N.I. Lobachevsky State University
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Probability theory and mathematical statistics:

Random variables — Practice

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Problem. Probability distribution of a random variable is given by a table

a	1	2	3
$\mathrm{P}(X = a)$	0.1	?	0.3

What is $\mathrm{P}(X = 2)$?

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Solution. Probabilities should sum up to 1. So,
$\mathrm{P}(X = 2) = 1 - 0.1 - 0.3 = 0.6.$

Problem. A lottery contains 100 tickets. A ticket costs 70 roubles. There's one prize ticket of 15 thousand roubles, two prize tickets of 10 thousand roubles each, and 5 prize tickets of 5 thousand roubles. Find the distribution of profit from buying one ticket.

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Solution. Enumerate all tickets from 1 to 100 so that ticket #1 is the 15 thousand prize, #2 and #3 are 10 thousand prizes, #4 thru #8 are 5 thousand prizes. A ticket is an elementary outcome, $\Omega = \{1, 2, \ldots, 100\}$. Let random variable $X = X(\omega)$ be the profit from buying one ticket. Then $X(1) = 15\,000 - 70 = 14\,930$,
$X(2) = X(3) = 10\,000 - 70 = 9\,930$,
$X(4) = X(5) = X(6) = X(7) = X(8) = 5\,000 - 70 = 4\,930$,
$X(9) = X(10) = \ldots = X(100) = -70$.

profit in roubles	14 930	9 930	4 930	−70
probability	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{5}{100}$	$\frac{92}{100}$

Problem. A car crosses 3 crossroads controlled by traffic lights. Each traffic light has green light on with the probability 0,5. Find the probability distribution for: a) the number of green lights before the first red light, b) the total number of green lights.

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Solution 1. a) Event A_i occurs if ith traffic light is green, $i = 1, 2, 3$. Events A_1, A_2, A_3 are independent, $\mathrm{P}(A_i) = \frac{1}{2}$. Denote by X the number of green lights before the first red light. Then

$$\{X = 0\} = \bar{A}_1, \quad \mathrm{P}(X = 0) = \frac{1}{2}$$

$$\{X = 1\} = A_1 \cap \bar{A}_2, \quad \mathrm{P}(X = 1) = \mathrm{P}(A_1)\mathrm{P}(\bar{A}_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4},$$

$$\{X = 2\} = A_1 \cap A_2 \cap \bar{A}_3, \quad \mathrm{P}(X = 2) = \frac{1}{8},$$

$$\{X = 3\} = A_1 \cap A_2 \cap A_3, \quad \mathrm{P}(X = 3) = \frac{1}{8}$$

a	0	1	2	3
$\mathrm{P}(X = a)$	0.5	0.25	0.125	0.125

a	0	1	2	3
$P(X = a)$	0.5	0.25	0.125	0.125

b) Denote by Y the total number of green lights. Y has the binomial distribution with $n = 3, p = \frac{1}{2}$. So,

$$P(Y = k) = C_3^k \left(\frac{1}{2}\right)^3$$

a	0	1	2	3
$P(X = a)$	0.125	0.375	0.375	0.125

Solution 2. Let's construct the probability space and define all random variable point-wise. "0" means that a traffic light is red, "1" means green light.

ω	$p(\omega) = \mathrm{P}(\{\omega\})$	$X(\omega)$	$Y(\omega)$
$(0,0,0)$	$\left(\frac{1}{2}\right)^2 = \frac{1}{8}$	0	0
$(0,0,1)$	$\frac{1}{8}$	0	1
$(0,1,0)$	$\frac{1}{8}$	0	1
$(0,1,1)$	$\frac{1}{8}$	0	2
$(1,0,0)$	$\frac{1}{8}$	1	1
$(1,0,1)$	$\frac{1}{8}$	1	2
$(1,1,0)$	$\frac{1}{8}$	2	2
$(1,1,1)$	$\frac{1}{8}$	3	3

$\mathrm{P}(X=0) = \mathrm{P}(\{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\}) = 4 \cdot \frac{1}{4} = \frac{1}{2}$,
$\mathrm{P}(Y=0) = \mathrm{P}(\{(0,0,0)\} = \frac{1}{8}$, etc.

Problem. A hunter has four bullets. He begins shooting and a target runs away, so the probability of hitting it decreases. Assume the probability for the first shot to be successful 0.8, for the second shot 0.7, for the third shot 0.6 and for the forth shot 0.5. Random variable X is the number of shots until a success or all bullets run out. Find probability distribution of X.

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Solution. Event A_i means that ith shot is successful, events A_1, A_2, A_3 are independent. $\{X = 1\} = A_1$, $\{X = 2\} = \bar{A}_1 \cap A_2$, $\{X = 3\} = \bar{A}_1 \cap \bar{A}_2 \cap A_3$, $\{X = 4\} = \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$,

a	1	2	3	4
$\mathrm{P}(X = a)$	0.8	$0.2 \cdot 0.7$	$0.2 \cdot 0.3 \cdot 0.6$	$0.2 \cdot 0.3 \cdot 0.4$

Problem. There are 2 broken electronic devices among 20. 4 devices are selected. Find the probability distribution of the number of broken devices among selected.

