N.I. Lobachevsky State University
of Nizhni Novgorod

Probability theory and mathematical statistics:

Classical Probability

Associate Professor
A.V. Zorine

Outline

1. Definition of classical probability.
2. Properties of classical probability.
3. Examples: balls, dice, cards, etc. Basic tricks and sample spaces.

Classical probability

When an event A occurs together with the elementary outcomes $E' = \{\omega'\}, E'' = \{\omega''\}, \dots, E^{(r)} = \{\omega^{(r)}\}$, we write $A = \{\omega', \omega'', \dots, \omega^{(r)}\}$.

Definition

If all elementary outcomes are equally likely to happen (this can be deduced from observations or symmetry considerations), define *probability* of an event A as

$$\mathrm{P}(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega} = \frac{N(A)}{N(\Omega)}$$

Laplace: *”La probabilité d’un événement est le rapport du nombre des cas qui lui sont favorables au nombre de tout les cas possibles...”* (1812, Théorie analytique des probabilités)

Corollary

For any elementary outcome $E = \{\omega\}$, $\mathrm{P}(E) = 1/n$.

Corollary

For any event A,

$$\mathrm{P}(\bar{A}) = 1 - \mathrm{P}(A)$$

Proof.

$$N(\bar{A}) = N(\Omega) - N(A),$$
$$\frac{N(\bar{A})}{N(\Omega)} = \frac{N(\Omega)}{N(\Omega)} - \frac{N(A)}{N(\Omega)},$$
$$\mathrm{P}(\bar{A}) = 1 - \mathrm{P}(A)$$

Corollary

For any events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B)$$

Proof. From the elementary sets theory,

$$N(A \cup B) = N(A) + N(B) - N(A \cap B).$$

Now dividing by $N(\Omega)$, obtain the equations above.

Example. There are 5 red and 3 blue balls in an urn. One ball is taken at random. What's the probability it is a red ball?

Example. There are 5 red and 3 blue balls in an urn. One ball is taken at random. What's the probability it is a red ball?

Solution. Since the numbers of red and blue balls are different, ball colors are not equally possible. To bring symmetry into situation, let's enumerate all the balls like this:

●	●	●	●	●	●	●	●
1	2	3	4	5	6	7	8

Now we have 8 balls differing only in their number. Hence elementary outcome can be $\{\omega\}$, where ω is the number of the ball taken out, $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$, event $A = \{1, 2, 3, 4, 5\}$ means a red ball is taken, $\mathrm{P}(A) = \frac{5}{8}$.

Example. A symmetric coin is tossed twice. What's the probability to have at least one Tail?
Solution. Four outcomes $\{HH\}$, $\{HT\}$, $\{TH\}$, $\{TT\}$ are equally possible (from experience!). Here HT means that the first coin is Head while the second is Tail. Then $\Omega = \{HH, HT, TH, TT\}$, $A = \{TT, TH, HT\}$ is the event under question, $\mathrm{P}(A) = \frac{3}{4}$.

Example. A symmetric coin is tossed twice. What's the probability to have at least one Tail?
Solution. Four outcomes $\{HH\}$, $\{HT\}$, $\{TH\}$, $\{TT\}$ are equally possible (from experience!). Here HT means that the first coin is Head while the second is Tail. Then $\Omega = \{HH, HT, TH, TT\}$, $A = \{TT, TH, HT\}$ is the event under question, $\mathrm{P}(A) = \frac{3}{4}$.

Wrong solution. A Tail can appear either in the first toss, or in the second toss, or never. Hence there are two favorable outcomes out of three and the probability is $\frac{2}{3}$. In fact tossing a coin we can see that the first outcome is about twice as frequent as the second or the third!. These outcomes are not equally probable.

Example. There are M defective among N manufactured articles. S articles are taken to a test. What's the probability to have exactly K defected articles in the sample (event A_K)?

Solution. To make bring symmetry into the situation, enumerate all articles from 1 to N so that numbers from 1 to M correspond to defective articles. An elementary outcome is a combination $\omega = \{x_1, x_2, \ldots, x_S\}$ without repetitions of numbers from 1 to N. From lecture 1 we know that $N(\Omega) = C_N^S$. It is convenient to assume $x_1 < x_2 < \ldots < x_S$. For any favorable elementary outcome, numbers $x_1, x_2, \ldots, x_K$ belong to a set $\{1, 2, \ldots, M\}$, and numbers x_{K+1}, x_{K+2}, $\ldots$, x_S belong to a set $\{M+1, M+2, \ldots, N\}$. Hence $N(A_K) = C_M^K C_{N-M}^{S-K}$,

$$\mathrm{P}(A_K) = \frac{C_M^K C_{N-M}^{S-K}}{C_N^S}$$

Example. A die is rolled once. What's the probability that the number of point on top of it cannot be divided by 3 (event A)?

Solution. $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 4, 5\}$, $\mathrm{P}(A) = 4/6 = 2/3$.

Example. A die is rolled once. What's the probability that the number of point on top of it cannot be divided by 3 (event A)?

Solution. $\Omega = \{1, 2, 3, 4, 5, 6\}, A = \{1, 2, 4, 5\}, \mathrm{P}(A) = 4/6 = 2/3.$

Example. Three dice are rolled, what's the probability to have different numbers of points on their tops (event A)?

Solution. Here $\omega = (x_1, x_2, x_3)$, where x_1 is the number of points on the first die, x_2 is the number of points on the second die, x_3 is the number of points on the third die, $N(\Omega) = 6^3$. An elementary outcome $\{(x_1, x_2, x_3)\}$ is favorable to A if $x_1 \neq x_2$. $x_1 \neq x_3$, and $x_2 \neq x_3$, thus,

$$N(A) = 6 \cdot 5 \cdot 4 \qquad \mathrm{P}(A) = \frac{6 \cdot 5 \cdot 4}{6^3} = \frac{5}{9} \approx 0{,}556.$$

Example. Two cards are picked from a pile of 52 cards. What is the probability to get two portraits (i. e. Jacks, Queens, or Kings)?

