$		21/5	-1/5 Smallest	22/5	2/5	0	0	7/5	0	Table-2

Basis	C_B^t	c_j	3	4	1	7	0	0	0	Ratio of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_o$ & $\underline{P}_j$
$\underline{P}_1$	3	16/9	1	9/38	1/2	0	5/38	-1/38	0	
$\underline{P}_4$	7	5/19	0	21/19	0	1	-1/19	4/19	0	
$\underline{P}_7$	0	126/19	0	59/38	9/2	0	-1/38	-15/38	1	
$z_j - c_j$		83/19	0	169/38	1/2	0	1/38	53/38	0	Table-3

Table-3 gives the optimal solution $x_1 = 16/9$, $x_2 = 0$, $x_3 = 0$, $x_4 = 5/19$ and $z_{max} = 83/19$

<u>Second part</u>: The given new constraint is

$$2x_1 + 3x_2 + x_3 + 5x_4 \leq 2$$

Adding slack variable $x_8 \geq 0$, we get

$$2x_1 + 3x_2 + x_3 + 5x_4 + x_8 = 2$$

We add the constraint at the bottom of the table-3, we get

Basis	C_B^t	c_j	3	4	1	7	0	0	0	0	
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	
$\underline{P}_1$	3	16/9	1	9/38	1/2	0	5/38	-1/38	0	0	
$\underline{P}_4$	7	5/19	0	21/19	0	1	-1/19	4/19	0	0	
$\underline{P}_7$	0	126/19	0	59/38	9/2	0	-1/38	-15/38	1	0	
$\underline{P}_8$	0	2	2	3	1	5	0	0	0	1	
$z_j - c_j$		83/19	0	169/38	1/2	0	1/38	53/38	0	0	Table-4

Since x_1, x_4 are in the basis, the corresponding coefficients of x_1 and x_4 in the new constraint must be zero. This may be achieved by the appropriate row operation. Doing $R_4 - 2R_1$, we get

Basis	C_B^t	c_j	3	4	1	7	0	0	0	0	
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	
$\underline{P}_1$	3	16/9	1	9/38	1/2	0	5/38	-1/38	0	0	
$\underline{P}_4$	7	5/19	0	21/19	0	1	-1/19	4/19	0	0	
$\underline{P}_7$	0	126/19	0	59/38	9/2	0	-1/38	-15/38	1	0	
$\underline{P}_8$	0	6/19	0	96/38	0	5	-5/19	1/19	0	1	
$z_j - c_j$		83/19	0	169/38	1/2	0	1/38	53/38	0	0	Table-5

Doing $R_4 - 5R_2$, we get

Basis	C_B^t	c_j	3	4	1	7	0	0	0	0	
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	
$\underline{P}_1$	3	16/9	1	9/38	1/2	0	5/38	-1/38	0	0	
$\underline{P}_4$	7	5/19	0	21/19	0	1	-1/19	4/19	0	0	
$\underline{P}_7$	0	126/19	0	59/38	9/2	0	-1/38	-15/38	1	0	
$\underline{P}_8$	0	-1	0	-3	0	0	0	-1	0	1	
$z_j - c_j$		83/19	0	169/38	1/2	0	1/38	53/38	0	0	Table-6

Since a component of $\underline{P}_0$ is negative, we have to apply dual simplex method to get the optimal solution.

Basis	C_B^t	c_j	3	4	1	7	0	0	0	0	-ve of
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	$\underline{P}_0$
$\underline{P}_1$	3	16/9	1	9/38	1/2	0	5/38	-1/38	0	0	
$\underline{P}_4$	7	5/19	0	21/19	0	1	-1/19	4/19	0	0	
$\underline{P}_7$	0	126/19	0	59/38	9/2	0	-1/38	-15/38	1	0	
$\underline{P}_8$	0	-1	0	-3	0	0	0	-1	0	1	-1 *
$z_j - c_j$		83/19	0	169/38	1/2	0	1/38	53/38	0	0	Table-7
$\frac{z_j - c_j}{y_{4j}}$; $y_{4j}<0$				-169/114				-53/38*			

Basis	C_B^t	c_j	3	4	1	7	0	0	0	0	
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	$\underline{P}_7$	$\underline{P}_8$	
$\underline{P}_1$	3	33/38	1	6/19	1/2	0	5/38	0	0	-1/38	
$\underline{P}_4$	7	1/19	0	9/19	0	1	-1/19	0	0	4/19	
$\underline{P}_7$	0	267/38	0	67/16	9/2	0	-1/38	0	1	-15/38	
$\underline{P}_8$	0	1	0	3	0	0	0	1	0	-1	
$z_j - c_j$		119/38	0	5/19	1/2	0	1/38	0	0	53/38	Table-8

Table-8 gives the optimal solution $x_1 = 33/38$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1/19$ and $z_{max} = 119/38$.

6.6. **Some done examples**: (কিছু কৃত উদাহরণ)

Example (6.11): Solve the following LP problem by simplex method and discuss the variation of c_j ($j = 1, 2$) for which the optimal solution does not change.

Maximize $z = 3x_1 + 9x_2$

Subject to $x_1 + 4x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

Solution: To solve the problem by simplex method, we introduce slack variables $x_3 \geq 0$ and $x_4 \geq 0$ to 1st and 2nd constraints respectively. Then we get,

Maximize $z = 3x_1 + 9x_2$

Subject to $x_1 + 4x_2 + x_3 = 8$

$x_1 + 2x_2 + x_4 = 4$

$x_1, x_2, x_3, x_4 \geq 0$

Making initial simplex table and taking necessary iterations, we get the following tables.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	3 / $\underline{P}_1$	9 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Min Ratio θ
$\underline{P}_3$	0	8	1	4	1	0	8/4=2 Min. arbitrarily
$\underline{P}_4$	0	4	1	2	0	1	4/2=2
$z_j - c_j$		0	-3	-9	0	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	3 / $\underline{P}_1$	9 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Min Ratio θ
$\underline{P}_2$	9	2	1/4	1	1/4	0	
$\underline{P}_4$	0	0	1/2	0	-1/2	1	0/(1/2) = 0
$z_j - c_j$		18	-3/4	0	9/4	0	

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	3 / $\underline{P}_1$	9 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	Min Ratio θ
$\underline{P}_2$	9	2	0	1	1/2	-1/2	
$\underline{P}_1$	3	0	1	0	-1	2	
$z_j - c_j$		18	0	0	3/2	3/2	

Since in the 3rd table all $z_j - c_j \geq 0$, the optimality conditions are satisfied. Hence the optimum solution is $x_1 = 0$ $x_2 = 2$ and $z_{max} = 18$.

Variation of c_1: Since c_1 is the coefficient of the 2nd basic variable in the objective function, the changing range Δc_1 of c_1 is given by

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} > 0\right\} \leq \Delta c_1 \leq \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} < 0\right\}$$

for all non-basic vectors $\underline{P}_j$.

Or, $\quad Max\left\{\frac{-3/2}{2}\right\} \leq \Delta c_1 \leq Min\left\{\frac{-3/2}{-1}\right\}$

Or, $\quad -3/4 \leq \Delta c_1 \leq 3/2$

So, the range of c_1 is $3 - 3/4 \leq c_1 \leq 3 + 3/2$ or, $9/4 \leq c_1 \leq 9/2$ for which the optimal solution does not change.

Variation of c_2: Since c_2 is the coefficient of the 1st basic variable in the objective function, the changing range Δc_2 of c_2 is given by

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{1j}}; y_{1j} > 0\right\} \leq \Delta c_2 \leq \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{1j}}; y_{1j} < 0\right\}$$

for all non-basic vectors $\underline{P}_j$.

Or, $\quad Max\left\{\frac{-3/2}{1/2}\right\} \leq \Delta c_2 \leq Min\left\{\frac{-3/2}{-1/2}\right\}$

Or, $\quad -3 \leq \Delta c_2 \leq 3$

So, the range of c_2 is $9 - 3 \leq c_2 \leq 9 + 3$ or, $-6 \leq c_2 \leq 12$, for which the optimal solution does not change.

Example (6.12): A furniture company makes tables and chairs. Each table takes 5 hours of carpentry and 10 hours in painting and varnishing shop. Each chair requires 20 hours in carpentry and 15 hours in painting and varnishing. During the current production period, 400 hours of carpentry and 450 hours of painting and varnishing time are available. Each table sold yields a profit of \$45 and each chair yields a profit of \$80. Using simplex method determine the number of tables and chairs to be made to maximize the profit. Also find the range of profit for a table that the optimum solution does not change.

Solution: Let x_1 be the number of tables and x_2 be the number of chairs. So, the total profit $45x_1 + 80x_2$ which is the objective function. The objective function $z = 45x_1+80x_2$ is to be maximized. The required carpentry hours are $5x_1 + 20x_2$. Since 400 carpentry hours are available, $4x_1 + 3x_2 \leq 240$.

Similarly, for painting and varnishing, we have $10x_1 + 15x_2 \leq 450$.

The non-negativity conditions $x_1, x_2 \geq 0$

So, the linear programming (LP) form of the given problem is

$$\begin{aligned} &\text{Maximize } z = 45x_1 + 80\, x_2 \\ &\text{Subject to } \; 5x_1 + 20x_2 \leq 400 \\ &\qquad\qquad\quad 10x_1 + 15x_2 \leq 450 \\ &\qquad\qquad\qquad x_1, x_2 \geq 0 \end{aligned}$$

Introducing slack variables x_3, x_4, we can rewrite the problem as follows:

$$\begin{aligned} &\text{Maximize } z = 45x_1 + 80x_2 \\ &\text{Subject to } \; x_1 + 4x_2 + x_3 = 80 \\ &\qquad\qquad\quad 2x_1 + 3x_2 + x_4 = 90 \\ &\qquad\qquad\quad x_j, \geq 0;\ j = 1, 2, 3, 4 \end{aligned}$$

Basis	C_B^t	$\underline{P}_o$	45	80	0	0	Ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	θ
$\underline{P}_3$	0	80	1	4	1	0	$80/4 = 20 = \theta_o$
$\underline{P}_4$	0	90	2	3	0	1	$90/3 = 30$
$z_j - c_j$		0	-45	-80	0	0	
$\underline{P}_2$	80	20	1/4	1	1/4	0	$20/(1/4) = 80$
$\underline{P}_4$	0	30	5/4	0	-3/4	1	$30/(5/4)=24=\theta_o$
$z_j - c_j$		1600	-25	0	20	0	
$\underline{P}_2$	80	14	0	1	2/5	-1/5	
$\underline{P}_1$	45	24	1	0	-3/5	4/5	
$z_j - c_j$		2200	0	0	5	20	

The above tableau gives the extreme point (24, 14), i.e., $x_1 = 24$, $x_2 = 14$. So, the company will earn maximum profit \$2200 if 24 tables and 14 chairs are made.

