Idiosyncratic Asset Return Risk and Portfolio Choice - When does Social Security lead to Crowding IN of Capital?

Raphael Abiry*

December 19, 2021

Abstract

When studying the welfare effects of pay-as-you-go social security systems, efficiency gains due to risk-sharing are contrasted to welfare losses due to distortions. In related literature distorted saving decisions leading to crowding out of capital are identified as a major source for welfare losses. By and large many studies find that the costs of introducing social security outweigh the benefits.

But to my knowledge the literature so far disregards positive welfare effects of social security due to shifts in the asset portfolio of households. I study an overlapping generations model featuring idiosyncratic asset return risk and portfolio choice and find that social security benefits stipulate households to shift their savings to riskier assets with higher returns. On the one hand social security lowers the marginal propensity to save, but on the other hand it boosts returns on savings. The overall effect on savings and hence capital depends on which of these two effects dominates. Whether we encounter crowding in or out depends on the level of the risk-free rate and its elasticity with respect to changes in the quantity of the safe asset. Near-zero inelastic risk-free rates guarantee that crowding in prevails.

*Goethe University Frankfurt. Email: abiry.econ@gmail.com

1 Introduction

When studying the welfare effects of pay-as-you-go (PAYG) social security systems, efficiency gains due to risk-sharing are contrasted to welfare losses due to distortions. In the related literature distorted saving decisions leading to crowding out of capital are identified as a major source for welfare losses. By and large many studies find that the costs of introducing social security outweigh its benefits.

However, to my knowledge the literature disregards so far the effects on the portfolio choice of households. Safe future social security benefits stipulate households to shift their savings to riskier assets with higher returns. In the following I will refer to this as the *portfolio shift mechanism / effect*. On the one hand social security lowers the marginal propensity to save, but on the other hand it boosts returns on savings. The overall effect on savings and hence capital and output depends on which of these two effects dominates. Social security may also lead to crowding in of capital and this may tilt the welfare findings in favour of social security. Moreover, whether social security leads to crowding in or out of capital is also crucial for determining its effect on aggregate output.

The safe real rate is an important quantity when determining whether social security leads to crowding in or out of capital. Low safe rates (which are also relatively inelastic to changes in demand for safe assets) increase the strength of the portfolio shift effect. Hence, the currently prevailing low interest rates in industrialized countries, particularly low safe rates, increase the likelihood that an expansion of a PAYG social security system may lead to capital crowding in and thus to an increase in GDP.

I investigate the portfolio shift effect when a social security system is introduced or expanded (at the margin). I do this within a simplified incomplete market overlapping generations (OLG) model with idiosyncratic asset return risk. Households receive returns from a safe and a risky asset and social security benefits are financed by a consumption tax.[1] Higher benefits have two effects. First, they reduce the marginal saving rate. This is the classical savings crowding out effect. Second, social security benefits are safe assets within this model (as aggregate risk is disregarded). Thus, social security benefits crowd out the safe assets and households increase their portfolio share in the risky asset. As risky assets have a higher return, the total portfolio return increases. The overall effect depends on which of the two effects dominates. This determines if an increase in the consumption tax - corresponding to the contribution rate in this model - leads to a decrease or an increase in aggregate savings and thus capital.

[1] An extension with benefits financed via payroll taxes is currently in the works.

Furthermore, a determining factor for the direction of the overall effect is the safe rate. The reason is that households discount future social security benefits by the gross safe rate to determine their current value. The lower the safe rate, the lower the discounting and hence the higher the current value of benefits. A higher current value of discounted benefits means a larger share of the safe assets is crowded out and also a larger shift to risky assets. I find that in the case of a safe rate equal to zero the increase in the portfolio return outweighs the decrease in the savings rate and raising social security benefits (financed via consumption taxes) leads to a crowding in of capital and therefore to an expansion of output.

The above described findings characterize the partial equilibrium. In general equilibrium the elasticity of the safe rate with respect to the demand for safe assets also plays a role. The more inelastic it is, the stronger the portfolio shift effect.

The finding that social security can insure against risks, by partially completing financial markets, thereby increasing economic efficiency, dates back to (Diamond 1965) and (Merton 1981). In terms of risks, aggregate and idiosyncratic risks can be distinguished. Aggregate risk is typically wage and asset return risk. In that setting social security improves risk-sharing by pooling these risks across generations, i.e. *inter*generational risk sharing. Idiosyncratic risk is typically modeled as income or survival risk and social security provides *intra*generational insurance by redistributing ex-post from high- to low-income households or providing substitutes for annuity contracts.

Quantitative papers calibrated to the US economy include Krueger and Kubler (2006) considering aggregate risk and İmrohoroglu, Imrohoroglu, and Joines (1995) and Storesletten, Telmer, and Yaron (1999) with idiosyncratic risk. These studies find that the adverse effects due to crowding out of capital more than outweigh welfare benefits from risk-sharing in the case of the US social security system.[2] Harenberg and Ludwig (2019) combine aggregate and idiosyncratic risks and find welfare gains from the US social security system. The reason is that welfare gains from both risks are larger than the sum of the benefits from insurance against isolated risks.[3]

My analysis is based on the assumption that asset returns differ substantially across individuals, such that social security can act as a partial insurance instrument against these idiosyncratic risks. And indeed, recent empirical literature reports substantial heterogeneity of returns across individuals (c.f. Fagereng, Guiso, Malacrino, and Pistaferri (2020) and Bach, Calvet, and Sodini (2020)).

[2] İmrohoroglu, Imrohoroglu, and Joines (1995) find positive welfare effects in the baseline scenario, in which they use a discount factor above unity. When using a more usual value for the discount factor below unity or when incorporating exogenous technical progress into the model the welfare effects are negative.

[3] A second factor is the countercyclical cross sectional variance of the idiosyncratic risk. In a negative aggregate state idiosyncratic shocks have a higher variance.

From a theoretical standpoint my paper is closely related to Obstfeld (1992). He shows that international trade allows each country to diversify their portfolio of risky investments and thus to insure themselves against country specific idiosyncratic risks. This insurance through trade encourages countries to simultaneously shift their investment portfolios from safe but lower yielding investments to riskier but higher yielding ones. As higher yielding investments are more productive, this spurs economic growth. Within my setup, idiosyncratic risks are partly socially insured through pooling of consumption taxes, combined with lump-sum social security benefits. This insurance through pooling strengthens the safe asset crowding out property of social security that I identified above.

The paper proceeds as follows. Section 2 develops the model setup and studies it in partial equilibrium. Section 3 studies exemplary cases in general equilibrium and finally section 4 concludes.

2 Partial Equilibrium

2.1 Model Setup

Time is discrete and runs from $t = 0$ to infinity. In each period a generation is born and lives for two periods. Let $j = 0$ indicate young and $j = 1$ old households, with the mass of each generation in each period being 1.

