Chapter 13

Partial Differentiation

13.1 Functions of Several Variables

In single-variable calculus we were concerned with functions that map the real numbers $\mathbb{R}$ to $\mathbb{R}$, sometimes called "real functions of one variable", meaning the "input" is a single real number and the "output" is likewise a single real number. Now we turn to functions of several variables, where several input variables are mapped to one value: functions $f : \mathbb{R}^n \to \mathbb{R}$. We will deal primarily with $n = 2$ and to a lesser extent $n = 3$; in fact many of the techniques we discuss can be applied to larger values of n as well.

A function $f : \mathbb{R}^2 \to \mathbb{R}$ maps a pair of values (x, y) to a single real number. The three-dimensional coordinate system we have already used is a convenient way to visualize such functions: above each point (x, y) in the xy-plane we graph the point (x, y, z), where of course $z = f(x, y)$.

Example 13.1: Plane

Describe the function $f(x, y) = 3x + 4y - 5$.

Solution. Writing this as $z = 3x + 4y - 5$ and then $3x + 4y - z = 5$ we recognize the equation of a plane. In the form $f(x, y) = 3x + 4y - 5$ the emphasis has shifted: we now think of x and y as independent variables and z as a variable dependent on them, but the geometry is unchanged. ♣

Example 13.2: Sphere

Describe the equation $x^2 + y^2 + z^2 = 4$.

Solution. We have seen that $x^2 + y^2 + z^2 = 4$ represents a sphere of radius 2. We cannot write this in the form $f(x, y)$, since for each x and y in the disk $x^2 + y^2 < 4$ there are two corresponding points on the sphere. As with the equation of a circle, we can resolve this equation into two functions, $f_1(x, y) = \sqrt{4 - x^2 - y^2}$ and $f_2(x, y) = -\sqrt{4 - x^2 - y^2}$, representing the upper and lower hemispheres, respectively. Each of these is an example of a function with a restricted domain: only certain values of x and y make sense (namely, those for which $x^2 + y^2 \leq 4$) and the graphs of these functions are limited to a small region of the plane. ♣

Example 13.3: Square Root

Describe the function $f(x, y) = \sqrt{x} + \sqrt{y}$.

Solution. This function is defined only when both x and y are non-negative. When $y = 0$ we get $f(x, y) = \sqrt{x}$, the familiar square root function in the xz-plane, and when $x = 0$ we get the same curve in the yz-plane. Generally speaking, we see that starting from $f(0, 0) = 0$ this function gets larger in every direction in roughly the same way that the square root function gets larger. For example, if we restrict attention to the line $x = y$, we get $f(x,y) = 2\sqrt{x}$ and along the line $y = 2x$ we have $f(x,y) = \sqrt{x} + \sqrt{2x} = (1+\sqrt{2})\sqrt{x}$. ♣

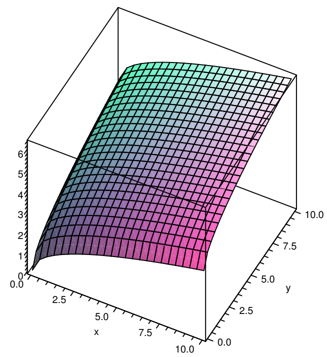


Figure 13.1: $f(x,y) = \sqrt{x} + \sqrt{y}$.

A computer program that plots such surfaces can be very useful, as it is often difficult to get a good idea of what they look like. Still, it is valuable to be able to visualize relatively simple surfaces without such aids. As in the previous example, it is often a good idea to examine the function on restricted subsets of the plane, especially lines. It can also be useful to identify those points (x,y) that share a common z-value.

Example 13.4: Elliptic Paraboloid

Describe the graph of $f(x,y) = x^2 + y^2$.

Solution. When $x = 0$ this becomes $f = y^2$, a parabola in the yz-plane; when $y = 0$ we get the "same" parabola $f = x^2$ in the xz-plane.

Finally, picking a value $z = k$, at what points does $f(x,y) = k$? This means $x^2 + y^2 = k$, which we recognize as the equation of a circle of radius $\sqrt{k}$. So the graph of $f(x,y)$ has parabolic cross-sections, and the same height everywhere on concentric circles with center at the origin. This fits with what we have already discovered. ♣

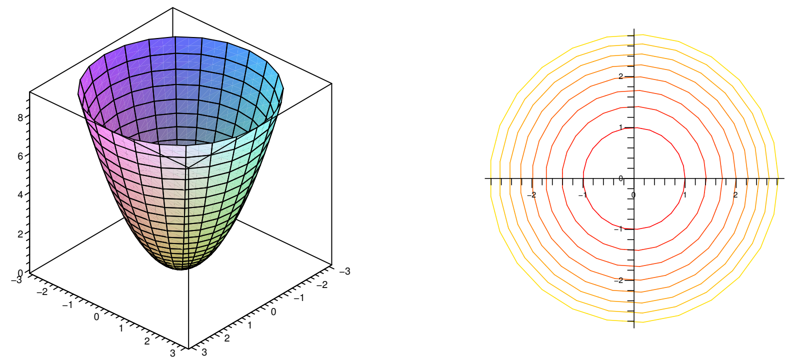


Figure 13.2: $f(x,y) = x^2 + y^2$.

As in this example, the points (x,y) such that $f(x,y) = k$ usually form a curve, called a **level curve** of the function. A graph of some level curves can give a good idea of the shape of the surface; it looks much like a topographic map of the surface. In Figure 13.2 both the surface and its associated level curves are shown. Note that, as with a topographic map, the heights corresponding to the level curves are evenly spaced, so that where curves are closer together the surface is steeper.

Functions $f : \mathbb{R}^n \to \mathbb{R}$ behave much like functions of two variables; we will on occasion discuss functions of three variables. The principal difficulty with such functions is visualizing them, as they do not "fit" in the three dimensions we are familiar with. For three variables there are various ways to interpret functions that make them easier to understand. For example, $f(x,y,z)$ could represent the temperature at the point (x,y,z), or the pressure, or the strength of a magnetic field. It remains useful to consider those points at which $f(x,y,z) = k$, where k is some constant value. If $f(x,y,z)$ is temperature, the set of points (x,y,z) such that $f(x,y,z) = k$ is the collection of points in space with temperature k; in general this is called a **level set**; for three variables, a level set is typically a surface, called a **level surface**.

Example 13.5: Level Surfaces

Suppose the temperature at (x,y,z) is $T(x,y,z) = e^{-(x^2+y^2+z^2)}$. This function has a maximum value of 1 at the origin, and tends to 0 in all directions. If k is positive and at most 1, the set of points for which $T(x,y,z) = k$ is those points satisfying $x^2 + y^2 + z^2 = -\ln k$, a sphere centered at the origin. The level surfaces are the concentric spheres centered at the origin.

Exercises for 13.1

Exercise 13.1.1 *Let $f(x,y) = (x-y)^2$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.*

Exercise 13.1.2 *Let $f(x,y) = |x| + |y|$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.*

Exercise 13.1.3 *Let $f(x,y) = e^{-(x^2+y^2)}\sin(x^2+y^2)$. Determine the equations and shapes of the cross-sections when $x=0$, $y=0$, $x=y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.*

Exercise 13.1.4 *Let $f(x,y) = \sin(x-y)$. Determine the equations and shapes of the cross-sections when $x=0$, $y=0$, $x=y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.*

Exercise 13.1.5 *Let $f(x,y) = (x^2-y^2)^2$. Determine the equations and shapes of the cross-sections when $x=0$, $y=0$, $x=y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.*

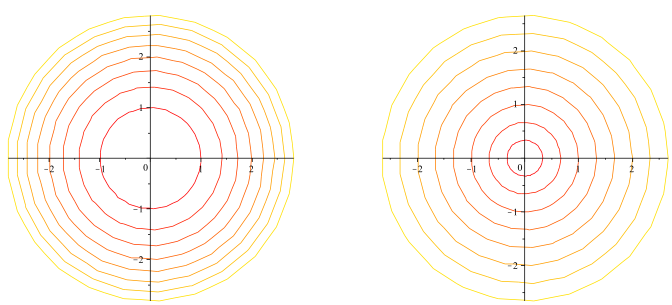
Exercise 13.1.6 *Find the domain of each of the following functions of two variables:*

(a) $\sqrt{9-x^2}+\sqrt{y^2-4}$

(b) $\arcsin(x^2+y^2-2)$

(c) $\sqrt{16-x^2-4y^2}$

Exercise 13.1.7 *Below are two sets of level curves. One is for a cone, one is for a paraboloid. Which is which? Explain.*



13.2 Limits and Continuity

To develop calculus for functions of one variable, we needed to make sense of the concept of a limit, which was used in the definition of a continuous function and the derivative of a function. Limits involving functions of two variables can be considerably more difficult to deal with; fortunately, most of the functions we encounter are fairly easy to understand.

The potential difficulty is largely due to the fact that there are many ways to "approach" a point in the xy-plane. If we want to say that $\lim_{(x,y)\to(a,b)} f(x,y) = L$, we need to capture the idea that as (x,y) gets close to (a,b) then $f(x,y)$ gets close to L. For functions of one variable, $f(x)$, there are only two ways that x can approach a: from the left or right. But there are an infinite number of ways to approach (a,b): along any one of an infinite number of straight lines, or even along a curved path in the xy-plane. We might hope that it's really not so bad—suppose, for example, that along every possible line through (a,b) the value of $f(x,y)$ gets close to L; surely this means that "$f(x,y)$ approaches L as (x,y) approaches (a,b)". Sadly, no.

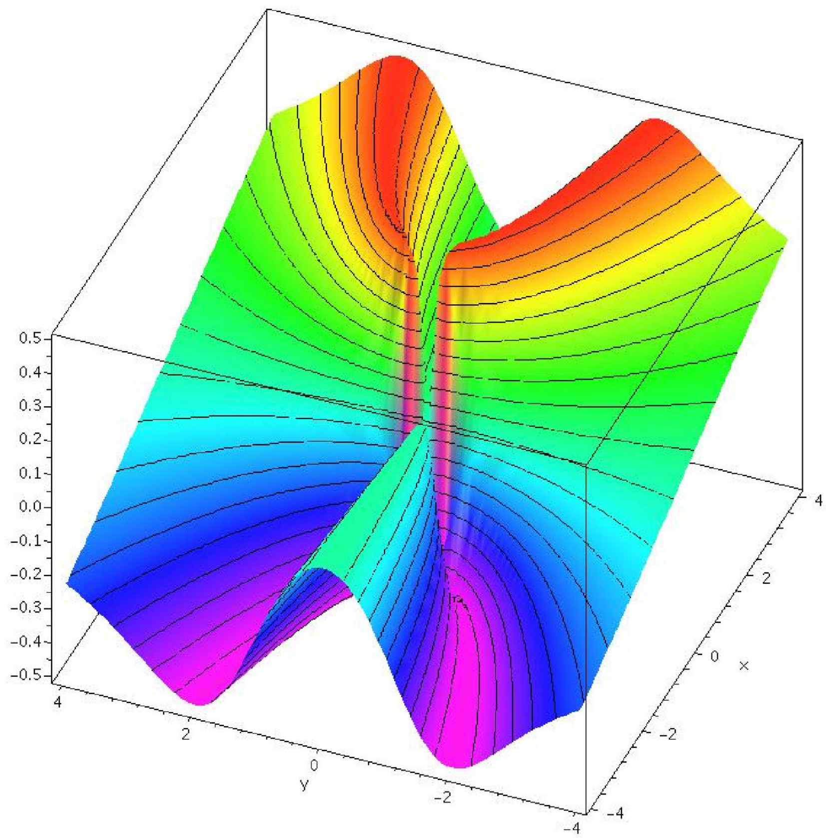


Figure 13.3: $f(x,y) = \dfrac{xy^2}{x^2+y^4}$.