Here, the profit coefficient of the table in the objective function, $c_1 = 45$. Since c_1 is the coefficient of the 2nd basic variable, the variation range Δc_1 of c_1 is given by

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} > 0\right\} \le \Delta c_1 \le \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} < 0\right\}$$

for all non-basic vectors $\underline{P}_j$.

Or, $\quad Max\left\{\frac{-20}{4/5}\right\} \le \Delta c_1 \le Min\left\{\frac{-5}{-3/5}\right\}$

Or, $\quad -25 \le \Delta c_1 \le 25/3$

So, the range of c_1 is $45 - 25 \le c_1 \le 45 + 25/3$ or, $20 \le c_1 \le 160/3$ for which the optimal solution does not change. That is, if the profit of a table varies from \$20 to \$160/3, the current solution remains optimal.

<u>Example (6.13)</u>: Solve the following LP problem by simplex method.

$$\begin{aligned} \text{Maximize } z &= 2x_1 + x_2 + 3x_3 \\ \text{Subject to } \quad & x_1 + x_2 + 2x_3 \le 5 \\ & 2x_1 + 3x_2 + 4x_3 = 12 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

(i) If the objective function changes to $z = 3x_1 + 2x_2 + 2x_3$ then find the optimum solution.

(ii) If the objective function changes to $z = x_1 + x_2 + 5x_3$ then find the optimum solution.

Solution: Introducing slack variable $x_4 \ge 0$ to 1st constraint and artificial variable $x_5 \ge 0$ to 2nd constraint, we get the problem as follows:

$$\begin{aligned} \text{Maximize } z &= 2x_1 + x_2 + 3x_3 + 0x_4 - Mx_5 \\ \text{Subject to } \quad & x_1 + x_2 + 2x_3 + x_4 + 0x_5 = 5 \\ & 2x_1 + 3x_2 + 4x_3 + 0x_4 + x_5 = 12 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

We find the following initial and iterative simplex tables:

Basis	C_B^t	c_j / P_o	2 P_1	1 P_2	3 P_3	0 P_4	-M P_5	Ratio θ
P_4	0	5	1	1	2	1	0	$5/2 = \theta_o$
P_5	-M	12	2	3	4	0	1	$12/4 = 3$
$z_j - c_j$		0	-2	-1	-3	0	0	
		-12	-2	-3	-4 Smallest	0	0	Table-1

Basis	C_B^t	c_j / P_o	2 P_1	1 P_2	3 P_3	0 P_4	-M P_5	Ratio θ
P_3	3	5/2	1/2	1/2	1	1/2	0	5
P_5	-M	2	0	1	0	-2	1	$2/1=2 = \theta_o$
$z_j - c_j$		15/2	-1/2	1/2	0	3/2	0	
		-2	0	-1 Smallest	0	2	0	Table-2

Basis	C_B^t	c_j / P_o	2 P_1	1 P_2	3 P_3	0 P_4	-M P_5	Ratio θ
P_3	3	3/2	1/2	0	1	3/2	-	$3 = \theta_o$
P_2	1	2	0	1	0	-2	-	
$z_j - c_j$		13/2	-1/2 Smallest	0	0	5/2	-	
		0	0	0	0	0	-	Table-3

Basis	C_B^t	c_j / P_o	2 P_1	1 P_2	3 P_3	0 P_4	-M P_5	Ratio θ
P_1	2	3	1	0	2	3	-	
P_2	1	2	0	1	0	-2	-	
$z_j - c_j$		8	0	0	1	4	-	Table-4

Since all $z_j - c_j \geq 0$ the table is optimal. The above tableau gives us the solution of the problem: $x_1 = 3$, $x_2 = 2$, $x_3 = 0$ and $z_{max} = 8$.

(i) The changed objective function is

$$z = 3x_1 + 2x_2 + 2x_3$$

The effect on the optimal solution be determined by checking whether the $\bar{z}_j = z_j - c_j$ row in table-4 remains non-negative, that is,

$$\bar{z}_1 = \bar{z}_2 = 0$$

$$\bar{z}_3 = (3, 2)\begin{pmatrix} 2 \\ 0 \end{pmatrix} - 3 = 6 - 3 = 3 \geq 0$$

$$\bar{z}_4 = (3, 2)\begin{pmatrix} 3 \\ -2 \end{pmatrix} - 0 = 5 - 0 = 5 \geq 0$$

Hence, the optimal solution does not change. The optimal solution remains $x_1 = 3$, $x_2 = 2$, $x_3 = 0$ and $z_{max} = 3\times3 + 2\times2 + 2\times0 = 13$.

(ii) The new objective function is

$$z = x_1 + x_2 + 5x_3$$

The effect on the optimal solution be determined by checking whether the $\bar{z}_j = z_j - c_j$ row in table-4 remains non-negative, that is,

$$\bar{z}_1 = \bar{z}_2 = 0$$

$$\bar{z}_3 = (1, 1)\begin{pmatrix} 2 \\ 0 \end{pmatrix} - 5 = 2 - 5 = -3 \ngeq 0$$

$$\bar{z}_4 = (1, 1)\begin{pmatrix} 3 \\ -2 \end{pmatrix} - 0 = 1 - 0 = 1 \geq 0$$

As $\bar{z}_3$ becomes negative, the solution given in table-4 no longer remains optimal. Non-basic variable x_3 enters into the basis. This is shown as follows:

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	1	1	5	0	-M	Ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_1$	1	3	1	0	2	3		
$\underline{P}_2$	1	2	0	1	0	-2		
$z_j - c_j$		5	0	0	-3	1		

Table-5

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	1	1	5	0	-M	Ratio
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	θ
$\underline{P}_3$	5	3/2	1/2	0	1	3/2		
$\underline{P}_2$	1	2	0	1	0	-2		
$z_j - c_j$		19/2	3/2	0	0	11/2		Table-6

Table-6 gives the optimal solution $x_1 = 0$, $x_2 = 2$, $x_3 = 3/2$ and $z_{max} = 0 + 1\times 2 + 5\times 3/2 = 19/2$.

Example (6.14): Solve the following LP problem by simplex method and discuss the variation of c_j ($j = 1, 2$) for which the optimal solution does not change.

Maximize $z = 3x_1 + 5x_2$ **[NUH-06]**

Subject to $x_1 + x_2 \leq 1$

$2x_1 + 3x_2 \leq 1$

$x_1, x_2 \geq 0$

Solution: To solve the problem by simplex method, we introduce slack variables $x_3 \geq 0$ and $x_4 \geq 0$ to 1st and 2nd constraints respectively. Then we get,

Maximize $z = 3x_1 + 5x_2 + 0x_3 + 0x_4$

Subject to $x_1 + x_2 + x_3 + 0x_4 = 1$

$2x_1 + 3x_2 + 0x_3 + x_4 = 1$

$x_1, x_2, x_3, x_4 \geq 0$

Making initial simplex table and taking necessary iterations, we get the following tables.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	3	5	0	0	Ratio of
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_o$ & $\underline{P}_2$
$\underline{P}_3$	0	1	1	1	1	0	1
$\underline{P}_4$	0	1	2	3	0	1	1/3 min.
$z_j - c_j$		0	-3	-5 Smallest	0	0	Table-1

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	3 $\underline{P}_1$	5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Ratio of $\underline{P}_o$ & $\underline{P}_2$
$\underline{P}_3$	0	1	1/3	0	1	-1/3	
$\underline{P}_2$	5	1/3	2/3	1	0	1/3	
$z_j - c_j$		5/3	1/3	0	0	5/3	Table-2

Since in the 2nd table all $z_j - c_j \geq 0$, the optimality conditions are satisfied. Hence the optimum solution is $x_1 = 0$ $x_2 = 1/3$ and $z_{max} = 5/3$.

Variation of c_1: Since $c_1 = 3$ is the coefficient of a non-basic variable in the objective function, the changing range Δc_1 of c_1 is given by

$$\Delta c_1 \leq z_1 - c_1 \text{ and } \Delta c_1 \text{ has no lower bound.}$$

Or, $\Delta c_1 \leq 1/3$

So, the range of c_1 is $c_1 \leq 3 + 1/3$ or, $c_1 \leq 10/3$ for which the optimal solution does not change.

Variation of c_2: Since $c_2 = 5$ is the coefficient of the 2nd basic variable in the objective function, the changing range Δc_2 of c_2 is given by

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} > 0\right\} \leq \Delta c_2 \leq \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} < 0\right\}$$

for all non-basic vectors $\underline{P}_j$.

Or, $Max\left\{\frac{-1/3}{2/3}, \frac{-5/3}{1/3}\right\} \leq \Delta c_2$ and no upper bound as no $y_{2j} < 0$.

Or, $-1/2 \leq \Delta c_2$

So, the range of c_2 is $5 - 1/2 \leq c_2$ or, $c_2 \geq 9/2$, for which the optimal solution does not change.

<u>Example (6.15)</u>: Solve the following LP problem by simplex method and discuss the variation of c_j (j = 1, 2) for which the optimal solution does not change. Maximize $z = 3x_1 + 5x_2$

Subject to $x_1 + x_2 \le 1$

$2x_1 + 3x_2 \le 1$

$x_1, x_2 \ge 0$

Solution: To solve the problem by simplex method, we introduce slack variables $x_3 \ge 0$ and $x_4 \ge 0$ to 1st and 2nd constraints respectively. Then we get,

$$\text{Maximize } z = 3x_1 + 5x_2 + 0x_3 + 0x_4$$

$$\text{Subject to } x_1 + x_2 + x_3 + 0x_4 = 1$$

$$2x_1 + 3x_2 + 0x_3 + x_4 = 1$$

$$x_1, x_2, x_3, x_4 \ge 0$$

Making initial simplex table and taking necessary iterations, we get the following tables.

Basis	C_B^t	c_j / $\underline{P}_o$	3 $\underline{P}_1$	5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Ratio of $\underline{P}_o$ & $\underline{P}_2$
$\underline{P}_3$	0	1	1	1	1	0	1
$\underline{P}_4$	0	1	2	3	0	1	1/3 min.
$z_j - c_j$		0	-3	-5 Smallest	0	0	Table-1

Basis	C_B^t	c_j / $\underline{P}_o$	3 $\underline{P}_1$	5 $\underline{P}_2$	0 $\underline{P}_3$	0 $\underline{P}_4$	Ratio of $\underline{P}_o$ & $\underline{P}_2$
$\underline{P}_3$	0	1	1/3	0	1	-1/3	
$\underline{P}_2$	5	1/3	2/3	1	0	1/3	
$z_j - c_j$		5/3	1/3	0	0	5/3	Table-2

Since in the 2nd table all $z_j - c_j \ge 0$, the optimality conditions are satisfied. Hence the optimum solution is $x_1 = 0$ $x_2 = 1/3$ and $z_{max} = 5/3$.