2.1.1 Households

Households hold their financial wealth in bonds b_{tj} and stocks s_{tj}. Bonds pay a safe return of r_t^f and stocks a risky return of $1 + r_t^s + \varepsilon$, where $r_t^s > \max\{0, r_t^f\}$ constitutes the deterministic part and ε is a mean zero random variable with cumulative distribution function $F(\varepsilon)$ with $F(-(1 + \min_t\{r_t^s\})) = 0$, i.e. the probability that the realization of the gross return of the risky asset is zero or negative is zero. The risky asset never looses all its value. The expected stock return is r_t^s.

The ε's are neither serially nor across households correlated. After accruing interest from the assets households consume c_{tj} and reinvest the rest into bonds and stocks, $b_{t+1,j+1}$ and $s_{t+1,j+1}$. Households make investment decisions and yield returns both when young and old. Young households are given an endowment of h to invest in b_{t0} and s_{t0}. Old households may not exit the model with debt, implying that $b_{t1} + s_{t1} \geq 0$. I abstract from labor supply, such that all household income is from asset income.

When old, households receive social security benefits p_t, which are independent of their wealth and asset income and which are financed by taxes on consumption with a tax rate of

$\tau \in [0, \infty)$. The tax rate is constant across periods. Finally, households derive utility from consumption and I assume log-utility. Thus, the household maximization problem is given by

$$\begin{aligned}
\max_{\substack{c_{t0}, c_{t+1,1}, \\ b_0, s_0, b_{t+1,1}, s_{t+1,1}}} \quad & \mathbb{E}_{\varepsilon_{t0}}[\ln c_{t0}] + \beta \mathbb{E}_{\varepsilon_{t+1,1}|\varepsilon_{t0}}[\ln c_{t+1,1}] \\
\text{s.t. } & b_{t+1,1} + s_{t+1,1} + (1+\tau) \cdot c_{t0} = (1 + r_t^f) \cdot b_{t0} + (1 + r_t^s + \varepsilon) \cdot s_{t0} \\
& (1+\tau) \cdot c_{t+1,1} = (1 + r_{t+1}^f) \cdot b_{t+1,1} + (1 + r_{t+1}^s + \varepsilon) \cdot s_{t+1,1} + p_{t+1} \\
& b_{t0} + s_{t0} = h,
\end{aligned} \tag{1}$$

where $\mathbb{E}_{\varepsilon_{t0}}[\ln c_{t0}]$ is the unconditional expectation of log-consumption over ε_{t0} and $\mathbb{E}_{\varepsilon_{t+1,1}|\varepsilon_{t0}}[\ln c_{t+1,1}]$ the conditional expectation of log-consumption over $\varepsilon_{t+1,1}$ given ε_{t0}. As ε_{t0} and are $\varepsilon_{t+1,1}$ are uncorrelated I will only write unconditional expectation terms from here on.

2.1.2 Social Security System

The social security system is fully financed by taxes on consumption. As both generations have a measure of one and idiosyncratic shocks cancel each other out due to the law of large numbers, aggregate consumption is given by

$$C_t = \mathbb{E}[c_{t0}] + \mathbb{E}[c_{t1}].$$

The social security system is exclusively and fully financed through consumption taxes, implying

$$p_t = \tau C_t. \tag{2}$$

2.1.3 Transformation of the Household Problem

The household problem fits into the framework of Merton (1969) and Samuelson (1969). To see this, specify (beginning of period) wealth as

$$x_{tj} = b_{tj} + s_{tj} + \frac{p_{t+1-j}}{\prod_{i=0}^{1-j}(1 + r_{t+i}^f)}, \tag{3}$$

the risky portfolio share as

$$\theta_{tj} = \frac{s_{tj}}{x_{tj}}$$

and the portfolio return as

$$r_{tj}^{P} = r_t^f + \theta_{tj} \cdot (r_t^s + \varepsilon - r_t^f). \tag{4}$$

Two points are noteworthy. First, discounted (by the risk-free rate) social security benefits are a component of wealth. Second, together with bonds they constitute risk-free asset holdings, $(1-\theta_{tj})x_{tj} = b_{tj} + \frac{p_{t+1-j}}{\prod_{i=0}^{1-j}(1+r_{t+i}^f)}$. From one period to the next, bonds and discounted benefits have the same return of r_t^f. Within this framework social security benefits represent surrogate bonds.

Given these transformations we can rewrite the household problem (1) as

$$\begin{aligned} \max_{\substack{c_{t0},c_{t+1,1},\\ \theta_{t0},\theta_{t+1,1}}} \quad & \mathbb{E}[\log c_{t0}] + \beta\mathbb{E}[\log c_{t+1,1}] \\ \text{s.t. } & x_{t+1,1} + (1+\tau)\cdot c_{t0} = (1 + r_{t0}^P(\theta_{t0}))\cdot x_{t0} \\ & (1+\tau)\cdot c_{t+1,1} = (1 + r_{t+1,1}^P(\theta_{t+1,1}))\cdot x_{t+1,1}. \end{aligned} \tag{5}$$

This structure of the household problem gives rise to a simple solution.

Proposition 1. *Optimal portfolio shares are independent of age, i.e.* $\theta_{tj} = \theta_t$ *and solve*

$$\mathbb{E}\left[\frac{r_t^s + \varepsilon - r_t^f}{1 + r_t^P(\theta)}\right] = 0.\text{[4]} \tag{6}$$

Conditional on the portfolio shares, optimal consumption is linear in wealth

$$c_{tj}(x_{t,j}, \varepsilon) = m_j(1 + r_t^P(\theta_t, \varepsilon))x_{tj}, \tag{7}$$

where marginal propensity to consume out of wealth cum interest is independent of wealth $x_{t,j}$*, the shock realization of* ε *and time. Furthermore it is given by*

$$m_0 = \frac{1}{1+\beta} \qquad \text{and} \qquad m_1 = 1.$$

Proof. See appendix A.1. □

[4]This condition does not rule out short selling of bonds or stocks. The corresponding condition ruling this out is $\theta = \arg\max_{\tilde{\theta}\in[0,1]} \mathbb{E}\big[ln(1 + r_t^P(\tilde{\theta}))\big]$.