Example 13.6: Weird Limit

Analyze $f(x,y) = xy^2/(x^2+y^4)$.

Solution. When $x = 0$ or $y = 0$, $f(x,y)$ is 0, so the limit of $f(x,y)$ approaching the origin along either the x- or y-axis is 0. Moreover, along the line $y = mx$, $f(x,y) = m^2x^3/(x^2+m^4x^4)$. As x approaches 0 this expression approaches 0 as well. So along every line through the origin $f(x,y)$ approaches 0. Now suppose we approach the origin along $x = y^2$. Then

$$f(x,y) = \frac{y^2y^2}{y^4+y^4} = \frac{y^4}{2y^4} = \frac{1}{2},$$

so the limit is $1/2$. Looking at Figure 13.3, it is apparent that there is a ridge above $x = y^2$. Approaching the origin along a straight line, we go over the ridge and then drop down toward 0, but approaching along the ridge the height is a constant $1/2$. Therefore the limit does not exist. ♣

Fortunately, we can define the concept of limit without needing to specify how a particular point is approached—indeed, in Definition 3.3, we didn't need the concept of "approach." Roughly, that definition says that when x is close to a then $f(x)$ is close to L; there is no mention of "how" we get close to a. We can adapt that definition to two variables quite easily:

Definition 13.7: Limit of a Multivariate Function

Suppose $f(x,y)$ is a function. We say that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, $|f(x,y)-L| < \varepsilon$.

This says that we can make $|f(x,y)-L| < \varepsilon$, no matter how small ε is, by making the distance from (x,y) to (a,b) "small enough".

Example 13.8: Multivariate Limit

Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$

Solution. Suppose $\varepsilon > 0$. Then

$$\left|\frac{3x^2y}{x^2+y^2}\right| = \frac{x^2}{x^2+y^2}3|y|.$$

Note that $x^2/(x^2+y^2) \le 1$ and $|y| = \sqrt{y^2} \le \sqrt{x^2+y^2} < \delta$. So

$$\frac{x^2}{x^2+y^2}3|y| < 1\cdot 3\cdot\delta.$$

We want to force this to be less than ε by picking δ "small enough." If we choose $\delta = \varepsilon/3$ then

$$\left|\frac{3x^2y}{x^2+y^2}\right| < 1\cdot 3\cdot\frac{\varepsilon}{3} = \varepsilon.$$

♣

Recall that a function $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$. We can say exactly the same thing about a function of two variables: $f(x,y)$ is continuous at (a,b) if $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

The function $f(x,y) = 3x^2y/(x^2+y^2)$ is not continuous at $(0,0)$, because $f(0,0)$ is not defined. However, we know that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$, so we can make a continuous function, by extending the definition of f so that $f(0,0) = 0$. This surface is shown in Figure 13.4.

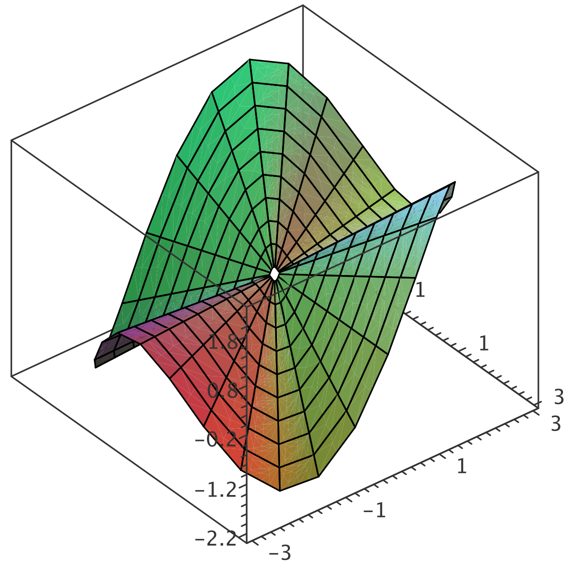


Figure 13.4: $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$.

Note that we cannot extend the definition of the function in Example 13.6 to create a continuous function, since the limit does not exist as we approach $(0,0)$.

Fortunately, the functions we will be working with will usually be continuous almost everywhere. As with single variable functions, two classes of common functions are particularly useful and easy to describe. A polynomial in two variables is a sum of terms of the form ax^my^n, where a is a real number and m and n are non-negative integers. A rational function is a quotient of polynomials.

Theorem 13.9: Continuity of Functions

Polynomials are continuous everywhere. Rational functions are continuous everywhere they are defined.

Exercises for 13.2

Determine whether each limit exists. If it does, find the limit and prove that it is the limit; if it does not, explain how you know.

Exercise 13.2.1 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

Exercise 13.2.2 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Exercise 13.2.3 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+y^2}$

Exercise 13.2.4 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^2+y^2}$

Exercise 13.2.5 $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

Exercise 13.2.6 $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{2x^2+y^2}}$

Exercise 13.2.7 $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{-x^2-y^2}-1}{x^2+y^2}$

Exercise 13.2.8 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2}$

Exercise 13.2.9 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+\sin^2 y}{2x^2+y^2}$

Exercise 13.2.10 $\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)^2\ln x}{(x-1)^2+y^2}$

Exercise 13.2.11 $\lim\limits_{(x,y)\to(1,-1)} 3x+4y$

Exercise 13.2.12 $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x^2y}{x^2+y^2}$

Exercise 13.2.13 *Does the function* $f(x,y)=\dfrac{x-y}{1+x+y}$ *have any discontinuities? What about* $f(x,y)=\dfrac{x-y}{1+x^2+y^2}$*? Explain.*

13.3 Partial Differentiation

The derivative of a function of a single variable tells us how quickly the value of the function changes as the value of the independent variable changes. Intuitively, it tells us how "steep" the graph of the function is. We might wonder if there is a similar idea for graphs of functions of two variables, that is, surfaces. It is not clear that this has a simple answer, nor how we might proceed. We will start with what seem to be very small steps toward the goal. Surprisingly, it turns out that these simple ideas hold the keys to a more general understanding.

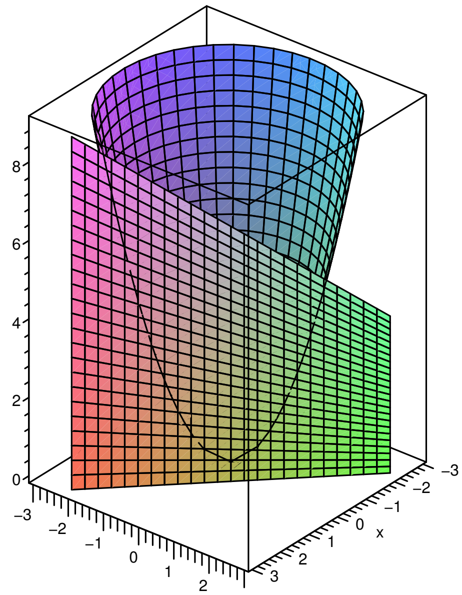


Figure 13.5: The intersection of a plane $x+y=1$ and the surface $f(x,y)=x^2+y^2$.

The derivative of a single-variable function $f(x)$ tells us how much $f(x)$ changes as x increases. The obvious analogue for a function of two variables $g(x,y)$ would be something that tells us how quickly $g(x,y)$ increases as x and y increase. However, in most cases this will depend on how quickly x and y are changing relative to each other.

Example 13.10

Analyze $f(x,y)=y^2$.

Solution. If we look at a point (x,y,y^2) on this surface, the value of a function does not change at all if we fix y and let x increase, but increases like y^2 if we fix x and let y increase. ♣

Now let us consider what happens to $f(x,y)$ when both x and y are increasing, perhaps at different rates. We can think of this as being a movement in a certain direction of a point in the xy-plane. A point and a direction defines a line in the xy-plane, and so we are asking how the function changes as we move along this line.

Let us then imagine a plane perpendicular to the xy-plane that intersects the xy-plane along this line. This plane will intersect the surface of f in a curve, so we can just look at the behaviour of this curve in the given plane.

Figure 13.5 shows the plane $x+y=1$, which is the plane above the line $x+y=1$ in the xy-plane. Observe that its intersection with the surface of f is a curve, in fact, a parabola. We will refer to such a curve as the cross-section of the surface above the line in the xy-plane.

We can now look at the rate of change (or slope) of f in a particular direction by looking at the slope of a curve in a plane — something we already have experience with.

Let's start by looking at some particularly easy lines: Those parallel to the x- or y-axis. Suppose we are interested in the cross-section of $f(x,y)$ above the line $y=b$. If we substitute b for y in $f(x,y)$, we get a function in one variable, describing the height of the cross-section as a function of x. Because $y=b$ is parallel to the x-axis, if we view it from a vantage point on the negative y-axis, we will see what appears to be simply an ordinary curve in the xz-plane.

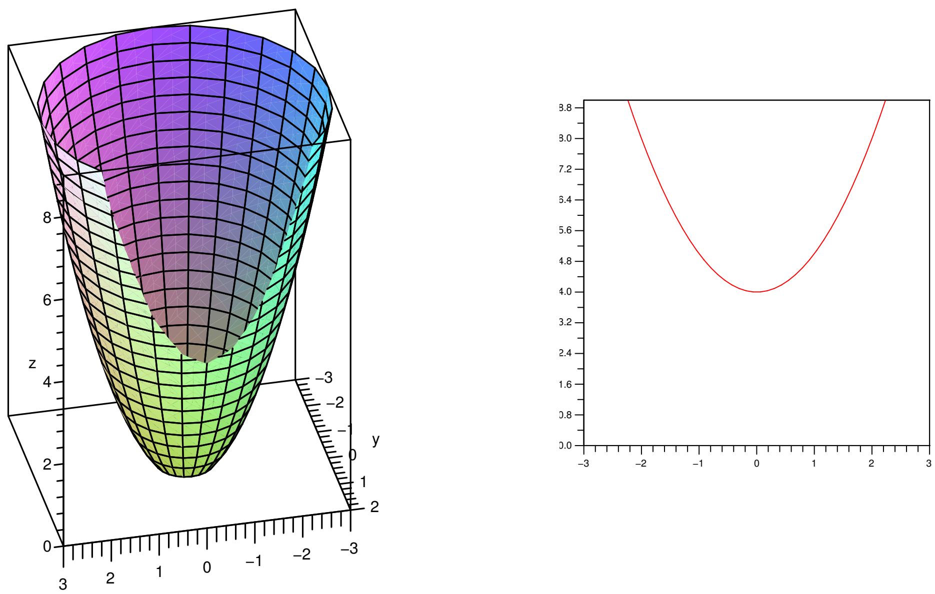


Figure 13.6: $f(x,y) = x^2 + y^2$**, cut by the plane** $y = 2$**.**

Consider again the parabolic surface $f(x,y) = x^2 + y^2$. The cross-section above the line $y = 2$ consists of all points $(x, 2, x^2 + 4)$. Looking at this cross-section we see what appears to be just the curve $f(x) = x^2 + 4$. At any point on the cross-section, $(a, 2, a^2 + 4)$, the slope of the surface *in the direction of the line* $y = 2$ is simply the slope of the curve $f(x) = x^2 + 4$, namely $2x$. Figure 13.6 shows the same parabolic surface as before, but now cut by the plane $y = 2$. The left graph shows the cut-off surface, the right shows just the cross-section.

