3.2 Linear independence

Definition 10: (1) A collection of vectors $\{v_1, v_2, \cdots, v_k\} \subseteq \mathbb{R}^n$ is called linearly dependent (線性相依) if $\exists\, a_1, a_2, \cdots, a_k \in \mathbb{R}$, not all zero, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0.$$

(2) $\{v_1, v_2, \cdots, v_k\} \subseteq \mathbb{R}^n$ is called linearly independent (線性獨立) if they are not linearly dependent.

Remark 11: $\{v_1, v_2, \cdots, v_k\} \subseteq \mathbb{R}^n$ is linearly independent.

$$\Longleftrightarrow \boxed{\begin{array}{l} a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0 \\ \Longrightarrow a_1 = a_2 = \cdots = a_k = 0 \end{array}}$$

這是檢驗 linearly independent 最好用的方法.

為什麼叫做線性相依呢?

$\{v_1, v_2, \cdots, v_k\} \subseteq \mathbb{R}^n$: linearly dependent.

$\Longrightarrow \exists\, a_1, a_2, \cdots, a_k \in \mathbb{R}$, not all zero, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0.$$

Suppose $a_1 \neq 0$, then

$$v_1 = -\frac{a_2}{a_1} v_2 - \frac{a_3}{a_1} v_3 - \cdots - \frac{a_k}{a_1} v_k$$

$\Longrightarrow$ v_1 is a linear combination of $\{v_2, v_3, \cdots, v_k\}$

而 linearly independent 則表示這組 vectors 相互之間沒有 linear combination 的關係.

Example 12: (1) Consider the set

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Check if S is linearly independent.

Solution: Let

$$a_1 \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 2 \\ -4 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix} + a_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\begin{cases} a_1 + 2a_2 + a_3 - a_4 = 0 \\ 3a_1 + 2a_2 - 3a_3 = 0 \\ -4a_1 - 4a_2 + 2a_3 + a_4 = 0 \\ 2a_1 - 4a_3 = 0 \end{cases}$$

$\Longrightarrow$ $a_1 = 2a_3$

$$a_2 = \frac{1}{2}(-3a_1 + 3a_3) = \frac{1}{2}(-6a_3 + 3a_3) = -\frac{3}{2}a_3$$

$$a_4 = a_1 + 2a_2 + a_3 = 2a_3 - 3a_3 + a_3 = 0$$

Thus, $a_1 = 4$. $a_2 = -3$. $a_3 = 2$. $a_4 = 0$ is one solution.

$\Longrightarrow$ S is linearly dependent.

(2) Consider the set

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$$

Check if S is linearly independent.

Solution: Consider

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

Thus,

$$\Longrightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \\ -a_1 - a_2 - a_3 + a_4 = 0 \end{cases}$$

$$\Longrightarrow a_1 = a_2 = a_3 = a_4 = 0$$

$\Longrightarrow$ S is linearly independent.

Remark 13: 一直利用定義來 check linear independence 是比較複雜的.
比較"簡單"的方法是利用 (reduced) row echelon form:

Let $S = \{v_1, v_2, \cdots, v_k\}$.

$$A = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_k^T \end{bmatrix} \xrightarrow[\text{利用 row operation}]{} \cdots \longrightarrow \text{row echelon form } R$$

Conclusion: R 中有一列為 0 $\Longleftrightarrow$ S is linearly dependent.

i.e., R 中沒有一列為 0 $\Longleftrightarrow$ S is linearly independent.

Example 14: (1) As in Example 12 (1):

$$\begin{bmatrix} 1 & 3 & -4 & 2 \\ 2 & 2 & -4 & 0 \\ 1 & -3 & 2 & -4 \\ -1 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -4 & 2 \\ 0 & -4 & 4 & -4 \\ 0 & -6 & 6 & -6 \\ 0 & 3 & -3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -4 & 2 \\ 0 & -4 & 4 & -4 \\ 0 & 3 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

不用做到底！只要有一列是 0 的出現，就可以停了.

$\Longrightarrow$ S is linearly dependent

(2) As in Example 12 (2)

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} : \text{此即為 row echelon form}$$

$\Longrightarrow S$ is linearly independent.

Remark 15: (1) $\{v\}$ is linearly independent $\Longleftrightarrow v \neq 0$.

(2) If $0 \in \{v_1, v_2, \cdots, v_k\}$, then $\{v_1, v_2, \cdots, v_k\}$ is linearly dependent.

Theorem 16: $\{v_1, v_2, \cdots, v_k\} \subseteq \{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_m\} \subseteq \mathbb{R}^n$

(1) $\{v_1, v_2, \cdots, v_k\}$: linearly dependent

$\Longrightarrow \{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_m\}$: linearly dependent.

(2) $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_m\}$: linearly independent.

$\Longrightarrow \{v_1, v_2, \cdots, v_k\}$: linearly independent.

proof: (1) $\{v_1, v_2, \cdots, v_k\}$: linearly dependent

$\Longrightarrow \exists\ a_1, a_2, \cdots, a_k \in \mathbb{R}$, not all zero, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0$$

$\Longrightarrow a_1 v_1 + a_2 v_2 + \cdots + a_k v_k + 0 \cdot v_{k+1} + \cdots + 0 \cdot v_m = 0$

$\Longrightarrow \{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_m\}$: linearly dependent.

(2) Suppose $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_m\}$: linearly independent.

Consider

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0$$

$\Longrightarrow a_1 v_1 + a_2 v_2 + \cdots + a_k v_k + 0 \cdot v_{k+1} + \cdots + 0 \cdot v_m = 0$

$\Longrightarrow a_1 = a_2 = \cdots = a_k = 0$

$\Longrightarrow \{v_1, v_2, \cdots, v_k\}$: linearly independent.