2.2 Equilibrium

Note that aggregate wealth of both cohorts in period t is given by

$$X_{t0} = x_{t0} = h + \frac{p_{t+1}}{\prod_{i=0}^{1}(1 + r^f_{t+i})}, \tag{8a}$$

$$X_{t1} = \mathbb{E}[x_{t1}] = (1 - m_0) \cdot \mathbb{E}[r^P_t(\theta_t, \varepsilon)] \cdot X_{t0}, \tag{8b}$$

and hence aggregate consumption is

$$C_t = \frac{m_0}{1+\tau} \cdot \mathbb{E}[r^P_t(\theta_t, \varepsilon)] \cdot X_{t0} + \frac{m_1}{1+\tau} \cdot \mathbb{E}[r^P_t(\theta_t, \varepsilon)] \cdot X_{t1}. \tag{9}$$

In the partial equilibrium the sequence of interest rates, r^f_t and r^s_t is given, but social security is financed within the model through consumption taxes. Given the parameters, including h and a tax rate τ the equilibrium definition is as follows

Definition 1 (Partial Equilibrium)**.** *A partial equilibrium are policy functions for the household, $\theta(t), m(t,j)$, $j \in \{1,2\}$, aggregate wealth holdings $X(t,j)$, $j \in \{1,2\}$, aggregate consumption $C(t)$ and benefits $p(t)$ such that for all t*

1. *$\theta(t)$ and $m(t,j)$ solve the household problem in* (5)*,*
2. *Aggregate wealth holdings $X(t,j)$ are given by* (8)*, where the portfolio return is given in* (4)*,*
3. *Aggregate consumption is given by* (9)*,*
4. *social security funding* (2) *is fulfilled*
5. *$r^f(t)$ and $r^s(t)$ are given and fulfill the conditions outlined in the model description.*

In the following let $R^f_t = 1 + r^f_t$ and $R^P_t(\varepsilon) = 1 + r^P_t(\varepsilon)$ denote the gross risk-free and portfolio returns.

Given (7) consumption is proportional to initial wealth:

$$c_{t0} = \frac{1}{1+\tau} \cdot \frac{1}{1+\beta} \cdot R^P_t \cdot x_{t0}$$

$$c_{t+1,1} = \frac{1}{1+\tau} \cdot \frac{\beta}{1+\beta} \cdot R^P_{t+1} \cdot R^P_t \cdot x_{t0}$$

Using the law of large numbers aggregate consumption is thus

$$\begin{aligned} C_t &= \mathbb{E}[c_{t0}] + \mathbb{E}[c_{t1}] \\ &= \mathcal{B}_t(\tau) \cdot h + \frac{1}{1+\tau} \cdot \mathcal{A}_t \cdot \left(\frac{p_{t+1}}{R^f_t R^f_{t+1}} + \beta \mathbb{E}[1 + r^P_{t-1}] \cdot \frac{p_t}{R^f_{t-1} R^f_t} \right), \end{aligned} \tag{10}$$

where h represents the endowed financial assets of the young generation and

$$\begin{aligned} \mathcal{A}_t &= \frac{\mathbb{E}[R^P_t]}{1+\beta} \\ \mathcal{J}_t &= 1 + \beta \mathbb{E}[R^P_{t-1}] \\ \mathcal{B}_t(\tau) &= \frac{\tau}{1+\tau} \cdot \mathcal{A}_t \mathcal{J}_t. \end{aligned}$$

These quantities have intuitive interpretations. $\mathcal{A}_t \cdot \mathcal{J}_t$ corresponds to the marginal change in aggregate expenditures in response to a change in wealth of the young generation across periods, i.e. $\mathcal{A}_t \cdot \mathcal{J}_t = \frac{\partial (1+\tau) C_t}{\partial x_0}$, where $dx_{t0} = dx_{t-1,0} = dx_0$. Similarly $\mathcal{B}_t(\tau)$ corresponds to the marginal change in tax revenues in response to a change in wealth of the young generation across periods, i.e. $\mathcal{B}_t(\tau) = \frac{\partial \tau C_t}{\partial x_0}$, where $dx_{t0} = dx_{t-1,0} = dx_0$. An increase of wealth of the young generation in periods $t-1$ and t by \$1 leads in period t to an increase of expenditures of \$$\mathcal{A}_t \mathcal{J}_t$ and of tax revenues of \$$\mathcal{B}_t(\tau)$ at the margin.

For a young household in period t the present value of future benefits as measured as (beginning-of-period) wealth is $\frac{p_{t+1}}{R^f_t R^f_{t+1}}$. Hence an increase in benefits by \$1 in periods t and $t+1$, i.e. $dp_t = dp_{t+1} = 1\$$, leads in period t to young households having a higher present value of wealth of \$$\frac{1}{R^f_t R^f_{t+1}}$ and old households having had a higher present value of wealth when young of \$$\frac{1}{R^f_{t-1} R^f_t}$. Thus at the margin aggregate expenditures increase by $\frac{\partial (1+\tau) C_t}{\partial p} = \frac{\mathcal{A}_t \cdot \mathcal{J}_t}{R^f_t R^f_{t+1}} + \frac{\mathcal{A}_t \beta \mathbb{E}[1 + r^P_{t-1}](r^f_{t+1} - r^f_{t-1})}{R^f_{t-1} R^f_t R^f_{t+1}}$ and tax revenues by $\frac{\partial \tau C_t}{\partial p} = \frac{\mathcal{B}_t(\tau)}{R^f_t R^f_{t+1}} + \frac{\mathcal{B}_t(\tau)/\mathcal{J}_t \cdot \beta \mathbb{E}[1 + r^P_{t-1}](r^f_{t+1} - r^f_{t-1})}{R^f_{t-1} R^f_t R^f_{t+1}}$, where $dp_t = dp_{t+1} = dp$. It is the first term in both quantities that is of interest. The second term drops when the risk-free rate is constant across periods, i.e. $r^f_{t-1} = r^f_t = r^f_{t+1} = r^f$, as is the case in steady state. In this case the increase in present value wealth due to an increase in future benefits is the same across generations. Both quantities $\frac{\mathcal{A}_t \mathcal{J}_t}{(1+r^f)^2}$ and $\frac{\mathcal{B}_t(\tau)}{(1+r^f)^2}$ are crucial when determining existence of a steady state and its comparative statics, as analyzed later.

Using equation (10) in the condition for closed social security funding (2) and solving for p_t, benefits are given by

$$p_t = \tau C_t = \frac{1}{1 - \frac{\mathcal{B}_t(\tau)}{R^f_{t-1} R^f_t} \frac{\beta \mathbb{E}[R^P_{t-1}]}{\mathcal{J}_t}} \cdot \left[\mathcal{B}_t(\tau) \cdot h \ + \ \frac{\mathcal{B}_t(\tau)}{R^f_t R^f_{t+1}} \frac{1}{\mathcal{J}_t} p_{t+1} \right]. \tag{11}$$

Note that social security benefits evolve according to a linear difference equations. Equation (11) pins down under what condition the partial equilibrium is unique as specified in the following proposition:

Proposition 2 (Existence and uniqueness of the equilibrium)**.** *If there exists a $\eta < 1$ such that $\frac{\partial \tau C_t}{\partial p} = \frac{\mathcal{B}_t(\tau)}{R^f_t R^f_{t+1}} + \frac{\mathcal{B}_t(\tau)/\mathcal{J}_t \cdot \beta \mathbb{E}[1 + r^P_{t-1}](r^f_{t+1} - r^f_{t-1})}{R^f_{t-1} R^f_t R^f_{t+1}} \leq \eta$ for all t a unique partial equilibrium with a bounded sequence of benefits exists.*[5] *In this case benefits and aggregate consumption are given by*

$$\begin{aligned} p_t &= h \cdot \sum_{i=0}^{\infty} \gamma_{t+1} \prod_{\ell=0}^{i-1} \lambda_{t+\ell} \\ C_t &= \tau \cdot p_t, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \gamma_t &= \frac{\mathcal{B}_t(\tau)}{1 - \frac{\mathcal{B}_t(\tau)}{R^f_{t-1} R^f_t} \frac{\beta \mathbb{E}[R^P_{t-1}]}{\mathcal{J}_t}} \\ \lambda_t &= \frac{\frac{\mathcal{B}_t(\tau)/\mathcal{J}_t}{R^f_t R^f_{t+1}}}{1 - \frac{\mathcal{B}_t(\tau)}{R^f_{t-1} R^f_t} \frac{\beta \mathbb{E}[R^P_{t-1}]}{\mathcal{J}_t}}. \end{aligned}$$

