4 Eigenvalues and Eigenvectors (固有值與固有向量)

eigenvalues 與 eigenvectors 和矩陣的對角線化有相當大的關連.
矩陣的對角線化:

$$A \text{ 可寫成 } Q^{-1}DQ. \text{ 其中 } D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \text{ 為對角線矩陣.}$$

Question: (1) 所有 matrices 都可以對角線化呢?

Answer: No.

(2) 如何檢驗矩陣可對角線化?

(3) 若 A 為 diagonalizable, 如何求 Q, D?

4.1 Eigenvalues and eigenvectors.

$A \in M_{n\times n}(\mathbb{F})$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition 1: A nonzero vector $v \in \mathbb{F}^n$ is called an eigenvector (固有向量) of A if

$$Av = \lambda v \quad \text{for some } \lambda \in \mathbb{F}.$$

The scalar λ is called the eigenvalue of A corresponding to eigenvector v.

Example 2: $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

(1) Consider $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\Rightarrow Av_1 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-3 \\ 4-2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = (-2)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\Rightarrow v_1$ is an eigenvector of A, and -2 is the eigenvalue of A corresponding to v_1.

(2) Consider $v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$Av_2 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3+12 \\ 12+8 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$\Rightarrow$ v_2 is an eigenvector of A, and 5 is the eigenvalue of A corresponding to v_2.

(3) Consider $v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$Av_3 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$\Longrightarrow$ v_3 is not an eigenvector of A.

但每次用這種方法求 eigenvalues 及 eigenvectors 實在太麻煩了.
有沒有別的方法?

Theorem 3: $\lambda \in \mathbb{F}$ is an eigenvalue of $A \Longleftrightarrow \det(A - \lambda I_n) = 0$.

proof: λ is an eigenvalue of A

$\Longleftrightarrow \exists\, v \in \mathbb{F}^n,\ v \neq 0$ s.t. $Av = \lambda v$

$\Longleftrightarrow \exists\, v \in \mathbb{F}^n,\ v \neq 0$ s.t. $(A - \lambda I_n)v = 0$.

($A - \lambda I_n \in M_{n\times n}(\mathbb{F})$.
$(A - \lambda I_n)x = 0$ 有非零解)

$\Longleftrightarrow A - \lambda I_n$ is not invertible

$\Longleftrightarrow \det(A - \lambda I_n) = 0$.

Theorem 3 是算 eigenvalues 最簡單的方法.

Example 4: (1) $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

$$\det(A-\lambda I_2) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 3\cdot 4$$

$$= \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5) = 0$$

$\Longrightarrow$ eigenvalues of A : -2, 5.

(2) $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

$$\det(A-\lambda I_2) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$$

$$= (\lambda-3)(\lambda+1) = 0$$

$\Longrightarrow$ eigenvalues of A : 3, -1.

Definition 5: The polynomial $f(t) = \det(A - tI_n)$ is called the characteristic polynomial (特徵多項式, char. poly.) of A.

Example 6: (1) $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$.

$\Longrightarrow$ the char. poly of A

$$f(t) = \det(A - tI_2) = t^2 - 3t - 10.$$

(Example 4 (1)).

(2) $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

$\Longrightarrow$ the char. poly. of A:

$$f(t) = \det(A - tI_2) = t^2 - 2t - 3.$$

(Example 4 (2)).

Theorem 7: The characteristic polynomial of $A \in M_{n\times n}(\mathbb{F})$ is a polynomial of degree n with leading coefficient $(-1)^n$

(n次多項式, 首項係數 $(-1)^n$).

Corollary 8: $A \in M_{n\times n}(\mathbb{F})$ with characteristic polynomial $f(t)$.

(1) λ: eigenvalue of $A \iff f(\lambda) = 0$.

(2) A has at most n distinct eigenvalues.

但如何找 eigenvectors？我們用例子來看.

Example 9: (1) $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

由 Example 4(2) 已知 A 有 2 eigenvalues：$\lambda_1 = 3$, $\lambda_2 = -1$.

Find the eigenvectors of A:

(i) $\lambda_1 = 3$: The corresponding eigenvectors of A satisfies

$$A v = \lambda_1 v$$

i.e., $$(A - \lambda_1 I) v = 0$$

$$\Longrightarrow \quad v \in N(A - \lambda_1 I).$$

Since

$$A - \lambda_1 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}.$$

Suppose $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector of A corr. to $\lambda_1 = 3$,

$\iff$ $v \neq 0$ and

$$0 = (A - \lambda_1 I) v = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2v_1 + v_2 \\ 4v_1 - 2v_2 \end{pmatrix}$$

$\iff$ $v \neq 0$ and $2v_1 = v_2$

$\iff$ eigenvectors $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for $v_1 \neq 0$.

(eigenvectors 並非唯一)

(ii) $\lambda_2 = -1$: since

$$A - \lambda_2 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$$

$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$ is an eigenvector corresponding to $\lambda_2 = -1$

$\Longleftrightarrow$ $v \neq 0$ and

$$0 = Av - \lambda_2 v = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 + v_2 \\ 4v_1 + 2v_2 \end{pmatrix}$$

i.e., $2v_1 + v_2 = 0$.

