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Probability theory and mathematical statistics:

Determinism and Randomness

Experiments, events, outcomes

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Outline

1. Experiment types.
2. Events and outcomes.
3. Logical relations among events.

Any science has certain intuitive foundations which are not strictly defined. For example points, lines, and planes are the fundamentals in geometry. A force, a mass, a path etc are the fundamentals in physics.
In the theory of probability such intuitive fundamentals are *an experiment* (*an observation*, *a trial*), *outcomes* (*events*), *statistical stability*, and, finally, *probability*.
To describe an experiment we need to state all of its conditions.
Depending on the conditions we can deduce a set of possible (admissible) outcomes.

Example.

Experiment is tossing a coin three times. Its conditions are:

1. It's a 5 roubles coin
2. We hit the coin with a thumb
3. We toss three times
4. The coin falls on a flat solid desk.
5. We do in a well-lighted room to see what happens
6. ...

The most important conditions are 1–5. Less important conditions are in "...". Evident outcomes are HHH, HHT, HTH, ..., TTT, "at least one heads", "even number of tails", ...

1. *Deterministic* experiments. If we know the conditions, we can predict what happens, which outcome to expect. Repeating an experiment several times we can observe a result well-known *a priori*. E.g., classical mechanics investigates movements of systems of bodies. Initial positions of the bodies and their velocities determine uniquely their position at any time instant in the future.

   Semi-deterministic experiments. We can predict the future but not the past. E.g., heat distribution.

2. *Statistically stable random* experiments. We cannot predict results of such experiments. Let's toss a coin several times, say n times. Denote by $\mu(T,n)$ the number of the tails side occurrences in n tossings.

Who	n	$\mu(T,n)$	$\frac{\mu(T,n)}{n}$
G.-L. Buffon,	4040	2048	0,5080
(XVIII c., French naturalist)			
K. Pearson,	12000	6019	0,5016
(XIX-XX c., English statistician)	24000	12012	0,5005

We see that the ratio $\frac{\mu(T,n)}{n}$ *oscillates* near 0,5. This ratio is called *frequency* of tails.

Experiment types (ended)

3. *Badly random* experiments. E.g., a certain candidate will be elected as the next President of Russia on March, 2nd. It will rain on May, 9 2008. We still can't make a prediction, but we cannot repeat these experiments many times. Another reason could be bad behaviour of frequencies.

Dice tossing example

Two dice are tossed. Possible events (outcomes):

- Both dice are ⚀'s
- The first die is ⚀ and the second is ⚁
- The first die is ⚀ and the second is not ⚁
- There's at least one of ⚁, ⚃.
- The sum of points is even.
- The greatest number of points is 2
- One of the following combinations: ⚀⚁, ⚁⚀, ⚁⚁
- …

Last two are the same! There are $2^{36} = 68\,719\,476\,736$ possible outcomes with two dice (explanations later :-))

Denoting events and outcomes

Along with descriptions or long names for events we use shorter names. We use capital letters of the Latin alphabet (with upper and lower indices sometimes) for that. For example:

- A = "Both dice are ⚀'s"
- B = "First die is ⚀ and the second is ⚁"
- C = "First die is ⚀ and the second is not ⚁"
- D = "There's a least one of ⚁, ⚃"
- D_1 = "The sum of points is even"
- D_2 = "The maximum number of points on the dice is 2"
- D_3 = "Occurs either ⚀⚁, or ⚁⚀, or ⚁⚁"

We see that events D_2 and D_3 are equivalent.

Elementary outcomes

To deal with huge number of possible events (sometimes an infinite number of events), we select a smaller number of outcomes which we call *elementary* outcomes. A set of elementary outcomes should satisfy the following axioms:

- Some elementary outcome happens every time we repeat the experiment or an observation
- If an elementary outcome happened, no other elementary outcome is possible *in this repetition* of the experiment.
- Every event in our experiment is defined someway with use of elementary outcomes

Elementary outcomes (cont.)

