3. Subspaces and Their Properties.

考慮 $\mathbb{R}^n$ 及一般的向量加法, 常數乘法.

3.1 Subspaces. (子空間)

在線性代數中會提到一般的 vector space (向量空間). 但因為我們這門課只討論矩陣. 因此現在只用 $\mathbb{R}^n$. 先定義什麼叫 subspace (子空間)

Definition 1: A set $W \subseteq \mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if

(i) (closed under addition) $u, v \in W \implies u+v \in W$

(ii) (closed under scalar multiplication) $u \in W,\ c \in \mathbb{R} \implies cu \in W$

直覺想法: subspace 就是通過原點的直線, 平面, ...

Example 2: (1) $W = \{0\}$: called the zero subspace (零子空間)

(2) $W = \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3 : 3w_1 - 2w_2 + w_3 = 0 \right\}$

(i) $u, v \in W$

$\implies 3u_1 - 2u_2 + u_3 = 0$ and $3v_1 - 2v_2 + v_3 = 0$.

For $u + v = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$,

$3(u_1+v_1) - 2(u_2+v_2) + (u_3+v_3)$

$= (3u_1 - 2u_2 + u_3) + (3v_1 - 2v_2 + v_3) = 0 + 0 = 0$

$\implies u + v \in W$.

(ii) $u \in W,\ c \in \mathbb{R}$

$\implies 3u_1 - 2u_2 + u_3 = 0$.

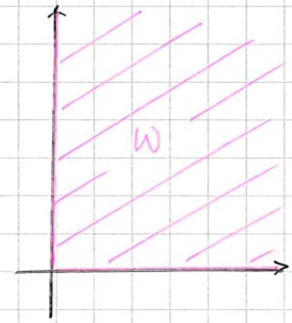
$cu = \begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \end{pmatrix}$

$3(cu_1) - 2(cu_2) + cu_3 = c(3u_1 - 2u_2 + u_3) = 0$

$\Rightarrow cu \in W$

(i) + (ii) $\Rightarrow$ W is a subspace of $\mathbb{R}^3$.

(3) $W = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2 : u_1 \geq 0,\ u_2 \geq 0 \right\}$



Since $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in W$, $(-2) \in \mathbb{R}$.

$$(-2)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \notin W$$

Thus, W is <u>not</u> a subspace.

(4) The set $W = \left\{ \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} : s, t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$

For $x, y \in W$, $c \in \mathbb{R}$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix},\quad y = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}$$

$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ 0 \end{bmatrix} \in W$$

$$cx = c\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ 0 \end{bmatrix} \in W$$

$\Rightarrow$ W is a subspace of $\mathbb{R}^3$.

<u>Remark 3</u>: If W is a subspace of $\mathbb{R}^n$, then

(1) $0 \in W$

(2) $v \in W \Rightarrow -v \in W$.

這個結果最大的用處在於判斷 W "不是" 個 subspace.

If $0 \notin W$. W cannot be a subspace of $\mathbb{R}^n$.

Example 4: $W = \left\{ \begin{pmatrix} 2s+1 \\ 0 \\ 3s \end{pmatrix} : s \in \mathbb{R} \right\}$ is not a subspace of $\mathbb{R}^3$,

since $0 \notin W$.

Recall: Given $v_1, v_2, \cdots, v_k \in \mathbb{R}^n$, $c_1, c_2, \cdots, c_k \in \mathbb{R}$,

then $c_1v_1 + c_2v_2 + \cdots + c_kv_k$ is a linear combination (線性組合) of $v_1, v_2, \cdots, v_k$

Definition 5: The collection of all linear combinations of $v_1, v_2, \cdots, v_k$ is denoted by $\operatorname{span}\{v_1, v_2, \cdots, v_k\}$ and is called the subset of $\mathbb{R}^n$ spanned by $v_1, v_2, \cdots, v_k$.

Remark 6: (1) If $u \neq 0$, $\operatorname{span}\{u\} = \{cu : c \in \mathbb{R}\}$ 一條直線.

(2) If $u, v \neq 0$,

(i) $u = cv$: $\operatorname{span}\{u, v\} = \{kv : k \in \mathbb{R}\}$ 一條直線.

(ii) $u \neq cv$: $\operatorname{span}\{u, v\} = \{c_1u + c_2v : c_1, c_2 \in \mathbb{R}\}$ 一個平面.

Example 7:

$$u = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$$

$$\operatorname{span}\{u, v\} = \left\{ c_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} c_1 + 5c_2 \\ -2c_1 - 13c_2 \\ 3c_1 - 3c_2 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

Then span$\{u, v\}$ is a subspace of $\mathbb{R}^3$, since for $x, y \in$ span$\{u, v\}$

$\Longrightarrow \exists\ c_1, c_2, d_1, d_2 \in \mathbb{R}$ s.t.

$$x = c_1 u + c_2 v, \qquad y = d_1 u + d_2 v$$

$$x + y = (c_1 u + c_2 v) + (d_1 u + d_2 v) = (c_1 + d_1) u + (c_2 + d_2) v \in \text{span}\{u, v\}$$

$$kx = k(c_1 u + c_2 v) = (kc_1) u + (kc_2) v \in \text{span}\{u, v\}$$

同樣的方法,我們可得下列的結果.

Theorem 8: $u_1, u_2, \cdots, u_k \in \mathbb{R}^n$, then span$\{u_1, u_2, \cdots, u_k\}$ is a subspace of $\mathbb{R}^n$.

Example 9: (1) Let

$$W = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

Since

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Let

$$u = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$W = \{au + bv : a, b \in \mathbb{R}\} = \text{span}\{u, v\}.$$

$\Longrightarrow$ W is a subspace of $\mathbb{R}^4$.

(2) Let

$$u = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad v = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix}, \quad w = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$$

Find h such that $y \in \operatorname{span}\{u, v, w\}$.

Solution: If $y \in \operatorname{span}\{u, v, w\}$, $\exists\, c_1, c_2, c_3 \in \mathbb{R}$ s.t.

$$c_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + c_3 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$$

$$\begin{cases} c_1 + 5c_2 - 3c_3 = -4 & (1) \\ -c_1 - 4c_2 + c_3 = 3 & (2) \\ -2c_1 - 7c_2 = h & (3) \end{cases}$$

$(1)+(2)$: $c_2 - 2c_3 = -1 \implies c_2 = 2c_3 - 1$

代入 (1): $c_1 = -4 - 5c_2 + 3c_3 = -4 - 5(2c_3 - 1) + 3c_3$

$= -7c_3 + 1$

(3): $h = -2c_1 - 7c_2 = -2(-7c_3 + 1) - 7(2c_3 - 1) = 5$

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