<u>Variation of c_1</u>: Since $c_1 = 3$ is the coefficient of a non-basic variable in the objective function, the changing range Δc_1 of c_1 is given by

$\Delta c_1 \leq z_1 - c_1$ and Δc_1 has no lower bound.

Or, $\Delta c_1 \leq 1/3$

So, the range of c_1 is $c_1 \leq 3 + 1/3$ or, $c_1 \leq 10/3$ for which the optimal solution does not change.

Variation of c_2: Since $c_2 = 5$ is the coefficient of the 2nd basic variable in the objective function, the changing range Δc_2 of c_2 is given by

$$\underset{j}{Max}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} > 0\right\} \leq \Delta c_2 \leq \underset{j}{Min}\left\{\frac{-(z_j - c_j)}{y_{2j}}; y_{2j} < 0\right\}$$

for all non-basic vectors $\underline{P}_j$.

Or, $Max\left\{\frac{-1/3}{2/3}, \frac{-5/3}{1/3}\right\} \leq \Delta c_2$ and no upper bound as no $y_{2j} < 0$.

Or, $-1/2 \leq \Delta c_2$

So, the range of c_2 is $5 - 1/2 \leq c_2$ or, $c_2 \geq 9/2$, for which the optimal solution does not change.

<u>Example (6.16)</u>: Solve the following LP problem by simplex method and hence find the optimal solution if $c_1 = 1$ changes by $c_1 = 5$. Maximize $z = x_1 + x_2 + x_3$

Subject to $2x_1 + x_2 + 2x_3 \leq 3$

$4x_1 + 2x_2 + x_3 \leq 2$

$x_1, x_2, x_3 \geq 0$

Solution: Introducing slack variables $x_4 \geq 0$ and $x_5 \geq 0$ to 1st and 2nd constraints respectively, we have

Maximize $z = x_1 + x_2 + x_3$

Subject to $2x_1 + x_2 + 2x_3 + x_4 = 3$

$4x_1 + 2x_2 + x_3 + x_5 = 2$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	1 $\underline{P}_1$	1 $\underline{P}_2$	1 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_4$	0	3	2	1	2	1	0	$3/2=\theta_0$
$\underline{P}_5$	0	2	4	2	1	0	1	2/1=2
$z_j - c_j$		0	-1	-1	-1	0	0	Table-1
$\underline{P}_3$	1	3/2	1	1/2	1	1/2	0	3
$\underline{P}_5$	0	1/2	3	3/2	0	-1/2	1	$1/3=\theta_0$
$z_j - c_j$		3/2	0	-1/2	0	1/2	0	Table-2
$\underline{P}_3$	1	4/3	0	0	1	2/3	-1/3	
$\underline{P}_2$	1	1/3	2	1	0	-1/3	2/3	
$z_j - c_j$		5/3	1	0	0	1/3	1/3	Table-3

Since all $z_j - c_j \geq 0$, the optimality conditions are satisfied. Hence the optimal basic feasible solution is $x_1 = 0$, $x_2 = 1/3$, $x_3 = 4/3$ and $z_{max} = 5/3$.

<u>Second part</u>: Now $c_1 = 5$ and it is the coefficient of a non-basic variable. Then $\bar{z}_3 = (1, 1)\begin{pmatrix}0\\2\end{pmatrix} - 5 = 2 - 5 = -3$ is negative, so the current solution is not optimal. To find the new solution, we do the following simplex tableau

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	5 $\underline{P}_1$	1 $\underline{P}_2$	1 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio
$\underline{P}_3$	1	4/3	0	0	1	2/3	-1/3	
$\underline{P}_2$	1	1/3	2	1	0	-1/3	2/3	
$z_j - c_j$		5/3	-3	0	0	1/3	1/3	Table-4

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	5 $\underline{P}_1$	1 $\underline{P}_2$	1 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio
$\underline{P}_3$	1	4/3	0	0	1	2/3	-1/3	
$\underline{P}_1$	5	1/6	1	1/2	0	-1/6	1/3	
$z_j - c_j$		13/6	0	3/2	0	-1/6	4/3	Table-5

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	5 $\underline{P}_1$	1 $\underline{P}_2$	1 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Min Ratio θ
$\underline{P}_4$	0	2	0	0	3/2	1	-1/2	
$\underline{P}_1$	5	1/2	1	1/2	1/4	0	1/4	
$z_j - c_j$		5/2	0	3/2	1/4	0	5/4	Table-6

Table-6 gives the new optimal solution $x_1 = 1/2$, $x_2 = 0$, $x_3 = 0$ and $z_{max} = 5/2$.

Example (6.17): Solve the following LPP:

Maximize $z = 5x_1 + 12x_2 + 4x_3$ **[NU-07, DU-95]**

Subject to $x_1 + 2x_2 + x_3 \leq 5$

$2x_1 - x_2 + 3x_3 = 2$

$x_1, x_2, x_3 \geq 0$

and describe the effect of changing the requirement vector form $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$, on the optimum solution.

Solution: Table-4 of Example (6.6) gives the optimal solution $x_1 = 9/5$, $x_2 = 8/5$, $x_3 = 0$ and $z_{max} = 141/5$

Here, $\underline{x} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$, $\underline{B}^{-1} = \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix}$.

Second part: Given that $\underline{b} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$. So the new values of the current basic variables are given by

$$\underline{x} = \underline{B}^{-1}\underline{b} = \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} -3/5 \\ 21/5 \end{pmatrix} \not\geq \underline{0}.$$

Since, x_2 becomes negative; the current optimal solution becomes infeasible but the dual feasible. Hence modifying Table-4 of Example (6.6), we solve the new problem using dual simplex method as follows:

Basis	$\underline{C}_B^t$	c_j	5	12	4	0	-M	Negative of $\underline{P}_o$
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	
$\underline{P}_2$	12	-3/5	0	1	-1/5	2/5	-1/5	
$\underline{P}_1$	5	21/5	1	0	7/5	1/5	2/5	
$z_j - c_j$		69/5	0	0	3/5	29/5		
$\frac{z_j - c_j}{y_{1j}}$; $y_{1j} \leq 0$					-3			Table-5

Basis	$\underline{C}_B^t$	c_j	5	12	4	0	-M	Negative of $\underline{P}_o$
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	
$\underline{P}_3$	4	3	0	-5	1	-2	1	
$\underline{P}_1$	5	0	1	7	0	3	-1	
$z_j - c_j$		12	0	3	0	7		Table-6

Table-6 gives the optimal solution as follows: $x_1 = 0$, $x_2 = 0$, $x_3 = 4$ and new $z_{max} = 12$

Example (6.18): Solve the following LP problem by simplex method

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$
$$\text{Subject to} \quad x_1 + 2x_2 + x_3 \leq 430$$
$$x_1 + 4x_2 \leq 420$$
$$3x_1 + 2x_3 \leq 460$$
$$x_1, x_2, x_3 \geq 0$$

(i) Describe the effect of changing the requirement vector form $\begin{pmatrix} 430 \\ 420 \\ 460 \end{pmatrix}$ to $\begin{pmatrix} 400 \\ 500 \\ 800 \end{pmatrix}$, on the optimum solution.

(ii) Describe the effect of changing the requirement vector form

$$\begin{pmatrix}430\\420\\460\end{pmatrix} \text{ to } \begin{pmatrix}200\\500\\300\end{pmatrix} \text{ and } \begin{pmatrix}200\\420\\460\end{pmatrix}$$, on the optimum solution.

Solution: Introducing slack variables $x_4 \geq 0$, $x_5 \geq 0$ and $x_6 \geq 0$ to 1st, 2nd and 3rd constraints respectively, we get

$$\begin{aligned} &\text{Maximize } z = 3x_1 + 2x_2 + 5x_3 + 0x_4 + 0x_5 + 0x_6 \\ &\text{Subject to } x_1 + 2x_2 + x_3 + x_4 + 0x_5 + 0x_6 = 430 \\ &\qquad x_1 + 4x_2 + 0x_3 + 0x_4 + x_5 + 0x_6 = 420 \\ &\qquad 3x_1 + 0x_2 + 2x_3 + 0x_4 + 0x_5 + x_6 = 460 \\ &\qquad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	C_B^t	c_j / $\underline{P}_o$	3	2	5	0	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_4$	0	430	1	2	1	1	0	0	430
$\underline{P}_5$	0	420	1	4	0	0	1	0	
$\underline{P}_6$	0	460	3	0	2	0	0	1	230 Min.
$z_j - c_j$		0	-3	-2	-5	0	0	0	Table-1
$\underline{P}_4$	0	200	-1/2	2	0	1	0	-1/2	100 Min.
$\underline{P}_5$	0	420	1	4	0	0	1	0	105
$\underline{P}_3$	5	230	3/2	0	1	0	0	1/2	
$z_j - c_j$		1150	9/2	-2	0	0	0	5/2	Table-2
$\underline{P}_2$	2	100	-1/4	1	0	1/2	0	-1/4	
$\underline{P}_5$	0	20	2	0	0	-2	1	1	
$\underline{P}_3$	5	230	3/2	0	1	0	0	1/2	
$z_j - c_j$		1350	4	0	0	1	0	2	Table-3

Since in the 3rd table all $z_j - c_j \geq 0$, the optimality conditions are satisfied. Hence the optimum solution is $x_1 = 0$, $x_2 = 100$, $x_3 = 230$ and $z_{max} = 1350$.

Discussion of variation of $\underline{\mathbf{b}}$:

Here, $\underline{x}_B = \begin{pmatrix} 100 \\ 20 \\ 230 \end{pmatrix}$ and $\underline{B}^{-1} = \begin{pmatrix} 1/2 & 0 & -1/4 \\ -2 & 1 & 1 \\ 0 & 0 & 1/2 \end{pmatrix}$

(i) Given that $\underline{b} = \begin{pmatrix} 400 \\ 500 \\ 800 \end{pmatrix}$. So, the new values of the current basic variables are given by

$$\underline{x} = \underline{B}^{-1}\underline{b} = \begin{pmatrix} 1/2 & 0 & -1/4 \\ -2 & 1 & 1 \\ 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 400 \\ 500 \\ 800 \end{pmatrix} = \begin{pmatrix} 0 \\ 500 \\ 400 \end{pmatrix} \geq \underline{0}.$$

Hence, the current basic solution consisting of x_2, x_5, x_3 remains feasible and optimal at the new values $x_1 = 0$, $x_2 = 0$ and $x_3 = 400$. The new $z_{max} = 3\times0 + 2\times0 + 5\times400 = 2000$.