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Solution. There can be 0, 1, or 2 brocken devices. An outcome is an unordered sample of 4 items out of 20. Use classical probability:

a	0	1	2
$\mathrm{P}(X=a)$	$\frac{C_2^0 C_{18}^4}{C_{20}^4}=\frac{12}{19}$	$\frac{C_2^1 C_{18}^3}{C_{20}^4}=\frac{32}{95}$	$\frac{C_2^2 C_{18}^2}{C_{20}^4}=\frac{3}{95}$

The number of broken devices has hyper-geometric distribution.

Problem. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathrm{P}(\{\omega_1\}) = \mathrm{P}(\{\omega_2\}) = \mathrm{P}(\{\omega_3\}) = \frac{1}{3}$. Define

$$X(\omega_1) = 1, \quad X(\omega_2) = 2, \quad X(\omega_3) = 3$$

$$Y(\omega_1) = 2, \quad Y(\omega_2) = 3, \quad Y(\omega_3) = 1$$

$$Z(\omega_1) = 3, \quad Z(\omega_2) = 1, \quad Z(\omega_3) = 2$$

Show that all random variables are identically distributed, Find probability distributions for variables $X + Y$, $Y + Z$, and $Z + X$.

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Show that all random variables are identically distributed, Find probability distributions for variables $X + Y$, $Y + Z$, and $Z + X$.

Solution. From definition,

a	1	2	3
$\mathrm{P}(X = a)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\mathrm{P}(Y = a)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\mathrm{P}(Z = a)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

ω	$X(\omega)$	$Y(\omega)$	$Z(\omega)$	$X(\omega)+Y(\omega)$	$Y(\omega)+Z(\omega)$	$Z(\omega)+X(\omega)$
ω_1	1	2	3	3	5	4
ω_2	2	3	1	5	4	3
ω_3	3	1	2	4	3	5

So,

a	3	4	5
$\mathrm{P}(X+Y=a)$	$\mathrm{P}(\{\omega_1\})=\frac{1}{3}$	$\mathrm{P}(\{\omega_2\})=\frac{1}{3}$	$\mathrm{P}(\{\omega_3\})=\frac{1}{3}$
$\mathrm{P}(Y+Z=a)$	$\mathrm{P}(\{\omega_3\})=\frac{1}{3}$	$\mathrm{P}(\{\omega_2\})=\frac{1}{3}$	$\mathrm{P}(\{\omega_1\})=\frac{1}{3}$
$\mathrm{P}(Z+X=a)$	$\mathrm{P}(\{\omega_2\})=\frac{1}{3}$	$\mathrm{P}(\{\omega_1\})=\frac{1}{3}$	$\mathrm{P}(\{\omega_3\})=\frac{1}{3}$

Problem. The first urn contains 5 blue balls and 5 red balls, the second urn contains 4 blue balls and 5 red balls. One ball is moved from the first urn to the second urn. Then 3 balls are sample with replacement from the second urn. Find the distribution of the number of blue balls among 3 taken from the second urn.

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Solution. We need to introduce hypotheses and use the law of total probability. Hypothesis H_r occurs if a ball from the first urn was red, H_b occurs if a ball from the first urn was blue. $\mathrm{P}(H_r) = \mathrm{P}(H_b) = \frac{5}{10}$. Denote the number of blue balls taken from the second urn by X. Given H_r occured, the number of blue balls has binomial distribution with $n = 3$ and $p = \frac{4}{10}$, and under hypothesis H_b binomial distribution with $n = 3$ and $p = \frac{5}{10}$.

By law of total probability,

$$\begin{aligned}
\mathrm{P}(X=0) &= \mathrm{P}(H_r)\mathrm{P}(X=0|H_r)+\mathrm{P}(H_b)\mathrm{P}(X=0|H_b) = \\
&= \frac{5}{10}C_3^0\left(\frac{4}{10}\right)^0\left(\frac{6}{10}\right)^3+\frac{5}{10}C_3^0\left(\frac{5}{10}\right)^0\left(\frac{5}{10}\right)^3 = 0.1705, \\
\mathrm{P}(X=1) &= \mathrm{P}(H_r)\mathrm{P}(X=1|H_r)+\mathrm{P}(H_b)\mathrm{P}(X=1|H_b) = \\
&= \frac{5}{10}C_3^1\left(\frac{4}{10}\right)^1\left(\frac{6}{10}\right)^2+\frac{5}{10}C_3^1\left(\frac{5}{10}\right)^1\left(\frac{5}{10}\right)^2 = 0.4035, \\
\mathrm{P}(X=2) &= \mathrm{P}(H_r)\mathrm{P}(X=2|H_r)+\mathrm{P}(H_b)\mathrm{P}(X=2|H_b) = \\
&= \frac{5}{10}C_3^2\left(\frac{4}{10}\right)^2\left(\frac{6}{10}\right)^1+\frac{5}{10}C_3^2\left(\frac{5}{10}\right)^2\left(\frac{5}{10}\right)^1 = 0.3315, \\
\mathrm{P}(X=3) &= \mathrm{P}(H_r)\mathrm{P}(X=3|H_r)+\mathrm{P}(H_b)\mathrm{P}(X=3|H_b) = \\
&= \frac{5}{10}C_3^3\left(\frac{4}{10}\right)^3\left(\frac{6}{10}\right)^0+\frac{5}{10}C_3^0\left(\frac{5}{10}\right)^3\left(\frac{5}{10}\right)^0 = 0.0945,
\end{aligned}$$