If, for example, we're interested in the point $(-1, 2, 5)$ on the surface, then the slope in the direction of the line $y = 2$ is $2x = 2(-1) = -2$. This means that starting at $(-1, 2, 5)$ and moving on the surface, above the line $y = 2$, in the direction of increasing x values, the surface goes down; of course moving in the opposite direction, toward decreasing x values, the surface will rise.

If we're interested in some other line $y = k$, there is really no change in the computation. The equation of the cross-section above $y = k$ is $x^2 + k^2$ with derivative $2x$. We can save ourselves the effort, small as it is, of substituting k for y: all we are in effect doing is temporarily assuming that y is some constant. With this assumption, the derivative $\frac{d}{dx}(x^2 + y^2) = 2x$. To emphasize that we are only temporarily assuming y is constant, we use a slightly different notation: $\frac{\partial}{\partial x}(x^2 + y^2) = 2x$; the "$\partial$" reminds us that there are more variables than x, but that only x is being treated as a variable. We read the equation as "the partial derivative of $(x^2 + y^2)$ with respect to x is $2x$." A convenient alternate notation for the partial derivative of $f(x,y)$ with respect to x is $f_x(x,y)$.

Example 13.11: Partial Derivative with respect to x

Find the partial derivative with respect to x of $x^3 + 3xy$.

Solution. The partial derivative with respect to x of $x^3 + 3xy$ is $3x^2 + 3y$. Note that the partial derivative includes the variable y, unlike the example x^2+y^2. It is somewhat unusual for the partial derivative to depend on a single variable; this example is more typical. ♣

Of course, we can do the same sort of calculation for lines parallel to the y-axis. We temporarily hold x constant, which gives us the equation of the cross-section above a line $x=k$. We can then compute the derivative with respect to y; this will measure the slope of the curve in the y direction.

Example 13.12: Partial Derivative with respect to y

Find the partial derivative with respect to y of $f(x,y)=\sin(xy)+3xy$.

Solution. The partial derivative with respect to y of $f(x,y)=\sin(xy)+3xy$ is

$$f_y(x,y)=\frac{\partial}{\partial y}(\sin(xy)+3xy)=\cos(xy)\frac{\partial}{\partial y}(xy)+3x=x\cos(xy)+3x.$$

♣

So far, using no new techniques, we have succeeded in measuring the slope of a surface in two quite special directions. For functions of one variable, the derivative is closely linked to the notion of tangent line. For surfaces, the analogous idea is the tangent plane—a plane that just touches a surface at a point, and has the same slope as the surface in all directions. Even though we haven't yet figured out how to compute the slope in all directions, we have enough information to find tangent planes. Suppose we want the plane tangent to a surface at a particular point (a,b,c). If we compute the two partial derivatives (with respect to x and y) of the function for that point, we get enough information to determine two lines tangent to the surface, both through (a,b,c) and both tangent to the surface in their respective directions. These two lines determine a plane, that is, there is exactly one plane containing the two lines: the tangent plane. Figure 13.7 shows (part of) two tangent lines at a point, and the tangent plane containing them.

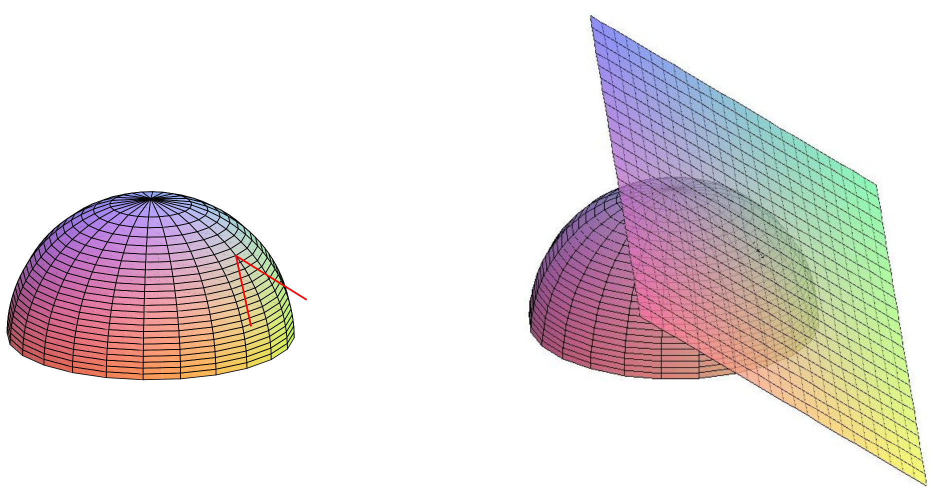


Figure 13.7: Tangent vectors and tangent plane.

How can we discover an equation for this tangent plane? We know a point on the plane, (a,b,c); we need a vector normal to the plane. If we can find two vectors, one parallel to each of the tangent lines we know how to find, then the cross product of these vectors will give the desired normal vector.

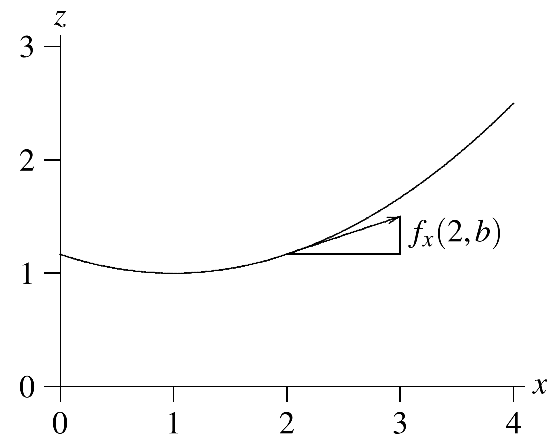


Figure 13.8: A tangent vector.

How can we find vectors parallel to the tangent lines? Consider first the line tangent to the surface above the line $y = b$. A vector $\langle u, v, w\rangle$ parallel to this tangent line must have y component $v = 0$, and we may as well take the x component to be $u = 1$. The ratio of the z component to the x component is the slope of the tangent line, precisely what we know how to compute. The slope of the tangent line is $f_x(a,b)$, so

$$f_x(a,b) = \frac{w}{u} = \frac{w}{1} = w.$$

In other words, a vector parallel to this tangent line is $\langle 1, 0, f_x(a,b)\rangle$, as shown in Figure 13.8. If we repeat the reasoning for the tangent line above $x = a$, we get the vector $\langle 0, 1, f_y(a,b)\rangle$.

Now to find the desired normal vector we compute the cross product, $\langle 0, 1, f_y\rangle \times \langle 1, 0, f_x\rangle = \langle f_x, f_y, -1\rangle$. From our earlier discussion of planes, we can write down the equation we seek: $f_x(a,b)x + f_y(a,b)y - z = k$, and k as usual can be computed by substituting a known point: $f_x(a,b)(a) + f_y(a,b)(b) - c = k$. There are various more-or-less nice ways to write the result:

$$\begin{aligned}
&f_x(a,b)x + f_y(a,b)y - z = f_x(a,b)a + f_y(a,b)b - c\\
&f_x(a,b)x + f_y(a,b)y - f_x(a,b)a - f_y(a,b)b + c = z\\
&f_x(a,b)(x-a) + f_y(a,b)(y-b) + c = z\\
&f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b) = z
\end{aligned}$$

Example 13.13: Tangent Plane to a Sphere

Find the plane tangent to $x^2 + y^2 + z^2 = 4$ at $(1, 1, \sqrt{2})$.

Solution. The point $(1, 1, \sqrt{2})$ is on the upper hemisphere, so we use $f(x,y) = \sqrt{4 - x^2 - y^2}$. Then $f_x(x,y) = -x(4 - x^2 - y^2)^{-1/2}$ and $f_y(x,y) = -y(4 - x^2 - y^2)^{-1/2}$, so $f_x(1,1) = f_y(1,1) = -1/\sqrt{2}$ and the equation of the plane is

$$z = -\frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1) + \sqrt{2}.$$

The hemisphere and this tangent plane are pictured in Figure 13.7. ♣

So it appears that to find a tangent plane, we need only find two quite simple ordinary derivatives, namely f_x and f_y. This is true *if the tangent plane exists*. It is, unfortunately, not always the case that if f_x and f_y exist there is a tangent plane. Consider the function $xy^2/(x^2+y^4)$ with $f(0,0)$ defined to be 0, pictured in Figure 13.3. This function has value 0 when $x=0$ or $y=0$. Now it's clear that $f_x(0,0) = f_y(0,0) = 0$, because in the x and y directions the surface is simply a horizontal line. But it's also clear from the picture that this surface does not have anything that deserves to be called a tangent plane at the origin, certainly not the xy-plane containing these two tangent lines.

When does a surface have a tangent plane at a particular point? What we really want from a tangent plane, as from a tangent line, is that the plane be a "good" approximation of the surface near the point. Here is how we can make this precise:

Definition 13.14: Tangent Plane

Let $\Delta x = x - x_0$, $\Delta y = y - y_0$, and $\Delta z = z - z_0$ where $z_0 = f(x_0,y_0)$. The function $z = f(x,y)$ is differentiable at (x_0,y_0) if

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where both ε_1 and ε_2 approach 0 as (x,y) approaches (x_0,y_0).

This definition takes a bit of absorbing. Let's rewrite the central equation a bit:

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0) + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y. \tag{13.1}$$

The first three terms on the right are the equation of the tangent plane, that is,

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$$

is the z-value of the point on the plane above (x,y). Equation 13.1 says that the z-value of a point on the surface is equal to the z-value of a point on the plane plus a "little bit," namely $\varepsilon_1\Delta x + \varepsilon_2 \Delta y$. As (x,y) approaches (x_0,y_0), both Δx and Δy approach 0, so this little bit $\varepsilon_1\Delta x + \varepsilon_2 \Delta y$ also approaches 0, and the z-values on the surface and the plane get close to each other. But that by itself is not very interesting: since the surface and the plane both contain the point (x_0,y_0,z_0), the z values will approach z_0 and hence get close to each other whether the tangent plane is "tangent" to the surface or not. The extra condition in the definition says that as (x,y) approaches (x_0,y_0), the ε values approach 0—this means that $\varepsilon_1\Delta x + \varepsilon_2 \Delta y$ approaches 0 much, much faster, because $\varepsilon_1 \Delta x$ is much smaller than either ε_1 or Δx. It is this extra condition that makes the plane a tangent plane.