(ii) 1st part: Given that $\underline{b} = \begin{pmatrix} 200 \\ 500 \\ 300 \end{pmatrix}$. So the new values of the current basic variables are given by

$$\underline{x} = \underline{B}^{-1}\underline{b} = \begin{pmatrix} 1/2 & 0 & -1/4 \\ -2 & 1 & 1 \\ 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 200 \\ 500 \\ 300 \end{pmatrix} = \begin{pmatrix} 25 \\ 400 \\ 150 \end{pmatrix} \geq \underline{0}.$$

Hence, the current basic solution consisting of x_2, x_5, x_3 remains feasible and optimal at the new values $x_1 = 0$, $x_2 = 25$ and $x_3 = 150$. The new $z_{max} = 3\times0 + 2\times25 + 5\times150 = 800$.

2nd part: Given that $\underline{b} = \begin{pmatrix} 200 \\ 420 \\ 460 \end{pmatrix}$. So the new values of the current basic variables are given by

$$\underline{x} = \underline{B}^{-1}\underline{b} = \begin{pmatrix} 1/2 & 0 & -1/4 \\ -2 & 1 & 1 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 200 \\ 420 \\ 460 \end{pmatrix} = \begin{pmatrix} -15 \\ 480 \\ 230 \end{pmatrix} \not\geq \underline{0}.$$

Since, x_2 becomes negative; the current optimal solution becomes infeasible but the dual feasible. Hence modifying Table-3, we solve the new problem using dual simplex method as follows:

Basis	$\underline{C}_B^t$	c_j	3	2	5	0	0	0	Negative of $\underline{P}_o$
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_2$	2	-15	-1/4	1	0	1/2	0	-1/4	-15
$\underline{P}_5$	0	480	2	0	0	-2	1	1	
$\underline{P}_3$	5	230	3/2	0	1	0	0	1/2	
$z_j - c_j$		1120	4	0	0	1	0	2	
$\dfrac{z_j - c_j}{y_{1j}}$; $y_{1j} \leq 0$			-16 Smallest					-8	Table-4

Basis	$\underline{C}_B^t$	c_j	3	2	5	0	0	0	Negative of $\underline{P}_o$
		$\underline{P}_0$	$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_1$	3	60	1	-4	0	-2	0	1	
$\underline{P}_5$	0	360	0	8	0	2	1	-1	
$\underline{P}_3$	5	140	0	6	1	3	0	-1/2	
$z_j - c_j$		880	0	16	0	9	0	1/2	Table-5

Table-5 gives the optimal solution as follows: $x_1 = 60$, $x_2 = 0$, $x_3 = 140$ and new $z_{max} = 880$

Example (6.19): Solve the following linear programming problem using the simplex method:

Maximize $z = x_1 + 4x_2 + 5x_3$

Subject to $3x_1 + 3x_3 \leq 22$

$x_1 + 2x_2 + 3x_3 \leq 14$

$$3x_1 + 2x_2 \le 14$$
$$x_1, x_2, x_3 \ge 0$$

Find the effect of changing the objective function to $z = 2x_1 + 8x_2 + 5x_3$ on the current optimal solution.

Solution: Introducing slack variables $x_4 \ge 0$, $x_5 \ge 0$, $x_6 \ge 0$ to 1st, 2nd and 3rd constraints, we get

Maximize $z = x_1 + 4x_2 + 5x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to $3x_1 + 3x_3 + x_4 = 22$

$x_1 + 2x_2 + 3x_3 + x_5 = 14$

$3x_1 + 2x_2 + x_6 = 14$

$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	C_B^t	c_j / $\underline{P}_o$	1	4	5	0	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_4$	0	22	3	0	3	1	0	0	22/3
$\underline{P}_5$	0	14	1	2	3	0	1	0	14/3
$\underline{P}_6$	0	14	3	2	0	0	0	1	
$z_j - c_j$		0	-1	-4	-5 Smallest	0	0	0	Table-1

Basis	C_B^t	c_j / $\underline{P}_o$	1	4	5	0	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_4$	0	8	2	-2	0	1	-1	0	
$\underline{P}_3$	5	14/3	1/3	2/3	1	0	1/3	0	7
$\underline{P}_6$	0	14	3	2	0	0	0	1	7 Let min
$z_j - c_j$		70/3	2/3	-2/3 Smallest	0	0	5/3	0	Table-2

Basis	C_B^t	c_j / $\underline{P}_o$	1	4	5	0	0	0	Min Ratio θ
			$\underline{P}_1$	$\underline{P}_2$	$\underline{P}_3$	$\underline{P}_4$	$\underline{P}_5$	$\underline{P}_6$	
$\underline{P}_4$	0	22	5	0	0	1	-1	1	
$\underline{P}_3$	5	0	-2/3	0	1	0	1/3	-1/3	
$\underline{P}_2$	4	7	3/2	1	0	0	0	1/2	
$z_j - c_j$		28	5/3	0	0	0	5/3	2	Table-3

The 3rd tableau gives the optimal solution $x_1 = 0$, $x_2 = 7$, $x_3 = 0$ and $z_{max} = 28$.

Second part: The changed objective function is

$$z = 2x_1 + 8x_2 + 5x_3$$

The effect on the optimal solution be determined by checking whether the $\bar{z}_j = z_j - c_j$ row in table-3 remains non-negative, that is,

$$\bar{z}_2 = \bar{z}_3 = \bar{z}_4 = 0$$

$$\bar{z}_1 = (0, 5, 8)\begin{pmatrix} 5 \\ -2/3 \\ 3/2 \end{pmatrix} - 2 = 26/3 - 2 = 20/3 \geq 0$$

$$\bar{z}_5 = (0, 5, 8)\begin{pmatrix} -1 \\ 1/3 \\ 0 \end{pmatrix} - 0 = 5/3 - 0 = 5/3 \geq 0$$

$$\bar{z}_5 = (0, 5, 8)\begin{pmatrix} 1 \\ -1/3 \\ 1/2 \end{pmatrix} - 0 = 7/3 - 0 = 7/3 \geq 0$$

Hence, the optimal solution does not change. The optimal solution remains $x_1 = 0$, $x_2 = 7$, $x_3 = 0$ and $z_{max} = 2\times0 + 8\times7 + 5\times0 = 56$.

Example (6.20): Solve the following LPP:

Maximize $z = 4x_1 + 6x_2 + 2x_3$ **[JU-95]**

Subject to $x_1 + x_2 + x_3 \leq 3$

$x_1 + 4x_2 + 7x_3 \leq 9$

$x_1, x_2, x_3 \geq 0$

Find the effect of changing the objective function to

$z = 2x_1 + 8x_2 + 4x_3$

on the current optimal solution.

Solution: Adding slack variables $x_4 \geq 0$ and $x_5 \geq 0$ to first and second constraints respectively, we have

$$\text{Maximize } z = 4x_1 + 6x_2 + 2x_3 + 0x_4 + 0x_5$$
$$\text{Subject to } x_1 + x_2 + x_3 + x_4 + 0x_5 = 3$$
$$x_1 + 4x_2 + 7x_3 + 0x_4 + x_5 = 9$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Making initial simplex table and taking necessary iterations, we get the following tables

Basis	C_B^t	c_j / $\underline{P}_o$	4 $\underline{P}_1$	6 $\underline{P}_2$	2 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Ratio of $\underline{P}_o$ & $\underline{P}_2$
$\underline{P}_4$	0	3	1	1	1	1	0	3
$\underline{P}_5$	0	9	1	4	7	0	1	9/4 min.
$z_j - c_j$		0	-4	-6 Smallest	-2	0	0	Table-1

Basis	C_B^t	c_j / $\underline{P}_o$	4 $\underline{P}_1$	6 $\underline{P}_2$	2 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Ratio of $\underline{P}_o$ & $\underline{P}_1$
$\underline{P}_4$	0	3/4	3/4	0	-3/4	1	-1/4	1 min.
$\underline{P}_2$	6	9/4	1/4	1	7/4	0	1/4	9
$z_j - c_j$		27/2	-5/2 Smallest	0	17/2	0	3/2	Table-2

Basis	C_B^t	c_j / $\underline{P}_o$	4 $\underline{P}_1$	6 $\underline{P}_2$	2 $\underline{P}_3$	0 $\underline{P}_4$	0 $\underline{P}_5$	Ratio of $\underline{P}_o$ & $\underline{P}_j$
$\underline{P}_1$	4	1	1	0	-1	4/3	-1/3	
$\underline{P}_2$	6	2	0	1	2	-1/3	1/3	
$z_j - c_j$		16	0	0	6	10/3	2/3	Table-3

Table-3 gives the optimal solution $x_1 = 1$, $x_2 = 2$, $x_3 = 0$ and $z_{max} = 16$

<u>Second part</u>: The changed objective function is

$$z = 2x_1 + 8x_2 + 4x_3$$

The effect on the optimal solution be determined by checking whether the $\bar{z}_j = z_j - c_j$ row in table-3 remains non-negative, that is, $\bar{z}_1 = \bar{z}_2 = 0$

$$\bar{z}_3 = (2, 8)\begin{pmatrix} -1 \\ 2 \end{pmatrix} - 4 = 14 - 4 = 10 \geq 0$$

$$\bar{z}_4 = (2, 8)\begin{pmatrix} 4/3 \\ -1/3 \end{pmatrix} - 0 = 8/3 - 8/3 = 0$$

$$\bar{z}_5 = (2, 8)\begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix} - 0 = 8/3 - 2/3 = 2 \geq 0$$

Hence, the optimal solution does not change. The optimal solution remains $x_1 = 1$, $x_2 = 2$, $x_3 = 0$ and $z_{max} = 2\times1 + 8\times2 + 4\times0 = 18$.

<u>Example (6.21)</u>: Solve the following linear programming problem by simplex method: Maximize $z = 3x_1 + 5x_2$ **[RU-92]**

Subject to $3x_1 + 2x_2 \leq 18$

$x_1 \leq 4$

$x_2 \leq 6$

$x_1, x_2 \geq 0$

Find the optimum solution if c_1 changes to 5.

Solution: Introducing slack variables $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$ to first, second and third constraints respectively, we get

Maximize $z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5$

Subject to $3x_1 + 2x_2 + x_3 + 0x_4 + 0x_5 = 18$

$x_1 + 0x_2 + 0x_3 + x_4 + 0x_5 = 4$

$0x_1 + x_2 + 0x_3 + 0x_4 + x_5 = 6$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Now taking the initial simplex table and then taking necessary iterations, we get the following tables.

