N.I. Lobachevsky State University
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Probability theory and mathematical statistics:

Mathematical expectation — Practice

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Problem. Given the probability distribution

a	0	1	2	3
$\mathrm{P}(X = a)$	x	$\frac{1}{3}$	y	$\frac{1}{6}$

of random variable X, $\mathbf{E}X = 1$. Compute $\mathbf{E}X^2$.

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Solution.

$$x + \frac{1}{3} + y + \frac{1}{6} = 1,$$

$$0 \cdot x + 1 \cdot \frac{1}{3} + 2 \cdot y + 3 \cdot \frac{1}{6} = \mathbf{E}X = 1.$$

Then, $y = \frac{1}{12}$, $x = \frac{5}{12}$. Now,

$$\mathbf{E}X^2 = 0^2 \cdot \frac{5}{12} + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{12} + 3^2 \cdot \frac{1}{6} = \frac{13}{6}.$$

Problem. A coin is tossed three times. Random variables are:
X — the total number of Heads,
Y — the total number of Tails,
Z — the total number of faces changes (for example, the sequence HTH contains 2 faces changes).
Compute $\mathbf{E}X$, $\mathbf{E}Y$, $\mathbf{E}Z$.

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Solution. X, Y are distributed as $b(3, \frac{1}{2})$; $\mathbf{E}X = \mathbf{E}Y = np = \frac{3}{2}$.

ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$\mathrm{P}(\{\omega\})$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$Z(\omega)$	0	1	2	1	1	2	1	0

$$\mathbf{E}Z = 0 \cdot \frac{2}{8} + 1 \cdot \frac{4}{8} + 2 \cdot \frac{2}{8} = 1$$

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Solution. Denote the random variable by X. Introduce events: A_i — the ith guess is correct. Then

$$\{X = 1\} = A_1, \quad \mathrm{P}(X = 1) = \frac{1}{5},$$

$$\{X = 2\} = \bar{A}_1 \cap A_2,$$

$$\mathrm{P}(X = 2) = \mathrm{P}(\bar{A}_1)\mathrm{P}(A_2|\bar{A}_1) = \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5},$$

$$\dots$$

$$\mathbf{E}X = 3.$$

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Solution. Denote by X the random variable. We have

$$\mathrm{P}(X = k) = \frac{C_{15}^{k} C_{10}^{4-k}}{C_{25}^{4}}, \quad k = 0, 1, 2, 3, 4;$$

$$\mathbf{E}X = 0 \cdot \frac{21}{1265} + 1 \cdot \frac{36}{253} + 2 \cdot \frac{189}{506} + 3 \cdot \frac{91}{253} + 4 \cdot \frac{273}{2530} = 4 \cdot \frac{15}{25} = \frac{12}{5}.$$

Problem. Given that $\mathbf{E}X = 1$, $\mathbf{E}Y = 0.4$, compute $\mathbf{E}(2X - Y)$.

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Solution.

$$\mathbf{E}(2X - Y) = 2\mathbf{E}X - \mathbf{E}Y = 2 \cdot 1 - 0.4 = 1.6$$

Problem. Three students get highest scores with the probabilities 0.9, 0.8 and 0.7 independently from each other. Denote by X the number of highest scores recieved. Compute $\mathbf{E}X$, $\mathbf{E}X^2$.

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Solution. Let Y_i equal 1 then the ith student gets the highest score, equal 0 otherwise. Then $X = Y_1 + Y_2 + Y_3$,

$$\begin{aligned}
\mathbf{E}X &= \mathbf{E}(Y_1 + Y_2 + Y_3) \\
&= \mathrm{P}(Y_1 = 1) + \mathrm{P}(Y_2 = 1) + \mathrm{P}(Y_3 = 1) \\
&= 0.9 + 0.8 + 0.7 = 2.4, \\
\mathbf{E}X^2 &= \mathbf{E}(Y_1 + Y_2 + Y_3)^2 \\
&= \mathbf{E}(Y_1^2 + Y_2^2 + Y_3^2 + 2Y_1Y_2 + 2Y_1Y_3 + 2Y_2Y_3) \\
&= \mathbf{E}Y_1 + \mathbf{E}Y_2 + \mathbf{E}Y_3 + 2(\mathbf{E}Y_1\mathbf{E}Y_2 + \mathbf{E}Y_1\mathbf{E}Y_3 + \mathbf{E}Y_2\mathbf{E}Y_3) \\
&= 0.9 + 0.8 + 0.7 + 2(0.9 \cdot 0.8 + 0.9 \cdot 0.7 + 0.8 \cdot 0.7) \\
&= 6.22
\end{aligned}$$