2.3 Applications of determinants.

2.3.1 Computation of A^{-1}

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$$

$$\Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)}\begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix}$$

$$= \frac{1}{\det(A)}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^T$$

For general $A \in M_{n\times n}(\mathbb{R})$, 看看 A 的 cofactor expansion

$$A_{11}C_{11} + A_{12}C_{12} + \cdots + A_{1n}C_{1n} = \det(A)$$

$$= \det\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

$$A_{21}C_{11} + A_{22}C_{12} + \cdots + A_{2n}C_{1n}$$

$$= \det\begin{pmatrix} A_{21} & A_{22} & \cdots & A_{2n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} = 0$$ (兩列相等)

In fact,

$$A_{p1}C_{q1} + A_{p2}C_{q2} + \cdots + A_{pn}C_{qn} = \begin{cases} \det(A) & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$$

Thus,

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} = \begin{pmatrix} \sum A_{1i}C_{1i} & \sum A_{1i}C_{2i} & \cdots & \sum A_{1i}C_{ni} \\ \sum A_{2i}C_{1i} & \sum A_{2i}C_{2i} & \cdots & \sum A_{2i}C_{ni} \\ \vdots & \vdots & \ddots & \vdots \\ \sum A_{ni}C_{1i} & \sum A_{ni}C_{2i} & \cdots & \sum A_{ni}C_{ni} \end{pmatrix}$$

$$= \begin{pmatrix} \det(A) & & & 0 \\ & \det(A) & & \\ & & \ddots & \\ 0 & & & \det(A) \end{pmatrix} = \det(A) \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

$$= \det(A)\, I_n$$

This implies that if A is invertible,

$$A \cdot \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T = I_n$$

Hence, we have the following result.

Proposition 19: Suppose $A \in M_{n\times n}(\mathbb{R})$ is invertible, then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$$

Example 20: (1) $A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \\ 3 & -1 & 0 \end{pmatrix}$

then

$$\det(A) = 6 - 12 + 4 = -2$$

$$A^{-1} = \frac{1}{-2} \begin{pmatrix} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \end{pmatrix}^T$$

$$= -\frac{1}{2}\begin{bmatrix} 2 & 6 & -3 \\ -4 & -12 & 5 \\ -2 & -4 & 2 \end{bmatrix}^T = \begin{bmatrix} -1 & -3 & 3/2 \\ 2 & 6 & -5/2 \\ 1 & 2 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 2 & 1 \\ -3 & 6 & 2 \\ 3/2 & -5/2 & -1 \end{bmatrix}$$

(2) $$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

then

$$\det(A) = (-1)^{3+3}\cdot 1 \cdot \det\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = 1 + 2 - 1 = 2.$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix}
\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} & \begin{vmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\
-\begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\
\begin{vmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\
-\begin{vmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 0 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}
\end{bmatrix}^T$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 & 2 \\ 1 & -1 & 0 & -1 \\ 0 & -2 & 2 & -2 \\ 1 & 1 & 0 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1 & -1/2 & -1 & 1/2 \\ 0 & 0 & 1 & 0 \\ 1 & -1/2 & -1 & -1/2 \end{bmatrix}$$