Solution. An elementary outcome is a combination $\{x_1, x_2\}$ of two cards, $N(\Omega) = C_{52}^2$. A deck contains 12 portraits, $N(A) = C_{12}^2$, $\mathrm{P}(A) = C_{12}^2 / C_{52}^2 \approx 0{,}049$.

Example. Two cards are picked from a pile of 52 cards. What is the probability to get two portraits (i. e. Jacks, Queens, or Kings)?

Solution. An elementary outcome is a combination $\{x_1, x_2\}$ of two cards, $N(\Omega) = C_{52}^2$. A deck contains 12 portraits, $N(A) = C_{12}^2$, $\mathrm{P}(A) = C_{12}^2 / C_{52}^2 \approx 0{,}049$.

Example. A shuffled deck of cards is divided into two parts. What is the probability to have the same number of red and black cards in both parts (event A)?

Solution. An elementary outcome is a combination of 18 cards in the first part, $N(\Omega) = C_{52}^{26}$. An elementary outcome favorable to A has 13 cards of red suits and 13 cards of black suits, $N(A) = C_{26}^{13} C_{26}^{13}$,

$$\mathrm{P}(A) = \frac{C_{26}^{13} C_{26}^{13}}{C_{52}^{26}} = \frac{16232365000}{74417546961} \approx 0{,}218126$$

Example. Two dice are rolled once. What is the probability that the sum of points is prime?
Solution. Here $\omega = (x, y)$, we compute $x + y$ for each elementary outcome in the following table:

$x\backslash y$	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

Prime numbers here are 2, 3, 5, 7, and 11.

Example. Two dice are rolled once. What is the probability that the sum of points is prime?
Solution. Here $\omega = (x, y)$, we compute $x + y$ for each elementary outcome in the following table:

$x \backslash y$	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

Prime numbers here are 2, 3, 5, 7, and 11. $N(A) = 15$,
$\mathrm{P}(A) = 15/36 \approx 0{,}417$.

Example. Two dice are rolled, what's the probability to have 6 points in sum (event A), 7 points in sum (event B)?
Solution

$$\begin{aligned} A &= \text{"The sum of points equals 6"} \\ &= \{(1,5),(2,4),(3,3),(4,2),(5,1)\}, \end{aligned}$$

$$\mathrm{P}(A) = \frac{5}{36}$$

Example. Two dice are rolled, what's the probability to have 6 points in sum (event A), 7 points in sum (event B)?
Wrong solution. An elementary outcome could be a combination $\{x, y\}$

$x \backslash y$	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁		4	5	6	7	8
⚂			6	7	8	9
⚃				8	9	10
⚄					10	11
⚅						12

$N(\Omega) = 21,$

Example. Two dice are rolled, what's the probability to have 6 points in sum (event A), 7 points in sum (event B)?

Wrong solution. An elementary outcome could be a combination $\{x, y\}$

$x \backslash y$	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁		4	5	6	7	8
⚂			6	7	8	9
⚃				8	9	10
⚄					10	11
⚅						12

$N(\Omega) = 21, N(A) = 3, N(B) = 3,$

$$\mathrm{P}(A) = \mathrm{P}(B) = 3/21.$$

These elementary outcomes are not equally possible!

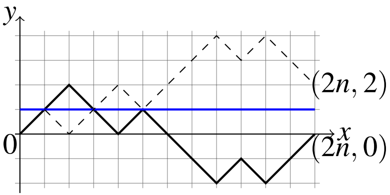
Change problem

Example. There are $2n$ people in a waiting line at a box office. n of them have \$ 1 coins while the others have 50 cents coins. A ticket costs 50 cents. Every one in the line needs exactly one ticket. The cashbox is empty initially. What is the probability that no person will wait for change?

Change problem

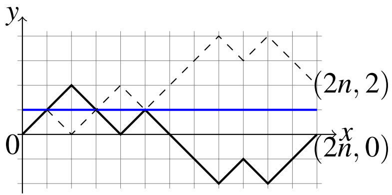
Example. There are $2n$ people in a waiting line at a box office. n of them have $ 1 coins while the others have 50 cents coins. A ticket costs 50 cents. Every one in the line needs exactly one ticket. The cashbox is empty initially. What is the probability that no person will wait for change?

Solution. All permutations of $2n$ people are equally likely to occur, thus there are C_{2n}^{n} elementary outcomes. Each outcome can be represented graphically.



Assume ticket buyers are situated at points 1, 2, …, $2n$ of the horizontal axis. The empty cashbox is at the origin. If a person has a dollar, the line goes one step up, otherwise it goes one step down. At both ends the line has zero y-coordinate. Favorable lines are those which do

Change problem (cont.)



Let's compute the number of lines which reach or cross the line $y = 1$. These are the only lines which are favorable to the opposite event, when someone will wait for the change.

For each of these lines draw a dummy line. It coincides with the original line till the first hit of the line $y = 1$, then it mirrors the original line. Any dummy line then starts at the origin and ends at the point $(2n, 2)$. It consists of $n + 1$ steps up and $n - 1$ steps down. Thus there are C_{2n}^{n-1} dummy lines. Finally there are $C_{2n}^{n} - C_{2n}^{n-1}$ lines favorable to our event, and the probability is

$$p = \frac{C_{2n}^{n} - C_{2n}^{n-1}}{C_{2n}^{n}} = \frac{1}{n+1}$$