Basis	$\underline{C}_B^t$	c_j / $\underline{P}_o$	3 / $\underline{P}_1$	5 / $\underline{P}_2$	0 / $\underline{P}_3$	0 / $\underline{P}_4$	0 / $\underline{P}_5$	Min Ratio θ
$\underline{P}_3$	0	18	3	2	1	0	0	18/2=9
$\underline{P}_4$	0	4	1	0	0	1	0	
$\underline{P}_5$	0	6	0	1 Pivot	0	0	1	6/1=6=θ_0
$z_j - c_j$		0	-3	-5 Smallest	0	0	0	Table-1

Basis	C_B^t	c_j / P_o	3 / P_1	5 / P_2	0 / P_3	0 / P_4	0 / P_5	
P_3	0	6	3 Pivot	0	1	0	-2	$6/3=2=\theta_0$
P_4	0	4	1	0	0	1	0	4/1 = 4
P_2	5	6	0	1	0	0	1	
$z_j - c_j$		30	-3 Smallest	0	0	0	5	Table-2

Basis	C_B^t	c_j / P_o	3 / P_1	5 / P_2	0 / P_3	0 / P_4	0 / P_5	Min Ratio θ
P_1	3	2	1	0	1/3	0	-2/3	
P_4	0	2	0	0	-1/3	1	2/3	
P_2	5	6	0	1	0	0	1	
$z_j - c_j$		36	0	0	1	0	3	Table-3

Table-3 gives the optimal solution as follows:

$x_1 = 2$, $x_2 = 6$ and $z_{max} = 36$..

<u>Second part</u>: When $c_1 = 5$, then

$$\bar{z}_1 = \bar{z}_2 = \bar{z}_4 = 0 \text{ as } x_1, x_2, x_4 \text{ are in basis.}$$

$$\bar{z}_3 = 5/3 - 0 = 5/3 \geq 0$$

$$\bar{z}_5 = 5/3 - 0 = 5/5 \geq 0$$

Hence, the optimal solution does not change. The optimal solution remains $x_1 = 2$, $x_2 = 6$ and $z_{max} = 5\times2 + 5\times6 = 40$.

6.7 <u>Exercises</u>: (প্রশ্নমালা)

1. What is sensitivity analysis?
2. What do you mean by parametric programming?
3. Solve the following LP problem and discuss the variation of c_j ($j = 1, 2, 3$) for which the optimal solution does not change.

$$\text{Minimize } z = -2x_1 - x_2 - 3x_3$$
$$\text{Subject to} \quad x_1 + x_2 + 2x_3 \leq 5$$
$$2x_1 + 3x_2 + 4x_3 = 12$$

$x_1, x_2, x_3 \geq 0$

4. Solve the following LP problem and discuss the variation of c_j (j = 1, 2, 3) for which the optimal solution does not change.

   Maximize $z = -2x_1 - 3x_2 - x_3$

   Subject to $\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \leq 1$

   $\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3}x_3 \leq 3$

   $x_1, x_2, x_3 \geq 0$

5. Solve the following LP problem by simplex method and discuss the variation of c_j (j = 1, 2) for which the optimal solution does not change. Maximize $z = -3x_1 - 5x_2$

   Subject to $x_1 + x_2 \leq 1$

   $2x_1 + 3x_2 \leq 1$

   $x_1, x_2 \geq 0$

6. Solve the following LP problem by simplex method and discuss the variation of c_j (j = 1, 2) for which the optimal solution does not change. Minimize $z = 3x_1 + 5x_2$

   Subject to $x_1 + x_2 \leq 1$

   $2x_1 + 3x_2 \leq 1$

   $x_1, x_2 \geq 0$

7. Solve the following LP problem by simplex method and discuss the variation of c_j (j = 1, 2) for which the optimal solution does not change. Minimize $z = -3x_1 - 5x_2$

   Subject to $x_1 + x_2 \leq 1$

   $2x_1 + 3x_2 \leq 1$

   $x_1, x_2 \geq 0$

8. Solve the following LP problem by simplex method and hence find the optimal solution if $c_2 = 1$ changes by $c_1 = 7$.

   Maximize $z = x_1 + x_2 + x_3$

   Subject to $2x_1 + x_2 + 2x_3 \leq 3$

   $4x_1 + 2x_2 + x_3 \leq 2$

   $x_1, x_2, x_3 \geq 0$

9. Solve the following LP problem by simplex method and hence find the optimal solution if $c_1 = 1$ changes by $c_1 = 4$.

$$\text{Maximize } z = x_1 + x_2 + x_3$$
$$\text{Subject to } 2x_1 + x_2 + 2x_3 \le 3$$
$$4x_1 + 2x_2 + x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

10. Solve the following LP problem by simplex method.

$$\text{Maximize } z = 2x_1 + x_2 + 3x_3$$
$$\text{Subject to } \quad x_1 + x_2 + 2x_3 \le 5$$
$$2x_1 + 3x_2 + 4x_3 = 12$$
$$x_1, x_2, x_3 \ge 0$$

If the objective function changes to $z = 5x_1 + 3x_2 + 6x_3$ then find the optimum solution.

11. A furniture company makes tables and chairs. Each table takes 5 hours of carpentry and 10 hours in painting and varnishing shop. Each chair requires 20 hours in carpentry and 15 hours in painting and varnishing. During the current production period, 400 hours of carpentry and 450 hours of painting and varnishing time are available. Each table sold yields a profit of \$45 and each chair yields a profit of \$80. Using simplex method determine the number of tables and chairs to be made to maximize the profit. Also find the range of profit for a chair that the optimum solution does not change.

12. After solving the following LP problem find the range of b_2 for which the optimal is unaffected.

$$\text{Maximize } z = 2x_1 + 3x_2 + x_3$$
$$\text{Subject to } \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \le 1$$
$$\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3}x_3 \le 3$$
$$x_1, x_2, x_3 \ge 0$$

13. Solve the following LP problem by simplex method and discuss the range of discrete changes in the requirement vector $\underline{b}$ so that the optimality not affected.

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$
$$\text{Subject to } x_1 + 2x_2 + x_3 \le 5$$
$$2x_1 - x_2 + 3x_3 = 2$$
$$x_1, x_2, x_3 \ge 0$$

14. Solve the following LPP:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$
$$\text{Subject to } x_1 + 2x_2 + x_3 \le 5$$
$$2x_1 - x_2 + 3x_3 = 2$$
$$x_1, x_2, x_3 \ge 0$$

and describe the effect of changing the requirement vector form $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 9 \\ 3 \end{bmatrix}$, on the optimum solution.

15. Solve the following LP problem by simplex method and discuss the range of discrete changes in the requirement vector $\underline{b}$ so that the optimality not affected.

$$\text{Maximize: } Z = 3x_1 + 5x_2$$
$$\text{Subject to } x_1 + x_2 \le 1$$
$$2x_1 + 3x_2 \le 1$$
$$x_1, x_2 \ge 0$$

16. Solve the following linear programming problem using the simplex method: Maximize $z = x_1 + 4x_2 + 5x_3$

$$\text{Subject to } 3x_1 + 3x_3 \le 22$$
$$x_1 + 2x_2 + 3x_3 \le 14$$
$$3x_1 + 2x_2 \le 14$$
$$x_1, x_2, x_3 \ge 0$$

Find the effect of changing the objective function to $z = 5x_1 + 2x_2 + 5x_3$ on the current optimal solution.

University Questions
N.U.H-2001
Subject: Linear Programming

1. (a) Show that $S = \{(x_1, x_2, x_3) \mid 2x_1 - x_2 + x_3 \le 4\} \subset \mathbf{R}^3$ is a convex set. **[Ex. 2.3]**

(b) Prove that closed half-spaces are convex sets. **[Th^m-2.3]**

2. (a) Formulate the general linear programming problem in n variables **[§1.4]** and explain what do you mean by (i) Solution; (ii) Feasible solution; (iii) Optimal solution; (iv) Slack variable **[§1.9]**.

(b) Show graphically that the following LP problem has no solution and state the reason.

Maximize $z = x_1 + \frac{1}{2}x_2$ **[Ex.3.24]**

$$\begin{aligned} \text{Subject to } 3x_1 + 2x_2 &\le 12 \\ 5x_1 &\le 10 \\ x_1 + x_2 &\ge 8 \\ -x_1 + x_2 &\ge 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

3. (a) Solve graphically the following linear programme:

Minimize $z = 3x_1 + 2x_2$ **[Ex. 3.21]**

$$\begin{aligned} \text{Subject to } x_1 + 2x_2 &\ge 4 \\ 2x_1 + x_2 &\ge 4 \\ x_1 + x_2 &\le 5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

(b) Transform the following linear programme to the standard form: Minimize $z = -3x_1 + 4x_2 - 2x_3 + 5x_4$ **[Ex-1.23]**

$$\begin{aligned} \text{Subject to } 4x_1 - x_2 + 2x_3 - x_4 &= -2 \\ x_1 + x_2 + 3x_3 - x_4 &\le 14 \\ -2x_1 + 3x_2 - x_3 + 2x_4 &\ge 2 \end{aligned}$$

$x_1, x_2 \ge 0$, $x_3 \le 0$ and x_4 unrestricted in sign

4. Explain terms: Unbounded solution **[§4.5.6]**, Alternative optimal solution **[§4.5.7]**. Solve the given problem by simplex method:

Maximize $z = 3x_1 + 2x_2 + 5x_3$ **[Ex-4.30]**

Subject to $x_1 + 2x_2 + x_3 \leq 430$

$x_1 + 4x_2 \geq 420$

$3x_1 + 2x_3 \leq 460$

$x_1, x_2, x_3 \geq 0.$

5. Define an artificial variable. What is the role of an artificial variable in simplex method of solving LP problem **[§4.8.1]**? Solve the LP problem by using big M method:

Maximize $z = 3x_1 + 2x_2$ **[Ex-4.58]**

Subject to $2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$

N.U.H-2002

Subject: Linear Programming

(Answer any three from the questions.)

1. (a) What is meant by a convex set **[§2.2]**? Show that hyperplane, closed half space and open half space are convex sets.

[Th$^{\mathbf{m}}$-2.1, 2.3, Corollary-2.1]

(b) Show that the following set is convex:

$S = \{(x_1 - x_2):\ x_1^2 + x_2^2 \leq 4\}$ **[Ex. 2.19]**

2. (a) Define feasible solution, basic feasible solution, non-degenerate basic feasible solution and optimal solution to a linear programming problem **[§1.9]**. Prove that the set of all feasible solutions to a linear program is a convex set. **[Th$^{\mathbf{m}}$-2.9]**

(b) Prove that if the set of all feasible solutions of a linear program is non-empty closed and bounded, then an optimal solution to the LP exists and it is attained at a vertex of the set. **[Th$^{\mathbf{m}}$-2.10]**

3. (a) What is a Transportation Problem? Formulate a general transportation problem as a linear programming problem.

(b) A company produces AM and AM-FM radios. A plan of the company can be operated 48 hours per week. Production of an AM radio will require 2 hours and production of AM-FM radio will require 3 hours each. AM radio yields Tk.40 as profit and an AM-FM radio yields Tk.80. The marketing department determined that a maximum of 15 AM and 10 AM-FM radios can be sold in a week. Formulate the problem as LP problem and solve it graphically. **[Ex. 3.12]**

4. Describe the simplex method for solving a linear programming problem **[§4.2, 4.7]**. Solve the following linear programming problem using the simplex method.

