1.4 Inverse and transpose (反矩陣與轉置矩陣)

這節要談的是反矩陣與轉置矩陣

Definition 30: $I_n \in M_{n\times n}(\mathbb{R})$ is called an identity matrix (單位矩陣) if

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j. \end{cases}$$

That is, I_n is of the form

$$I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \ 1 \end{bmatrix}$$

(對角線為 1. 其他都是 0)

Remark 31: For all $A \in M_{n\times n}(\mathbb{R})$, $AI_n = I_nA$.

In general, for $A \in M_{m\times n}(\mathbb{R})$, $AI_n = I_mA$.

Definition 32: $A \in M_{n\times n}(\mathbb{R})$ (必須是方陣才行)

The inverse (matrix) (反矩陣) of A is a matrix $B \in M_{n\times n}(\mathbb{R})$ satisfying

$$AB = BA = I_n.$$

We denoted by A^{-1}.

這是個很自然的想法，將 $\mathbb{R}$ 與 $M_{n\times n}(\mathbb{R})$ 做個對比

$\mathbb{R}$		$M_{n\times n}(\mathbb{R})$
0	$\longrightarrow$	$O_{n\times n}$ (零矩陣)
1	$\longrightarrow$	I_n
乘上任何東西都等於自己		
$\frac{1}{a}$ (倒數)	$\longrightarrow$	? (inverse matrix A^{-1})
		但注意符號.

一些很直觀的問題:

1. inverse matrix 一定會存在嗎?
2. 如果存在的話，是否唯一?
3. 如果存在的話，怎麼求?

關於第一個問題，其實是很簡單的。剛剛我們將 $\mathbb{R}$ 與 $M_{n\times n}(\mathbb{R})$ 做了個對照。$\mathbb{R}$ 中的倒數是否一定存在? <u>No</u>，0 沒有倒數。
同樣的想法，零矩陣 $O_{n\times n}$ 也沒有 inverse matrix, since

$$AO_{n\times n} = O_{n\times n}A = O_{n\times n} \quad \forall A \in M_{n\times n}(\mathbb{R})$$

所以 $O_{n\times n}$ 並沒有 inverse matrix，但這也衍生出另一個問題:

4. 有沒有辦法判別矩陣有沒有 inverse matrix?

3, 4 這兩個問題，後面再談，我們先談談 2.

<u>Lemma 33:</u> For given $A \in M_{n\times n}(\mathbb{R})$, there is at most one inverse matrix of A.

<u>proof of uniqueness:</u> 假設 B, C 都是 A 的 inverse matrices

$\Rightarrow$ $AB = BA = I_n$. $AC = CA = I_n$.

Then

$$B = I_nB = (CA)B = C(AB) = CI_n = C$$

$\Rightarrow$ $B = C$.

<u>Definition 34:</u> If A^{-1} exists, we say that A is <u>invertible</u> <u>(可逆的)</u>

<u>Remark 35:</u> (1) A is invertible $\Longleftrightarrow$ A^{-1} is invertible,
in fact, $(A^{-1})^{-1} = A$.

(2) If A is invertible, the one and only one solution to $Ax = b$ is $x = A^{-1}b$.

所以 $Ax = b$ 的左右兩邊各乘上 A^{-1} 即可得。

這是解線性聯立方程組的方法之一。

(3) If there is a nonzero vector x s.t. $Ax = 0$, then A cannot have an inverse.

這個結果可以由(2)得來. 若 A 為 invertible, $Ax=0$ 有唯一解 $x = A^{-1}0 = 0$, 顯然矛盾.

Lemma 36: (1) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible $\iff ad - bc \neq 0$.

Moreover,

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(這是 2×2 matrix 求反矩陣的公式)

(2) $A = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$ has an inverse $\iff d_1 \cdot d_2 \cdots d_n \neq 0$.

Moreover,

$$A^{-1} = \begin{bmatrix} 1/d_1 & & & 0 \\ & 1/d_2 & & \\ & & \ddots & \\ 0 & & & 1/d_n \end{bmatrix}$$

Example 37: (1) $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$. 則

$$A^{-1} = \frac{1}{1\times5 - 2\times3}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = -\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

(2) $A = \begin{bmatrix} 1 & & & 0 \\ & 3 & & \\ & & 5 & \\ 0 & & & -1 \end{bmatrix}$, then

$$A^{-1} = \begin{bmatrix} 1 & & & 0 \\ & 1/3 & & \\ & & 1/5 & \\ 0 & & & -1 \end{bmatrix}$$

(3) $B = \begin{bmatrix} 4 & & & 0 \\ & 0 & & \\ & & -1 & \\ 0 & & & 1 \end{bmatrix}$ is not invertible.

Proposition 38: $A, B \in M_{n\times n}(\mathbb{R})$: invertible $\Rightarrow$ AB: invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

proof: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$

$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

Corollary 39: $E_1, E_2, E_3, \cdots, E_m \in M_{n\times n}(\mathbb{R})$: invertible

$\Rightarrow E_1E_2\cdots E_m$: invertible and

$(E_1E_2\cdots E_m)^{-1} = E_m^{-1}\cdots E_2^{-1}E_1^{-1}$

注意順序，順序必須反過來.

Remark 40: non-singular (非奇異矩陣)

= invertible

= non-zero determinant (行列式 $\neq 0$)

至於一般 inverse matrix 的方法，我們晚点来說.

Definition 41: The transpose (轉置矩陣) A^T of an $m\times n$ matrix A is the $n\times m$ matrix obtained from A interchanging the rows with columns; that is,

$$(A^T)_{ij} = A_{ji} \quad \forall\ 1\le i\le n,\ 1\le j\le m.$$

(行列對調)

Example 42:
$$\begin{pmatrix} 1 & 3 & 4 & -1 \\ 2 & -1 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 4 & 0 \\ -1 & 1 \end{pmatrix}$$

Proposition 43: $A, B \in M_{m\times n}(\mathbb{R})$, $C \in M_{n\times p}(\mathbb{R})$. Then

(1) $(A+B)^T = A^T + B^T$

(2) $(A^T)^T = A$.

(3) $(AC)^T = C^T A^T$ (～請必須對調)

proof: (3) $((AC)^T)_{ij} = (AC)_{ji} = \sum_{k=1}^{n} A_{jk} C_{ki}$

$$= \sum_{k=1}^{n} (C^T)_{ik} (A^T)_{kj} = (C^T A^T)_{ij}$$

$$\forall\ 1 \leq i \leq p,\ 1 \leq j \leq m$$

$\Rightarrow (AC)^T = C^T A^T$

Proposition 44: $A \in M_{n\times n}(\mathbb{R})$. Then

$$(A^T)^{-1} = (A^{-1})^T,$$

i.e., the inverse of A^T = the transpose of A^{-1}.

proof: Claim: $A^T \cdot (A^{-1})^T = I_n$.

$$A^T \cdot (A^{-1})^T \underset{\text{Prop. 43(3)}}{=} (A^{-1}A)^T = I_n^T = I_n$$

Similarly, $(A^{-1})^T \cdot A^T = I_n$

$\Rightarrow (A^T)^{-1} = (A^{-1})^T$.