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2.3.2 The solution of $Ax = b$.

Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns.

Theorem 2.1 (Cramer's rule)

If $\det(A) \neq 0$, then the system $Ax = b$ has a unique solution and for each $k = 1, 2, 3, \cdots, n$,

$$x_k = \frac{\det(M_k)}{\det(A)}.$$

where

$$M_k = \begin{pmatrix} A_{11} & \cdots & A_{1k-1} & b_1 & A_{1k+1} & \cdots & A_{1n} \\ A_{21} & \cdots & A_{2k-1} & b_2 & A_{2k+2} & \cdots & A_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nk-1} & b_n & A_{nk+1} & \cdots & A_{nn} \end{pmatrix} = (A_1 \cdots A_{k-1}\ b\ A_{k+1} \cdots A_n)$$

(將 A 中的第 k 行換成 b)

proof: Let

$$Y_k = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & x_1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & & x_2 & & & \vdots \\ 0 & 0 & 1 & & & x_3 & & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 & \vdots & & & \vdots \\ \vdots & \vdots & & \ddots & 1 & \vdots & 0 & & \vdots \\ \vdots & \vdots & & & 0 & \vdots & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & & \vdots & \vdots & 0 & \ddots 1 & 0 \\ 0 & 0 & \cdots & & 0 & x_n & 0 & \cdots 0 & 1 \end{pmatrix}$$

= 將 identity matrix I_n 中的第 k 行改成 x.

$= (e_1\ e_2 \cdots e_{k-1}\ x\ e_{k+1} \cdots e_n).$

$\Longrightarrow AY_k = A(e_1\ e_2 \cdots e_{k-1}\ x\ e_{k+1} \cdots e_n)$

$= (Ae_1\ Ae_2 \cdots Ae_{k-1}\ Ax\ Ae_{k+1} \cdots Ae_n)$

$= (A_1\ A_2 \cdots A_{k-1}\ b\ A_{k+1} \cdots A_n) = M_k$

Moreover,

$$\det(Y_k) = x_k \det(I_{n-1}) = x_k$$

↑ the cofactor expansion of Y_k along the kth row

$$\Rightarrow \det(M_k) = \det(AY_k) = \det(A)\cdot\det(Y_k)$$

$$= \det(A)\cdot x_k$$

$$\Rightarrow x_k = \frac{\det(M_k)}{\det(A)}.$$

Example22: Solving

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 2 \\ x_1 + x_3 = 3 \\ x_1 + x_2 - x_3 = 1 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

i.e.,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

then

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 3 + 2 - 1 - (-2) = 6$$

$$\det(M_1) = \begin{vmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 9 + 2 - 2 - (-6) = 15$$

$$\det(M_2) = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -3+3+2-9-1-(-2) = -6$$

$$\det(M_3) = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 6+2-3-2 = 3$$

$$\Rightarrow \quad x_1 = \frac{\det(M_1)}{\det(A)} = \frac{15}{6} = \frac{5}{2}$$

$$x_2 = \frac{\det(M_2)}{\det(A)} = \frac{-6}{6} = -1$$

$$x_3 = \frac{\det(M_3)}{\det(A)} = \frac{3}{6} = \frac{1}{2}$$

$\Rightarrow$ solution: $(x_1, x_2, x_3) = \left(\frac{5}{2}, -1, \frac{1}{2}\right)$.