$v_2 = -2v_1$

$\Longleftrightarrow$ $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for some $v_1 \neq 0$.

Observe that $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$ consisting of eigenvectors of A.

(2) $A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & -3 \end{bmatrix}$

Then

$$f(t) = \det(A - tI_3) = \begin{vmatrix} -t & 1 & 3 \\ 0 & -t & 4 \\ 0 & 0 & -3-t \end{vmatrix} = -t^2(t+3)$$

λ: eigenvalue of A $\Longleftrightarrow$ $f(\lambda) = 0$

$\Longleftrightarrow$ $\lambda^2(\lambda+3) = 0$

$\Longleftrightarrow$ $\lambda = 0, 0, -3$.

(i) $\lambda = 0$:

$$0 = Av = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_2 + 3v_3 \\ 4v_3 \\ -3v_3 \end{bmatrix}$$

$\Longrightarrow v_2 = v_3 = 0$

$$\Longrightarrow \text{eigenvectors of } A: \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for some } v_1 \neq 0.$$

(ii) $\lambda = -3$:

$$0 = (A+3I)v = \begin{bmatrix} 3 & 1 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3v_1 + v_2 + 3v_3 \\ 3v_2 + 4v_3 \\ 0 \end{bmatrix}$$

$$\Longrightarrow v_2 = -\frac{4}{3} v_3, \quad v_1 = -\frac{5}{9} v_3$$

$$\text{eigenvectors of } A: \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{bmatrix} -5/9 \; v_3 \\ -4/3 \; v_3 \\ v_3 \end{bmatrix} = t \begin{bmatrix} -5 \\ -12 \\ 9 \end{bmatrix}$$

for some $t \neq 0$.

Note: 觀察一下 λ 的個數與對應 eigenvectors 個數的關係.
詳細的比較,我們在後面會講.

在這章,我們特別提到 $\mathbb{F}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, 會有差別嗎?

Example 10: (1) Consider $A \in M_{2\times 2}(\mathbb{R})$ (over $\mathbb{R}$) with

$$A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

the characteristic polynomial of A

$$f(t) = \det(A - tI_2) = \begin{vmatrix} 1-t & 1 \\ -2 & -1-t \end{vmatrix}$$

$$= (1-t)(-1-t) + 2 = t^2 + 1$$

λ: eigenvalue of $A \iff f(\lambda) = 0$

$\iff \lambda^2 + 1$

Since $\lambda \in \mathbb{R}$, $\lambda^2 + 1 = 0$ has no solution.

$\Longrightarrow$ A has no eigenvalue.

(2) Consider $A \in M_{2\times 2}(\mathbb{C})$ (over $\mathbb{C}$) with

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

λ: eigenvalue of $A \iff \lambda^2 + 1 = 0$

$\iff \lambda = \pm i$.

(i) $\lambda = i$:

$$(A - iI)v = \begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} (1-i)v_1 + v_2 \\ -2v_1 + (-1-i)v_2 \end{bmatrix} = 0$$

$\iff (1-i)v_1 + v_2 = 0$

$\iff v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{pmatrix} v_1 \\ -(1-i)v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ i-1 \end{pmatrix}$ for some $v_1 \neq 0$.

(ii) $\lambda = -i$:

$$(A + iI)v = \begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} (1+i)v_1 + v_2 \\ -2v_1 + (-1+i)v_2 \end{bmatrix} = 0$$

$\iff (1+i)v_1 + v_2 = 0$

$\iff v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -(1+i)v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -1-i \end{pmatrix}$ for some $v_1 \neq 0$.

<u>Summary</u>: 三個步驟

(1) Compute the characteristic polynomial $f(\lambda) = \det(A - \lambda I_n)$.

(2) Find the roots of $f(\lambda) = 0$ (見下列注意事項).

$\Longrightarrow$ find the eigenvalues of A.

(3) For each eigenvalue λ, solve the equation $(A - \lambda I)v = 0$, $v \neq 0$

$\Longrightarrow$ find the eigenvector of A.

<u>注意事項</u>: $A \in M_{n\times n}(\mathbb{F}) \Longrightarrow f(t)$: polynomial of degree n.

(i) If $\mathbb{F} = \mathbb{R}$, at most n distinct eigenvalues.

(ii) If $\mathbb{F} = \mathbb{C}$, exactly n roots of $f(t) = 0$.

但這裏需要注意的情況非常多:
n個根都不一樣, 有重根的, … 都需要再討論
有關 eigenvalues, eigenvectors 更多的性質, 與對角線化的問題,
我們稍後再談.

這幾章要談的問題: $A \in M_{n\times n}(\mathbb{F})$

1. A 是否 diagonalizable? 檢定的方法.
2. Q, D 的求法. (Theorem 11).
3. 若非 diagonalizable, 有無改進的方法.