Maximize $Z = 6x_1 + 5.5x_2 + 9x_3 + 8x_4$ **[Ex-4.29]**

Subject to
$$x_1 + x_2 + x_3 + x_4 \le 15$$
$$7x_1 + 5x_2 + 3x_3 + 2x_4 \le 120$$
$$3x_1 + 5x_2 + 10x_3 + 15x_4 \le 100$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

5. How do you recognize that an LP problem is unbounded **[§4.5.6]** and the optimal solution is unique **[§4.5.7]**, while using the simplex method? What are artificial variables and why do you need them **[§4.8.1]**? Use artificial variables to solve the linear program: Maximize $Z = 4x_1 + 5x_2 + 2x_3$

Subject to
$$-6x_1 + x_2 - x_3 \le 5$$
$$-2x_1 + 2x_2 - 3x_3 \ge 3$$
$$2x_2 - 4x_3 = 1$$
$$x_1, x_2, x_3 \ge 0.$$

N.U.H – 2003

Subject: Linear Programming

1. (a) Define convex hull of a set with example **[§2.14]**. Let S, T be convex sets in $\mathbf{R}^n$, show that for scalars α, β the set $\alpha S + \beta T$ is also convex. **[Th^{m}-2.6]**

(b) Define closure, interior and boundary of a convex set **[§2.15]**. Explain the terms:

(i) separating hyperplane **[§2.12]**

(ii) supporting hyperplane. **[§2.11]**

2. (a) State linear programming (LP) problem in canonical forms. **[§1.6]** Reduce the given LP problem into canonical form:

Maximize $z = x_1 + x_2$ **[Example 1.27]**

Subject to $x_1 + 2x_2 \leq 5$

$-3x_1 + x_2 \geq 3$

$x_1 \geq 0$; x_2 unrestricted.

(b) Solve the problem graphically:

Maximize $z = 4x_1 + 3x_2$ **[Ex. 3.25]**

Subject to $x_1 + x_2 \leq 50$

$x_1 + 2x_2 \leq 80$

$2x_1 + x_2 \geq 20$

$x_1, x_2 \geq 0$

3. Define slack, surplus **[§1.9]** and artificial variables **[§4.8.1]**. Solve the following LP problem using simplex method:

Maximize $z = 3x_1 + 2x_2 + 5x_3$ **[Ex-4.30]**

Subject to $x_1 + 2x_2 + x_3 \leq 430$

$x_1 + 4x_2 - 3x_3 \leq 420$

$3x_1 + 2x_3 \leq 460$

$x_1, x_2, x_3 \geq 0$

4. (a) Describe the role of a pivot element in simplex tableau.

(b) Use big M-method to solve the given LP problem: **[§4.5.5]**

Maximize $z = x_1 + 5x_2$ **[Ex-4.59]**

Subject to $3x_1 + 4x_2 \leq 6$

$x_1 + 3x_2 \geq 2$

$x_1, x_2 \geq 0$

5. Identify the unboundedness of the solution of the given LP problem using simplex algorithm:

Maximize $z = 107x_1 + x_2 + 3x_3$ **[Ex-4.31]**

Subject to $14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5$$

$3x_1 - x_2 - x_3 \leq 0$

$x_1, x_2, x_3, x_4 \geq 0$

N.U.H – 2004
Subject: Linear Programming

1. (a) Define convex set **[§2.2]**, hyperplane **[§2.9]**, half spaces **[§2.10]**, convex hull **[§2.6]**, convex polyhedron **[§2.7]** with examples.
(b) Prove that a set S $\subseteq$ $\mathbf{R}^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S. **[Th$^{\mathbf{m}}$-2.8]**
2. (a) Formulate the general linear programming problem in n variables **[§1.4]** and explain what do you mean by (i) solution; (ii) feasible solution; (iii) optimal solution; (iv) slack variable; (v) surplus variable. **[§1.9]**
(b) Prove that if the constraint set T of a Linear Program (LP) is non-empty closed and bounded, then an optimal solution to LP exists and it is attained of a vertex of T. **[Th$^{\mathbf{m}}$-2.10]**
3. (a) A company sells two products A and B. The company makes profit Tk.40 and Tk.30 per unit of each product. The two products are produced in a common process. The production process has capacity 30,000 man hours. It takes 3 hours to produce one unit of A and one hour per unit of B. The market has been surveyed and if feels that A can be sold 8,000 units, B of 12,000 units. Subject to above limitations form LP problem which maximizes the profit. **[Ex-3.32]**
(b) Solve graphically the following linear program:

Minimize $z = 3x_1 + 2x_2$ **[Ex. 3.21]**

$$\begin{aligned} \text{Subject to } \; x_1 + 2x_2 &\geq 4 \\ 2x_1 + x_2 &\geq 4 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

4. (a) Describe the simplex method for solving a linear programming problem. **[§4.2, 4.7]**

(b) Solve the following linear programming problem using the simplex method:

Maximize $z = x_1 + 4x_2 + 5x_3$ **[Ex-4.60]**

Subject to $3x_1 + 3x_3 \leq 22$

$x_1 + 2x_2 + 3x_3 \leq 14$

$3x_1 + 2x_2 \leq 14$

$x_1, x_2, x_3 \geq 0$

5. (a) State and prove the main or direct duality theorem in linear programming. **[Th^{m}-5.6]**

(b) Find the dual of the following linear programming problem:

Minimize $z = 3x_1 + 4x_2 + 2x_3$ **[Ex-5.17]**

Subject to $3x_1 + x_2 - 2x_3 = 4$

$x_1 - 2x_2 + 3x_3 \leq 1$

$2x_1 + x_2 - x_3 \geq 2$

$x_1, x_2 \geq 0, x_3 \leq 0$

N.U.H – 2005

Subject: Linear Programming

(N.B. - Answer any **four** questions.)

1. (a) Define convex set with example. Show that, $S = \{(x_1, x_2, x_3): 2x_1 - x_2 + x_3 \leq 4\} \subset \mathbf{R}^3$ is a convex set. **[Ex-2.3]**

(b) Define closure, interior and boundary of a convex set **[§2.15]**. Explain the terms:

(i) Separating hyper plane **[§2.12]**

(ii) Supporting hyper plane. **[§2.11]**

2. Vitamin A and B are found in two different Foods F_1 and F_2. One unit of Food F_1 contains 2 units of vitamin A and 5 units of vitamin B. One unit of Food F_2 contains 4 units of vitamin A and 2 units of vitamin B. One unit of Food F_1 and F_2 costs Tk.5 and Tk.3 respectively. The minimum daily requirements (for a man) of vitamin A and B are 40 and 50 units respectively. Assuming that anything is excess of daily minimum requirement of vitamin A and B is not harmful. Find out the optimum mixture of Food F_1 and F_2

at minimum cost which meets the daily minimum requirement of vitamin A and B. Formulate this as linear programming problem and solve it graphically. **[Ex-3.33]**

3. (a) Show graphically that the following LP problem has no solution. State the reason.

Maximize $z = x_1 + \frac{1}{2}x_2$ **[Ex-3.34]**

Subject to $3x_1 + 2x_2 \leq 12$

$5x_1 \geq 10$

$x_1 + x_2 \geq \frac{8}{3}$

$-x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

(b) Explain what you mean by-

(i) Feasible solution; **[§1.9]**

(ii) Optimal solution; **[§1.9]**

(iii) Slack variable; **[§1.9]**

(iv) Surplus variable; **[§1.9]**

(v) Linear programming problem in canonical form. **[§1.6]**

(c) Reduce the given LP problem into canonical forms:

Maximize $z = 2x_1 - 3x_2 + x_3$ **[Example 1.37]**

Subject to $2x_1 + 3x_2 \leq 5$

$x_1 - x_3 \geq 3$

$x_1 + 2x_2 = 1$

$x_3 \geq 0$, x_1 and x_2 are unrestricted in sign.

4. (a) Solve the following LP problem by simplex method:

Maximize $z = 4x_1 + 5x_2 + 9x_3 + 11x_4$ **[Ex-4.61]**

Subject to $x_1 + x_2 + x_3 + x_4 \leq 15$

$7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120$

$3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100$

$x_1, x_2, x_3, x_4 \geq 0.$

(b) Using artificial variable technique, find the optimal solution of the following program: Minimize $z = 4x_1 + 8x_2 + 3x_3$ **[Ex-4.62]**

$$\text{Subject to } x_1 + x_2 \geq 2$$
$$2x_1 + x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

5. (a) State and prove weak duality theorem **[Th^{m}-5.4]**. Construct the dual of the LP problem:

Minimize $z = x_1 + 2x_2 + 3x_3$ **[Ex-5.18]**

$$\text{Subject to } x_1 - x_2 + 2x_3 \geq 4$$
$$x_1 + x_2 + 2x_3 \leq 8$$
$$x_2 - x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

(b) Use dual simplex method to solve the LP problem stated in part (a).

6. (a) What is sensitivity analysis **[§6.1]**? Solve the following LPP:

Maximize $z = 5x_1 + 12x_2 + 4x_3$ **[Ex-6.6]**

$$\text{Subject to } x_1 + 2x_2 + x_3 \leq 5$$
$$2x_1 - x_2 + 3x_3 = 2$$
$$x_1, x_2, x_3 \geq 0$$

(b) Describe the effect of changing the requirement vector form $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$, on the optimum solution.

N.U.H – 2006

Subject: Linear Programming (Code-3733)

Time-3 hours **Total Marks-50**

(N.B. – All questions are of equal value. Answer any **4** questions.)

1. (a) Define convex hull of a set with example **[§2.14]**. Let S, T be convex sets in $\mathbf{R}^n$. Show that for scalars α, β, the set $\alpha S + \beta T$ is also convex. **[Th^{m}-2.6]**

(b) Define closure, interior and boundary of a convex set **[§2.15]**. Show that $S = \{(x_1, x_2, x_3): 2x_1 + x_2 - x_3 \leq 1, x_1 - 2x_2 + x_3 \leq 4\}$ is a convex set. **[Ex-2.22]**

2. A television manufacturer is concerned about what types of portable television sets should be produced during the next time

period to maximize the profit. Because on past demands, a minimum of 200, 250 and 100 units of type I, II and III respectively are required. In addition the manufacturer has available maximum of 1000 units of time and 2000 units of raw materials during the next time period. Table below gives the essential data:

Types	Raw materials	Time	Minimum requirement (unit)	Profits per unit
I	1.0	2.0	200	10
II	1.5	1.2	250	14
III	4.0	1.0	100	12
Available	2000	1000		

Formulate a LPP model and solve it graphically. **[Ex-3.36]**

3. (a) State Linear Programming (LP) problem in canonical forms **[§1.6]**. Reduce the given LPP into canonical form:

Max. $Z = x_1 + x_2$ **[Ex-1.27]**

Subject to $x_1 + 2x_2 \leq 5$

$-3x_1 + x_2 \geq 3$

$x_1 \geq 0$, x_2 unrestricted

(b) Show graphically that the following LPP has no solution. Give the reason:- Max. $Z = 3x + 2y$ **[Ex-3.35]**

Subject to $-2x + 3y \leq 9$

$3x - 2y \leq -20$

$x, y \geq 0$

4. State and prove complementary slackness theorem in linear programming **[Th^{m}-5.7]**. Write down the dual problem (D) of the primal problem: (P) Maximize: $Z = 3x_1 + 2x_2$ **[Ex-5.57]**

Subject to $-x_1 + 2x_2 \leq 4$

$3x_1 + 2x_2 \leq 14$

$x_1 - x_2 \leq 3$

$x_1, x_2 \geq 0$

Find the optimal solution of (P). Using complementary slackness conditions derive the optimal solution of (D).

