4.3.2 The matrix e^{tA}

Recall: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

Definition 35: $A \in M_{n\times n}(\mathbb{F})$.

The matrix exponential is defined by

$$\exp(tA) = e^{tA} = I + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \cdots + \frac{(tA)^n}{n!} + \cdots$$

Remark 36: $\exp(sA)\cdot\exp(tA) = \exp((s+t)A)$.

$\exp(tA)\cdot\exp(-tA) = I$

$\frac{d}{dt}\exp(tA) = Ae^{tA}$.

Remark 37: If A is diagonalizable

$\Longrightarrow$ $\exists$ eigenvector matrix Q and diagonal matrix D s.t.

$$A = QDQ^{-1}.$$

$$\Longrightarrow \exp(tA) = I + tA + \frac{t^2A^2}{2!} + \cdots + \frac{t^nA^n}{n!} + \cdots$$

$$= QQ^{-1} + tQDQ^{-1} + \frac{t^2}{2!}QD^2Q^{-1} + \cdots + \frac{t^n}{n!}QD^nQ^{-1} + \cdots$$

$$= Q\left[I + tD + \frac{t^2D^2}{2!} + \cdots + \frac{t^nD^n}{n!} + \cdots\right]Q^{-1}$$

$$= Qe^{tD}Q^{-1}$$

重点:

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}, \qquad e^{tD} = \begin{bmatrix} e^{t\lambda_1} & & & 0 \\ & e^{t\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{t\lambda_n} \end{bmatrix}$$

Example 38: $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

Compute e^{tA}.

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Solution: characteristic polynomial:

$$\det(A - tI) = \begin{vmatrix} -2-t & 1 \\ 1 & -2-t \end{vmatrix} = t^2 + 4t + 3 = (t+1)(t+3)$$

$\Longrightarrow$ eigenvalues: $-3, -1$

$\Longrightarrow$ A is diagonalizable.

$$E_{-1} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$E_{-3} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

$$\Longrightarrow \quad Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$$

and $A = QDQ^{-1}$

$$\Longrightarrow \quad e^{tA} = Q e^{tD} Q^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \cdot \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= -\frac{1}{2} \begin{pmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(e^{-t} + e^{-3t}) & \frac{1}{2}(e^{-t} - e^{-3t}) \\ \frac{1}{2}(e^{-t} - e^{-3t}) & \frac{1}{2}(e^{-t} + e^{-3t}) \end{pmatrix}$$