3.3 Basis and dimension.

W: subspace of $\mathbb{R}^n$.

Definition 17: A set S is called a basis (基底) of W if

(i) S is linearly independent.

(ii) $\text{span}(S) = W$.

簡單地說, basis S 是生成 W 的最小集合.

Example 18: (1) In $\mathbb{R}^n$, $\{e_1, e_2, \cdots, e_n\}$ is a basis for $\mathbb{R}^n$.

(i) Claim: $\{e_1, e_2, \cdots, e_n\}$: linearly independent.

$$a_1 e_1 + a_2 e_2 + \cdots + a_n e_n = 0$$

$$\Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0$$

$$\Rightarrow a_1 = a_2 = \cdots = a_n = 0.$$

(ii) Claim: $\text{span}\{e_1, e_2, \cdots, e_n\} = \mathbb{R}^n$

For $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \in \text{span}\{e_1, e_2, \cdots, e_n\}.$$

$$\Rightarrow \mathbb{R}^n \subseteq \text{span}\{e_1, e_2, \cdots, e_n\}.$$

Conversely, since $e_i \in \mathbb{R}^n$ $\forall i = 1, 2, \cdots, n$

$$\Rightarrow a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \in \mathbb{R}^n \quad \forall a_1, a_2, \cdots, a_n \in \mathbb{R}$$

$$\Rightarrow \text{span}\{e_1, e_2, \cdots, e_n\} \subseteq \mathbb{R}^n$$

Thus, $\text{span}\{e_1, e_2, \cdots, e_n\} = \mathbb{R}^n$.

(2) $$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$$

then $W = \operatorname{span}(S)$ is a subspace of $\mathbb{R}^4$

Moreover, S is a basis for W, since

(i) by Example 12(2), S is linearly independent.

(ii) clearly, $\operatorname{span}(S) = W$.

(3) $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$

then $W = \operatorname{span}(S)$ is a subspace of $\mathbb{R}^3$.

Moreover, S is a basis for W, since

$$a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = 0$$

$\Rightarrow a = b = 0$

$\Rightarrow S$ is linearly independent

(4) $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$

then $W = \operatorname{span}(S)$ is a subspace of $\mathbb{R}^2$, but S is not a basis for W, since S is not linearly independent.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

By Remark 13, S is linearly dependent.

$S' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for W.

Theorem 19: $S = \{v_1, v_2, \cdots, v_k\}$: basis for W

$\iff \forall x \in W,\ \exists!\ a_1, a_2, \cdots, a_k \in \mathbb{R}$ s.t.

$x = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$

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proof: "$\Rightarrow$" Suppose S is a basis for W

$\Rightarrow$ for $x \in W$, $\exists\ a_1, a_2, \cdots, a_k \in \mathbb{R}$ s.t.

$x = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$.

Suppose

$x = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = b_1 v_1 + b_2 v_2 + \cdots + b_k v_k$

$\Rightarrow (a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \cdots + (a_k - b_k) v_k = 0$

$\underset{S:\text{ lin. indep.}}{\Rightarrow} a_1 - b_1 = a_2 - b_2 = \cdots = a_k - b_k = 0$

$\Rightarrow a_1 = b_1,\ a_2 = b_2,\ \cdots,\ a_k = b_k$

$\Rightarrow$ uniqueness.

"$\Leftarrow$" 必須分成兩個部份來証明.

(i) Since $0 \in W$, and $\exists!\ a_1, a_2, \cdots, a_k \in \mathbb{R}$ s.t.

$0 = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$

$\Rightarrow a_1 = a_2 = \cdots = a_k = 0$

(ii) Since for $x \in W$, $\exists\ a_1, a_2, \cdots, a_k \in \mathbb{R}$ s.t.

$x = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$

$\Rightarrow x \in \text{span}(S)$

Theorem 20: Any basis for W consists the same number of elements.

Definition 21: The unique number of elements in each basis for W is called the dimension (維度) of W and is denoted by $\dim(W)$.

Example 22: (1) $\dim(\{0\}) = 0$

(2) $\dim(\mathbb{R}^n) = n$, since $\{e_1, e_2, \cdots, e_n\}$ is a basis for $\mathbb{R}^n$.

(3) $W = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$

then $\dim(W) = 2$.

(4) $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$

$W = \text{span}(S)$,

then $\dim(W) = 2$.