N.I. Lobachevsky State University
of Nizhni Novgorod

# Probability theory and mathematical statistics:

## Conditional probability. Independence. Law of total probability Practice

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# Main formulas

Classical probability:

$$\mathrm{P}(A) = \frac{N(A)}{N(\Omega)}$$

Conditional probability:

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

Total probability law:

$$\mathrm{P}(A) = \mathrm{P}(H_1)\mathrm{P}(A|H_1) + \mathrm{P}(H_2)\mathrm{P}(A|H_2) + \dots$$

We roll a die $n$ times. Let $A_{i,j}$ be the event that the $i$th and $j$th rolls produce the same number. Show that events $A_{i,j}$ and $A_{j,k}$, $i < j < k$, are independent. Show that events $A_{i,j}$, $A_{j,k}$, $A_{i,k}$ are not independent.

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Here

$$\Omega = \{(x_1, x_2, \dots, x_n) \colon x_1 = \overline{1,6}, x_2 = \overline{1,6}, \dots, x_n = \overline{1,6}\},$$

$$A_{i,j} = \{(x_1, x_2, \dots, x_n) \colon x_1 = \overline{1,6}, x_2 = \overline{1,6}, \dots, x_n = \overline{1,6}, x_i = x_j\},$$

$$N(\Omega) = 6^n, \quad N(A_{i,j}) = 6^{n-1}, \quad \mathrm{P}(A_{i,j}) = \frac{6^{n-1}}{6^n} = \frac{1}{6}.$$

$$A_{i,j} \cap A_{j,k} = \{(x_1, x_2, \ldots, x_n) \colon x_1 = \overline{1,6}, x_2 = \overline{1,6}, \ldots,$$
$$x_n = \overline{1,6}, x_i = x_j = x_k\},$$
$$N(A_{i,j} \cap A_{j,k}) = 6^{n-2}, \quad \mathrm{P}(A_{i,j} \cap A_{j,k}) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = \mathrm{P}(A_{i,j})\mathrm{P}(A_{j,k}).$$

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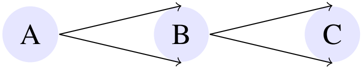
$$A = \{(x_1, x_2) \colon x_1 = r, x_2 = \overline{1,6}\}, \quad \mathrm{P}(A) = \frac{N(A)}{N(\Omega)} = \frac{6}{36} = \frac{1}{6},$$

$$B = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}, \quad \mathrm{P}(B) = \frac{1}{6}$$

$$A \cap B = \{(r, 7 - r)\}, \qquad \mathrm{P}(A \cap B) = \frac{1}{36} = \mathrm{P}(A)\mathrm{P}(B)$$

There are two roads from A to B and two roads from B to C, Each of the four roads is blocked by snow with probability $p$, independently of the others. Find the probability that there is an open road from A to B given that there is no open route from A to C.



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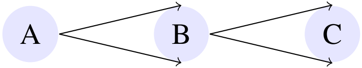


Let event $H_{AB}$ occur if there's an open road between $A$ and $B$, event $H_{BC}$ occur if there's an open road between $B$ and $C$, event $H_{AC}$ occur if there's an open road between $A$ and $C$. Observe that $H_{AB}$ and $H_{BC}$ are independent and $H_{AC} = H_{AB} \cap H_{BC}$. We have to compute

$$\mathrm{P}(H_{AB} | \overline{H_{AC}})$$

Let event $H_{AB,1}$ occur if the first road from A to B is open, $H_{AB,2}$ occur if the second road from A to B is open, then $H_{AB} = H_{AB,1} \cup H_{AB_2}$. Since $H_{AB,1}$ and $H_{AB,2}$ are independent,

$$\mathrm{P}(H_{AB}) = (1-p) + (1-p) - (1-p)^2 = 1 - p^2$$

Similary, $\mathrm{P}(H_{BC}) = 1 - p^2$.
By definition of conditioinal probability,

$$\mathrm{P}(H_{AB}|\overline{H_{AC}}) = \frac{\mathrm{P}(H_{AB} \cap \overline{H_{AC}})}{\mathrm{P}(\overline{H_{AC}})} = \frac{\mathrm{P}(H_{AB} \cap \overline{H_{BC}})}{1 - \mathrm{P}(H_{AC})},$$

$$\mathrm{P}(H_{AB} \cap \overline{H_{BC}}) = \mathrm{P}(H_{AB})\mathrm{P}(\overline{H_{BC}}) = (1-p^2)(1-(1-p^2)),$$

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$$\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A|B) = (1 - \mathrm{P}(\bar{B}))\mathrm{P}(A|B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

A doctor assumes that a patient has one of three diseases $d_1$, $d_2$, or $d_3$. Before any test, he assumes an equal probability for each disease. He carries out a test that will be positive with probability 0,8 if the patient has $d_1$, 0,6 if he has disease $d_2$, and 0,4 if he has disease $d_3$. What is the probability that the test it positive?

