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Probability theory and mathematical statistics:

Independent trials — Practice

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Main formulas

Binomial coefficient:

$$C_n^k = \frac{n!}{k!(n-k)!}$$

Binomial probabilities, Bernoulli's formula:

$$b(k; n, p) = C_n^k p^k (1-p)^{n-k}$$

Multinomial probabilities:

$$\text{multi}(n_1, \dots, n_r; n, p_1 \dots, p_r) = \frac{n!}{n_1! \cdots n_r!} p_1^{n_1} \cdots p_r^{n_r}$$

Hyper-geometric probabilities:

$$\frac{C_M^k C_{N-M}^{r-k}}{C_N^r}$$

Problem. Ten coins are flipped. What's the probability to have 7 Heads?

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Solution.

$$b\left(7; 10, \frac{1}{2}\right) = C_{10}^{7}\left(\frac{1}{2}\right)^{7}\left(\frac{1}{2}\right)^{10-7} = \frac{15}{128} \approx 0{,}1171875$$

Problem. Five dice are rolled. What is the probability that only one even number of points appears?

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Solution. Even number of points appears with the probability $\frac{3}{6} = \frac{1}{2}$.

$$b\left(1; 5, \frac{1}{2}\right) = C_5^1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{5-1} = \frac{5}{32} = 0{,}15625$$

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Solution. We have no successes with the probability

$$1 - 0{,}5904 = 0{,}4096.$$

On the other hand the probability of all failures equals $b(0; 4, p) = (1 - p)^4$. From equation

$$(1 - p)^4 = 0{,}4096$$

obtain $1 - p = 0{,}8$ and $p = 0{,}2$.
Now, $b(2; 4, 0{,}2) = 0{,}1536$, $b(3; 4, 0{,}2) = 0{,}0256$.

Problem. 3 missiles are launched at a target. Each missile hits the target with probability 0,4. A single missile disables the target with the probability 0,3. Two missiles disable the target with the probability 0,7, three or more missiles disable the target surely. What is the probability to disable the target?

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Solution. We don't know how many missiles hit the target. We need to introduce hypotheses. Denote the event that i missiles hit the target by H_i. By Bernoulli's formula,

$$\mathrm{P}(H_1) = C_3^1 \cdot 0{,}4 \cdot 0{,}6^2 = 0{,}432; \quad \mathrm{P}(H_2) = C_3^2 \cdot 0{,}4^2 \cdot 0{,}6^1 = 0{,}288;$$

$$\mathrm{P}(H_3) = C_3^3 \cdot 0{,}4^3 \cdot 0{,}6^0 = 0{,}064;$$

Event A occurs if the target is disabled.

$$\mathrm{P}(A|H_1) = 0{,}3, \quad \mathrm{P}(A|H_2) = 0{,}7, \quad \mathrm{P}(A|H_3) = 1.$$

By the law of total probability,

$$\mathrm{P}(A) = \mathrm{P}(H_1)\mathrm{P}(A|H_1) + \mathrm{P}(H_2)\mathrm{P}(A|H_2) + \mathrm{P}(H_3)\mathrm{P}(A|H_3) = 0{,}3952.$$

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Solution. We have Bernoulli trials with $n = 10$ and $p = 0{,}15$. Let event A occur if at least one device failed to pass the test, event B occur when exactly 3 devices failed to pass the test. Then

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} = \frac{b(3; 10, 0{,}15)}{1 - b(0; 10, 0{,}15)} = 0{,}1616605440738407$$

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Solution.
The probability that a single bus eats too much fuel is $p = 0{,}2$.

$$\sum_{k=15}^{100} b(k; n, p) = 1 - \sum_{k=0}^{14} b(k; n, p) = 0{,}919556278861954$$

$$\sum_{k=0}^{14} b(k; n, p) \approx b(14; 100, 0.2) \frac{(100 - 14 + 1) \cdot 0.2}{(100 + 1) \cdot 0{,}2 - 14}$$

$$= 0{,}0941044872910341$$

$$1 - \sum_{k=0}^{14} b(k; n, p) \approx 0{,}9058955127089658$$

Use spreadsheets (OpenOffice Calc, Microsoft Excel, etc)!

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Solution. We have $n = 12$ independent trials with 6 equiprobable outcomes.

$$\text{multi}(2, 2, 2, 2, 2, 2; 12, 1/6, 1/6, 1/6, 1/6, 1/6, 1/6) =$$
$$= \frac{12!}{(2!)^2}\left(\left(\frac{1}{6}\right)^2\right)^6 = \frac{1925}{34992} = 0{,}05501257430269776$$

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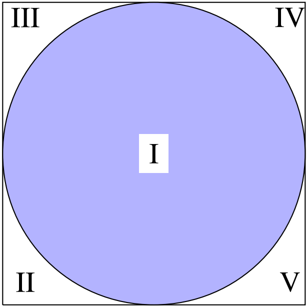
Solution.



Let the square's side equal a. A point belongs to area I with the probability

$$\frac{\pi a^2}{4a^2} = \frac{\pi}{4};$$

it belongs to the areas II, III, IV, V with equal probability

$$\frac{(a^2 - \pi a^2)/4}{4a^2} = \frac{1-\pi}{16}.$$

Finally the probability in question is

$$\text{multi}\left(1,1,1,1,1;5,\frac{\pi}{4},\frac{1-\pi}{16},\frac{1-\pi}{16},\frac{1-\pi}{16},\frac{1-\pi}{16}\right) = \frac{5!}{(1!)^5}\frac{\pi}{4}\left(\frac{1-\pi}{16}\right)^4$$