5. (a) Describe the role of a Pivot element in simplex tableau **[§4.5.5]**.

(b) Using simplex method solve the given LP problem:

$$\begin{aligned} \text{Minimize:} \quad & Z = x_1 - x_2 + x_3 + x_4 + x_5 - x_6 \\ \text{Subject to} \quad & x_1 + x_4 + 6x_6 = 9 \\ & 3x_1 + x_2 - 4x_3 + 2x_6 = 2 \\ & x_1 + 2x_3 + x_5 + 2x_6 = 6 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

[Ex-4.22]

6. (a) What is sensitivity analysis **[§6.1]**? Solve the following LPP:

$$\begin{aligned} \text{Maximize:} \quad & Z = 3x_1 + 5x_2 \\ \text{Subject to} \quad & x_1 + x_2 \leq 1 \\ & 2x_1 + 3x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

[Ex-6.7]

(b) Describe the effect of changing the requirement vector form $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$, on the optimum solution.

(বাংলা ভার্সন)

(দ্রষ্টব্য:- সকল প্রশ্নের মান সমান। যে কোন ৪ টি প্রশ্নের উত্তর দিতে হবে।)

১। (ক) কোন সেটের উত্তল খোসা (convex hull) উদাহরণসহ সংজ্ঞায়িত কর। মনে করি $S, T \in \mathbf{R}^n$ এর উত্তল সেট, যদি α, β স্কেলার রাশি হয় তবে দেখাও যে $\alpha S + \beta T$ উত্তল সেট হবে।

(খ) উত্তল সেটের সীমায়িত (closure) অন্তস্থ এবং সীমা বিন্দুর সংজ্ঞা দাও। দেখাও যে, $S = \{(x_1, x_2, x_3): 2x_1 + x_2 - x_3 \leq 1, x_1 - 2x_2 + x_3 \leq 4\}$ একটি উত্তল সেট।

২। একটি টেলিভিশন প্রস্তুতকারী প্রতিষ্ঠান সর্বোচ্চ লাভের জন্য সহজে বহনীয় কতকগুলো টেলিভিশন পরবর্তী সময়ের জন্য প্রস্তুত করতে চায়। কেননা পূর্বের চাহিদা মতে যথাক্রমে 200, 250 এবং 100 টি টাইপ-1, টাইপ-2 এবং

টাইপ-3 টেলিভিশন দরকার। পরবর্তী সময়ের জন্য প্রস্তুতকারী প্রতিষ্ঠানের সর্বোচ্চ 1000 একক সময় এবং 2000 একক কাঁচামাল বরাদ্দ আছে। নিম্নের উপাত্ত থেকে একটি লিনিয়ার প্রোগ্রাম গঠন কর এবং লেখচিত্রের সাহায্যে সমাধান কর।

ধরন	কাঁচামাল	সময়	কমপক্ষে প্রয়োজন (একক)	প্রতি এককে লাভ
I	1.0	2.0	200	10
II	1.5	1.2	250	14
III	4.0	1.0	100	12
বরাদ্দ	2000	1000		

৩। (ক) ক্যানোনিক্যাল ফর্মে যোগাশ্রয়ী প্রোগ্রামিং সমস্যার বর্ণনা দাও। প্রদত্ত LP সমস্যাকে ক্যানোনিক্যাল কাঠামোতে প্রকাশ করঃ

গরিষ্ঠকরণ কর: $Z = x_1 + x_2$

শর্ত হচ্ছে $x_1 + 2x_2 \leq 5$

$-3x_1 + x_2 \geq 3$

$x_1 \geq 0$, x_2 বাধাহীন

(খ) লেখচিত্র পদ্ধতিতে দেখাও যে, নিম্নের লিনিয়ার প্রোগ্রামের কোন সমাধান নাই। কারণ ব্যাখ্যা করঃ

গরিষ্ঠকরণ কর: $Z = 3x + 2y$

শর্ত হচ্ছে $-2x + 3y \leq 9$

$3x - 2y \leq -20$

$x, y \geq 0$

৪। যোগাশ্রয়ী প্রোগ্রামিং এর কমতি-পরিপূরক উপপাদ্যের বর্ণনা দাও এবং প্রমান কর। আদি যোগাশ্রয়ী প্রোগ্রামঃ

(P) গরিষ্ঠকরণ করঃ $Z = 3x_1 + 2x_2$

শর্ত হচ্ছে $-x_1 + 2x_2 \leq 4$

$$3x_1 + 2x_2 \le 14$$
$$x_1 - x_2 \le 3$$
$$x_1, x_2 \ge 0$$

এর দ্বৈত যোগাশ্রয়ী প্রোগ্রাম (D) লিখ। আদি প্রোগ্রাম (P) এর চরম অনুকূল সমাধান বের কর।

৫। (ক) সিমপে-ক্স তালিকায় পাইভোট (Pivot) উপাদানের ভূমিকা বর্ণনা কর।

(খ) সিমপে-ক্স পদ্ধতি ব্যবহার করে নিম্নের যোগাশ্রয়ী প্রোগ্রামটির সমাধান করঃ

লঘিষ্ঠকরণ করঃ $Z = x_1 - x_2 + x_3 + x_4 + x_5 - x_6$

শর্ত হচ্ছে $x_1 + x_4 + 6x_6 = 9$

$$3x_1 + x_2 - 4x_3 + 2x_6 = 2$$
$$x_1 + 2x_3 + x_5 + 2x_6 = 6$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0.$$

৬। (ক) সেনসিটিভিটি অ্যানালাইসিস কি? নিম্নের লিনিয়ার প্রোগ্রামিং সমস্যাটির সমাধান করঃ গরিষ্ঠকরণ করঃ $Z = 3x_1 + 5x_2$

শর্ত হচ্ছে $x_1 + x_2 \le 1$

$$2x_1 + 3x_2 \le 1$$
$$x_1, x_2 \ge 0$$

(খ) প্রয়োজনীয় ভেক্টর $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ হইতে $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$, তে পরিবর্তনের ফলে চরম অনুকূল সমাধানের উপর এর প্রভাব বর্ণনা কর।

N.U.H – 2007

Subject: Linear Programming (Code-3733)

Time-3 hours **Total Marks-50**

(N.B. – All questions are of equal value. Answer any **4** questions.)

1. (a) Define convex set **[§2.2]**, closed half-spaces, open half-spaces **[§2.10]**, convex hull **[§2.6]**, convex polyhedron **[§2.7]** and hyper plane **[§2.9]** with examples.

(b) Prove that a set $S \subseteq \mathbf{R}^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S. **[Th^m-2.8]**

2. (a) Formulate the general linear programming problem in n variables **[§1.4]** and explain the terms (i) solution **[§1.9.4]**, (ii) feasible solution **[§1.9.5]**, (iii) optimal solution **[§1.9.12]**, (iv) slack variable **[§1.9.13]**, (v) surplus variable **[§1.9.14]**.

(b) A farmer has 100 acres of land. He produces tomato, lettuce and radish and can sell them all. The price he can obtain is Tk.1 per kg. for tomato, Tk.0.75 a head for lettuce and Tk.2 per kg. for radish. The average yield per acre is 2000 kg. of tomato, 3000 heads of lettuce and 1000 kg. of radish. Fertilizer is available at Tk.5 per kg. and the amount required per acre is 100 kg. each for tomato and lettuce and 50 kg. for radish. Labour required for sowing, cultivation and harvesting per acre is 5 man-days for tomato and radish and 6 man-days for lettuce. The farmer has 400 man-days of labour are available at Tk.20 per man-day. Formulate a linear program for this problem to maximize the farmer's total profit. **[Ex-3.6]**

3.(a) Solve the following linear programming problem graphically.

Minimize $z = 3x_1 + 2x_2$ **[Ex-3.21]**

Subject to
$$x_1 + 2x_2 \geq 4$$
$$2x_1 + x_2 \geq 4$$
$$x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

(b) Reduce the following linear programming problem to its standard form: Maximize $z = x_1 + 4x_2 + 3x_3$

Subject to
$$2x_1 + 3x_2 - 5x_3 \leq 2$$
$$3x_1 - x_2 + 6x_3 \geq 1$$
$$x_1 + x_2 + x_3 = 4 \quad \textbf{[Ex-1.4]}$$
$$x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ unrestricted.}$$

4. (a) Describe the simplex method for solving a linear programming problem (LPP). **[§4.7]**
(b) Solve the following LPP by simplex method.

Maximize $z = 3x_1 + 2x_2 + 5x_3$ **[Ex-4.30]**

Subject to
$$x_1 + 2x_2 + x_3 \leq 430$$
$$x_1 + 4x_2 \leq 420$$
$$3x_1 + 2x_3 \leq 460$$
$$x_1, x_2, x_3 \geq 0$$

5. (a) State and prove the main duality theorem or fundamental theorem of duality in linear programming **[Th^m-5.6]**.
(b) Solve the following L.P.P by using its dual :

Maximize $z = 5x_1 - 2x_2 + 3x_3$

Subject to $2x_1 + 2x_2 - x_3 \geq 2$ **[Ex-5.53]**
$$3x_1 - 4x_2 \leq 3$$
$$x_2 + 3x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

6. (a) What do you mean by sensitivity analysis **[§6.1]**? Solve the following LPP:-

Maximize $z = 5x_1 + 12x_2 + 4x_3$ **[Ex-6.17]**

Subject to $x_1 + 2x_2 + x_3 \leq 5$
$$2x_1 - x_2 + 3x_3 = 2$$
$$x_1, x_2, x_3 \geq 0$$

and describe the effect of changing the requirement vector form $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ on the optimum solution.

N.U.M.Sc (Previous) – 2001

Subject: Linear Programming

(N.B.- Answer any **three** questions.)

1. (a) Define convex set, hyper plane, half spaces, convex hull and polyhedron.