Proof. See appendix A.2. □

Having established uniqueness of the partial equilibrium let us turn to the steady state equilibrium with constant risk-free and risky rates. In the following I state the central finding of the paper:

Corollary 1 (Stationary equilibrium)**.** *If risk-free and risky rates are constant across time, $r^f_t = r^f$ and $r^s_t = r^s$ for all $t > 0$ and $\frac{\partial \tau C}{\partial p} = \frac{\mathcal{B}(\tau)}{(R^f)^2} < 1$ then the unique equilibrium with*

[5] Note that when ignoring the condition of a bounded benefit sequence an infinity of equilibria exists of the form $p_t = h \cdot \sum_{i=0}^{\infty} \gamma_{t+1} \prod_{\ell=0}^{i-1} \lambda_{t+\ell} + \frac{z}{\prod_{i=0}^{t-1} \lambda_i}$ with some $z \in \mathbb{R}^+$. These equilibria have in common an unbounded growth of benefits. However, these equilibria can be easily ruled out by requiring non-negative bond holdings, $b_t \geq 0$ for all t.

bounded benefits is given by

$$p = \frac{\mathcal{B}(\tau)}{1 - \frac{\mathcal{B}(\tau)}{(R^f)^2}} \cdot h \tag{13}$$

and

$$C = \frac{p}{\tau} = \frac{\frac{\mathcal{B}(\tau)}{\tau}}{1 - \frac{\mathcal{B}(\tau)}{(R^f)^2}} \cdot h. \tag{14}$$

If $\frac{\mathcal{B}_t(\tau)}{(R^f)^2} \geq 1$ *no equilibrium exists.*

Furthermore, in equilibrium the comparative statics are

$$\frac{dp}{d\tau} > 0$$
$$\frac{dC}{d\tau} \begin{cases} < 0 & if \ \frac{\mathcal{A}\mathcal{J}}{(R^f)^2} < 1 \\ > 0 & if \ \frac{\mathcal{A}\mathcal{J}}{(R^f)^2} > 1, \end{cases}$$

and there exists a cutoff rate $\bar{r}^f \in (0, r^s)$ *such that*

$$\frac{dC}{d\tau} \begin{cases} > 0 & if \ r^f < \bar{r}^f \\ < 0 & if \ r^f > \bar{r}^f, \end{cases}$$

particularly $\frac{dC}{d\tau} > 0$ *if* $r^f = 0$.

Proof. See appendix A.3. □

Evidently, the steady state equilibrium is a special case of the general case in proposition 2. But what is interesting are the comparative static results. Benefits always increase with the tax rate, no matter the level of the tax rate. Interestingly, aggregate consumption may also increase monotonously with the tax rate. Furthermore there is an easy rule when aggregate consumption increases with τ and when it decreases. $\frac{\mathcal{A}\mathcal{J}}{(R^f)^2} = 1$ defines the watershed condition between an increasing and a decreasing reaction to tax increases. As mentioned earlier $\frac{\mathcal{A}\mathcal{J}}{(R^f)^2}$ equals the marginal change in aggregate expenditure in reaction to an increase in benefits. $\frac{\mathcal{A}\mathcal{J}}{(R^f)^2} > 1$ yields a security benefit multiplier larger than one. Any additional dollar taxed away and redistributed as benefit thus leads to aggregate expenditure increasing by more than one dollar. Hence, (i) consumption increases and (ii) the extra expenditure gets taxed and redistributed again, leading to a multiplier effect.

Furthermore there exists a cutoff-point for the risk-free rate, that lies somewhere between zero and the expected risky rate, such that $\frac{\mathcal{A}\mathcal{J}}{(R^f)^2} = 1$. For risk-free rates below (above) this cutoff point consumption increases (decreases) with the tax rate. Furthermore, a risk-free rate equal to zero guarantees that consumption increases with rising tax rates. The intuition is straightforward. The lower the risk-free rate, the less future benefits get discounted and the more valuable they are when young. Therefore, expenditures also react more sensitive when risk-free rates are low, i.e. $\frac{\mathcal{A}\mathcal{J}}{(R^f)^2} = \frac{\partial\,(1+\tau)C_t}{\partial p}$ increases when risk-free rates are lower.

One interesting implication of the existence of such a positive cutoff rate $\bar{r}^f > 0$ is that an increase in aggregate consumption - and as discussed within the general equilibrium setup also an increase in output - due to an increase in benefits may happen amid a dynamically efficient economy. This can be verified by checking that the sufficient conditions of Krueger and Kubler (2006) are fulfilled.[6]

In order to get some insights into the driving forces behind the findings of corollary 1, particularly the response of aggregate consumption to changes in the tax rate we have a closer look at the financial portfolio. As *financial portfolio/wealth* I understand the holdings of bonds and stocks, $b_j + s_j$, that is total wealth minus the present value of future benefits. This allows me to define a risky share, θ_j^{FP}, portfolio return, r_j^{FP}, and marginal propensity to consume, m_j^{FP}, with respect to financial wealth. This yields

$$\begin{aligned}
\theta_j^{FP} &= \frac{s_j}{b_j + s_j} = \theta \cdot \left(1 + \frac{\frac{p}{(R^f)^{2-j}}}{b_j + s_j}\right) \\
r_j^{FP} &= r^f + \theta_j^{FP} \cdot (r^s + \varepsilon_j - r^f) \\
m_0^{FP} &= m_0 \cdot \frac{1 + r^P}{1 + r_0^{FP}} \cdot \left(1 + \frac{\frac{p}{(R^f)^2}}{b_0 + s_0}\right).
\end{aligned}$$
[7]

Note that the risky share of the financial portfolio, θ_j^{FP}, and the rate of return on the financial portfolio, r_j^{FP}, are age dependent, in contrast to their equivalents with respect to total wealth.

This allows me to state financial wealth when old as

$$b_1 + s_1 = (1 - m_0^{FP}) \cdot (1 + r_0^{FP}) \cdot h.$$

[6] In the following I employ the variant of the sufficiency conditions of Krueger and Kubler (2006) as stated in Harenberg and Ludwig (2019): Let $0 < r^f < \bar{r}^f$. This r^f is by assumption larger than the implicit return of social security in this model, which is equal to zero. Furthermore, there exist ε and ε' such that $r^s + \varepsilon < r^f < r^s + \varepsilon'$. This suffices to show that the conditions are met.

