2. Determinants (行列式)

行列式是個你們在高中時已經學過的概念，在行列式的 order = 2, 3 時的性質，我想大部份的同學應該都沒有問題，但基本上，"大多數"的性質在所有的 order 都對。

為什麼要討論 determinants? determinants 在線性代數中的用處非常廣泛，底下列出一些基本的性質與應用。

1. Test for invertibility:

$$A: \text{invertible} \iff \det(A) = |A| \neq 0.$$

這是行列式最重要的應用。

2. The determinant of A = the volume of a box in an n-dimensional space
這個用法在你們以後學重積分時會用到，尤其是在學多變數的變數變換更是有用。

3. Solving the system of linear equations $Ax = b$
4. Find A^{-1}
5. Find the eigenvalues (固有值) of A.

由於 determinant of order 2 & order 3，對你們大多數人來講應該很熟了，我在這裏就只大略地一些定義與性質。

2.1 Properties of the determinants of order ≤ 3.

Definition 1: (1) $A \in M_{2\times 2}(\mathbb{R})$, the determinant of A is defined by

$$\det(A) = |A| = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

(2) $A \in M_{3\times 3}(\mathbb{R})$, the determinant of A is defined by

$$\det(A) = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}$$
$$- a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} - a_{33}a_{12}a_{21}$$

注意：在 order ≥ 4 時, determinant 的定法完全不一樣.

Remark 2: determinant of order 3 的記憶法：

method 1:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$-$ $\quad$ $+ \quad + \quad +$

method 2:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix}$$

前兩列重複再寫一次

$- \quad - \quad -$ $\quad$ $+ \quad + \quad +$

Example 3: (1) $\begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} = 1\times 4 - 2\times(-3) = 4-(-6) = 10$

(2) $\det\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -1 \\ 4 & 1 & 2 \end{pmatrix} = 1\times 0\times 2 + 3\times 1\times 2 + 4\times(-1)\times(-1) - 2\times 0\times 4$
$$- 3\times(-1)\times 2 - 1\times(-1)\times 1$$
$$= 6+4+6+1 = 17$$

Proposition 4: $A \in M_{n\times n}(\mathbb{R})$ for $n=2$ or 3.

(1) $\det(I_2) = \det(I_3) = 1$

(2) A: triangular matrix (上三角或下三角矩陣), then
$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

(3) If A has a row (column) of zeros, then $\det(A) = 0$.

(4) The determinant depends linearly on each row and each column, e.g.,

$$\det\begin{pmatrix} A_{11}+cB_{11} & A_{12}+cB_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + c\det\begin{pmatrix} B_{11} & B_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

但注意下列情況

$$\det(A+cB)$$

$$= \det\begin{pmatrix} A_{11}+cB_{11} & A_{12}+cB_{12} \\ A_{21}+cB_{21} & A_{22}+cB_{22} \end{pmatrix}$$

$$= \det(A) + c\det\begin{pmatrix} A_{11} & A_{12} \\ B_{21} & B_{22} \end{pmatrix} + c\det\begin{pmatrix} B_{11} & B_{12} \\ A_{21} & A_{22} \end{pmatrix} + c^2\det(B).$$

行列式並不保留加法, i.e., $\det(A+B) \neq \det(A) + \det(B)$.

(5) The determinant changes sign when two rows or two columns are exchanged.

($A \xrightarrow[(E1)]{} B$, then $\det(A) = -\det(B)$)

(6) Subtracting a multiple of one row (column) from another row (column) leaves the same determinant.

($A \xrightarrow[(E3)]{} B$, then $\det(A) = \det(B)$).

(7) If two rows or two columns of A are equal, then $\det(A) = 0$.

Example 5: (1) $\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(2) $\det\begin{bmatrix} 2 & 0.4185 & 6\times10^{23} \\ 0 & -1 & 3.241\pi \\ 0 & 0 & 3 \end{bmatrix} = 2\times(-1)\times3 = -6.$

(3) $\det\begin{bmatrix} 4 & 1 & 6 \\ 3 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} = 0.$

(4) $\det\begin{bmatrix} x+2a & y+2b & z+2c \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}$

$= \det\begin{bmatrix} x & y & z \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix} + 2\det\begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}$

(5) $\det\begin{bmatrix} 1 & 3 & 4 \\ -1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix} = -\det\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

第一、三列對調

$= (-1)^2\det\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 3 \\ 3 & 1 & 4 \end{bmatrix}$

第一、二行對調.

$\det\begin{bmatrix} 1 & 4 & 5 \\ -1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix} = (-1)^2\det\begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & 4 & 5 \end{bmatrix}$

兩列對調兩次.

(6) $\det\begin{bmatrix} 1024 & 2048 & 2149 \\ 1023 & 2047 & 2147 \\ 1024 & 2048 & 2148 \end{bmatrix}$

$= \det\begin{bmatrix} 1024 & 2048 & 2149 \\ -1 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix} = 1024 - 2048 = -1024$

(7) $\det \begin{bmatrix} 1 & 1 & 5 \\ 4 & 1 & 8 \\ 1 & 1 & 5 \end{bmatrix} = 0.$

Remark 6: $A, B \in M_{n\times n}(\mathbb{R})$. $c \in \mathbb{R}$. then

$\det(A+B) \neq \det(A) + \det(B)$

$\det(cA) = c^n \det(A).$

Proposition 7: $A, B \in M_{n\times n}(\mathbb{R})$. for $n = 2$ or 3.

(1) A: invertible $\Longleftrightarrow$ $\det(A) \neq 0.$

(2) $\det(A\cdot B) = \det(A)\cdot \det(B).$

In particular,

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

(3) $\det(A^T) = \det(A).$

proof: (2) $\det(A)\cdot\det(A^{-1}) = \det(A\cdot A^{-1}) = \det(I_n) = 1$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}.$$