(b) Prove that a set $S \subseteq \mathbf{R}^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S.
(c) Prove that if the constraint set T of a Linear Program (LP) is non-empty closed and bounded, then an optimal solution to LP exists and it is attained of a vertex of T.
2. A firm manufactures two types of electrical item A and B. Item A gives profit Tk.160 per unit; B gives Tk.240 per unit. Both items use two essential components a motor and a transformer. Each unit of A requires 3 motors and 2 transformers while each unit of B requires 2 motors and 4 transformers. Monthly total supply of motors is 210, that of transformers is 300 and item B has maximum market demand 65 unit per month. How many each of A and B item should be manufactured per month to maximize the profit? Formulate a linear program model of this problem and using graphical method find an optimal solution of the problem.
3. (a) Write in standard form of the following linear program:

$$\begin{aligned}
&\text{Minimize } z = 2x_1 - x_2 + x_3\\
&\text{Subject to } x_1 + 3x_2 - x_3 \le 20\\
&\qquad 2x_1 - x_2 + 3x_3 \ge 12\\
&\qquad -x_1 + 4x_2 + 4x_3 \le -2\\
&\qquad x_1 \ge 0,\ x_2 \le 0 \text{ and } x_3 \text{ unrestricted in sign.}
\end{aligned}$$

(b) Find all basic feasible solutions of the following linear system:

$$\begin{aligned}
&x_1 + 2x_2 + x_3 = 4\\
&2x_1 + x_2 + 5x_3 = 5\\
&x_1, x_2, x_3 \ge 0
\end{aligned}$$

4. Using simplex method solve the linear program:

$$\begin{aligned}
&\text{Minimize } z = x_1 - 3x_2 + 2x_3\\
&\text{Subject to } 3x_1 - x_2 + 2x_3 \le 7\\
&\qquad -2x_1 + 4x_2 \le 12\\
&\qquad -4x_1 + 3x_2 + 8x_3 \le 10\\
&\qquad x_1, x_2, x_3 \ge 0
\end{aligned}$$

5. Use two phase simplex method to solve

$$\begin{aligned}
&\text{Minimize } z = -3x_1 + x_2 + x_3\\
&\text{Subject to } x_1 - 2x_2 + x_3 \le 11
\end{aligned}$$

$$-4x_1 + x_2 + 2x_3 \geq 3$$
$$-2x_1 + x_3 = 1,\ x_1, x_2, x_3 \geq 0$$

N.U.M.Sc (Previous) – 2003

Subject: Linear Programming

1. (a) Define convex set and extreme points of a convex set. Prove that the set Q of all feasible solutions is convex.

(b) Show that:

(i) The set $Q = \{(x_1, x_2) \mid x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$ is a convex set.

(ii) The set $Q = \{(x_1, x_2) \mid x_1 \geq 1 \text{ or } x_2 \geq 2\}$ is not a convex set.

2. (a) State and prove the fundamental theorem of linear programming problems.

(b) By using graphical method maximize the objective function.

Maximize $z = 2x_1 + 3x_2$

Subject to constraints $x_1 + x_2 \leq 1$
$$3x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

3. (a) Describe simplex method for the solution of linear programming problem and using this method solve the following LPP. Maximize $z = 4x_1 + 10x_2$

Subject to the constraints $2x_1 + x_2 \leq 50$
$$2x_1 + 5x_2 \leq 100$$
$$2x_1 + 3x_2 \leq 90$$
$$x_1, x_2 \geq 0$$

(b) Explain duality theory of linear programming. Write down the dual of the following LPP:

Maximize $z = 7x_1 + 3x_2 + 8x_3$

Subject to $8x_1 + 2x_2 + x_3 \geq 7$
$$3x_1 + 6x_2 + 4x_3 \geq 4$$
$$4x_1 + x_2 + 5x_3 \geq 1$$
$$x_1 + 5x_2 + 2x_3 \geq 7$$
$$x_1, x_2, x_3 \geq 0$$

4. (a) What are slack and surplus in LPP? Write down the characteristics of canonical form and the standard form of LPP.
(b) State and prove the Minimax theorem. Verify the Minimax theorem for the function f(x) = {9, 7, 5, 3, 1}.
5. A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below:

Machines	Time per unit (minutes)			Machine capacity (minutes/day)
	Product-1	Product-2	Product-3	
M-1	2	3	2	440
M-2	4	-	3	470
M-3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is Rs.4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical model for the problem and solve the model by any method.

N.U.M.Sc (Previous) – 2004

Subject: Linear Programming

1. (a) Define convex set and convex polyhedron. Prove that convex polyhedron is a convex set. Also prove that any point of a convex polyhedron can be expressed as a convex combination of its extreme points.
(b) Define linear programming problem and its feasible solution. Prove that the set of all feasible solutions of a linear programming problem is a convex set.
2. (a) Transform the following linear programme to standard form:

Minimize $z = -3x_1 + 4x_2 - 2x_3 + 5x_4$

Subject to $4x_1 - x_2 + 2x_3 - x_4 = -2$

$$x_1 + x_2 + 3x_3 - x_4 \leq 14$$
$$-2x_1 + 3x_2 - x_3 + 2x_4 \geq 12$$
$$x_1, x_2 \geq 0, x_3 \leq 0, x_4 \text{ unrestricted in sign.}$$

(b) Solve graphically:

Maximize $z = 3x_1 + 2x_2$

Subject to $-2x_1 + 3x_2 \leq 9$

$2x_1 - 2x_3 \leq -20$

$x_1, x_2 \geq 0$

3. (a) Describe simplex method for the solution of linear programming problem and using this method solve the following LPP. Maximize $z = 3x_1 + 4x_2$

Subject to the constraints $x_1 + x_2 \leq 450$

$2x_1 + x_2 \leq 600$

$x_1, x_2 \geq 0$

(b) Explain the duality theory of linear programming problem. Construct the dual of the following LPP:

Minimize $z = 3x_1 - 2x_2 + x_3$

Subject to the constraints $2x_1 - 3x_2 + x_3 \leq 5$

$4x_1 - 2x_2 \geq 9$

$-8x_1 + 4x_2 + 3x_3 = 8$

$x_1, x_2 \geq 0$, x3 unrestricted.

4. (a) Define slack and surplus variables. Solve the following LP problem using simplex method:

Maximize $z = 3x_1 + 4x_2 + x_3 + 5x_4$

Subject to $8x_1 + 3x_2 + 2x_3 + 2x_4 \leq 10$

$2x_1 + 5x_2 + x_3 + 4x_4 \leq 5$

$x_1 + 2x_2 + 5x_3 + x_4 \leq 6$

$x_1, x_2, x_3, x_4 \geq 0.$

(b) What are the characteristics of the standard form of LPP. Express the following LPP in standard form:

Maximize $z = 3x_1 + 2x_2 + 5x_3$

Subject to constraints $2x_1 - 3x_2 \leq 3$

$x_1 + 2x_2 + 3x_3 \geq 5$

$3x_1 + 2x_3 \leq 2$

$$x_1, x_2 \geq 0$$

5. A firm manufactures three products – A, B and C. Time to manufacture product A is twice that for B and thrice that for C and they are to be produced in the ratio 3 : 4 : 5. The relevant data is given in the table below. If the whole labour is engaged in manufacturing product A, 1600 units of this product can be produced. There are demand for at least 300, 250 and 200 units of products A, B and C and the profit earned per unit is Rs.59, Rs.40 and Rs.70 respectively. Formulate the problem as a linear programming problem and solve the problem by any method:

Raw Materials	Requirement per unit of product (kg)			Total availability (kg)
	A	B	C	
P	6	5	9	5000
Q	4	7	8	6000

Model Question

(For the students of Dhaka University.**)**

Subject: Linear Programming (MTH-308)

Time: 3 hours Total Marks: 50

(N.B. – All questions are of equal value. Answer any **4** questions.)

1. (a) Define convex set, hyper plane, half spaces with examples. Prove that the set of all feasible solutions of a linear programming problem is a convex set.

(b) Find all basic solutions of the following system of simultaneous linear equations:

$$4x_1 + 2x_2 + 3x_3 – 8x_4 = 12$$
$$3x_1 + 5x_2 + 4x_3 – 6x_4 = 16$$

2. (a) A farmer has 20 acres of land. He produces tomato and potato and can sell them all. The price he can obtain is Tk.8 per kg. for tomato and Tk.12 per kg. for potato. The average yield per acre is 2000 kg. of tomato and 1500 kg. of potato. Fertilizer is available

at Tk.30 per kg. and the amount required per acre is 100 kg. each for tomato and 50 kg. for potato. Labour required for sowing, cultivation and harvesting per acre is 20 man-days for tomato and 15 man-days for potato. The farmer has 180 man-days of labour are available at Tk.80 per man-day. Formulate a linear program for this problem to maximize the farmer's total profit and then solve it graphically.

(b) Show graphically that the following LP problem has no solution. State the reason.

$$\text{Maximize } z = x_1 + \frac{1}{2}x_2$$

$$\text{Subject to } 3x_1 + 2x_2 \leq 12$$
$$5x_1 \leq 10$$
$$x_1 + x_2 \geq \frac{8}{3}$$
$$-x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

3. (a) State and prove the fundamental theorem of linear programming

(b) Explain dual simplex algorithm and hence solve the following LP problem by dual simplex algorithm.

$$\text{Minimize } z = x_1 + 5x_2$$
$$\text{Subject to } 3x_1 + 4x_2 \leq 6$$
$$x_1 + 3x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

4. (a) Describe simplex method for the solution of linear programming problem and using this method solve the following LPP. $\text{Maximize } z = 4x_1 + 10x_2$

$$\text{Subject to } 2x_1 + x_2 \leq 50$$
$$2x_1 + 5x_2 \leq 100$$
$$2x_1 + 3x_2 \leq 90$$
$$x_1, x_2 \geq 0$$

(b) Using artificial variable technique, find the optimal solution of the following program:

$$\text{Minimize } z = 4x_1 + 8x_2 + 3x_3$$
$$\text{Subject to } x_1 + x_2 \geq 2$$
$$2x_1 + x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

5. (a) State and prove the main duality theorem in linear programming.

(b) Construct the dual of the following LP problem. Solving the primal problem by simplex method, find the solution of the dual also.

$$\text{Minimize } z = x_1 + 2x_2 + 3x_3$$
$$\text{Subject to } x_1 - x_2 + 2x_3 \geq 4$$
$$x_1 + x_2 + 2x_3 \leq 8$$
$$x_2 - x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

6. (a) What is sensitivity analysis? Solve the following LPP:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$
$$\text{Subject to } x_1 + 2x_2 + x_3 \leq 5$$
$$2x_1 - x_2 + 3x_3 = 2$$
$$x_1, x_2, x_3 \geq 0$$

(b) Describe the effect of changing the requirement vector form $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$, on the optimum solution.

Thank you for going through the book.