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Probability theory and mathematical statistics:

Conditional Probability, Independence

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Outline

1. Conditional experiment, conditional probabilities
2. Two ways to compute conditional probabilities
3. Properties of conditional probabilities
4. Algorithm for sampling random points from tricky plane figures
5. Multiplication theorem for probabilities
6. Examples of usage of multiplication theorem
7. Dependence and independence of random events

Suppose we have already a probability space $(\Omega, \mathfrak{F}, \mathrm{P})$ for our experiment, and we add additional condition: an event $B \in \mathfrak{F}$ occurs, $\mathrm{P}(B) > 0$. Now we need a probability model for the new situation. Options:

- Construct new probability space $(\Omega_B, \mathfrak{F}_B, \mathrm{P}_B)$.
- Keep Ω, $\mathfrak{F}$ and find relation between P and P_B.

P_B is called *conditional probability on the hypothesis B*, or *conditional probability assuming B*. P is called *absolute probability*.

Constructing new probability space

Let $\Omega_B = B$, $\mathfrak{F}_B = \{A \cap B \colon A \in \mathfrak{F}\} = \mathfrak{F} \cap B$, P_B is chosen anew (classical probability, geometric probability, etc.)

Example. Two dice are rolled. What's the probability that the first die throws even number of point (event A) given that the sum is 8 (event B)?

$$B = \{(2,6),(3,5),(4,4),(5,3),(6,2)\},$$

$$A \cap B = \{(2,6),(4,4),(6,2)\},$$

$$\mathrm{P}_B(A) = \frac{3}{5}$$

Relation between P and P_B

Keep Ω, $\mathfrak{F}$ and define $\mathrm{P}_B(A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Theorem

P_B is a probability (in Kolmogorov's sense).

Proof. 1) $\mathrm{P}_B(A) \geqslant 0$. 2) $\mathrm{P}_B(\Omega) = \frac{\mathrm{P}(\Omega \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(B)}{\mathrm{P}(B)} = 1$. 3) Let $A_1, A_2, \ldots$ be a sequence of mutually exclusive events,

$$\mathrm{P}_B(A_1 \cup A_2 \cup \ldots) = \frac{\mathrm{P}((A_1 \cup A_2 \cup \ldots) \cap B)}{\mathrm{P}(B)}$$

$$= \frac{\mathrm{P}((A_1 \cap B) \cup (A_2 \cap B) \cup \ldots)}{\mathrm{P}(B)}$$

$$= \frac{\mathrm{P}(A_1 \cap B) + \mathrm{P}(A_2 \cap B) + \ldots}{\mathrm{P}(B)} = \mathrm{P}_B(A_1) + \mathrm{P}_B(A_2) + \ldots$$

In certain cases the concordance between the two approaches can be proved. Let Ω consist of finite number of equally likely elementary outcomes. Then

$$\mathrm{P}_B(A) = \frac{N(A \cap B)}{N(B)} = \frac{N(A \cap B)/N(\Omega)}{N(B)/N(\Omega)} = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

where $\mathrm{P}(\cdot)$ is a classical probability.

$P_B(A)$ is often denoted $P(A|B)$. It's useful when B is a complicated expression. For example, $P(A|\bar{B})$ looks better than $P_{\bar{B}}(A)$.
All properties of general probability are true for conditional probability. For example,

$$P(A|B) + P(\bar{A}|B) = 1,$$

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B).$$

It's not true that

$$P(A|B) + P(A|\bar{B}) = 1! \quad :-)$$

When $P(B) = 0$, conditional probability $P(A|B)$ is **undefined**.

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Solution. Draw a square G around the triangle G_1. Let a side of the square equal a. Select a point in the square: generate random number x between 0 and a (`double x=(a*rand())/RAND_MAX`), generate random number y between 0 and a, check if the point belongs to the triangle. If the point lies outside the triangle, get another point.
Let $\Omega =$ "the point belongs to the square",
$A =$ "the point belongs to a region g inside the triangle",
$B =$ "the point is inside the triangle", inside the triangle,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{\text{area of } g}{\text{area of } G}}{\frac{\text{area of } G_1}{\text{area of } G}} = \frac{\text{area of } g}{\text{area of } G_1}$$

From

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

immediately obtain a multiplication formula for probabilities

$$\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A|B)$$

Let $\mathrm{P}(B \cap C) > 0$, then

$$\mathrm{P}(A \cap B \cap C) = \mathrm{P}(A|B \cap C)\mathrm{P}(B \cap C) = \mathrm{P}(A|B \cap C)\mathrm{P}(B|C)\mathrm{P}(C).$$

Here $\mathrm{P}(B|C)$ is defined since $B \cap C \subset C$ and $0 < \mathrm{P}(B \cap C) \leqslant \mathrm{P}(C)$.