[7] In contrast to the corresponding portfolio quantities (including the discounted benefits) the financial portfolio quantities are path dependent as they depend on $\frac{\frac{p}{(R^f)^{2-j}}}{b_j+s_j}$. $\theta_1^{FP}(\varepsilon_0)$ and $m_0^{FP}(\varepsilon_0)$ depend on the shock when young and $r_1^{FP}(\varepsilon_1, \varepsilon_0)$ on both shocks, when young and old. In a model in which households live for more than two periods the quantities would also be fully path dependent.

Aggregate consumption and stock and bond holdings in the economy are given by $C_t = \mathbb{E}[c_{t0}] + \mathbb{E}[c_{t1}]$, $S_t = s_0 + \mathbb{E}[s_{t1}]$ and $B_t = b_0 + \mathbb{E}[b_{t1}]$. Summing up the budget constraints in household problem (1) we get the aggregate budget constraint

$$C_t + S_{t+1} + B_{t+1} = (1 + r^s)S_t + (1 + r^f)B_t + h,$$

which simplifies in steady state to

$$C = r^s S + r^f B + h.$$

Consumption in steady state is equal to returns in both assets plus the initial endowment. We can decompose this equation into

$$C = \left(\mathbb{E}[r_1^{FP} \cdot (1 - m_0^{FP}) \cdot (1 + r_0^{FP})] + \mathbb{E}[1 + r_0^{FP}]\right) \cdot h.$$

This shows clearly that the savings rate out of financial wealth when young, $(1 - m_0^{FP})$, and the age-profiles of returns on financial wealth, r_j^{FP}, drive aggregate consumption. Hence, the response of these quantities with respect to changes in the tax rate determine how aggregate consumption C reacts to these tax rate changes. The following proposition attaches a sign to these responses:

Proposition 3. *In steady state it holds that for the risky share in the financial portfolio*

$$\frac{d\theta_j^{FP}}{d\tau} > 0,$$

for the return of the financial portfolio

$$\mathbb{E}\left[\frac{dr_j^{FP}}{d\tau}\right] > 0,$$

and for the savings rate of the financial portfolio

$$\frac{d(1 - m_0^{FP})}{d\tau} < 0.$$

Proof. See appendix A.3. □

The sign of the response of each of the quantities with respect to an increase in the tax rate is unambiguous and independent of the parametrization. As stated in corollary 1, social security benefits increase unambiguously with with the tax rate. This has two effects. First, higher benefits mean higher discounted wealth for households, thus crowding out the holding of other assets and leading to a decline in the savings rate in financial wealth. Second, as benefits are risk-free within this model they act as surrogate bonds, thus particularly crowding out bond holdings. This leads to an increase in the risky share in the financial portfolio $\frac{d\theta_j^{FP}}{d\tau} > 0$. A higher risky share also means higher expected returns $\mathbb{E}\left[\frac{dr_j^{FP}}{d\tau}\right] > 0$. The first effect - crowding out of overall savings - is well established in the literature. The seconds effect however - the shift in the portfolio towards riskier higher yielding assets - has not been given attention to yet. Yet, this portfolio shift can mitigate or even overturn the crowding out of overall savings.

3 General Equilibrium

The partial equilibrium findings in the last section were of course subject to interest rates that are unresponsive to changes in demand of stocks and bonds by households. In general equilibrium on the other hand, we would expect risk-free rates to rise and risky rates to fall in response to a shift of demand from bonds to stocks. This would weaken the findings in partial equilibrium. Particularly, additional restrictions are needed for an increase in the tax rate to still lead to an increase in consumption.

In the following I will go through some examples for stock and bond supply and investigate whether an increase in taxes may lead to capital crowding in and hence an increase in output. Before getting to these concrete examples let us state generic stock and bond supply functions as

$$r^f = \rho^f(S, B) \tag{15a}$$

$$r^s = \rho^s(S, B), \tag{15b}$$

with ρ^f (weakly) increasing in S and (weakly) decreasing in B, and ρ^s (weakly) decreasing in S and (weakly) increasing in B.

Note that aggregate bond and stock holdings by households are given by

$$B_t = b_{t0} + \mathbb{E}[b_{t1}] = (1 - \theta_t) \cdot (X_{t0} + X_{t1}) - \frac{p_{t+1}}{\prod_{i=0}^{1}(1 + r_{t+i}^f)} - \frac{p_t}{1 + r_t^f} \tag{16a}$$

$$S_t = s_{t0} + \mathbb{E}[s_{t1}] = \theta_t \cdot (X_{t0} + X_{t1}) \tag{16b}$$

Given this, we can define the general equilibrium as follows.

Definition 2 (General Equilibrium)**.** *A general equilibrium are policy functions for the household, $\theta(t), m(t,j)$, $j \in \{1,2\}$, aggregate wealth holdings $X(t,j)$, $j \in \{1,2\}$, aggregate consumption $C(t)$, benefits $p(t)$, rate of returns $r^f(t)$ and $r^s(t)$ and aggregate bond and stock holdings $B(t)$ and $S(t)$ such that for all t*

1. *conditions 1-4 in definition 1 are fulfilled and*

2. *bond and stock supply* (15) *and demand* (16) *equate.*

In the following I will go through three exemplary cases. In all cases I compare stationary equilibria for tax rate values $\tau \in [0, 0.1]$. The first two cases, (1) AK-production & debt rollover and (2) perfect substitutability of stocks and bonds in production, are equivalent to the partial equilibrium setup, as interest rates are insensitive to changing demand in bonds and stocks, when comparing stationary equilibria. The third case, Cobb Douglas production in stocks and bonds, deviates from that as interest rates respond to changes in asset demands. In all exemplary cases I choose constant returns to scale (CRS) production functions, such that $Y = r^s S + r^f B$.

Calibration. All models have the same household model in common. In order to get an (approximately) closed form solution of the household problem I choose $\log(1 + r^s + \varepsilon) \sim \mathcal{N}(\log(1 + r^s) - \frac{\sigma^2}{2}, \sigma^2)$. Note, that this implies $\mathbb{E}[1 + r^s + \varepsilon] = 1 + r^s$. Following Campbell and Viceira (2002, ch. 2) I can then approximate the risky portfolio share of households by

$$\theta = \frac{\log(1 + r^s) - \log(1 + r^f)}{\sigma^2}.$$

Up to four parameters need to be set. All models contain parameters β, σ and A, where A is total factor productivity (TFP). On top of these three parameters models two and three contain a forth technology parameter, α.

