1.7 Elementary matrices, triangular factors, and row exchanges.

在上一節談 Gaussian elimination 時，我們每一次都會做方程式運算（基本運算）。如在 Example 54 及 Remark 55 中，若用矩陣的方法來寫的話，會變成

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}$$

整個過程中，只是經過某些轉換，而這個轉換是在矩陣方程式的左右兩邊同時乘上個矩陣。在這節，我們就是將會介紹這類的矩陣，我們稱之為 elementary matrix（基本矩陣）

1.7.1 Elementary matrices.

Recall: Definition 53：三种 elementary operations.

(E1) interchange any two equations in the system.

(E2) multiplying any equation in the system by a nonzero constant

(E3) adding a multiple of one equation to another.

利用 elementary operations 就可以定義 elementary matrices.

Definition 63: An $n \times n$ elementary matrix (基本矩陣, 基礎矩陣) 是將 identity matrix I_n 做一次 elementary operation (或者說, 列運算).

Example 64: $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ elementary matrices of type 1.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ elementary matrices of type 2.

$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ elementary matrices of type 3.

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ not elementary matrices

Theorem 65: $E \in M_{m\times m}(\mathbb{R})$: elementary matrix

$A \in M_{m\times n}(\mathbb{R})$

$\Longrightarrow$ EA: the matrix obtained from A by performing the same elementary (row) operation as that produces E from I_m

或用解釋為:

$A \in M_{m\times n}(\mathbb{R})$

$$I_m \xrightarrow[\text{some elementary operation}]{} E \in M_{m\times m}(\mathbb{R})$$

$\Longrightarrow$ EA 為 $A \xrightarrow[\text{the same elementary operation.}]{} EA$

Example 66:

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{=x} = \underbrace{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}}_{=b}$$

i.e., $Ax = b$.

之前我們已經看過整個矩陣變換過程

Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$Ax = b$

$\Longrightarrow$ $E_1 A x = E_1 b$, i.e.,

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

$\Longrightarrow$ $E_2 E_1 A x = E_2 E_1 b$, i.e.,

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}$$

$\Longrightarrow$ $E_3 E_2 E_1 A x = E_3 E_2 E_1 b$.

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \quad (=: c)$$

Moreover, let

$$M = E_3 E_2 E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \quad \text{(必須硬算)}$$

$$U = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\Rightarrow \quad MA = U$$

$$\Rightarrow \quad MAx = Ux = Mb =: c$$

這一小節的重點在於介紹 elementary matrices，elementary matrices 與 elementary operations 的關係，應用於 Gaussian elimination 的作法。
最後的 Example 66 在後面會有後續的結果，在這之前先介紹 LU 分解法。