We can see that the extra condition on ε_1 and ε_2 is just what is needed if we look at partial derivatives. Suppose we temporarily fix $y = y_0$, so $\Delta y = 0$. Then the equation from the definition becomes

$$\Delta z = f_x(x_0,y_0)\Delta x + \varepsilon_1 \Delta x$$

or

$$\frac{\Delta z}{\Delta x} = f_x(x_0,y_0) + \varepsilon_1.$$

Now taking the limit of the two sides as Δx approaches 0, the left side turns into the partial derivative of z with respect to x at (x_0,y_0), or in other words $f_x(x_0,y_0)$, and the right side does the same, because as (x,y) approaches (x_0,y_0), ε_1 approaches 0. Essentially the same calculation works for f_y.

Exercises for 13.3

Exercise 13.3.1 *Find f_x and f_y where $f(x,y) = \cos(x^2y) + y^3$.*

Exercise 13.3.2 *Find f_x and f_y where $f(x,y) = \dfrac{xy}{x^2+y}$.*

Exercise 13.3.3 *Find f_x and f_y where $f(x,y) = e^{x^2+y^2}$.*

Exercise 13.3.4 *Find f_x and f_y where $f(x,y) = xy\ln(xy)$.*

Exercise 13.3.5 *Find f_x and f_y where $f(x,y) = \sqrt{1-x^2-y^2}$.*

Exercise 13.3.6 *Find f_x and f_y where $f(x,y) = x\tan(y)$.*

Exercise 13.3.7 *Find f_x and f_y where $f(x,y) = \dfrac{1}{xy}$.*

Exercise 13.3.8 *Find an equation for the plane tangent to $2x^2 + 3y^2 - z^2 = 4$ at $(1,1,-1)$.*

Exercise 13.3.9 *Find an equation for the plane tangent to $f(x,y) = \sin(xy)$ at $(\pi, 1/2, 1)$.*

Exercise 13.3.10 *Find an equation for the plane tangent to $f(x,y) = x^2 + y^3$ at $(3,1,10)$.*

Exercise 13.3.11 *Find an equation for the plane tangent to $f(x,y) = x\ln(xy)$ at $(2,1/2,0)$.*

Exercise 13.3.12 *Find an equation for the line normal to $x^2 + 4y^2 = 2z$ at $(2,1,4)$.*

Exercise 13.3.13 *Explain in your own words why, when taking a partial derivative of a function of multiple variables, we can treat the variables not being differentiated as constants.*

Exercise 13.3.14 *Consider a differentiable function, $f(x,y)$. Give physical interpretations of the meanings of $f_x(a,b)$ and $f_y(a,b)$ as they relate to the graph of f.*

Exercise 13.3.15 *In much the same way that we used the tangent line to approximate the value of a function from single variable calculus, we can use the tangent plane to approximate a function from multivariable calculus. Consider the tangent plane found in Exercise 13.3.11. Use this plane to approximate $f(1.98, 0.4)$.*

Exercise 13.3.16 *The volume of a cylinder is given by $V = \pi r^2 h$. Suppose that the current values of r and h are $r = 7$ cm and $h = 3$ cm. Is the volume more sensitive to a small change in radius or the same amount of change in height? Why?*

Exercise 13.3.17 *Suppose that one of your colleagues has calculated the partial derivatives of a given function, and reported to you that $f_x(x,y) = 2x + 3y$ and that $f_y(x,y) = 4x + 6y$. Do you believe them? Why or why not? If not, what answer might you have accepted for f_y?*

Exercise 13.3.18 *Suppose $f(t)$ and $g(t)$ are single variable differentiable functions. Find $\partial z/\partial x$ and $\partial z/\partial y$ for each of the following two variable functions.*

(a) $z = f(x)g(y)$

(b) $z = f(xy)$

(c) $z = f(x/y)$

13.4 The Chain Rule

Consider the surface $z = x^2y + xy^2$, and suppose that $x = 2+t^4$ and $y = 1-t^3$. We can think of the latter two equations as describing how x and y change relative to, say, time. Then

$$z = x^2y + xy^2 = (2+t^4)^2(1-t^3) + (2+t^4)(1-t^3)^2$$

tells us explicitly how the z-coordinate of the corresponding point on the surface depends on t. If we want to know dz/dt we can compute it more or less directly, but it's actually a bit simpler to use product and chain rules:

$$\begin{aligned}\frac{dz}{dt} &= x^2y' + 2xx'y + x2yy' + x'y^2\\ &= (2xy+y^2)x' + (x^2+2xy)y'\\ &= (2(2+t^4)(1-t^3) + (1-t^3)^2)(4t^3) + ((2+t^4)^2 + 2(2+t^4)(1-t^3))(-3t^2)\end{aligned}$$

If we look carefully at the middle step, $dz/dt = (2xy+y^2)x' + (x^2+2xy)y'$, we notice that $2xy+y^2$ is $\partial z/\partial x$, and x^2+2xy is $\partial z/\partial y$. This turns out to be true in general, and gives us a new chain rule:

Theorem 13.15: Multivariate Chain Rule

Suppose that $z = f(x,y)$, f is differentiable, $x = g(t)$, and $y = h(t)$. Assuming that the relevant derivatives exist,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Proof. If f is differentiable, then

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y,$$

where ε_1 and ε_2 approach 0 as (x,y) approaches (x_0,y_0). Then

$$\frac{\Delta z}{\Delta t} = f_x\frac{\Delta x}{\Delta t} + f_y\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}. \tag{13.2}$$

As Δt approaches 0, (x,y) approaches (x_0,y_0) and so

$$\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t} = \frac{dz}{dt}$$

$$\lim_{\Delta t \to 0} \varepsilon_1 \frac{\Delta x}{\Delta t} = 0 \cdot \frac{dx}{dt}$$
$$\lim_{\Delta t \to 0} \varepsilon_2 \frac{\Delta y}{\Delta t} = 0 \cdot \frac{dy}{dt}$$

and so taking the limit of (13.2) as Δt goes to 0 gives

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt},$$

as desired. ♣

We can write the chain rule in a way that is somewhat closer to the single variable chain rule:

$$\frac{df}{dt} = \langle f_x, f_y \rangle \cdot \langle x', y' \rangle,$$

or (roughly) the derivatives of the outside function "times" the derivatives of the inside functions. Not surprisingly, essentially the same chain rule works for functions of more than two variables, for example, given a function of three variables $f(x,y,z)$, where each of x, y and z is a function of t,

$$\frac{df}{dt} = \langle f_x, f_y, f_z \rangle \cdot \langle x', y', z' \rangle.$$

We can even extend the idea further. Suppose that $f(x,y)$ is a function and $x = g(s,t)$ and $y = h(s,t)$ are functions of two variables s and t. Then f is "really" a function of s and t as well, and

$$\frac{\partial f}{\partial s} = f_x g_s + f_y h_s \qquad \frac{\partial f}{\partial t} = f_x g_t + f_y h_t.$$

The natural extension of this to $f(x,y,z)$ works as well.

Recall that we used the ordinary chain rule to do implicit differentiation. We can do the same with the new chain rule.

Example 13.16: Equation of a Sphere

Find the partial derivative of $x^2 + y^2 + z^2 = 4$.

Solution. The equation $x^2 + y^2 + z^2 = 4$ defines a sphere, which is not a function of x and y, though it can be thought of as two functions, the top and bottom hemispheres. We can think of z as one of these two functions, so really $z = z(x,y)$, and we can think of x and y as particularly simple functions of x and y, and let $f(x,y,z) = x^2 + y^2 + z^2$. Since $f(x,y,z) = 4$, $\partial f/\partial x = 0$, but using the chain rule:

$$\begin{aligned} 0 = \frac{\partial f}{\partial x} &= f_x \frac{\partial x}{\partial x} + f_y \frac{\partial y}{\partial x} + f_z \frac{\partial z}{\partial x} \\ &= (2x)(1) + (2y)(0) + (2z)\frac{\partial z}{\partial x}, \end{aligned}$$

noting that since y is temporarily held constant its derivative $\partial y/\partial x = 0$. Now we can solve for $\partial z/\partial x$:

$$\frac{\partial z}{\partial x} = -\frac{2x}{2z} = -\frac{x}{z}.$$

In a similar manner we can compute $\partial z/\partial y$. ♣

Exercises for 13.4

Exercise 13.4.1 *Use the chain rule to compute dz/dt for $z = \sin(x^2 + y^2)$, $x = t^2 + 3$, $y = t^3$.*

Exercise 13.4.2 *Use the chain rule to compute dz/dt for $z = x^2 y$, $x = \sin(t)$, $y = t^2 + 1$.*

Exercise 13.4.3 *Use the chain rule to compute $\partial z/\partial s$ and $\partial z/\partial t$ for $z = x^2 y$, $x = \sin(st)$, $y = t^2 + s^2$.*

Exercise 13.4.4 *Use the chain rule to compute $\partial z/\partial s$ and $\partial z/\partial t$ for $z = x^2 y^2$, $x = st$, $y = t^2 - s^2$.*

Exercise 13.4.5 *Use the chain rule to compute $\partial z/\partial x$ and $\partial z/\partial y$ for $2x^2 + 3y^2 - 2z^2 = 9$.*

Exercise 13.4.6 *Use the chain rule to compute $\partial z/\partial x$ and $\partial z/\partial y$ for $2x^2 + y^2 + z^2 = 9$.*

Exercise 13.4.7 *Chemistry students will recognize the* ideal gas law*, given by $PV = nRT$ which relates the Pressure, Volume, and Temperature of n moles of gas. (R is the ideal gas constant). Thus, we can view pressure, volume, and temperature as variables, each one dependent on the other two.*

(a) *If pressure of a gas is increasing at a rate of $0.2Pa/min$ and temperature is increasing at a rate of $1K/min$, how fast is the volume changing?*

(b) *If the volume of a gas is decreasing at a rate of $0.3L/min$ and temperature is increasing at a rate of $0.5K/min$, how fast is the pressure changing?*

(c) *If the pressure of a gas is decreasing at a rate of $0.4Pa/min$ and the volume is increasing at a rate of $3L/min$, how fast is the temperature changing?*

Exercise 13.4.8 *Verify the following identity in the case of the ideal gas law:*

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$$

Exercise 13.4.9 *The previous exercise was a special case of the following fact, which you are to verify here: If $F(x,y,z)$ is a function of 3 variables, and the relation $F(x,y,z) = 0$ defines each of the variables in terms of the other two, namely $x = f(y,z)$, $y = g(x,z)$ and $z = h(x,y)$, then*

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$$

13.5 Directional Derivatives

We still have not answered one of our first questions about the steepness of a surface: starting at a point on a surface given by $f(x,y)$, and walking in a particular direction, how steep is the surface? We are now ready to answer the question.