I normalize initial wealth endowment $h = 1$ and assume that each period to correspond to 30 years. Following Ludwig and Vogel (2010) I set $\beta = 0.99^{30} = 0.74$. I calibrate the other up to three parameters by targeting mean and variance moments of returns as documented in Piazzesi, Schneider, and Tuzel (2007) in a model with a zero tax rate $\tau = 0$. Piazzesi, Schneider, and Tuzel (2007) report an average yearly risk-free return of 0.75%, and average yearly stock and housing returns of 6.94% and 2.52%, respectively. Following Glover, Heathcote, Krueger, and Rios-Rull (2014), Geppert, Ludwig, and Abiry (2016) and Abiry (2020), I set the yearly risky return to be the return of an equally-weighted portfolio of stocks and housing, yielding 4.73%. A crude estimate of the 30-year returns on safe and risky assets is hence $1.0075^{30} - 1 = 25.1\%$ and $1.0473^{30} - 1 = 300.1\%$ (implying a 30 year risk-premium of 275.0%). I use these numbers to calibrate A and α if present. Taking the numbers from Piazzesi, Schneider, and Tuzel (2007) again, an equally-weighted portfolio has a yearly standard deviation of roughly 8.4%. As a very rough estimate I take the 30 year standard deviation to be $30 * 8.4\% = 252\%$. By setting $\sigma^2 = \log\left(1 + \frac{2.52}{1+2.75}\right) = 0.514$ I can meet this target for the variance of the portfolio return in all three models. For a derivation see section B.1 in the appendix.

Given the calibrated model I increase τ from 0 up to 0.1 and report resulting aggregate statistics.

Model 1: AK-Production and Debt Rollover. Production is according to

$$Y_t = A \cdot S_t,$$

where A is a TFP parameter. The bond issuing authority simply rolls over its debt in every period,[8] i.e.

$$B_{t+1} = (1 + r_t^f)B_t.$$

In steady state this implies that $r^s = A$ and $r^f = 0$. Hence, when comparing steady states, interest rates do not respond to aggregate changes in stocks or bonds and we are thus in the partial equilibrium world. Furthermore, as the risk-free rate is zero, an increase in the tax rate τ leads to an increase in consumption and output unambiguously. This follows directly from corollary 1 as the risk-free rate is equal to zero. I set $A = 2.75$ so that I match the risk premium mentioned above.

Model 2: CRS-Production with S and B being perfect substitutes. Production is according to

$$Y_t = A\left(\alpha S_t + (1-\alpha)B_t\right),$$

leading to $r^s = A\alpha$ and $r^f = A(1-\alpha)$ in steady state. In order to ensure $r^s > r^f$ I assume $\alpha > 0.5$. At $\alpha = 0.5$ stocks and bonds are not only perfect substitutes, but also have the same productivity and thus same return. As stocks incur risks households choose to hold zero stocks for $\alpha \leq 0.5$, independent of τ.[9]

[8] I assume that the reason for taking on this debt in the past has no bearing on current economic outcomes.

[9] Even though zero stocks are held in the economy in the case of $\alpha \leq 0.5$ social security benefits still crowd out bonds, due to the first effect - a decrease in the savings rate.

Using $r^s = 300\%$ and $r^f = 25\%$ as targets pins down α and A as

$$A = \frac{r^s}{\alpha} = 3.26$$
$$\alpha = \frac{1}{1 + \frac{r^f}{r^s}} = 0.92$$

Model 3: CRS-Cobb-Douglas production with full depreciation Production is according to

$$Y_t = A S_t^{\alpha} B_t^{1-\alpha} - S_t - B_t,$$

where I assume both forms of capital, stocks and bonds, to depreciate fully every period. Calibration is described in section B.2 in the appendix and results in $\alpha = 0.42$ and $A = 4.01$.

Results. Models 1 and 2 are equivalent to a pure partial equilibrium model with fixed interest rates. Hence, equilibria can be computed in closed form using equation (14) for aggregate consumption. Note that in all models it holds by Walras's law that $C = Y + h$. Model 3 is solved numerically. Figure 1 shows aggregate consumption, output, bond and stocks holdings stationary equilibria change with the tax rate τ. All variables are normalized to one for $\tau = 0$. Note that the scale for models 1 and 2 are the same, but aggregate statistics in model 3 exhibit much lower movements (for $\tau \in [0, 0.1]$) such that a finer scaling is needed.

As discussed before, model 1 features a zero risk-free rate in a partial equilibrium setup and thus by corollary 1, higher taxes increase output and consumption. In model 2 - that is also equivalent to a partial equilibrium setup - the risk-free rate is not zero, but positive. However as the results in column 2 in figure 1 show the risk free rate is still low enough for higher taxation leading to higher output and consumption (see corollary 1). Model 3, the Cobb Douglas case in column 3, paints a more diversified picture. For the tax rate τ large enough a further increase in the tax rate will eventually lead to output losses. However, interestingly, at low tax rate levels, increasing the tax rate has an output expanding effect even in this case.

The economic intuition for these findings goes as follow: Due to the imperfect market setup households cannot insure against idiosyncratic risks in the stock returns. Hence they invest too little into stocks than what would be socially optimal. In a socially optimal equilibrium, idiosyncratic risk could be diversified away in the aggregate and returns on stocks and bonds would equate. As insurance is not available households demand higher expected returns from risky stocks, which make stocks a more productive asset in the economy, compared to bonds. Given this background a higher tax rate increases social security benefits

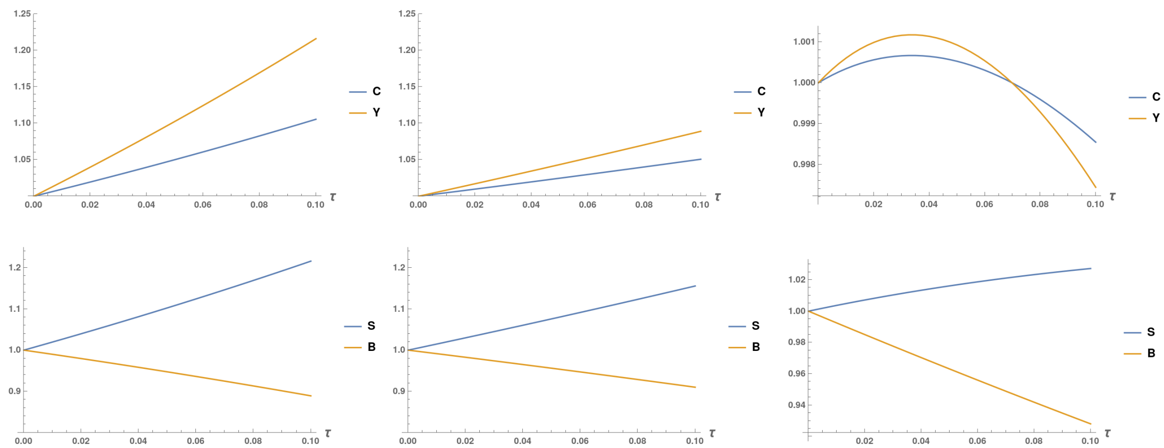

Figure 1: Consumption, Output, Stocks and Bonds for three different stocks and bonds supply functions

Notes: Left column: AK production & debt rollover, Middle column: Perfect substitutes, Right column: Cobb Douglas. Top Row: aggregate Consumption C and Output Y, Bottom Row: aggregate Stocks S and Bonds B. All variables normalized to 1 for $\tau = 0$. Left and middle column (AK production & debt rollover and perfect substitutes) have the same scales for both figures. Both scales in the Cobb Douglas case are different.

which act as surrogate bonds, inducing households to shift their portfolio away from safe bonds to risky stocks. This increases average productivity of assets. Of course, this output expending effect is counterbalanced by the crowding out of overall savings. As above exemplary results show the former effect might however dominate, particularly at low tax rates.

