3.4 Column space, null space, rank, and nullity

Definition 23: $A \in M_{m\times n}(\mathbb{R})$

The column space (行空間) contains all linear combinations of the columns of A. Explicitly, if

$$A = (A_1\ A_2 \cdots A_n) \quad \text{with} \quad A_i = \begin{bmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{mi} \end{bmatrix}$$

$$\begin{aligned} \text{the column space of } A &= \{ c_1A_1 + c_2A_2 + \cdots + c_nA_n : c_1, c_2, \cdots, c_n \in \mathbb{R} \} \\ &= \text{span}\{A_1, A_2, \cdots, A_n\} \subseteq \mathbb{R}^m. \end{aligned}$$

很顯然，$A \in M_{m\times n}(\mathbb{R})$ 的 column space 是 $\mathbb{R}^m$ 的 subspace.

Definition 24: The dimension of column space of a matrix is called the rank (秩) of the matrix. 若 A 為矩陣，A 的 rank 記成 $\text{rank}(A)$.

Example 25: Consider

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

則 A 的 column space 為 $\text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 3 \\ 9 \end{bmatrix} \right\}$

尋找 A 的 column space 的 basis 或求 $\text{rank}(A)$ 的方法：

(i) 做行運算

(ii) 對 A^T 做列運算，化成 row echelon form

$$A^T = \begin{array}{c} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & -2 & 4 & -6 \\ -1 & 1 & -3 & 2 \\ 2 & 2 & 2 & 0 \\ 1 & 3 & 0 & 3 \\ 2 & 6 & 3 & 9 \end{bmatrix} \end{array} \begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \end{array} \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 4 & -2 & 6 \\ 0 & 4 & -2 & 6 \\ 0 & 8 & -1 & 15 \end{bmatrix} \begin{array}{c} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \end{array}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix} \begin{array}{c} ① \\ ④ \\ ③ \\ ② \\ ⑤ \\ ⑥ \end{array} \longrightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} ① \\ ④ \\ ③ \\ ② \\ ⑤ \\ ⑥ \end{array}$$

A 的 column space $= \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} \right\}$

$= \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix} \right\} = R(A)$

$\text{rank}(A) = 3.$

Definition 26: $A \in M_{m\times n}(\mathbb{R})$

(1) The null space (零空間.零核空間) of A:

$\text{null}(A) = \{ x \in \mathbb{R}^n : Ax = 0 \} = N(A)$

(2) The dimension of the null space of A is called the nullity (核數.零度.零化度) of A, 記成 nullity(A).

Example 27: Consider

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

(1) Find $N(A)$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N(A) = \{ x \in \mathbb{R}^6 : Ax = 0 \}$$

$$= \{ x \in \mathbb{R}^6 : x_1 = -2x_2 + x_5 + 5x_6,\ x_3 = 3x_6,\ x_4 = -x_5 - 2x_6 \}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_5 + 5x_6 \\ x_2 \\ 3x_6 \\ -x_5 - 2x_6 \\ x_5 \\ x_6 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 5 \\ 0 \\ 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(2) nullity$(A) = 3$.

Example 28: Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 & -1 \\ 0 & 0 & 1 & 3 & 2 \\ 2 & 4 & 7 & 0 & 1 \\ 3 & 6 & 11 & 1 & 2 \end{bmatrix} \in M_{4\times 5}(\mathbb{R})$$

(1) Find $R(A)$ and rank(A).

$$A \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 2 & 0 & 1 & 4 & 3 \\ 3 & 0 & 2 & 7 & 5 \end{bmatrix} \\ \begin{matrix} 1 & \ 2 & \ 3 & \ 4 & \ 5 \end{matrix} \end{array} \longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 \end{bmatrix} \\ \begin{matrix} 1 & \ 3 & \ 4 & \ 5 & \ 2 \end{matrix} \end{array}$$

$$\longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \end{bmatrix} \\ \begin{matrix} 1 & \ 3 & \ 4 & \ \ & \ \ \end{matrix} \end{array}$$

$$R(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$R(A)$ 的 basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \text{亦或} \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(basis 並非唯一)

$\Longrightarrow \text{rank}(A) = 3.$

(2) Find $N(A)$ and nullity(A)

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 & -1 \\ 0 & 0 & 1 & 3 & 2 \\ 2 & 4 & 7 & 0 & 1 \\ 3 & 6 & 11 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & -2 & -1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 2 & 7 & 5 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 & -2 & -1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$N(A) = \{x \in \mathbb{R}^5 : x_1 + 2x_2 + 4x_5 = 0,\ x_3 - x_5 = 0,\ x_4 + x_5 = 0\}$

For $x = (x_1, x_2, x_3, x_4, x_5)^T \in \mathbb{R}^5$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2-4x_5 \\ x_2 \\ x_5 \\ -x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$N(A) = \text{span}\underbrace{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}}_{\text{basis for } N(A).} \text{ and nullity}(A) = 2.$$

仔細觀察 rank(A) 與 nullity(A). 可得

Theorem 29 (rank-nullity theorem / dimension theorem)

For $A \in M_{m\times n}(\mathbb{R})$,

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Example 30: (1) As in Example 28. $A \in M_{4\times 5}(\mathbb{R})$.

$\text{rank}(A) = 3$, $\text{nullity}(A) = 2$

$\text{rank}(A) + \text{nullity}(A) = 5 = n$.

(2) As in Example 27. $A \in M_{4\times 6}(\mathbb{R})$.

$\text{nullity}(A) = 3$

$\Longrightarrow \text{rank}(A) = n - \text{nullity}(A) = 6 - 3 = 3$

As in Example 25.

$\text{rank}(A) = 3$