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Probability theory and mathematical statistics:

Law of Total Probability. Method of recurrent equations

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Outline

1. Law of total probability
2. Application to problems in classical probability (comparision)
3. Use of recurrent relations in connection with law of total probability
4. Ruin problem
5. Problem in runs

Sometimes an experiment consists of two stages and the outcome of the first stage is not observed directly. We are interested only in the result of the second stage. We are watching for an event A. The following theorem shows how to compute the probability of A, $\mathrm{P}(A)$, averaging over all possible outcomes of the intermediate stage.

Theorem

Let $H_1, H_2, \ldots, H_n, \ldots$ be a finite or denumerable sequence of mutualually exclusive events with positive probability each, and $A \subset H_1 \cup H_2 \cup \ldots \cup H_n \cup \ldots$. Then

$$\mathrm{P}(A) = \mathrm{P}(H_1)\mathrm{P}(A|H_1) + \mathrm{P}(H_2)\mathrm{P}(A|H_2) + \ldots + \mathrm{P}(H_n)\mathrm{P}(A|H_n) + \ldots$$

Proof. Events $A \cap H_1, A \cap H_2, \ldots, A \cap H_n, \ldots$ are mutually exclusive and $A = (A \cap H_1) \cup (A \cap H_2) \cup \ldots \cup (A \cap H_n) \cup \ldots$. Then

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}((A \cap H_1) \cup (A \cap H_2) \cup \ldots (A \cap H_n) \cup \ldots) \\ &= \mathrm{P}(A \cap H_1) + \mathrm{P}(A \cap H_2) + \ldots + \mathrm{P}(A \cap H_n) + \ldots \\ &= \mathrm{P}(H_1)\mathrm{P}(A|H_1) + \mathrm{P}(H_2)\mathrm{P}(A|H_2) + \ldots \mathrm{P}(H_n)\mathrm{P}(A|H_n) + \ldots \end{aligned}$$

Example. There are m red balls and n blue balls in an urn. Two balls are taken out one by one without replacement. What's the probability the second ball is blue (event A)?

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Solution by classical formula

We can use classical formula here. An outcome is an arrangement of two balls without repetitions. Favorable outcomes are those with the second ball of the blue color. The first ball can be either blue or red.

$$N(\Omega) = A^2_{m+n} = (m+n)(m+n-1), \quad N(A) = n(n-1) + mn,$$

So,

$$\mathrm{P}(A) = \frac{n(n-1)+mn}{(m+n)(m+n-1)} = \frac{n}{m+n}$$

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Then

$$\begin{aligned}\mathrm{P}(A) &= \mathrm{P}(H_1)\mathrm{P}(A|H_1) + \mathrm{P}(H_2)\mathrm{P}(A|H_2) \\ &= \frac{m}{n+m} \cdot \frac{n}{m+n-1} + \frac{n}{m+n} \cdot \frac{n-1}{m+n-1} = \frac{n}{m+n}\end{aligned}$$

Example. There 5 urns, two of them contain 2 red balls and 1 blue ball, another two contain 3 red balls and 1 blue ball, and the last one contains 10 blue balls. An urn is picked at random and one ball is taken out. What's the probability that the ball is red?

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Solution.
Let event H_1 occur when one of the first two urns is chosen, H_2 occur when the third or the fourth urs is chosen, H_3 occur when the fifth urn is chosen. $\mathrm{P}(H_1) = \frac{2}{5}$, $\mathrm{P}(H_2) = \frac{2}{5}$, $\mathrm{P}(H_3) = \frac{1}{5}$, $\mathrm{P}(A|H_1) = \frac{2}{3}$, $\mathrm{P}(A|H_2) = \frac{3}{4}$, $\mathrm{P}(A|H_3) = 0$. Then

$$\mathrm{P}(A) = \frac{2}{5} \cdot \frac{2}{3} + \frac{2}{5} \cdot \frac{3}{4} + \frac{1}{5} \cdot 0 = \frac{17}{30}$$

There are m red balls and n blue balls in an urn. r balls are taken out one by one without replacement, $r \leqslant m+n$. Prove that the probability for the last ball to be blue is $\frac{n}{m+n}$.