We already know roughly what has to be done: as shown in Figure 13.5, we extend a line in the xy-plane to a vertical plane, and we then compute the slope of the curve that is the cross-section of the surface in that plane. The major stumbling block is that what appears in this plane to be the horizontal axis, namely the line in the xy-plane, is not an actual axis—we know nothing about the "units" along the axis. Our goal is to make this line into a t-axis; then we need formulas to write x and y in terms of this new variable t; then we can write z in terms of t since we know z in terms of x and y; and finally we can simply take the derivative.

So we need to somehow "mark off" units on the line, and we need a convenient way to refer to the line in calculations. It turns out that we can accomplish both by using the vector form of a line. Suppose that $\mathbf{u}$ is a unit vector $\langle u_1, u_2\rangle$ in the direction of interest. A vector equation for the line through (x_0,y_0) in this direction is $\mathbf{v}(t) = \langle u_1t + x_0, u_2t + y_0\rangle$. The height of the surface above the point $(u_1t + x_0, u_2t + y_0)$ is $g(t) = f(u_1t + x_0, u_2t + y_0)$. Because $\mathbf{u}$ is a unit vector, the value of t is precisely the distance along the line from (x_0,y_0) to $(u_1t + x_0, u_2t + y_0)$; this means that the line is effectively a t-axis, with origin at the point (x_0,y_0), so the slope we seek is

$$\begin{aligned} g'(0) &= \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \langle u_1,u_2\rangle \\ &= \langle f_x, f_y\rangle \cdot \mathbf{u} \\ &= \nabla f \cdot \mathbf{u} \end{aligned}$$

Here we have used the chain rule and the derivatives $\frac{d}{dt}(u_1t + x_0) = u_1$ and $\frac{d}{dt}(u_2t + y_0) = u_2$. The vector $\langle f_x, f_y\rangle$ is very useful, so it has its own symbol, ∇f, pronounced "del f"; it is also called the **gradient** of f.

Example 13.17: Slope

Find the slope of $z = x^2 + y^2$ at $(1,2)$ in the direction of the vector $\langle 3,4\rangle$.

Solution. We first compute the gradient at $(1,2)$: $\nabla f = \langle 2x, 2y\rangle$, which is $\langle 2,4\rangle$ at $(1,2)$. A unit vector in the desired direction is $\langle 3/5, 4/5\rangle$, and the desired slope is then $\langle 2,4\rangle \cdot \langle 3/5,4/5\rangle = 6/5 + 16/5 = 22/5$. ♣

Example 13.18: Tangent Vector

Find a tangent vector to $z = x^2 + y^2$ at $(1,2)$ in the direction of the vector $\langle 3,4\rangle$ and show that it is parallel to the tangent plane at that point.

Solution. Since $\langle 3/5,4/5\rangle$ is a unit vector in the desired direction, we can easily expand it to a tangent vector simply by adding the third coordinate computed in the previous example: $\langle 3/5, 4/5, 22/5\rangle$. To see that this vector is parallel to the tangent plane, we can compute its dot product with a normal to the plane. We know that a normal to the tangent plane is

$$\langle f_x(1,2), f_y(1,2), -1\rangle = \langle 2,4,-1\rangle,$$

and the dot product is $\langle 2,4,-1\rangle \cdot \langle 3/5,4/5,22/5\rangle = 6/5+16/5-22/5 = 0$, so the two vectors are perpendicular. (Note that the vector normal to the surface, namely $\langle f_x, f_y, -1\rangle$, is simply the gradient with a -1 tacked on as the third component.) ♣

The slope of a surface given by $z = f(x,y)$ in the direction of a (two-dimensional) vector $\mathbf{u}$ is called the **directional derivative** of f, written $D_{\mathbf{u}}f$. The directional derivative immediately provides us with some additional information. We know that

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

if $\mathbf{u}$ is a unit vector; θ is the angle between ∇f and $\mathbf{u}$. This tells us immediately that the largest value of $D_{\mathbf{u}}f$ occurs when $\cos\theta = 1$, namely, when $\theta = 0$, so ∇f is parallel to $\mathbf{u}$. In other words, the gradient ∇f points in the direction of steepest ascent of the surface, and $|\nabla f|$ is the slope in that direction. Likewise, the smallest value of $D_{\mathbf{u}}f$ occurs when $\cos\theta = -1$, namely, when $\theta = \pi$, so ∇f is anti-parallel to $\mathbf{u}$. In other words, $-\nabla f$ points in the direction of steepest descent of the surface, and $-|\nabla f|$ is the slope in that direction.

Example 13.19: Direction of Steepest Ascent and Descent

Investigate the direction of steepest ascent and descent for $z = x^2+y^2$.

Solution. The gradient is $\langle 2x, 2y\rangle = 2\langle x,y\rangle$; this is a vector parallel to the vector $\langle x,y\rangle$, so the direction of steepest ascent is directly away from the origin, starting at the point (x,y). The direction of steepest descent is thus directly toward the origin from (x,y). Note that at $(0,0)$ the gradient vector is $\langle 0,0\rangle$, which has no direction, and it is clear from the plot of this surface that there is a minimum point at the origin, and tangent vectors in all directions are parallel to the xy-plane. ♣

If ∇f is perpendicular to $\mathbf{u}$, $D_{\mathbf{u}}f = |\nabla f|\cos(\pi/2) = 0$, since $\cos(\pi/2) = 0$. This means that in either of the two directions perpendicular to ∇f, the slope of the surface is 0; this implies that a vector in either of these directions is tangent to the level curve at that point. Starting with $\nabla f = \langle f_x, f_y\rangle$, it is easy to find a vector perpendicular to it: either $\langle f_y, -f_x\rangle$ or $\langle -f_y, f_x\rangle$ will work.

If $f(x,y,z)$ is a function of three variables, all the calculations proceed in essentially the same way. The rate at which f changes in a particular direction is $\nabla f \cdot \mathbf{u}$, where now $\nabla f = \langle f_x, f_y, f_z\rangle$ and $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ is a unit vector. Again ∇f points in the direction of maximum rate of increase, $-\nabla f$ points in the direction of maximum rate of decrease, and any vector perpendicular to ∇f is tangent to the level surface $f(x,y,z) = k$ at the point in question. Of course there are no longer just two such vectors; the vectors perpendicular to ∇f describe the tangent plane to the level surface, or in other words ∇f is a normal to the tangent plane.

Example 13.20: Gradient

Suppose the temperature at a point in space is given by $T(x,y,z) = T_0/(1+x^2+y^2+z^2)$; *at the origin the temperature in Kelvin is* $T_0 > 0$, *and it decreases in every direction from there. It might be, for example, that there is a source of heat at the origin, and as we get farther from the source, the temperature decreases. The gradient is*

$$\begin{aligned}\nabla T &= \langle \frac{-2T_0x}{(1+x^2+y^2+z^2)^2}, \frac{-2T_0y}{(1+x^2+y^2+z^2)^2}, \frac{-2T_0z}{(1+x^2+y^2+z^2)^2} \rangle \\ &= \frac{-2T_0}{(1+x^2+y^2+z^2)^2}\langle x,y,z\rangle.\end{aligned}$$

The gradient points directly at the origin from the point (x,y,z)*—by moving directly toward the heat source, we increase the temperature as quickly as possible.*

Example 13.21: Tangent Plane

Find the points on the surface defined by $x^2+2y^2+3z^2 = 1$ *where the tangent plane is parallel to the plane defined by* $3x-y+3z = 1$.

Solution. Two planes are parallel if their normals are parallel or anti-parallel, so we want to find the points on the surface with normal parallel or anti-parallel to $\langle 3,-1,3\rangle$. Let $f = x^2+2y^2+3z^2$; the gradient of f is normal to the level surface at every point, so we are looking for a gradient parallel or anti-parallel to $\langle 3,-1,3\rangle$. The gradient is $\langle 2x,4y,6z\rangle$; if it is parallel or anti-parallel to $\langle 3,-1,3\rangle$, then

$$\langle 2x,4y,6z\rangle = k\langle 3,-1,3\rangle$$

for some k. This means we need a solution to the equations

$$2x = 3k \qquad 4y = -k \qquad 6z = 3k$$

but this is three equations in four unknowns—we need another equation. What we haven't used so far is that the points we seek are on the surface $x^2+2y^2+3z^2 = 1$; this is the fourth equation. If we solve the first three equations for x, y, and z and substitute into the fourth equation we get

$$\begin{aligned}1 &= \left(\frac{3k}{2}\right)^2 + 2\left(\frac{-k}{4}\right)^2 + 3\left(\frac{3k}{6}\right)^2 \\ &= \left(\frac{9}{4}+\frac{2}{16}+\frac{3}{4}\right)k^2 \\ &= \frac{25}{8}k^2\end{aligned}$$

so $k = \pm\frac{2\sqrt{2}}{5}$. The desired points are $\left(\frac{3\sqrt{2}}{5}, -\frac{\sqrt{2}}{10}, \frac{\sqrt{2}}{5}\right)$ and $\left(-\frac{3\sqrt{2}}{5}, \frac{\sqrt{2}}{10}, -\frac{\sqrt{2}}{5}\right)$. ♣

Exercises for 13.5

Exercise 13.5.1 *Find $D_{\mathbf{u}}f$ for $f = x^2 + xy + y^2$ in the direction of $\mathbf{u} = \langle 2,1 \rangle$ at the point $(1,1)$.*

Exercise 13.5.2 *Find $D_{\mathbf{u}}f$ for $f = \sin(xy)$ in the direction of $\mathbf{u} = \langle -1,1 \rangle$ at the point $(3,1)$.*

Exercise 13.5.3 *Find $D_{\mathbf{u}}f$ for $f = e^x \cos(y)$ in the direction 30 degrees from the positive x-axis at the point $(1,\pi/4)$.*

Exercise 13.5.4 *The temperature of a thin plate in the xy-plane is $T = x^2 + y^2$. How fast does temperature change at the point $(1,5)$ moving in a direction 30 degrees from the positive x-axis?*

Exercise 13.5.5 *Suppose the density of a thin plate at (x,y) is $1/\sqrt{x^2+y^2+1}$. Find the rate of change of the density at $(2,1)$ in a direction $\pi/3$ radians from the positive x-axis.*

Exercise 13.5.6 *Suppose the electric potential at (x,y) is $\ln\sqrt{x^2+y^2}$. Find the rate of change of the potential at $(3,4)$ toward the origin and also in a direction at a right angle to the direction toward the origin.*

Exercise 13.5.7 *A plane perpendicular to the xy-plane contains the point $(2,1,8)$ on the paraboloid $z = x^2 + 4y^2$. The cross-section of the paraboloid created by this plane has slope 0 at this point. Find an equation of the plane.*

Exercise 13.5.8 *A plane perpendicular to the xy-plane contains the point $(3,2,2)$ on the paraboloid $36z = 4x^2 + 9y^2$. The cross-section of the paraboloid created by this plane has slope 0 at this point. Find an equation of the plane.*

Exercise 13.5.9 *Suppose the temperature at (x,y,z) is given by $T = xy + \sin(yz)$. In what direction should you go from the point $(1,1,1)$ to decrease the temperature as quickly as possible? What is the rate of change of temperature in this direction?*

Exercise 13.5.10 *Suppose the temperature at (x,y,z) is given by $T = xyz$. In what direction can you go from the point $(1,1,1)$ to maintain the same temperature?*

Exercise 13.5.11 *Find an equation for the plane tangent to $x^2 - 3y^2 + z^2 = 7$ at $(1,1,3)$.*

Exercise 13.5.12 *Find an equation for the plane tangent to $xyz = 6$ at $(1,2,3)$.*

Exercise 13.5.13 *Find an equation for the line normal to $x^2 + 2y^2 + 4z^2 = 26$ at $(2,-3,-1)$.*

Exercise 13.5.14 *Find an equation for the line normal to $x^2 + y^2 + 9z^2 = 56$ at $(4,2,-2)$.*

Exercise 13.5.15 *Find an equation for the line normal to $x^2 + 5y^2 - z^2 = 0$ at $(4,2,6)$.*

Exercise 13.5.16 *Find the directions in which the directional derivative of $f(x,y) = x^2 + \sin(xy)$ at the point $(1,0)$ has the value 1.*

Exercise 13.5.17 *Show that the curve* $\mathbf{r}(t) = \langle \ln(t), t\ln(t), t\rangle$ *is tangent to the surface* $xz^2 - yz + \cos(xy) = 1$ *at the point* $(0,0,1)$.