One might argue that above models, especially models 1 and 2, are not particularly realistic or relevant in quantitative analysis. I would like to counter this argument with two examples. First, AK type of models are often used in studies on endogenous growth or might be used in models with dedicated human capital accumulation, cf. e.g. Krebs (2003), Krebs and Wilson (2004), Geppert, Ludwig, and Abiry (2016) and Abiry (2020). Second, models with CES production functions and high substitution elasticity might get sufficiently close to model 1 above to find similar effects.

4 Conclusion

In this paper I show that social security might in fact lead to a crowding in of capital and an expansion of output. I illustrate this finding in a stylized life-cycle model with incomplete markets and asset return risk, featuring a risky and a safe asset. As is well known, social security leads to crowding out of private capital within this framework, but crowding out is not uniform across assets. Instead, crowding out of safer assets with lower average returns is stronger than that of riskier assets with higher average returns, which might even be crowded in. This leads to a shift of the average asset portfolio held by households towards riskier higher yielding assets. Higher yielding assets are more productive, such that this shift might more than counterbalance the decline in the savings rate. The overall effect on output depends on which of these two effects is stronger - the decline in savings versus the shift to more productive forms of capital.

I show that in partial equilibrium a clear watershed condition can be identified. For safe interest rates below a certain threshold, an increase in social security benefits (financed through consumption taxes) leads to an increase in output. This threshold is positive such that a zero safe rate guarantees this positive relationship between benefits and output. In models with productivity and/or population growth a safe rate equal to the exogenous growth rates suffices already. In three exemplary models with a production sector I show that the finding of a positive relationship between social security benefits and output can also carry over to a general equilibrium setup.

While in partial equilibrium theoretical findings regarding capital crowding in/out are well established in this paper, future research can investigate to what extent these theoretical findings carry over to the general equilibrium setup and what further assumptions are necessary to establish similar relationships. I also abstract from any welfare analysis within the setup analyzed in this paper. This might also present itself as a fruitful path to follow on.

While this study here focuses on social security, similar findings should extend to public debt. Krueger and Ludwig (2019) for example show equivalence of welfare effects of public debt and social security in a simple setup with labor income risk. The investigation of public debt within the framework of this paper is particularly interesting as Olivier Blanchard sparked a vivid debate on the costs of public debt (Blanchard 2019). He argued that in a low interest rate environment, welfare costs from increasing public debt are small and that these stem from lower capital accumulation: public debt crowds out private capital holdings. This crowding out he is referring to is the same effect that is attributed to social security in the literature I am building on. Contrasting this decrease in private capital holdings with the shift towards higher yielding more productive assets should strengthen Blanchard's argumentation of low costs of public debt. This should particularly be the case in the current low interest environment as the safe rate is more likely within the range where public debt expansion might lead to an expansion of output.

A Partial Equilibrium

A.1 Solution of the Household Problem

Proof of proposition 1. The proof is by guess and verify.

First we state the first order conditions of the household problem (5), which are given by

$$\frac{1}{c_{t0}} = \beta \cdot \mathbb{E}\left[\frac{1 + r_t^P(\theta_{t+1,1}, \varepsilon_{t+1,1})}{c_{t+1,1}}\right] \quad \forall t \geq 0 \tag{17}$$

$$\mathbb{E}\left[\frac{r_t^s + \varepsilon_{tj} - r_t^f}{c_{tj}(\varepsilon_{tj})}\right] = 0^{10} \quad j \in \{0,1\} \text{ and } \forall t \geq 0. \tag{18}$$

Now let's guess that $c_{tj} = m_j(1 + r_t^P(\theta, \varepsilon))x_{tj}$ for some $m_j \in \mathbb{R}$. Plugging this into the portfolio condition (18) yields equation (6).

Note that our guess implies that

$$\begin{aligned} x_{t+1,1} &= (1 - m_0) \cdot (1 + r^P(\varepsilon_{t0})) \cdot x_{t0} \\ (1+\tau) \cdot c_{t+1,1} &= m_1 \cdot (1 + r^P(\varepsilon_{t1})) \cdot x_{t+1,1} \\ &= m_1 \cdot (1 + r^P(\varepsilon_{j+1})) \cdot (1 - m_0) \cdot (1 + r^P(\varepsilon_j)) \cdot x_{t0}. \end{aligned}$$

Using this together with the guess for consumption of the young in the Euler equation (17) verifies that the guess is indeed a solution of the Euler equation with

$$m_0 = \frac{m_1}{m_1 + \beta}.$$

As the old generation consumes all resources available to them $m_1 = 1$ and $m_0 = 1/(1 + \beta)$. □

A.2 Uniqueness of the Partial Equilibrium

Proof of proposition 2. The proof follows appendix A.2 in Sargent (1987). The metric used is $d_\infty(x, y) = \sup_t |x_t - y_t|$.

Given the definitions of γ_t and λ_t equation (11) can be rewritten as

$$p_t = \gamma_t h + \lambda_t p_{t+1} \tag{19}$$

[10] The corresponding condition ruling out short selling is $\theta = \arg\max_{\tilde{\theta} \in [0,1]} \mathbb{E}\big[ln(c_{tj}(\tilde{\theta}, \varepsilon_{tj}))\big]$.

Note that $\mathcal{B}_t(\tau) > 0, \mathcal{J}_t > 0, R_t^f > 0$ and $\mathbb{E}[R_t^P] > 0$. Thus $\frac{d\,\tau C_t}{dp} = \frac{\mathcal{B}_t(\tau)}{R_t^f R_{t+1}^f} + \frac{\mathcal{B}_t(\tau)/\mathcal{J}_t \cdot \beta \mathbb{E}[1+r_{t-1}^P](r_{t+1}^f - r_{t-1}^f)}{R_{t-1}^f R_t^f R_{t+1}^f} \leq \eta$ implies that the denominator of λ_t is greater than zero:

$$
\begin{aligned}
&\frac{\mathcal{B}_t(\tau)}{R_t^f R_{t+1}^f} + \frac{\mathcal{B}_t(\tau)/\mathcal{J}_t \cdot \beta \mathbb{E}[R_{t-1}^P](r_{t+1}^f - r_{t-1}^f)}{R_{t-1}^f R_t^f R_{t+1}^f} \leq \eta \\
&\Leftrightarrow \frac{\mathcal{B}_t(\tau)}{R_{t-1}^f R_t^f} \frac{\beta \mathbb{E}[R_{t-1}^P]}{\mathcal{J}_t} \cdot \left(\frac{R_{t-1}^f}{R_{t+1}^f} \frac{1}{\beta \mathbb{E}[R_{t-1}^P]} + 1 \right) \leq \eta \\
&\Leftrightarrow \frac{\mathcal{B}_t(\tau)}{R_{t-1}^f R_t^f} \frac{\beta \mathbb{E}[R_{t-1}^P]}{\mathcal{J}_t} < \eta < 1.
\end{aligned}
$$

Hence, $\lambda_t > 0$.

