1.8 Gaussian-Jordan method

這節我們要談談如何計算 inverse matrix A^{-1}.

我們要介紹 Gaussian-Jordan method. 以 3×3 矩陣為例.

$$A = (a_{ij})_{1 \le i, j \le 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Consider

$$A^{-1} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = (x_1 \ x_2 \ x_3),$$

where

$$x_i = \begin{pmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \end{pmatrix}$$

Then

$$AA^{-1} = A \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (e_1 \ e_2 \ e_3)$$

where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ ith component

在 $\mathbb{R}^3$ 的情況中

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\Longrightarrow Ax_i = e_i$ for $i = 1, 2, 3$.

解出 x_i :

$$x_i = A^{-1}e_i$$

$$\Rightarrow (x_1 \; x_2 \; x_3) = (A^{-1}e_1 \; A^{-1}e_2 \; Ae_3) = A^{-1}$$

解出 $x_i = A^{-1}e_i$ 的過程可用 elementary row operation 來看.

觀察一下 $Ax_1 = e_1$ 的過程.

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 1 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right) \xrightarrow[\text{row operation}]{\text{elementary}} \cdots \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & x_{11} \\ 0 & 1 & 0 & x_{21} \\ 0 & 0 & 1 & x_{31} \end{array}\right)$$

reduced row echelon form

每組 (x_i, e_i) 的作法都一樣, 所以可以全部寫在一起, 此即為所謂的 Gaussian-Jordan method:

$$(A \mid I_n) \xrightarrow[\text{elementary row operation}]{} (I_n \mid A^{-1})$$

Example 83: (1) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad-bc \neq 0$, $a \neq 0$.

$$(A \mid I_2) = \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc} a & 0 & 1 + \frac{bc}{ad-bc} & -\frac{ab}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) = (I_2 \mid A^{-1})$$

$$\Longrightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(2) $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

$$(A \mid I_3) = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1/2 & -1/2 \\ 0 & 1 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 3/2 & -1/2 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right] = (I_3 \mid A^{-1})$$

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$$A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$