1.5 Some special matrices.

這節介紹幾個特殊矩陣.

Definition 45: A symmetric matrix (對稱矩陣) is a matrix $A \in M_{n\times n}(\mathbb{R})$ with $A^T = A$, i.e.,

$$A_{ij} = A_{ji} \quad \forall \ 1 \le i, j \le n.$$

Example 46: $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 4 & 3 \\ 3 & -2 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 3 \end{pmatrix}$ are symmetric.

Lemma 47: $A, B \in M_{n\times n}(\mathbb{R})$: symmetric

$\Longrightarrow A+B,\ cA$: symmetric for $c \in \mathbb{R}$.

這個 Lemma 並不難証明. 同學們可自行証明.

Proposition 48: $R \in M_{m\times n}(\mathbb{R}) \Longrightarrow R^T R$ and RR^T: symmetric

proof: $(R^T R)^T = R^T (R^T)^T = R^T R$

$\Longrightarrow R^T R$: symmetric.

Example 49: $R = (2 \ \ 4) \Longrightarrow R^T = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$$R^T R = \begin{pmatrix} 2 \\ 4 \end{pmatrix} (2 \ \ 4) = \begin{pmatrix} 4 & 8 \\ 8 & 16 \end{pmatrix}$$

$$R R^T = (2 \ \ 4) \begin{pmatrix} 2 \\ 4 \end{pmatrix} = (20).$$

Definition 50: (1) $A \in M_{n\times n}(\mathbb{R})$ is called upper triangular (上三角矩陣) if

$$A_{ij} = 0 \quad \text{for } i > j$$

(2) $A \in M_{n\times n}(\mathbb{R})$ is called lower triangular (下三角矩陣) if

$$A_{ij} = 0 \quad \text{for } i < j.$$

(3) $A \in M_{n\times n}(\mathbb{R})$ is called diagonal (對角線矩陣) 若

$$A_{ij} = 0 \quad \text{for } i \neq j.$$

Example 51: $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{pmatrix}$: upper triangular

$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 0 \\ -3 & 0 & 0 \\ -2 & 2 & 1 \end{pmatrix}$: lower triangular.

$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$: diagonal

看形式就知道.

Remark 52: (1) A : upper triangular $\Longleftrightarrow$ A^T : lower triangular.

(2) A : diagonal

$\Longleftrightarrow$ A : both upper and lower triangular.

(3) $A, B \in M_{n\times n}(\mathbb{R})$: upper (lower) triangular

$\Longrightarrow$ AB : upper (lower) triangular.