2.2 Determinants of general order.

這一節要介紹的是一般 $n\times n$ 矩陣行列式的定義與性質.

Given $A \in M_{n\times n}(\mathbb{R})$.

Notation 8: Define $\hat{A}_{ij} \in M_{(n-1)\times(n-1)}(\mathbb{R})$ obtained from A by deleting ith row and jth column.

($\hat{A}_{ij}$ 就是在 A 中去掉 A_{ij} 所在那一行, 那一列)

不要把這個符號想得太難. 用例子來看比較簡單.

Example 9: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \in M_{3\times 3}(\mathbb{R})$.

$$\hat{A}_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$$

$$\hat{A}_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$\hat{A}_{32} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$$

Definition 10: $A \in M_{n\times n}(\mathbb{R})$.

For $n=1$. we define $\det(A) = A_{11}$

For $n \geq 2$, we define $\det(A)$ recursively by

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\hat{A}_{1j})$$

注意這個定義, 對第一列展開.

Remark 11: $n=1$: $\det(A) = A_{11}$

$n=2$: $\det(A) = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$

$$= (-1)^{1+1} A_{11} \det(\hat{A}_{11}) + (-1)^{1+2} A_{12} \det(\hat{A}_{12})$$

$$= (-1)^{1+1} A_{11} A_{22} + (-1)^{1+2} A_{12} A_{21}$$

$$= A_{11} A_{22} - A_{12} A_{21}$$

$n=3$: $\det(A) = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$

$$= (-1)^{1+1} A_{11} \det(\hat{A}_{11}) + (-1)^{1+2} A_{12} \det(\hat{A}_{12}) + (-1)^{1+3} A_{13} \det(\hat{A}_{13})$$

$$= A_{11} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13} \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}$$

$$= A_{11}(A_{22} A_{33} + A_{23} A_{32}) - A_{12}(A_{21} A_{33} - A_{23} A_{31}) + A_{13}(A_{21} A_{32} - A_{22} A_{31})$$

$$= A_{11} A_{22} A_{33} + A_{12} A_{23} A_{31} + A_{13} A_{32} A_{21} - A_{11} A_{32} A_{23} - A_{22} A_{13} A_{31} - A_{33} A_{12} A_{21}.$$

$n \geq 4$ 時，定法就和之前完全不同.

Remark 12: $n \geq 4$ 的 determinant 必須按照定義來算，用"降階"的方法來計算. In particular,

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix} \neq A_{11} A_{22} A_{33} A_{44} + A_{12} A_{23} A_{34} A_{41} + \cdots$$

不能採用和之前 $n=2$ 及 $n=3$ 相同的方法來計算.

Definition 13: The scalar

$$c_{ij} = (-1)^{i+j} \det(\hat{A}_{ij})$$

denote the cofactor (餘因子) of the matrix A in ith row, jth column.

Remark 14: $\det(A) = A_{11} c_{11} + A_{12} c_{12} + \cdots + A_{1n} c_{1n}$

This formula is called cofactor expansion (餘因子展開) along the first row of A.

Example 15: $A = \begin{bmatrix} 2 & -3 & 4 & 6 \\ -1 & 0 & 2 & 5 \\ 3 & 0 & -1 & -2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$

then the cofactor of A in row 1, column 2 is

$$C_{12} = (-1)^{1+2} \cdot \det \begin{bmatrix} -1 & 2 & 5 \\ 3 & -1 & -2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$= (-1) \cdot (5 + 30 - 4 - 30) = -1$$

and

$$\det(A) = \det \begin{bmatrix} 2 & -3 & 4 & 6 \\ -1 & 0 & 2 & 5 \\ 3 & 0 & -1 & -2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

$$= 2 \cdot \det \begin{bmatrix} 0 & 2 & 5 \\ 0 & -1 & -2 \\ 0 & 2 & 5 \end{bmatrix} - (-3) \det \begin{bmatrix} -1 & 2 & 5 \\ 3 & -1 & -2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$+ 4 \det \begin{bmatrix} -1 & 0 & 5 \\ 3 & 0 & -2 \\ 0 & 0 & 5 \end{bmatrix} - 6 \cdot \det \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= 2 \times 0 + 3 \times 1 + 4 \times 0 - 6 \times 0 = 3.$$

但一定要計算一列展開嗎?

Theorem 16: 方陣的 cofactor expansion 可對任何一行或任何一列展開.

i.e., for $A \in M_{n \times n}(\mathbb{R})$, then for any $1 \le i \le n$,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\hat{A}_{ij})$$

$$= \sum_{j=1}^{n} (-1)^{j+i} A_{ji} \det(\hat{A}_{ji})$$

Example 17: As in Example 15.

$$\det \begin{pmatrix} 2 & -3 & 4 & 6 \\ -1 & 0 & 2 & 5 \\ 3 & 0 & -1 & -2 \\ 0 & 0 & 2 & 5 \end{pmatrix}$$

= the cofactor expansion along the 2nd column

$$= (-1)^{1+2}(-3) \det \begin{pmatrix} -1 & 2 & 5 \\ 3 & -1 & -2 \\ 0 & 2 & 5 \end{pmatrix} = -1 \cdot (-3) \cdot 1 = 3.$$

Remark 18: Proposition 4 and Proposition 7 hold for any $n \geq 1$.