Exercise 13.5.18 *A bug is crawling on the surface of a hot plate, the temperature of which at the point x units to the right of the lower left corner and y units up from the lower left corner is given by* $T(x,y) = 100 - x^2 - 3y^3$.

(a) *If the bug is at the point* $(2,1)$, *in what direction should it move to cool off the fastest? How fast will the temperature drop in this direction?*

(b) *If the bug is at the point* $(1,3)$, *in what direction should it move in order to maintain its temperature?*

Exercise 13.5.19 *The elevation on a portion of a hill is given by* $f(x,y) = 100 - 4x^2 - 2y$. *From the location above* $(2,1)$, *in which direction will water run?*

Exercise 13.5.20 *Suppose that* $g(x,y) = y - x^2$. *Find the gradient at the point* $(-1,3)$. *Sketch the level curve to the graph of g when* $g(x,y) = 2$, *and plot both the tangent line and the gradient vector at the point* $(-1,3)$. *(Make your sketch large). What do you notice, geometrically?*

Exercise 13.5.21 *The gradient* ∇f *is a vector valued function of two variables. Prove the following gradient rules. Assume* $f(x,y)$ *and* $g(x,y)$ *are differentiable functions.*

(a) $\nabla(fg) = f\nabla(g) + g\nabla(f)$

(b) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$

(c) $\nabla((f(x,y))^n) = nf(x,y)^{n-1}\nabla f$

13.6 Higher Order Derivatives

In single variable calculus we saw that the second derivative is often useful: in appropriate circumstances it measures acceleration; it can be used to identify maximum and minimum points; it tells us something about how sharply curved a graph is. Not surprisingly, second derivatives are also useful in the multi-variable case, but again not surprisingly, things are a bit more complicated.

It's easy to see where some complication is going to come from: with two variables there are four possible second derivatives. To take a "derivative," we must take a partial derivative with respect to x or y, and there are four ways to do it: x then x, x then y, y then x, y then y.

Example 13.22: Second Derivatives

Compute all four second derivatives of $f(x,y) = x^2y^2$.

Solution. Using an obvious notation, we get:

$$f_{xx} = 2y^2 \qquad f_{xy} = 4xy \qquad f_{yx} = 4xy \qquad f_{yy} = 2x^2$$

♣

You will have noticed that two of these are the same, the "mixed partials" computed by taking partial derivatives with respect to both variables in the two possible orders. This is not an accident—as long as the function is reasonably nice, this will always be true.

Theorem 13.23: Clairaut's Theorem

If the mixed partial derivatives are continuous, they are equal.

Example 13.24: Mixed Partials

Compute the mixed partials of $f = xy/(x^2 + y^2)$.

Solution. The mixed partial f_{xy} is found by first taking the partial derivative with respect to x:

$$f_x = \frac{y^3 - x^2 y}{(x^2 + y^2)^2},$$

then with respect to y:

$$f_{xy} = -\frac{x^4 - 6x^2y^2 + y^4}{(x^2 + y^2)^3}.$$

We leave f_{yx} as an exercise. ♣

Exercises for 13.6

Exercise 13.6.1 *Let $f = xy/(x^2 + y^2)$; compute f_{xx}, f_{yx}, and f_{yy}.*

Exercise 13.6.2 *Find all first and second partial derivatives of $x^3y^2 + y^5$.*

Exercise 13.6.3 *Find all first and second partial derivatives of $4x^3 + xy^2 + 10$.*

Exercise 13.6.4 *Find all first and second partial derivatives of $x \sin y$.*

Exercise 13.6.5 *Find all first and second partial derivatives of $\sin(3x)\cos(2y)$.*

Exercise 13.6.6 *Find all first and second partial derivatives of e^{x+y^2}.*

Exercise 13.6.7 *Find all first and second partial derivatives of $\ln\sqrt{x^3 + y^4}$.*

Exercise 13.6.8 *Find all first and second partial derivatives of z with respect to x and y if $x^2 + 4y^2 + 16z^2 - 64 = 0$.*

Exercise 13.6.9 *Find all first and second partial derivatives of z with respect to x and y if $xy + yz + xz = 1$.*

Exercise 13.6.10 *Let α and k be constants. Prove that the function $u(x,t) = e^{-\alpha^2 k^2 t}\sin(kx)$ is a solution to the heat equation $u_t = \alpha^2 u_{xx}$.*

Exercise 13.6.11 *Let a be a constant. Prove that $u = \sin(x - at) + \ln(x + at)$ is a solution to the wave equation $u_{tt} = a^2 u_{xx}$.*

Exercise 13.6.12 *How many third-order derivatives does a function of 2 variables have? How many of these are distinct?*

Exercise 13.6.13 *How many nth order derivatives does a function of 2 variables have? How many of these are distinct?*

13.7 Maxima and Minima

Suppose a surface given by $f(x,y)$ has a local maximum at (x_0,y_0,z_0); geometrically, this point on the surface looks like the top of a hill. If we look at the cross-section in the plane $y = y_0$, we will see a local maximum on the curve at (x_0,z_0), and we know from single-variable calculus that $\frac{\partial z}{\partial x} = 0$ at this point. Likewise, in the plane $x = x_0$, $\frac{\partial z}{\partial y} = 0$. So if there is a local maximum at (x_0,y_0,z_0), both partial derivatives at the point must be zero, and likewise for a local minimum. Thus, to find local maximum and minimum points, we need only consider those points at which both partial derivatives are 0. As in the single-variable case, it is possible for the derivatives to be 0 at a point that is neither a maximum or a minimum, so we need to test these points further.

You will recall that in the single variable case, we examined three methods to identify maximum and minimum points; the most useful is the second derivative test, though it does not always work. For functions of two variables there is also a second derivative test; again it is by far the most useful test, though it doesn't always work.

Theorem 13.25: Extrema Test for Multivariate Functions

Suppose that the second partial derivatives of $f(x,y)$ are continuous near (x_0,y_0), and $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$. We denote by D the **discriminant** *$D(x_0,y_0) = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2$. If $D > 0$ and $f_{xx}(x_0,y_0) < 0$ there is a local maximum at (x_0,y_0); if $D > 0$ and $f_{xx}(x_0,y_0) > 0$ there is a local minimum at (x_0,y_0); if $D < 0$ there is neither a maximum nor a minimum at (x_0,y_0); if $D = 0$, the test fails.*

Example 13.26: Extrema on an Elliptic Paraboloid

Verify that $f(x,y) = x^2 + y^2$ has a minimum at $(0,0)$.

Solution. First, we compute all the needed derivatives:

$$f_x = 2x \qquad f_y = 2y \qquad f_{xx} = 2 \qquad f_{yy} = 2 \qquad f_{xy} = 0.$$

The derivatives f_x and f_y are zero only at $(0,0)$. Applying the second derivative test there:

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 2\cdot 2 - 0 = 4 > 0,$$

so there is a local minimum at $(0,0)$, and there are no other possibilities. ♣

Example 13.27: Extrema on a Hyperbolic Paraboloid

Find all local maxima and minima for $f(x,y) = x^2 - y^2$.

Solution. The derivatives:

$$f_x = 2x \qquad f_y = -2y \qquad f_{xx} = 2 \qquad f_{yy} = -2 \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0,0)$, and

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 2 \cdot -2 - 0 = -4 < 0,$$

so there is neither a maximum nor minimum there, and so there are no local maxima or minima. The surface is shown in Figure 13.9. ♣

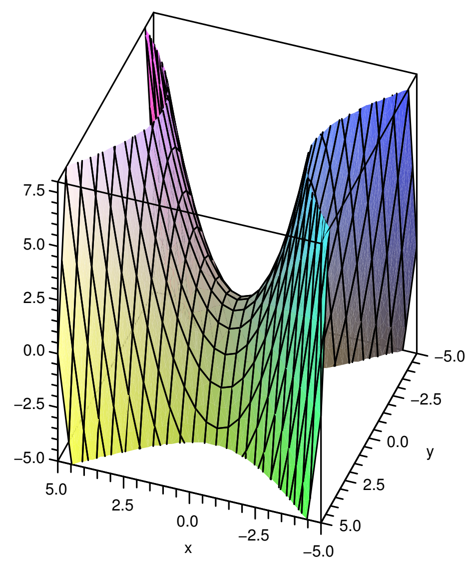


Figure 13.9: A saddle point, neither a maximum nor a minimum.

Example 13.28: Finding Extrema

Find all local maxima and minima for $f(x,y) = x^4 + y^4$.

Solution. The derivatives:

$$f_x = 4x^3 \qquad f_y = 4y^3 \qquad f_{xx} = 12x^2 \qquad f_{yy} = 12y^2 \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0,0)$, and

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0 \cdot 0 - 0 = 0,$$

so we get no information. However, in this case it is easy to see that there is a minimum at $(0,0)$, because $f(0,0) = 0$ and at all other points $f(x,y) > 0$. ♣

Example 13.29: Finding Extrema

Find all local maxima and minima for $f(x,y) = x^3 + y^3$.

Solution. The derivatives:

$$f_x = 3x^2 \qquad f_y = 3y^2 \qquad f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = 0.$$

Again there is a single critical point, at $(0,0)$, and

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0 \cdot 0 - 0 = 0,$$

so we get no information. In this case, a little thought shows there is neither a maximum nor a minimum at $(0,0)$: when x and y are both positive, $f(x,y) > 0$, and when x and y are both negative, $f(x,y) < 0$, and there are points of both kinds arbitrarily close to $(0,0)$. Alternately, if we look at the cross-section when $y = 0$, we get $f(x,0) = x^3$, which does not have either a maximum or minimum at $x = 0$. ♣

Example 13.30: Optimizing Dimensions of a Box

Suppose a box with no top is to hold a certain volume V. Find the dimensions for the box that result in the minimum surface area.

