N.I. Lobachevsky State University
of Nizhni Novgorod

Probability theory and mathematical statistics:

Combinatorics — Practice

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3. In a 'Everything for Tea' store there are 5 different teacups and 3 different tea-spoons. How many ways are there to buy a teacup and a teaspoon? Answer: $5 \cdot 3 = 15$.
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4. We call a number 'nice' if only odd digits are in it (e.g. 113, 599, etc.). How many nice numbers are there with 2 digits? with 5 digits? Answer: since there are only 5 odd digits 1, 3, 5, 7, and 9, there are $5 \cdot 5 = 25$ nice numbers with 2 digits, there are $5^5 = 3125$ nice numbers with 4 digits.

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5. Martian alphabet has 3 letters in it: A, B, and C. Any word has 4 letters or less. How many words are there in Martian language? Answer: We compute separately one-lettered words, two-lettered words, three-lettered words, and four-lettered words. Totally, $3 + 3^2 + 3^3 + 3^4 = 120$ words.

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7. How many ways are there to place one white rook and one black rook on a chessboard, if the rooks don't attack each other? Answer: black rook can take any of 64 squares. It attacks 15 more squares. Now, only $64 - 15 = 49$ spare squares for the white rook. There are $64 \cdot 49 = 3136$ possible positions.

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8. A *word* is a string of letters. How many words can you make with the letters V, E, C, T, O, R? Answer: Since all the letters differ from each other, any permutation of the letters makes a word. There are $6! = 720$ possible words.

9. How many words can you make with the letters L, E, T, T, E, R?

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Solution 1: We start with placing T's, then we place E's, then the rest. There are $C_6^2 = 15$ ways to choose places for T's. Each choice gives $C_4^2 = 6$ ways to choose places for E's. Finally there are 2! ways to assign places for the letters L and R. Totally, there are $15 \cdot 6 \cdot 2 = 180$ words.

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Solution 2: Denote by N the number of words in quistion. Enumerate for now both E's and T's as E_1, E_2, T_1 and T_2 to make them unique. Now the letters are all different. $6! = 720$ words can be made of these letters. These words can be divided into N classes. Each class contains words which correspond to the same word without subscripts. For example, the class for ERLTTE contains the words $E_1RLT_2T_1E_2$, $E_2RLT_2T_1E_1$, etc. There are $2! \cdot 2! = 4$ words in each class, we have an equation $4N = 720$, whence $N = 180$.

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9. How many words can you make with the letters L, E, T, T, E, R?

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10. In how many ways can four letters be put into four envelopes, one into each? Answer: Each possible way to do that is a permutation of four letters. The total number of permutations of four letters is $4! = 24$.

11. In how many ways can six children form themselves into a ring, to dance round a may-pole? (May-pole is a wooden pole with many long coloured ribbons suspended from the top)

We have not to assign the six children to particular places absolutely, but only to arrange them relatively to one another. We may make all possible arrangements, by placing the first child in any fixed position and disposing others in in different ways with respect to him. Now the five children can be arranged in $5! = 120$ ways. This, therefore, is the whole number of ways in which a ring can be formed.

12. There are 20 cities in the Far-Far-Away Land and every two of them are connected with an airline. How many airlines are there in the Far-Far-Away Land?

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Solution 1: An airline connects exactly two cities. There are $C_{20}^{2} = \frac{20 \cdot 19}{2 \cdot 1} = 190$ ways to take 2 cities out of 20.

Solution 2: Every city holds 19 airlines, there 20 cities. If we sum airlines in all cities, we'll have $19 \cdot 20 = 380$ airlines, but every airline is counted twice here, thus there are $380/2 = 190$ airlines.

13. Three schoolboys must be delegated from a class of 30 pupils to a Math Contest. How many teams can be chosen?

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14. Ann has 6 DVDs and Bart has another 10 DVDs. They decide to exchange 3 DVDs. How many ways are there to do that?

13. Three schoolboys must be delegated from a class of 30 pupils to a Math Contest. How many teams can be chosen? Answer: $C_{30}^{3} = \frac{30\cdot29\cdot28}{3\cdot2\cdot1} = 4060.$

14. Ann has 6 DVDs and Bart has another 10 DVDs. They decide to exchange 3 DVDs. How many ways are there to do that? Answer: There are two stages. First Ann selects her DVDs to give them away, $C_6^3 = 20$ possibilities. Second Bart selects his DVDs to give them away, $C_{10}^3 = 120$ possibilities. By multiplication theorem there are $C_6^3 C_{10}^3 = 2400$ ways to exchange 3 DVDs.

15. 2 girls and 7 boys visit a chess club. To participate in a tournament a team of 4 persons must be elected. At least one girl must be included in the team. How many ways are there to select a team?

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Answer: divide all possible teams into to classes. Let the first class have teams with only one girl while the second has teams with both girls. There are $C_2^1 C_7^3 = 2 \cdot 35 = 70$ teams with one girl and $C_2^2 C_7^2 = 21$ teams with two girls. The total number of ways to elect a team is $70 + 21 = 91$.

16. Two chess players play 10 games. In how many ways is it possible that 3 games are won by the first player, 5 games are won by the second player and 2 games are draws?

16. Two chess players play 10 games. In how many ways is it possible that 3 games are won by the first player, 5 games are won by the second player and 2 games are draws? Answer: first we choose games among 10 games where the first player wins, C_{10}^{3} ways to do that, next we choose games among $10 - 3 = 7$ games where the second player wins, C_{7}^{5} ways to do that, so that 2 games left for a draw. In total, there are $C_{10}^{3}C_{7}^{5} = 2520$ possible game histories.