Let $E_1, E_2, \ldots, E_n, \ldots$ be the elementary outcomes in our experiment. We can write down these events in many different ways. In the dice example, we used drawings of points, we could also use Arabic numbers 1, 2, 3, 4, 5, 6, or Roman numbers I, II, III, IV, V, VI, or Chinese hieroglyphs, or English words ONE, TWO, ...
Let E_1 be encoded as ω_1, E_2 as ω_2, ..., and we'll write

$$E_1 = \{\omega_1\}, \quad E_2 = \{\omega_2\}, \quad \ldots.$$

(We're using mathematical sets here!). The set of all codes denote by

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n, \ldots\}.$$

Coin tossing example

A coin is tossed three times and it falls on a desk.

$$\begin{aligned}\Omega = \{\omega_1 = (H,H,H), \quad & \omega_2 = (H,H,T), \quad \omega_3 = (H,T,H), \\ \omega_4 = (H,T,T), \quad & \omega_5 = (T,H,H), \quad \omega_6 = (T,H,T), \\ \omega_7 = (T,T,H), \quad & \omega_8 = (T,T,T)\}.\end{aligned}$$

Here $\omega_5 = (T,H,H)$ means the only tails was in the first tossing.

Lottery example

There are 100 tickets in a lottery, only one ticket wins a Porsche. An elementary outcome is encoded with ticket number, $\omega_n = n$ where $n = 1, 2, \ldots, 100$,

$$\Omega = \{1, 2, \ldots, 100\}.$$

Two dice example

Two dice are tossed and the number of points on the top of each is observed.

$$\begin{aligned}\Omega = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ &(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\end{aligned}$$

Here $\omega_i = (x, y)$ for each $i = 1, 2, \ldots, 36$, x is the number of point on the top of the first die, and y is the number of points on the top of the second die. An elementary outcome here is **an arrangement** of two numbers.

Infinite coin tossing

A coin is tossed until a Heads comes out.

$$\Omega = \{\omega_0 = \mathrm{TT}\ldots, \omega_1 = H, \omega_2 = TH, \ldots, \omega_n = \underbrace{TTT\ldots TT}_{n-1 \text{ times}} H, \ldots\}$$

Here $\omega_0 = \mathrm{TT}\ldots$ means Heads never occurs, $\omega_2 = \mathrm{TH}$ means Heads comes out in the second toss, etc. We do not expect the outcome $\{\omega_0\}$ to occur in reality but it's necessary due to its logical possibility.

Cards game

A pile of 52 cards is divided between four players. We observe 13 cards received by the first player. We can enumerate all the cards:

Suit / rank	Ace	2	3	...	King
$\clubsuit$	1	2	3	...	13
$\diamondsuit$	14	15	16	...	26
$\heartsuit$	27	28	29	...	39
$\spadesuit$	40	41	42	...	52

Each elementary outcome here is **a combination** of 13 cards out of 52:

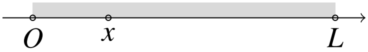
$$\omega_n = \{x_1, x_2, \dots, x_{13}\}, \quad x_i \in \{1, 2, \dots, 52\}, \quad i = 1, 2, \dots, 13.$$

There are $C_{52}^{13} = 635\,013\,559\,600$ elementary outcomes,

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_{635\,013\,559\,600}\}$$

A pole of length L is broken at a random point. Denote by x the distance from the left end of the pole to the place of break. Then we can put $\omega = x$,

$$\Omega = \{\omega = x \colon 0 \leqslant x \leqslant L\}.$$



Three balls: red (R), green (G) and blue (B) are distributed in three boxes. An arrangement of balls is observed. Let $\omega_1 = (RGB; -; -)$ denote such an arrangement that all three balls are in the first box.