Solution. The surface area of the box is $A = 2hw + 2hl + lw$, and the volume is $V = lwh$, so we can write the area as a function of two variables,

$$A(l,w) = \frac{2V}{l} + \frac{2V}{w} + lw.$$

Then

$$A_l = -\frac{2V}{l^2} + w \quad \text{and} \quad A_w = -\frac{2V}{w^2} + l.$$

If we set these equal to zero and solve, we find $w = (2V)^{1/3}$ and $l = (2V)^{1/3}$, and the corresponding height is $h = V/(2V)^{2/3}$.

The second derivatives are

$$A_{ll} = \frac{4V}{l^3} \qquad A_{ww} = \frac{4V}{w^3} \qquad A_{lw} = 1,$$

so the discriminant is

$$D = \frac{4V}{l^3}\frac{4V}{w^3} - 1 = 4 - 1 = 3 > 0.$$

Since A_{ll} is 2, there is a local minimum at the critical point. Is this a global minimum? It is, but it is difficult to see this analytically; physically and graphically it is clear that there is a minimum, in which case it must be at the single critical point. ♣

Recall that when we did single variable global maximum and minimum problems, the easiest cases were those for which the variable could be limited to a finite closed interval, for then we simply had to check all critical values and the endpoints. The previous example is difficult because there is no finite boundary to the domain of the problem—both w and l can be in $(0,\infty)$. As in the single variable case, the problem is often simpler when there is a finite boundary.

Theorem 13.31: Multivariate Absolute Extrema

If $f(x,y)$ is continuous on a closed and bounded subset of $\mathbb{R}^2$, then it has both a maximum and minimum value.

As in the case of single variable functions, this means that the maximum and minimum values must occur at a critical point or on the boundary; in the two variable case, however, the boundary is a curve, not merely two endpoints.

Example 13.32: Optimizing Volume of a Box

The length of the diagonal of a box is to be 1 meter; find the maximum possible volume.

Solution. If the box is placed with one corner at the origin, and sides along the axes, the length of the diagonal is $\sqrt{x^2+y^2+z^2}$, and the volume is

$$V = xyz = xy\sqrt{1-x^2-y^2}.$$

Clearly, $x^2+y^2 \leq 1$, so the domain we are interested in is the quarter of the unit disk in the first quadrant. Computing derivatives:

$$V_x = \frac{y-2yx^2-y^3}{\sqrt{1-x^2-y^2}}$$
$$V_y = \frac{x-2xy^2-x^3}{\sqrt{1-x^2-y^2}}$$

If these are both 0, then $x=0$ or $y=0$, or $x=y=1/\sqrt{3}$. The boundary of the domain is composed of three curves: $x=0$ for $y\in[0,1]$; $y=0$ for $x\in[0,1]$; and $x^2+y^2=1$, where $x\geq 0$ and $y\geq 0$. In all three cases, the volume $xy\sqrt{1-x^2-y^2}$ is 0, so the maximum occurs at the only critical point $(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})$. See Figure 13.10. ♣

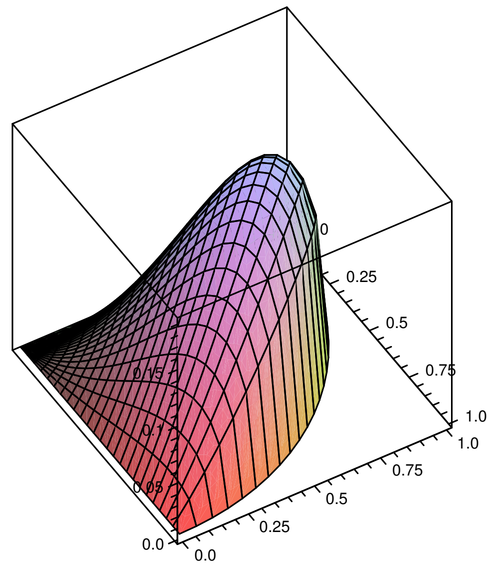


Figure 13.10: The volume of a box with fixed length diagonal.

Exercises for 13.7

Exercise 13.7.1 *Find all local maximum and minimum points of* $f = x^2 + 4y^2 - 2x + 8y - 1$.

Exercise 13.7.2 *Find all local maximum and minimum points of* $f = x^2 - y^2 + 6x - 10y + 2$.

Exercise 13.7.3 *Find all local maximum and minimum points of* $f = xy$.

Exercise 13.7.4 *Find all local maximum and minimum points of* $f = 9 + 4x - y - 2x^2 - 3y^2$.

Exercise 13.7.5 *Find all local maximum and minimum points of* $f = x^2 + 4xy + y^2 - 6y + 1$.

Exercise 13.7.6 *Find all local maximum and minimum points of* $f = x^2 - xy + 2y^2 - 5x + 6y - 9$.

Exercise 13.7.7 *Find the absolute maximum and minimum points of* $f = x^2 + 3y - 3xy$ *over the region bounded by* $y = x$, $y = 0$, *and* $x = 2$.

Exercise 13.7.8 *A six-sided rectangular box is to hold* $1/2$ *cubic meter; what shape should the box be to minimize surface area?*

Exercise 13.7.9 *The post office will accept packages whose combined length and girth is at most 130 inches. (Girth is the maximum distance around the package perpendicular to the length; for a rectangular box, the length is the largest of the three dimensions.) What is the largest volume that can be sent in a rectangular box?*

Exercise 13.7.10 *The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape for a given volume that will minimize cost.*

Exercise 13.7.11 *Using the methods of this section, find the shortest distance from the origin to the plane* $x+y+z=10$.

Exercise 13.7.12 *Using the methods of this section, find the shortest distance from the point* (x_0, y_0, z_0) *to the plane* $ax+by+cz=d$. *You may assume that* $c \neq 0$; *use of Sage or similar software is recommended.*

Exercise 13.7.13 *A trough is to be formed by bending up two sides of a long metal rectangle so that the cross-section of the trough is an isosceles trapezoid. If the width of the metal sheet is 2 meters, how should it be bent to maximize the volume of the trough?*

Exercise 13.7.14 *Given the three points* $(1,4)$, $(5,2)$, *and* $(3,-2)$, $(x-1)^2+(y-4)^2+(x-5)^2+(y-2)^2+(x-3)^2+(y+2)^2$ *is the sum of the squares of the distances from point* (x,y) *to the three points. Find* x *and* y *so that this quantity is minimized.*

Exercise 13.7.15 *Suppose that* $f(x,y)=x^2+y^2+kxy$. *Find and classify the critical points, and discuss how they change when* k *takes on different values.*

Exercise 13.7.16 *Find the shortest distance from the point* $(0,b)$ *to the parabola* $y=x^2$.

Exercise 13.7.17 *Find the shortest distance from the point* $(0,0,b)$ *to the paraboloid* $z=x^2+y^2$.

Exercise 13.7.18 *Consider the function* $f(x,y)=x^3-3x^2y+y^3$.

(a) *Show that* $(0,0)$ *is the only critical point of* f.

(b) *Show that the discriminant test is inconclusive for* f.

(c) *Determine the cross-sections of* f *obtained by setting* $y=kx$ *for various values of* k.

(d) *What kind of critical point is* $(0,0)$*?*

Exercise 13.7.19 *Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid* $2x^2+72y^2+18z^2=288$.

13.8 Lagrange Multipliers

Many applied max/min problems take the following form: we want to find an extreme value of a function, like $V=xyz$, subject to a constraint, like $1=\sqrt{x^2+y^2+z^2}$. Often this can be done, as we have, by explicitly combining the equations and then finding critical points. There is another approach that is often convenient, the method of **Lagrange multipliers**.

It is somewhat easier to understand two variable problems, so we begin with one as an example. Suppose the perimeter of a rectangle is to be 100 units. Find the rectangle with largest area. This is a fairly straightforward problem from single variable calculus. We write down the two equations: $A=xy$, $P=100=2x+2y$, solve the second of these for y (or x), substitute into the first, and end up with a one-variable maximization problem. Let's now think of it differently: the equation $A=xy$ defines a surface, and the equation $100 = 2x + 2y$ defines a curve (a line, in this case) in the xy-plane. If we graph both of these in the three-dimensional coordinate system, we can phrase the problem like this: what is the highest point on the surface above the line? The solution we already understand effectively produces the equation of the cross-section of the surface above the line and then treats it as a single variable problem. Instead, imagine that we draw the level curves (the contour lines) for the surface in the xy-plane, along with the line.

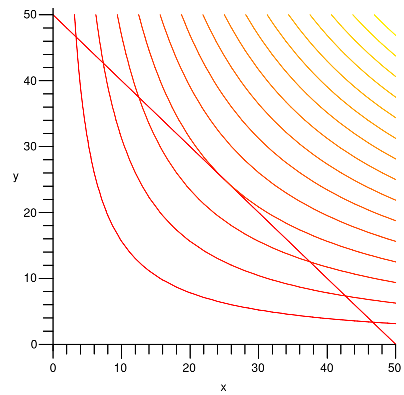


Figure 13.11: Constraint line with contour plot of the surface xy.

Imagine that the line represents a hiking trail and the contour lines are, as on a topographic map, the lines of constant altitude. How could you estimate, based on the graph, the high (or low) points on the path? As the path crosses contour lines, you know the path must be increasing or decreasing in elevation. At some point you will see the path just touch a contour line (tangent to it), and then begin to cross contours in the opposite order—that point of tangency must be a maximum or minimum point. If we can identify all such points, we can then check them to see which gives the maximum and which the minimum value. As usual, we also need to check boundary points; in this problem, we know that x and y are positive, so we are interested in just the portion of the line in the first quadrant, as shown. The endpoints of the path, the two points on the axes, are not points of tangency, but they are the two places that the function xy is a minimum in the first quadrant.

How can we actually make use of this? At the points of tangency that we seek, the constraint curve (in this case the line) and the level curve have the same slope—their tangent lines are parallel. This also means that the constraint curve is perpendicular to the gradient vector of the function; going a bit further, if we can express the constraint curve itself as a level curve, then we seek the points at which the two level curves have parallel gradients. The curve $100 = 2x + 2y$ can be thought of as a level curve of the function $2x + 2y$; Figure 13.12 shows both sets of level curves on a single graph. We are interested in those points where two level curves are tangent—but there are many such points, in fact an infinite number, as we've only shown a few of the level curves. All along the line $y = x$ are points at which two level curves are tangent. While this might seem to be a show-stopper, it is not.

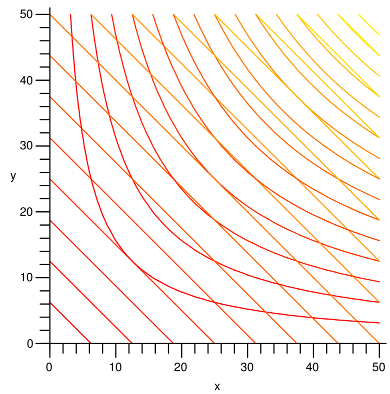


Figure 13.12: Contour plots for $2x+2y$ and xy.

