4.3 Applications

4.3.1 Powers and products: A^k and AB.

Lemma 29: $\lambda_1, \lambda_2, \cdots, \lambda_k$: eigenvalues of A. Then

(1) $\lambda_1^m, \lambda_2^m, \cdots, \lambda_k^m$: eigenvalues of A^m.

(2) every eigenvector of A is also an eigenvector of A^m.

proof: Let v: eigenvector of A corresponding to the eigenvalue λ

then

$$A^m v = A^{m-1}(Av) = A^{m-1}(\lambda v) = \lambda A^{m-1} v = \lambda A^{m-2}(Av)$$
$$= \lambda A^{m-2}(\lambda v) = \lambda^2 A^{m-2} v = \cdots = \lambda^m v$$

Lemma 30: A: invertible $\Longrightarrow$ all of its eigenvalues $\neq 0$.

Moreover, the eigenvalue of $A^{-1} = \frac{1}{\lambda}$

proof: A : invertible $\Longrightarrow \det(A) \neq 0$

$\Longrightarrow$ 0 cannot be an eigenvalue of A.

Furthermore,

$$Av = \lambda v \Longrightarrow v = A^{-1}(\lambda v) = \lambda A^{-1} v$$
$$\Longrightarrow A^{-1} v = \frac{1}{\lambda} v$$

($\Rightarrow A, A^{-1}$ 的 eigenvectors 相同)

Lemma 31: A: diagonalizable $\Longrightarrow A^m, A^{-1}$: diagonalizable.

proof: 這裏只証 A^m 的 case.

A: diagonalizable $\Longrightarrow \exists$ eigenvector matrix Q and diagonal matrix D such that

$$A = QDQ^{-1}$$

$$\Rightarrow A^m = (QDQ^{-1})(QDQ^{-1})\cdots(QDQ^{-1})$$
$$= QD(Q^{-1}Q)D(Q^{-1}Q)D\cdots DQ^{-1} = QDD\cdots DQ^{-1}$$
$$= QD^mQ^{-1}$$

and D^m is a diagonal matrix.

$\Rightarrow A^m$: diagonalizable.

Remark 32: A, B have the same eigenvector v corresponding to the eigenvalues λ and μ, respectively, i.e.,

$$Av = \lambda v, \quad Bv = \mu v.$$

then

(1) $A+B$ has an eigenvalue $\lambda + \mu$.

(2) AB has an eigenvalue $\lambda\mu$.

注意: 若 A, B "都有"相同的 eigenvector, 這個結果會是錯的.

Example 33: Consider

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

A:

eigenvalue 5. $E_5 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

eigenvalue -1. $E_{-1} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

B:

eigenvalue 4. $E_4 = \text{span}\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$

eigenvalue -1. $E_{-1} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

$$A + B = \begin{pmatrix} 3 & 5 \\ 6 & 4 \end{pmatrix}.$$

eigenvalue 9. $\quad E_9 = \text{span}\left\{\begin{pmatrix} 5 \\ 6 \end{pmatrix}\right\}$

eigenvalue -2 $\quad E_{-2} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

$$AB = \begin{pmatrix} 11 & 10 \\ 9 & 10 \end{pmatrix}$$

eigenvalue 20. $\quad E_{20} = \text{span}\left\{\begin{pmatrix} 10 \\ 9 \end{pmatrix}\right\}$

eigenvalue 1 $\quad E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

比較一下這些性質與 Remark 32 之間的關連.

如果只有一組 eigenvectors 相同的話，有沒有別的狀況

<u>Lemma 34</u>: A, B : diagonalizable

A, B share the same eigenvector matrix $Q \iff AB = BA$.

<u>proof</u>: "$\Rightarrow$" Suppose $A = QD_1Q^{-1}$. $B = QD_2Q^{-1}$

where D_1, D_2, diagonal matrices.

Since $D_1D_2 = D_2D_1$.

$$\begin{aligned} AB &= (QD_1Q^{-1})(QD_2Q^{-1}) = QD_1(Q^{-1}Q)D_2Q^{-1} \\ &= QD_1D_2Q^{-1} = QD_2D_1Q^{-1} = (QD_2Q^{-1})(QD_1Q^{-1}) \\ &= BA. \end{aligned}$$

"$\Leftarrow$" 省略