1.6 Gaussian elimination (高斯消去法)

Gaussian elimination (高斯消去法) 是用来解線性联立方程組的一個方法。所謂的高斯消去法是利用某些運算將線性联立方程組寫成"**上三角 (upper triangular)**"的方法。能夠用的運算只有三到三种基本運算 (elementary operation)

Definition 53: Elementary operation (基本運算)

(E1) interchanging any two equations in the system.
(將兩個式子对調)

(E2) multiplying any equation in the system by a nonzero constant.
(將某式乘個常数倍)

(E3) adding a multiple of one equation to another
(將某式乘個常数加到另一個方程式)

Example 54:

$$\begin{cases} 2x + 4y - 2z = 2 & (1) \\ 4x + 9y - 3z = 8 & (2) \\ -2x - 3y + 7z = 10 & (3) \end{cases}$$

Solving x, y, z.

Solution: 想辦法換成上三角矩陣

$(2) + (1) \times (-2)$:

$$\begin{cases} 2x + 4y - 2z = 2 \\ y + z = 4 & (4) \\ -2x - 3y + 7z = 10 \end{cases}$$

$(3) + (1) \times 1$

$$\begin{cases} 2x + 4y - 2z = 2 \\ y + z = 4 \\ y + 5z = 12 \end{cases} \qquad (5)$$

$(5)+(4)\times(-1)$:

$$\begin{cases} 2x + 4y - 2z = 2 \\ y + z = 4 \\ 4z = 8 \end{cases}$$

這种形式即為 Gaussian elimination 要達成的目標：將線性聯立方程組寫成上三角的形式.

This system is solved backward:

$$z = 2$$
$$y = 4 - 2 = 2$$
$$x = \frac{1}{2}(2 - 4y + 2z) = -1$$

This process is called back-substitution (反向代回法)

但這個寫法有点麻烦，每次都要將全部的 equations 寫出来.
有沒有辦法做修改？

Remark 55: 用矩陣的方式来表示

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -8 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right]$$

記得要用箭頭，不可以用等號

有些書上會加，有些書不加

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

row echelon form (階梯形矩陣)

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 4 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

reduced row echelon form
(化簡後的階梯形矩陣)

Definition 56: (1) A matrix is in row echelon form (階梯形) if it has the following three properties

(i) all non-zero rows are above any rows of zeros.

所有的非零列必須在整列為 0 的列之上.

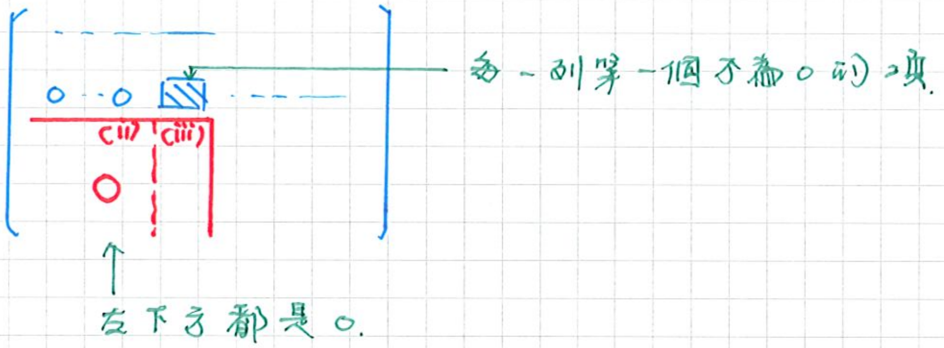
如

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

這种情況不能發生.

(ii) each leading entry of a row is in a column to the right of the leading entry of the row above it.

每一列中第一個不為零的項，其左下方的任何 entries 都是 0. 用圖形來看.



(iii) all entries in a column below a leading entry are zeros

每一列中第一個不為0的項，其下方任何的 entry 都是 0.

記住，就是個梯形就是了.

(2) If a matrix in a row echelon form satisfies the following additional conditions, then it is in reduced row echelon form (化簡後的階梯形)

(iv) the leading entry in each non-zero row is 1

每一列中第一個不為 0 的項為 1.

(v) each leading 1 is the only non-zero entry in its column.

每一列中第一個不為 0 的數同一行中其他數全為 0.

用圖表來看比較快.

Remark 57: row echelon form

$$\begin{bmatrix} \square & * & * & * \\ 0 & \square & * & * \\ 0 & 0 & \square & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \square & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \square & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \square & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \square & * \end{bmatrix}$$

reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Example 58: 化成 (reduced) row echelon form 最大的好處是：方解好方.

(1) $\begin{cases} x + w = 2 \\ y + z + w = 3 \\ z + 2w = 4 \\ 2w = 4 \end{cases}$

矩陣形式為

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right]$$ (為 row echelon form)

利用 back-substitution:

$w = 2$

$z = 4 - 2w = 0$

$y = 3 - z - w = 1$

$x = 2 - w = 0.$

$\Longrightarrow$ solution: $(x, y, z, w) = (0, 1, 0, 2)$

(2) $$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

寫成線性聯立方程組.

$$\begin{cases} x + w = 2 \\ y + z + w = 3 \\ z + 2w = 4 \end{cases}$$

有三個方程式, 但卻有四個未知數. 所以不可能有唯一解, 這時候會將其中一個變數固定, 但用那個變數固定呢?

規則很簡單: 在 (reduced) row echelon form 中不是 non-zero leading entries 當成定數. 例如在此例中, 固定 w:

$$z = 4 - 2w$$

$$y = 3 - z - w = 3 - (4 - 2w) - w = -1 + w$$

$$x = 2 - w$$

Thus, the solution is given by

$$\begin{cases} x = 2 - w \\ y = -1 + w, \qquad w \in \mathbb{R} \\ z = 4 - 2w \end{cases}$$

Definition 59: A pivot position (領先元) in a matrix A is a location in A that corresponds to a leading 1 in the reduced row echelon form of A.

Example 60:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 5 & 0 \\ 0 & 0 & 1 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot position.

Example 61: (1) Solve

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

Solution: $\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 8 & 8 \end{array}\right)$

row echelon form

$$\longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right)$$

reduced row echelon form

$\Longrightarrow$ solution: $(x, y) = (3, 1)$

(2) Solve

$$\begin{cases} x + y + z = 6 \\ x + 2y + 2z = 9 \\ x + 2y + 3z = 10 \end{cases}$$

Solution: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 9 \\ 1 & 2 & 3 & 10 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{array}\right]$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

row echelon form

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

reduced row echelon form

$\Longrightarrow$ solution: $x = 3,\ y = 2.\ z = 1.$

Remark 62: 若 $A \in M_{n\times n}(\mathbb{R})$，且為 non-singular $\Rightarrow$ 可用 Gaussian elimination.

若 A 不是 non-singular, Gaussian elimination 會出現不一樣的狀況

(i) <u>no solution:</u>

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 11 \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & -6 & 11 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 8 \end{array}\right)$$

但 $0\cdot x + 0\cdot y = 0 \neq 8$

$\Rightarrow$ 無解.

(ii) <u>infinite many solution:</u>

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 3 \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & -6 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

兩個變數，一個方程式.

$\Rightarrow$ infinite many solutions.

<u>Solution:</u> $x = 2y + 1$, $y \in \mathbb{R}$.

至於一般 $A \in M_{m\times n}(\mathbb{R})$，探討的方法類似，一樣會有 unique solution, no solution, infinite many solutions 三種狀況.