Theorem

Let $A_1, A_2, \ldots, A_n$ be events from $\mathfrak{F}$, and $\mathrm{P}(A_1 \cap A_2 \cap \ldots \cap A_n) > 0$. Then

$$\mathrm{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathrm{P}(A_1)\mathrm{P}(A_2|A_1)\mathrm{P}(A_3|A_1 \cap A_2) \cdots \mathrm{P}(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}).$$

To prove the theorem it is sufficient to have a weaker condition: $\mathrm{P}(A_1 \cap A_2 \cap \ldots \cap A_{n-1}) > 0$. But when $\mathrm{P}(A_1 \cap A_2 \cap \ldots \cap A_n) > 0$, all $n!$ different forms hold. For example,

$$\mathrm{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathrm{P}(A_n)\mathrm{P}(A_{n-1}|A_n)\mathrm{P}(A_{n-2}|A_n \cap A_{n-1}) \cdots \mathrm{P}(A_1|A_n \cap A_{n-1} \cap \ldots \cap A_2).$$

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Finally,

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B|A) = \frac{ab}{(a+b)(a+b-1)}.$$

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Let the balls be numbered 1 through a be blue while the others be red, $\omega = (x, y)$ where x is the number on the first ball, y is the number on the second ball,

$$\Omega = \{(1,2), (1,3), \dots, (a+b, a+b-1)\},$$

$$A \cap B = \{(a+1,1), (a+1,2), \dots, (a+b, a)\},$$

$$N(\Omega) = (a+b)(a+b-1), \quad N(A \cap B) = ba,$$

$$\mathrm{P}(A \cap B) = \frac{ab}{(a+b)(a+b-1)}$$

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Let

$$A_1 = \text{"The first card holds the letter I"},$$

$$A_2 = \text{"The second card holds the letter C"},$$

$$A_3 = \text{"The third card holds the letter E"},$$

$$\mathrm{P}(A_1 \cap A_2 \cap A_3) = \mathrm{P}(A_1)\mathrm{P}(A_2|A_1)\mathrm{P}(A_3|A_1 \cap A_2) = \frac{1}{7} \cdot \frac{1}{6} \cdot \frac{1}{5} = \frac{1}{210}$$

Urn models for aftereffect

Consider an industrial plant liable to accidents. The occurrence of an accident might be pictured as the result of a superhuman game of chance: Fate has in storage an urn containing red and black balls; at regular time intervals a ball is drawn at random, a red ball signifying an accident. If each accident has an aftereffect it corresponds to an urn whose composition changes according to certain rules that depend on the outcome of the successive drawings.

Polya's urn model for aftereffects

An urn contains b black and r red balls. A ball is drawn at random. It is replaced and, moreover, c balls of the color drawn are added. A new random drawing is made from the urn (now containing $r+b+c$ balls), and this procedure is repeated. This a rough model of phemonena such as *contagious deseases*, where each occurrence increases the probability of further occurrences.
The probability that of $n = n_1 + n_2$ drawings the first n_1 ones result in black balls and the remaining n_2 ones in red balls is given by

$$\frac{b}{b+r}\cdot\frac{b+c}{b+r+c}\cdots\frac{b+(n_1-1)c}{b+r+(n_1-1)c}\cdot\frac{r}{b+r+n_1c}\cdots\frac{r+(n_2-1)c}{b+r+nc}$$

Statistical independence of two events

Definition

Events A and B are called *independent* if $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.

What is *indeed* independent of what?

Let $\mathrm{P}(B) > 0$. Then

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} =$$

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$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A)\mathrm{P}(B)}{\mathrm{P}(B)} = \mathrm{P}(A)$$

In other words, occurrence of B doesn't change the probability of A, i.e. conditional and absolute probabilities of A are equal.

Theorem

If A and B are independent, then A and $\bar{B}$, $\bar{A}$ and B, $\bar{A}$ and $\bar{B}$ are independent.

