N.I. Lobachevsky State University
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Probability theory and mathematical statistics:

Variance and friends — Practice

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Problem. From the table of joint probability distribution of random variables X and Y compute

i) $\mathbf{E}\,X$, $\operatorname{Var} X$;

ii) $\mathbf{E}\,Y$, $\operatorname{Var} Y$;

iii) $\mathbf{E}(X-2Y)$, $\operatorname{Var}(X-2Y)$;

iv) $\operatorname{cov}(X,Y)$.

$X \setminus Y$	-1	0	1
-1	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{7}{24}$
1	$\frac{5}{24}$	$\frac{1}{6}$	$\frac{1}{8}$

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Solution: see example in the lecture

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Solution. We recognize Bernoulli trials. The number of hits follows binomials distribution with parameters $n = 3$, $p = \frac{3}{5}$. By formulas from Lecture 11 we have

$$\mathbf{E}\,X = np = 3 \cdot \frac{3}{5} = 1.8,$$
$$\operatorname{Var} X = np(1-p) = 0.72$$

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Solution. Let us denote by B_i the event “the target is hit at i-th shot“. Let X be the number of shots before hit. Then

$$\mathrm{P}(X=0)=\mathrm{P}(B_1)=\frac{3}{5}, \quad \mathrm{P}(X=1)=\mathrm{P}(\bar{B}_1\cap B_2)=\frac{2}{5}\cdot\frac{3}{5},$$
$$\mathrm{P}(X=2)=\mathrm{P}(\bar{B}_1\cap\bar{B}_2\cap B_3)=\left(\frac{2}{5}\right)^2\cdot\frac{3}{5},$$
$$\mathrm{P}(X=3)=\mathrm{P}(\bar{B}_1\cap\bar{B}_2\cap\bar{B}_3)=\left(\frac{2}{5}\right)^2$$

Then $\mathbf{E}\,X$ and $\operatorname{Var}X$ can be found by regular formulas.

Problem. Let $\mathrm{P}(X = 0) = \frac{1}{3}$, $\mathrm{P}(X = 1) = \frac{1}{2}$, $\mathrm{P}(X = -1) = \frac{1}{6}$. Find the correlation coefficient of X and X^2.

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Solution. We have

$$\mathbf{E}\,X=-1\cdot\frac{1}{6}+0\cdot\frac{1}{3}+1\cdot\frac{1}{2}=\frac{1}{3},$$

$$\mathbf{E}\,X^2=(-1)^2\cdot\frac{1}{6}+0^2\cdot\frac{1}{3}+1^2\cdot\frac{1}{2}=\frac{2}{3},$$

$$\mathbf{E}(X\cdot X^2)=\mathbf{E}\,X^3=\mathbf{E}\,X,\qquad \mathbf{E}\,X^4=\mathbf{E}\,X^2.$$

$$\mathrm{Var}(X^2)=\mathbf{E}(X^2)^2-(\mathbf{E}\,X^2)^2=\mathbf{E}(X^4)-(\mathbf{E}\,X^2)^2=\ldots,$$

$$\mathrm{cov}(X,X^2)=\mathbf{E}(X\cdot X^2)-(\mathbf{E}\,X)(\mathbf{E}\,X^2)=\cdots,$$

$$\mathrm{corr}(X,X^2)=\frac{\mathrm{cov}(X,X^2)}{\sqrt{\mathrm{Var}\,X\cdot\mathrm{Var}\,X^2}}.$$

Problem. Let X and Y be independent random variables, $P(X=1)=P(X=-1)=\frac{1}{2}$, $P(Y=1)=P(Y=-1)=\frac{1}{4}$, $P(Y=0)=\frac{1}{2}$. Are the random variables XY and Y

i) independent?

ii) uncorrelated?

Solution. XY can be -1, 0, or 1. For instance,

$$P(XY=-1, Y=-1)=P(X=1, Y=-1)=P(X=1)P(Y=-1)=\frac{1}{8},$$

etc. The joint distribution of XY and Y is

$XY \backslash Y$	-1	0	1
-1	$\frac{1}{8}$	0	$\frac{1}{8}$
0	0	$\frac{1}{2}$	0
1	$\frac{1}{8}$	0	$\frac{1}{8}$

Since $P(XY=-1, Y=0)=0 \neq P(XY=-1)P(Y=0)=\frac{2}{8}\cdot\frac{1}{2}>0$, the random variables XY and Y are statistically dependent.

Then compute the

$$\begin{aligned}\operatorname{cov}(XY, Y) = \mathbf{E}(XY \cdot Y) - \mathbf{E}(XY)\,\mathbf{E}\,Y = \mathbf{E}\,X\,\mathbf{E}(Y^2) - \mathbf{E}\,X(\mathbf{E}\,Y)^2 = \\ = \mathbf{E}\,X\,\mathbf{E}\,Y(1 - \mathbf{E}\,Y) = \dots\end{aligned}$$

If this equals zero then XY and Y are uncorrelated, otherwise correlated.

Problem. A coin is tossed three times. Random variables are:
X — the total number of Heads,
Y — the total number of Tails,
Z — the total number of faces changes (for example, the sequence HTH contains 2 faces changes).
Compute $\operatorname{Var} X$, $\operatorname{Var} Y$, $\operatorname{Var} Z$, $\operatorname{cov}(X, Y)$, $\operatorname{cov}(X, Z)$.

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With three tosses there are 8 elementary outcomes: HHH, HHT, ..., TTT. Calculate the values of X, Y, and Z for each outcome. It gives you the possible values of the random variables and favorable cases. Then write down tables with single-variable probability distributions and joind probability distributione. From these compute finally what is required.