1.7.3 Row exchanges and permutation matrix

之前我們談的情況都是"不准" row exchange 的狀況，在這一節要談的則是允許 row exchange.

Definition 77: A permutation matrix (排列矩陣) is a matrix which has the same rows as the identity matrix.

Example 78: (1) In $M_{2\times 2}(\mathbb{R})$, two permutation matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(2) In $M_{3\times 3}(\mathbb{R})$, 6 permutation matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Remark 79: (1) The product of two permutation matrices is again a permutation matrix.

(2) elementary matrix $\not\Rightarrow$ permutation matrix, elementary matrix $\not\Leftarrow$ permutation matrix

e.g. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$: permutation matrix, but not an elementary matrix.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: elementary matrix, but not a permutation matrix.

為什麼要討論 permutation matrices? 主要的原因是：若遇到 Gaussian elimination 中，上一列第一個非零出現得比下一列來得晚 (或對角線為 0) 的情形，可用 permutation matrix 來對調.

Remark 80: 方陣的 singular 與 non-singular 與最後完成的上三角矩陣有關. 若最後完成的上三角矩陣 U 的對角線全不為 0, 稱為 non-singular, 只要其中有一個是 0. 我們稱此矩陣為 singular.

遇到需要 row exchange 的狀況, 可先對調.

Proposition 81: In the non-singular case, there is a permutation matrix P such that

$$PA = LU,$$

where L : lower triangular

U : upper triangular.

Example 82:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 4 & 6 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 4 \end{pmatrix}$$

$$\xrightarrow[\text{對調}]{} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix} = U$$

則取 permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

現在的問題是 L = ?

直接計算 PA. 其他依照之前的作法.

$$PA = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 6 & 8 \\ 2 & 2 & 5 \end{pmatrix} \xrightarrow[E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 2 & 2 & 5 \end{pmatrix}$$

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$$\xrightarrow{E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix} = U$$

$\Longrightarrow E_2 E_1 P A = U$

$\Longrightarrow PA = E_1^{-1} E_2^{-1} U = LU$, where

$$L = E_1^{-1} E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$