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Denote by $\pi(r; m, n)$ the probability in question. It's obvious that

$$\pi(1; m, n) = \frac{n}{m+n}.$$

Now assume that

$$\pi(2; m, n) = \pi(3; m, n) = \ldots = \pi(r-1; m, n) = \frac{n}{m+n}$$

We have to prove that

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If H_1 occurs, then only $n+m-1$ balls left, n of them are blue, and we still have to take $r-1$ balls out, so the conditional probability that the last ball is blue given that the first is red equals $\pi(r-1; m-1, n)$.

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If H_2 occurs, then only $n+m-1$ balls left, $n-1$ of them are blue, and we are to take $r-1$ balls out, so the conditional probability that the last ball is blue given that the first is blue equals $\pi(r-1; m, n-1)$.

By assumption,

$$\pi(r-1;m-1,n)=\frac{n}{m+n-1},$$

$$\pi(r-1;m,n-1)=\frac{n-1}{m+n-1}.$$

According to the law of total probability,

$$\begin{aligned}\pi(r;m,n)&=\mathrm{P}(H_1)\pi(r-1;m-1,n)+\mathrm{P}(H_2)\pi(r-1;m,n-1)\\&=\frac{m}{n+m}\cdot\frac{n}{m+n-1}+\frac{n}{m+n}\cdot\frac{n-1}{m+n-1}=\frac{n}{m+n}\end{aligned}$$

Ruin problem

Assume Ann and Bart play Heads and Tails. If Ann wins, she pays one rouble to Bart. If Bart wins, he pays one rouble to Ann. In the beginning Ann has a roubles, Bart has b roubles. What's the probability Ann loses all her money? Ann's ruin means that Ann has no more roubles. The same holds for Bart's ruin.

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The total sum of roubles is $a + b$. Denote by $\pi(x)$ probabilty of Ann's ruin having x roubles initially. If she wins a toss, her fortune becomes $x + 1$ roubles, otherwise $x - 1$ roubles. She wins a toss with probability $\frac{1}{2}$ and loses with probability $\frac{1}{2}$. If has x roubles and she wins a toss, whether she gets ruined or not is defined only by the future tosses which are independent of the past tosses, so the ruin probability after that is $\pi(x + 1)$.

By the total probability formula,

$$\pi(a) = \frac{1}{2}\pi(x+1) + \frac{1}{2}\pi(x-1) \tag{1}$$

besides that,

$$\pi(0) = 1 \quad \text{and} \quad \pi(a+b) = 0. \tag{2}$$

From (1),

$$\pi(x) - \pi(x-1) = \pi(x+1) - \pi(x) = c$$

for a constant c, and

$$\pi(x) = \pi(0) + xc.$$

Using (2), obtain

$$\pi(x) = 1 - \frac{x}{a+b}$$

Thus, Ann gets ruined with probability $\pi(a) = \frac{b}{a+b}$.

A problem in runs

In any ordered sequence of elements of two kinds, each maximal subsequence of elements of like kind is called *a run*. For example, the sequence $xxxyxxyyyx$ opens with an x run of length 3; it is followed by runs of length 1, 2, 3, 1, respectively.

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Let α and β be positive integers, and consider a potentially unlimited sequence in independent trials, such as tossing a coin or throwing dice. Symbol x occurs in a single trial with probability p, symbol y occurs in a single trial with probability $q = 1 - p$. What's the probability that a run of α consecutive x's will occur before a run of β consecutive y's?

Since the trials are independent, the probability of the sequence

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equals

$$p \cdot p \cdot p \cdot q \cdot p \cdot p \cdot q \cdot q \cdot q \cdot p = p^6 q^4.$$

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Let A be the event that *a run of α consecutive x's occurs before a run of β consecutive y's*. Denote $x = \mathrm{P}(A)$. Let

- $u = \mathrm{P}(A|$"the first trial results in x"$)$,
- $v = \mathrm{P}(A|$"the first trial results in y"$)$.

Then

$$x = pu + qv.$$

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Hence

$$u = p^{\alpha-1} + qv(1 + p + \ldots + p^{\alpha-2})$$

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We have two equations for the two unknown u and v:

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For example, in tossing a coin ($p = \frac{1}{2}$) the probability that a run of two heads appears before a run of three tails is 0,7.