$$\begin{aligned}\Omega = \{&(RGB; -; -), (-; RGB; -), (-; -; RGB), (RG; B; -),\\ &(RB; G; -), (GB; R; -), (RG; -; G), (RB; -; G), (GR; -; R),\\ &(R; GB; -), (G; RB; -), (B; RG; -), (R; -; GB), (G; -; RB),\\ &(B; -; RG), (-; RG; B), (-; RB; G), (-; GB; R), (-; R; GB)\\ &(-; G; RB), (-; B; RG), (R; G; B), (R; B; G), (G; R; B),\\ &(G; B; R), (B; R; G), (B; G; R)\}.\end{aligned}$$

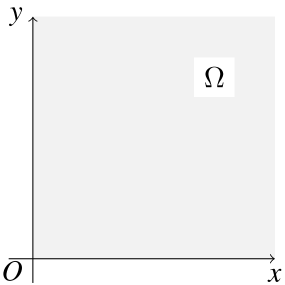
Three indistinguishable balls are distributed in three boxes.

$$\Omega = \{(BBB; -; -), (-; BBB; -), (-; -; BBB), (BB; B; -), (BB; -; B), (B; BB; -), (B; -; BB), (-; BB; B), (-; B; BB), (B; B; B)\}.$$

Three indistinguishable balls are distributed in tree indistinguishable boxes.

$$\Omega = \{\{BBB; -; -\}, \{BB; B; -\}, \{B; B; B\}\}$$

An insurance company is interested in the age distribution of couples. Let x stand for the age of the husband, y for the age of the wife. Elementary outcome is described with the pair (x, y) and Ω is the first quadrant of the x, y-plane.



Event as a set

If all of elementary outcomes $E'_1, E'_2, \dots$ with codes $\omega'_1, \omega'_2, \dots$ can be observed together with an event A, write $A = \{\omega'_1, \omega'_2, \dots\}$.

Definition

A subset $A = \{\omega'_1, \omega'_2, \dots\}$ of Ω is called *random event*.
Ω is called *the space of elementary outcomes, or* the certain event (event which is observed with certainty).
An empty set $\varnothing$ is called *the impossible event* (it never can be observed!).

Logical relations among events

Definition

If in every repetition of an experiment an event B is always observed together with event A, we say that A *implies* B, A is a *particular case* of B, A is a *part* of B, and write $A \subset B$.

Example. We roll one die. A is “we have ⚁”, B is ”we have even number of points”, then A implies B, $A \subset B$.

Definition

If A implies B and B implies A, then A and B are *equivalent*, write $A = B$.

Example. In 6 shots, $A =$ ”three hits” is equivalent to $B =$ ”three missed”,

Definition

Two events are said to be *mutually exclusive*, *disjoint*, if they can never be observed in the same repetition of an experiment.

Example. Any two elementary outcomes $\{\omega\}$, $\{\omega'\}$ are mutually exclusive.

Definition

An event which happens if and only if an event A hasn't happened is called *opposite* to A, write $\bar{A}$.

Definition

A *union* of events A and B is an event C which is observed if at least one of A and B occurs, i.e. either A, or B, or both occur. Write $C = A \cup B$

Definition

A *intersection* of events A and B is an event D which occurs if and only if both A and B occur. write $D = A \cap B$.

Verify!

$$A \cup B = B \cup A, \quad A \cap B = B \cap A, \tag{1}$$

$$\bar{\bar{A}} = A, \quad \bar{\Omega} = \varnothing, \quad \bar{\varnothing} = \Omega, \tag{2}$$

$$A \cup A = A, \quad A \cup \Omega = \Omega, \quad A \cup \varnothing = A, \tag{3}$$

$$A \cap A = A, \quad A \cap \Omega = A, \quad A \cup \varnothing = \varnothing, \tag{4}$$

$$A \cup B = \overline{\bar{A} \cap \bar{B}}, \quad A \cap B = \overline{\bar{A} \cup \bar{B}} \tag{5}$$