N.I. Lobachevsky State University
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Probability theory and mathematical statistics:

Random vectors — Practice

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Problem. From the table of the joint distribution

$x_i \setminus y_j$	-1	0	1
-2	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
2	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{12}$

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Solution.

$$\begin{aligned}\mathrm{P}(X=-2) &= \mathrm{P}(X=-2, Y=-1) + \mathrm{P}(X=-2, Y=0) \\ &+ \mathrm{P}(X=-2, Y=1) = \frac{1}{2} \\ \mathrm{P}(X=2) &= \mathrm{P}(X=2, Y=-1) + \mathrm{P}(X=2, Y=0) \\ &+ \mathrm{P}(X=2, Y=1) = \frac{1}{2}\end{aligned}$$

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\mathrm{P}(X + Y = -3) &= \mathrm{P}(X = -3, Y = -1) = \frac{1}{8}, \\
\mathrm{P}(X + Y = -2) &= \mathrm{P}(X = -2, Y = 0) = \frac{1}{4}, \\
\mathrm{P}(X + Y = -1) &= \mathrm{P}(X = -2, Y = 1) = \frac{1}{8}, \\
&\cdots
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The joint probability distribution of random variables X and Y is given in the following table:

$x_i \setminus y_j$	0	1	2	3
-1	0.1265625	0.1265625	0.0421875	0.0046875
1	0.2953125	0.2953125	0.0984375	0.0109375

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Solution. Marginal probability distributions are:

y_j	0	1	2	3
$\mathrm{P}(Y = y_j)$	0.421875	0.421875	0.140625	0.015625

x_i	-1	1
$\mathrm{P}(X = x_i)$	0.3	0.7

With means of exhausting calculations we verify that

$$\mathrm{P}(X = x_i, Y = y_j) = \mathrm{P}(X = x_i)\mathrm{P}(Y = y_j).$$

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Solution. From definition of X and Y,

ω	⚀	⚁	⚂	⚃	⚄	⚅
$X(\omega)$	1	0	0	0	0	0
$Y(\omega)$	0	0	0	0	0	1

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So,

$$\mathrm{P}(X = 0, Y = 0) = \frac{4}{6},$$

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For example, $\mathrm{P}(X = 1, Y = 1) \neq \mathrm{P}(X = 1)\mathrm{P}(Y = 1)$, thus the random variables are dependent.

Problem. Let X_1, X_2, …, X_k be independent identically distributed (i.i.d) random variables with uniform distribution on the set $\{1, 2, \ldots, n\}$. Find the probability distribution of $Y = \min\{X_1, X_2, \ldots, X_k\}$.

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Solution.
From

$$\{Y = m\} = \{Y > m - 1\} \setminus \{Y > m\}$$

and

$$\{Y > m\} \subset \{Y > m - 1\}$$

conclude that

$$\mathrm{P}(Y = m) = \mathrm{P}(Y > m - 1) - \mathrm{P}(Y > m).$$

For $m = 0, 1, \ldots, m$.

$$\begin{aligned}
P(Y > m) &= P(\min\{X_1, X_2, \ldots, X_k\} > m) \\
&= P(X_1 > m, X_2 > m, \ldots, X_k > m) \\
&= P(X_1 > m)P(X_2 > m) \cdots P(X_k > m) \\
&= \left(\frac{n-m}{n}\right)^k
\end{aligned}$$

Hence

$$P(Y = m) = \frac{(n-m+1)^k - (n-m)^k}{n^k}.$$

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Solution.

$$\mathrm{P}(X=m)=\mathrm{P}(Y=m)=(1-p)^{m-1}p, \quad m=1,2,\ldots$$

Now, for $n=2,3,\ldots$

$$\begin{aligned}\mathrm{P}(X+Y=n)&=\sum_{k=1}^{n-1}\mathrm{P}(X=k)\mathrm{P}(Y=n-k)\\&=\sum_{k=1}^{n-1}p^2(1-p)^{k-1}(1-p)^{n-k-1}=(n-1)p^2(1-p)^{n-2},\end{aligned}$$

$$\mathrm{P}(X = k | X + Y = n) = \frac{\mathrm{P}(X = k, X + Y = n)}{\mathrm{P}(X + Y = n)} =$$

$$= \frac{p(1-p)^{k-1} p(1-p)^{n-k-1}}{(n-1)p^2(1-p)^{n-2}} = \frac{1}{n-1}.$$

Conditional distribution of X given $X + Y = n$ is uniform on the set $\{1, 2, \dots, n-1\}$.

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Solution. We have 3 independent trials with 3 events of interest: E_1 —a die shows 6 points, E_2 — a die shows 2 or 4 points, E_3 — a die shows 1, or 3, or 5 points, with probabilities $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$ in a single trial. For $0 \leqslant k \leqslant m \leqslant 3$. Event $\{X = k, Y = m\}$ means that there are k occurrences of event E_1, $m - k$ occurrences of event E_2, and $3 - m$ occurrences of E_3.

$$\mathrm{P}(X = k, Y = m) = \frac{3!}{k!(m-k)!(3-m)!}\left(\frac{1}{6}\right)^{k}\left(\frac{2}{6}\right)^{m-k}\left(\frac{3}{6}\right)^{3-m}$$

due to multinomial formula.

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Obviously, Y can be either 0, or 1, or 2 and X can be any number from 0 to $3 - X$. We'll use classical probability here. There are C_{12}^3 elementary outcomes in total Consider event $\{X = k, Y = m\}$, $0 \leqslant m \leqslant 3, 0 \leqslant k \leqslant 3 - m$.