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Hypothesis $H_i$ means that disease $d_i$ is present.
$\mathrm{P}(H_1) = \mathrm{P}(H_2) = \mathrm{P}(H_3) = \frac{1}{3}$. Event $A$ occurs if the test is positive.
$\mathrm{P}(A|H_1) = 0{,}8$, $\mathrm{P}(A|H_2) = 0{,}6$, $\mathrm{P}(A|H_3) = 0{,}6$.

$$\mathrm{P}(A) = \frac{1}{3} \cdot 0{,}8 + \frac{1}{3} \cdot 0{,}6 + \frac{1}{3} \cdot 0{,}4 = 0{,}6.$$

You have two urns. There are 5 red balls and 5 blue balls in one urn, 6 red balls and 10 blue balls in the other urn. One ball is popped from the second urn and put into the first urn. Then two balls were taken from the first urn with replacement. What's the probability that both balls were red?

We need two hypotheses: $H_1$ means that a red ball was popped from the secondurn, $H_2$ means that a blue ball was popped from the second urn,

$$\mathrm{P}(H_1) = \frac{6}{16}, \qquad \mathrm{P}(H_2) = \frac{10}{16}.$$

Let event $A$ occur if two balls taken from the first urn are red.

$$\mathrm{P}(A|H_1) = \frac{6 \cdot 6}{11 \cdot 11}, \quad \mathrm{P}(A|H_2) = \frac{5 \cdot 5}{11 \cdot 11},$$

$$\mathrm{P}(A) = \frac{6}{16} \cdot \frac{36}{121} + \frac{10}{16} \cdot \frac{25}{121} = \frac{233}{968}$$

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$H_{rr}$ — the two balls were both red, $H_{rb}$ — one of the two balls was red and the other was blue, $H_{bb}$ — both balls were blue,

$$\mathrm{P}(H_{rr}) = \frac{C_6^2}{C_{16}^2}, \quad \mathrm{P}(H_{rb}) = \frac{C_6^1 C_{10}^1}{C_{16}^2}, \quad \mathrm{P}(H_{bb}) = \frac{C_{10}^2}{C_{16}^2}$$

$$\mathrm{P}(A|H_{rr}) = \frac{7}{12}, \quad \mathrm{P}(A|H_{rb}) = \frac{6}{12}, \quad \mathrm{P}(A|H_{bb}) = \frac{5}{12},$$

$$\mathrm{P}(A) = \frac{23}{48}.$$

A target consists of three parts of areas $S_1$, $S_2$, and $S_3$, correspondingly. The probability for a missile to hit a particular part is proportional to the part's area. If the first part is hit the probability of damage is $p_1$, if the second part is hit the probability of damage is $p_2$ and finally the probability is $p_3$ for the third part. What is the probability of damaging the target?

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Event $H_i$ occurs if the $i$th part is hit, $A$ if the target is damaged.

$$\mathrm{P}(H_i) = \frac{S_i}{S_1 + S_2 + S_3}.$$

# Bayes's formula

**Theorem**

Let $H_1, H_2, \dots$ be hypotheses, then

$$P(H_i|A) = \frac{P(H_i)P(A|H_i)}{P(A)}$$

Prove the theorem.

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Prove the theorem.

**Proof.**

$$P(H_i|A) = \frac{P(H_i \cap A)}{P(A)} = \frac{P(H_i)P(A|H_i)}{P(A)}$$

In London, half of the days have some rain. The weather forecaster is correct 2/3 of the time, i.e., the probability that it rains, given that she has predicted rain, and the probabiltiy that it does not rain, given that she has predicted that it won't rain, are both equal to 2/3. When rain is forecast, Mr. Pickwick takes his umbrells. When rain is not forecast, he takes it with probability 1/3. Find
(a) the probability that Pickwick has no umbrella, given that it rains
(b) the probability that it doesn't rain, given that he brings his umbrella.