The gradient of $2x+2y$ is $\langle 2,2\rangle$, and the gradient of xy is $\langle y,x\rangle$. They are parallel when $\langle 2,2\rangle = \lambda\langle y,x\rangle$, that is, when $2=\lambda y$ and $2=\lambda x$. We have two equations in three unknowns, which typically results in many solutions (as we expected). A third equation will reduce the number of solutions; the third equation is the original constraint, $100=2x+2y$. So we have the following system to solve:

$$2=\lambda y \qquad 2=\lambda x \qquad 100=2x+2y.$$

In the first two equations, λ can't be 0, so we may divide by it to get $x=y=2/\lambda$. Substituting into the third equation we get

$$\begin{aligned} 2\frac{2}{\lambda}+2\frac{2}{\lambda} &= 100 \\ \frac{8}{100} &= \lambda \end{aligned}$$

so $x=y=25$. Note that we are not really interested in the value of λ—it is a clever tool, the Lagrange multiplier, introduced to solve the problem. In many cases, as here, it is easier to find λ than to find everything else without using λ.

The same method works for functions of three variables, except of course everything is one dimension higher: the function to be optimized is a function of three variables and the constraint represents a surface—for example, the function may represent temperature, and we may be interested in the maximum temperature on some surface, like a sphere. The points we seek are those at which the constraint surface is tangent to a level surface of the function. Once again, we consider the constraint surface to be a level surface of some function, and we look for points at which the two gradients are parallel, giving us three equations in four unknowns. The constraint provides a fourth equation.

Example 13.33: Optimization with Constraints

Maximize the function xyz given the constraint $1=\sqrt{x^2+y^2+z^2}$.

Solution. The constraint is $1=\sqrt{x^2+y^2+z^2}$, which is the same as $1=x^2+y^2+z^2$. The function to maximize is xyz. The two gradient vectors are $\langle 2x,2y,2z\rangle$ and $\langle yz,xz,xy\rangle$, so the equations to be solved are

$$yz=2x\lambda$$

$$\begin{aligned} xz &= 2y\lambda \\ xy &= 2z\lambda \\ 1 &= x^2+y^2+z^2 \end{aligned}$$

If $\lambda = 0$ then at least two of x, y, z must be 0, giving a volume of 0, which will not be the maximum. If we multiply the first two equations by x and y respectively, we get

$$\begin{aligned} xyz &= 2x^2\lambda \\ xyz &= 2y^2\lambda \end{aligned}$$

so $2x^2\lambda = 2y^2\lambda$ or $x^2 = y^2$; in the same way we can show $x^2 = z^2$. Hence the fourth equation becomes $1 = x^2+x^2+x^2$ or $x = 1/\sqrt{3}$, and so $x = y = z = 1/\sqrt{3}$ gives the maximum volume. This is of course the same answer we obtained previously. ♣

Another possibility is that we have a function of three variables, and we want to find a maximum or minimum value not on a surface but on a curve; often the curve is the intersection of two surfaces, so that we really have two constraint equations, say $g(x,y,z) = c_1$ and $h(x,y,z) = c_2$. It turns out that at points on the intersection of the surfaces where f has a maximum or minimum value,

$$\nabla f = \lambda \nabla g + \mu \nabla h.$$

As before, this gives us three equations, one for each component of the vectors, but now in five unknowns, x, y, z, λ, and μ. Since there are two constraint functions, we have a total of five equations in five unknowns, and so can usually find the solutions we need.

Example 13.34: Intersection of a Plane with a Cylinder

The plane $x+y-z=1$ intersects the cylinder $x^2+y^2=1$ in an ellipse. Find the points on the ellipse closest to and farthest from the origin.

Solution. We want the extreme values of $f = \sqrt{x^2+y^2+z^2}$ subject to the constraints $g = x^2+y^2 = 1$ and $h = x+y-z = 1$. To simplify the algebra, we may use instead $f = x^2+y^2+z^2$, since this has a maximum or minimum value at exactly the points at which $\sqrt{x^2+y^2+z^2}$ does. The gradients are

$$\nabla f = \langle 2x, 2y, 2z\rangle \qquad \nabla g = \langle 2x, 2y, 0\rangle \qquad \nabla h = \langle 1, 1, -1\rangle,$$

so the equations we need to solve are

$$\begin{aligned} 2x &= \lambda 2x + \mu \\ 2y &= \lambda 2y + \mu \\ 2z &= 0 - \mu \\ 1 &= x^2+y^2 \\ 1 &= x+y-z. \end{aligned}$$

Subtracting the first two we get $2y-2x = \lambda(2y-2x)$, so either $\lambda = 1$ or $x = y$. If $\lambda = 1$ then $\mu = 0$, so $z = 0$ and the last two equations are

$$1 = x^2+y^2 \qquad \text{and} \qquad 1 = x+y.$$

Solving these gives $x = 1$, $y = 0$, or $x = 0$, $y = 1$, so the points of interest are $(1,0,0)$ and $(0,1,0)$, which are both distance 1 from the origin. If $x = y$, the fourth equation is $2x^2 = 1$, giving $x = y = \pm 1/\sqrt{2}$, and from the fifth equation we get $z = -1 \pm \sqrt{2}$. The distance from the origin to $(1/\sqrt{2}, 1/\sqrt{2}, -1+\sqrt{2})$ is $\sqrt{4-2\sqrt{2}} \approx 1.08$ and the distance from the origin to $(-1/\sqrt{2}, -1/\sqrt{2}, -1-\sqrt{2})$ is $\sqrt{4+2\sqrt{2}} \approx 2.6$. Thus, the points $(1,0,0)$ and $(0,1,0)$ are closest to the origin and $(-1/\sqrt{2}, -1/\sqrt{2}, -1-\sqrt{2})$ is farthest from the origin. ♣

Exercises for 13.8

Exercise 13.8.1 *A six-sided rectangular box is to hold* $1/2$ *cubic meter; what shape should the box be to minimize surface area?*

Exercise 13.8.2 *The post office will accept packages whose combined length and girth are at most 130 inches (girth is the maximum distance around the package perpendicular to the length). What is the largest volume that can be sent in a rectangular box?*

Exercise 13.8.3 *The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape for a given volume that will minimize cost.*

Exercise 13.8.4 *Using Lagrange multipliers, find the shortest distance from the point* (x_0, y_0, z_0) *to the plane* $ax + by + cz = d$.

Exercise 13.8.5 *Find all points on the surface* $xy - z^2 + 1 = 0$ *that are closest to the origin.*

Exercise 13.8.6 *The material for the bottom of an aquarium costs half as much as the high strength glass for the four sides. Find the shape of the cheapest aquarium that holds a given volume* V.

Exercise 13.8.7 *The plane* $x - y + z = 2$ *intersects the cylinder* $x^2 + y^2 = 4$ *in an ellipse. Find the points on the ellipse closest to and farthest from the origin.*

Exercise 13.8.8 *Find three positive numbers whose sum is 48 and whose product is as large as possible.*

Exercise 13.8.9 *Find all points on the plane* $x + y + z = 5$ *in the first octant at which* $f(x,y,z) = xy^2z^2$ *has a maximum value.*

Exercise 13.8.10 *Find the points on the surface* $x^2 - yz = 5$ *that are closest to the origin.*

Exercise 13.8.11 *A manufacturer makes two models of an item, standard and deluxe. It costs \$40 to manufacture the standard model and \$60 for the deluxe. A market research firm estimates that if the standard model is priced at x dollars and the deluxe at y dollars, then the manufacturer will sell* $500(y - x)$ *of the standard items and* $45{,}000 + 500(x - 2y)$ *of the deluxe each year. How should the items be priced to maximize profit?*

Exercise 13.8.12 *A length of sheet metal is to be made into a water trough by bending up two sides as shown in Figure 13.13. Find x and* ϕ *so that the trapezoid–shaped cross section has maximum area, when the width of the metal sheet is 27 inches (that is,* $2x + y = 27$*).*

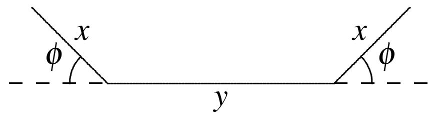


Figure 13.13: Cross-section of a trough.

Exercise 13.8.13 *Find the maximum and minimum values of* $f(x,y,z) = 6x+3y+2z$ *subject to the constraint* $g(x,y,z) = 4x^2+2y^2+z^2-70 = 0$.

Exercise 13.8.14 *Find the maximum and minimum values of* $f(x,y) = e^{xy}$ *subject to the constraint* $g(x,y) = x^3+y^3-16 = 0$.

Exercise 13.8.15 *Find the maximum and minimum values of* $f(x,y) = xy+\sqrt{9-x^2-y^2}$ *when* $x^2+y^2 \leq 9$.

Exercise 13.8.16 *Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.*

Exercise 13.8.17 *Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.*

Chapter 14

Multiple Integration

14.1 Volume and Average Height

Consider a surface $f(x,y)$; you might temporarily think of this as representing physical topography—a hilly landscape, perhaps. What is the average height of the surface (or average altitude of the landscape) over some region?

As with most such problems, we start by thinking about how we might approximate the answer. Suppose the region is a rectangle, $[a,b]\times[c,d]$. We can divide the rectangle into a grid, m subdivisions in one direction and n in the other, as indicated in Figure 14.1. We pick x values x_0, $x_1, \ldots$, x_{m-1} in each subdivision in the x direction, and similarly in the y direction. At each of the points (x_i,y_j) in one of the smaller rectangles in the grid, we compute the height of the surface: $f(x_i,y_j)$. Now the average of these heights should be (depending on the fineness of the grid) close to the average height of the surface:

$$\frac{f(x_0,y_0)+f(x_1,y_0)+\cdots+f(x_0,y_1)+f(x_1,y_1)+\cdots+f(x_{m-1},y_{n-1})}{mn}$$

As both m and n go to infinity, we expect this approximation to converge to a fixed value, the actual average height of the surface. For reasonably nice functions this does indeed happen.

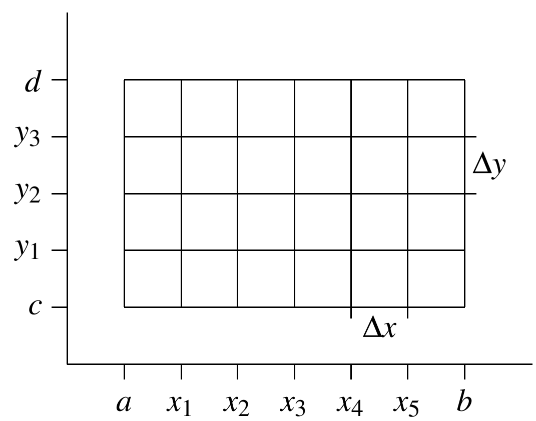


Figure 14.1: A rectangular subdivision of $[a,b]\times[c,d]$.

Using sigma notation, we can rewrite the approximation:

$$\begin{aligned}\frac{1}{mn}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j,y_i) &= \frac{1}{(b-a)(d-c)}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j,y_i)\frac{b-a}{m}\frac{d-c}{n}\\ &= \frac{1}{(b-a)(d-c)}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(x_j,y_i)\Delta x\Delta y.\end{aligned}$$

The two parts of this product have useful meaning: $(b-a)(d-c)$ is of course the area of the rectangle, and the double sum adds up mn terms of the form $f(x_j,y_i)\Delta x\Delta y$, which is the height of the surface at a