1.9 Partitioned matrices.

這節要談矩陣的分割. 先用個例子來看看.

Example 84: The matrix

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 4 & 1 & 3 \\ 2 & 4 & 0 & 2 & 1 & 5 \\ \hline -1 & -2 & 3 & 0 & 0 & 1 \end{array}\right]$$

can be written as the 2×3 partitioned (block) matrix

$$A = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$$

where

$$B_{11} = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 4 & 0 \end{bmatrix} \qquad B_{12} = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} \qquad B_{13} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$B_{21} = \begin{pmatrix} -1 & -2 & 3 \end{pmatrix}. \qquad B_{22} = \begin{pmatrix} 0 & 0 \end{pmatrix} \qquad B_{23} = \begin{pmatrix} 1 \end{pmatrix}$$

這在簡化矩陣的寫法上相當有用, 尤其是在下一章談行列式時更有用. 底下我們也順帶看看 partitioned matrix 的乘法.

Example 85:

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

The partitions of A and B are comfortable for block multiplication

$$AB = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{pmatrix}$$

$$A_{11}B_1 = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{pmatrix} = \begin{pmatrix} 15 & 12 \\ 2 & -5 \end{pmatrix}$$

$$A_{12}B_2 = \begin{pmatrix} 0 & -4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -20 & -8 \\ -8 & 7 \end{pmatrix}$$

$$\Longrightarrow A_{11}B_1 + A_{12}B_2 = \begin{pmatrix} 15 & 12 \\ 2 & -5 \end{pmatrix} + \begin{pmatrix} -20 & -8 \\ -8 & 7 \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ -6 & 2 \end{pmatrix}.$$

Similarly,

$$A_{21}B_1 + A_{22}B_2 = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

$$\Longrightarrow AB = \left[\begin{array}{cc} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right]$$

會用 partitioned matrix 的一個重要原因是矩陣太大時，求 inverse 及 determinant 都會很複雜，但利用 partitioned matrix 可以將討論的矩陣變小，求 inverse 及 determinant 會比較方便。

Example 86: A matrix of the form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}.$$

is said to be block upper triangular.

Assume that $B \in M_{p\times p}(\mathbb{R})$, $C \in M_{p\times q}(\mathbb{R})$, $D \in M_{q\times q}(\mathbb{R})$.

Find A^{-1}.

Solution: Suppose

$$A^{-1} = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}.$$

$$\Rightarrow AA^{-1} = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$$

$$\Rightarrow \begin{cases} BM + CP = I_p \\ BN + CQ = 0 \\ DP = 0 \\ DQ = I_q \end{cases} \qquad (B, C, D: \text{已知},\ \text{求}\ M, N, P, Q)$$

$DQ = I_q \Rightarrow D, Q:$ invertible and $Q = D^{-1}$

$DP = 0 \Rightarrow P = D^{-1}0 = 0.$

$BM + CP = I_p \Rightarrow BM = I_p \Rightarrow B, M:$ invertible and $M = B^{-1}$

$$\begin{aligned} BN + CQ = 0 &\Rightarrow BN + CD^{-1} = 0 \\ &\Rightarrow BN = -CD^{-1} \\ &\Rightarrow N = -B^{-1}CD^{-1} \end{aligned}$$

Thus,

$$A^{-1} = \begin{pmatrix} B^{-1} & -B^{-1}CD^{-1} \\ 0 & D^{-1} \end{pmatrix}.$$

Example 87: Let

$$A = \left(\begin{array}{cc|cc} 1 & 2 & 3 & 5 \\ 0 & 1 & 4 & 2 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow B^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow D^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$-B^{-1}CD^{-1} = -\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= -\begin{pmatrix} -5 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -7 \\ -2 & 0 \end{pmatrix}$$

Then

$$A^{-1} = \left(\begin{array}{cc|cc} 1 & -2 & 6 & -7 \\ 0 & 1 & -2 & 0 \\ \hline 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{array}\right)$$