Proof. Let's prove $\bar{A}$ and B are independent. We have:

$$\begin{aligned}
\bar{A} \cap B &= B \setminus (A \cap B), \\
\mathrm{P}(\bar{A} \cap B) &= \mathrm{P}(B) - \mathrm{P}(A \cap B) = \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B) \\
&= \mathrm{P}(B)(1 - \mathrm{P}(A)) = \mathrm{P}(B)\mathrm{P}(\bar{A}).
\end{aligned}$$

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\end{aligned}$$

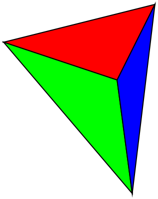
Let's prove A and $\bar{B}$ are independent in another way.

$$\begin{gathered}
B \cup \bar{B} = \Omega, \quad A \cap (B \cup \bar{B}) = A, \\
\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap \bar{B}) = \mathrm{P}(A)\mathrm{P}(B) + \mathrm{P}(A \cap \bar{B}), \\
\mathrm{P}(A \cap \bar{B}) = \mathrm{P}(A) - \mathrm{P}(A)\mathrm{P}(B) = \mathrm{P}(A)\mathrm{P}(\bar{B}).
\end{gathered}$$

Bernstein's example

Assume there are events A, B, and C, such that any two of them are independent. Is it true that all three of them are independent?

Example. Consider a regular tetrahedron (a pyramid whose faces are equal regular triangles). Let one face be blue, the second face red, the third face green and the fourth face have all three colors on it. The tetrahedron is dropped on a desk. We look at the lower face.



$$\begin{aligned} A &= \text{"the lower face contains red color"}, \\ B &= \text{"the lower face contains blue color"}, \\ C &= \text{"the lower face contains green color"}, \end{aligned}$$

$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ where ω_i is the i-th face of the tetrahedron,
$A = \{\omega_1, \omega_4\}$, $B = \{\omega_2, \omega_4\}$, $C = \{\omega_3, \omega_4\}$,

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$$\mathrm{P}(A) = \frac{2}{4}, \quad \mathrm{P}(B) = \frac{2}{4}, \quad \mathrm{P}(C) = \frac{2}{4},$$

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$$A \cap B = \{\omega_4\}, \quad \mathrm{P}(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{4} = \mathrm{P}(A)\mathrm{P}(B),$$

hence A and B are independant. Also A and C, B and C are independent.

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$$A \cap B \cap C = \{\omega_4\}, \quad \mathrm{P}(A \cap B \cap C) = \frac{1}{4} \neq \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C)$$

Definition

Events $A_1, A_2, \ldots, A_n$ are called *independent*, when for any $s = 2, 3, \ldots, n$ and for any events $A_{i_1}, A_{i_2}, \ldots, A_{i_s}$ from the set $\{A_1, A_2, \ldots, A_n\}$

$$\mathrm{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_s}) = \mathrm{P}(A_{i_1})\mathrm{P}(A_{i_2}) \cdots \mathrm{P}(A_{i_s})$$

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It is not sufficient for independence of the events $A_1, A_2, \ldots, A_n$ to assume that $\mathrm{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathrm{P}(A_1)\mathrm{P}(A_2)\ldots\mathrm{P}(A_n)$.

Example. Two dice are rolled. Events are: 1, 2, or 5 are thrown on the second die (event A), 4, 5, or 6 is thrown on the second die (event B), 9 is thrown on both dice (event C).

Then $\mathrm{P}(A) = \mathrm{P}(B) = \frac{18}{36}$, $\mathrm{P}(C) = \frac{4}{36}$, $\mathrm{P}(A \cap B) = \frac{6}{36} \neq \mathrm{P}(A)\mathrm{P}(B)$, $\mathrm{P}(A \cap B \cap C) = \frac{1}{36} = \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C)$.

What's the probability that a milling machine working at time t_0 won't stop until time $t_0 + t$, under the following assumptions: 1) this probability depends only on t. 2) probability that the milling machine stops during time slice Δt is asymptitically proportional to Δt (as $\Delta t \to 0$)?

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Denote $p(t)$ the probability in question. The probability that the milling machine stops during time slice $\Delta t \to 0$, equals

$$1 - p(\Delta t) = a\Delta t + o(\Delta t)$$

where a is a constant. By multiplication theorem,

$$p(t + \Delta t) = p(t) \cdot p(\Delta t) = p(t)(1 - a\Delta t - o(\Delta t)),$$

$$\frac{p(t+\Delta t)-p(t)}{\Delta t} = -ap(t) - o(1),$$

$$\frac{dp(t)}{dt} = -ap(t).$$

Solution of this differential equation is

$$p(t) = C\exp\{-at\}$$

To find C we use obvious condition $p(0) = 1$. We obtain

$$p(t) = \exp\{-at\}$$