Transforming the same condition we also get $\lambda_t << 1$:

$$
\begin{aligned}
&\frac{\mathcal{B}_t(\tau)}{R_t^f R_{t+1}^f} + \frac{\mathcal{B}_t(\tau)/\mathcal{J}_t \cdot \beta \mathbb{E}[R_{t-1}^P](r_{t+1}^f - r_{t-1}^f)}{R_{t-1}^f R_t^f R_{t+1}^f} \leq \eta \\
&\Leftrightarrow \mathcal{B}_t(\tau)/\mathcal{J}_t \cdot \left(\frac{1}{R_t^f R_{t+1}^f} + \frac{\beta \mathbb{E}[R_{t-1}^P]}{R_{t-1}^f R_t^f} \right) \leq \eta \\
&\Leftrightarrow \frac{\mathcal{B}_t(\tau)/\mathcal{J}_t}{R_t^f R_{t+1}^f} \leq 1 - \frac{\mathcal{B}_t(\tau)/\mathcal{J}_t}{R_{t-1}^f R_t^f} \beta \mathbb{E}[R_{t-1}^P] - (1-\eta) \\
&\Leftrightarrow \lambda_t = \frac{\frac{\mathcal{B}_t(\tau)/\mathcal{J}_t}{R_t^f R_{t+1}^f}}{1 - \frac{\mathcal{B}_t(\tau)}{R_{t-1}^f R_t^f} \frac{\beta \mathbb{E}[R_{t-1}^P]}{\mathcal{J}_t}} \leq 1 - \frac{1-\eta}{1 - \frac{\mathcal{B}_t(\tau)}{R_{t-1}^f R_t^f} \frac{\beta \mathbb{E}[R_{t-1}^P]}{\mathcal{J}_t}} < 1,
\end{aligned}
$$

as the denominator is greater than zero.

Now that we have established that $0 < \lambda_t << 1$ we show that the law of motion (19) is a contraction mapping as Blackwell's sufficient conditions hold:

To do so let us define an operator on the sequence p_t by

$$
[T(p)]_t = \gamma_t \cdot h + \lambda_t \cdot p_{t+1}. \tag{20}
$$

(a) T is monotone:

Take two sequences y_t and z_t with $z_t \geq y_t$. Then

$$
[T(z)]_t - [T(y)]_t = \lambda_t \cdot (z_t - y_t) \geq 0,
$$

as $\lambda_t > 0$.

(b) T discounts:

Let $\delta \in \mathbb{R}, \delta > 0$ and define the sequence $y + \delta$ such that $(y+\delta)_t = y_t + \delta$. Then

$$\begin{aligned}[T(y+\delta)]_t &= \gamma_t h + \lambda_t y_{t+1} + \lambda_t \cdot \delta \\ &= [T(y)]_t + \lambda_t \cdot \delta \\ &<< [T(y)]_t + \delta,\end{aligned}$$

as $\lambda_t << 1$ for all t.

As (19) is a contraction and we are considering a complete space it follows that $p = T(p)$ has a unique solution. Plugging in the solution (12) into (20) it can be verified that (12) constitutes the unique fix point.

□

A.3 Steady State

Proof of corollary 1.

(i) *Existence and uniqueness:* The uniqueness of the equilibrium and the specific equilibrium characterization are a result of proposition 2 with $r_t^f = r^f$ and $r_t^s = r^s$ for all t.

(ii) *Non-existence:* With $r_t^f = r^f$ and $r_t^s = r^s$ for all t the law of motion of the benefits gets

$$p_t = \frac{1}{1 - \frac{\mathcal{B}(\tau)}{(R^f)^2}\frac{\beta\mathbb{E}[R^P]}{\mathcal{J}}} \cdot \left[\mathcal{B}(\tau)h \; + \; \frac{\mathcal{B}(\tau)}{(R^f)^2}\frac{1}{\mathcal{J}}p_{t+1}\right]$$

or equivalently

$$p_{t+1} = \tilde{\gamma} \cdot h + \tilde{\lambda} \cdot p_t, \tag{21}$$

with

$$\begin{aligned}\tilde{\gamma} &= -(R^f)^2\mathcal{J} < 0 \\ \tilde{\lambda} &= \frac{1 - \frac{\mathcal{B}(\tau)}{(R^f)^2}\frac{\beta\mathbb{E}[R^P]}{\mathcal{J}}}{\frac{\mathcal{B}(\tau)}{(R^f)^2}\frac{1}{\mathcal{J}}}\end{aligned}$$

If $\frac{\mathcal{B}_t(\tau)}{(R^f)^2} > 1$ then $\tilde{\lambda} \leq 1$ and the general solution of the difference equation (21) is

$$p_t = \frac{\tilde{\gamma}}{1-\tilde{\lambda}} \cdot h + z \cdot \tilde{\lambda}^t,$$

for some $z \in \mathbb{R}$. As $\frac{\tilde{\gamma}}{1-\tilde{\lambda}} < 0$ and $\tilde{\lambda} < 1$, p_t will eventually become negative, no matter the z.

If $\frac{\mathcal{B}_t(\tau)}{(R^f)^2} = 1$ then $\tilde{\lambda} = 1$ and

$$p_t = t\tilde{\gamma} \cdot h + p_0,$$

for some $p_0 \in \mathbb{R}$. As $\tilde{\lambda} < 0$, also in this case p_t will eventually become negative, no matter the p_0.

In both cases means, that $C_t = \tau p_t$ would also become negative eventually, but this is not possible in equilibrium.

(iii) *Comparative Statics:* As $\mathcal{B}'(\tau) > 0$ for $\tau \in [0, 1)$, taking the derivative of (13) with respect to τ we get

$$\frac{dp}{d\tau} = \frac{\mathcal{B}'(\tau)}{\frac{\mathcal{B}(\tau)}{(R^f)^2}} > 0.$$

Rewriting (14) as

$$C = \frac{\mathcal{AJ}}{1 + \tau\left(1 - \frac{\mathcal{AJ}}{(R^f)^2}\right)}$$

we get

$$\frac{dC}{d\tau} = \left(\frac{\mathcal{AJ}}{(R^f)^2} - 1\right) \cdot \frac{\mathcal{AJ}}{\left[1 + \tau\left(1 - \frac{\mathcal{AJ}}{(R^f)^2}\right)\right]^2}$$

where the denominator is non-zero as $\frac{\mathcal{B}_t(\tau)}{(R^f)^2} < 1$, implying the case differentiation as in corollary 1.

(iv) *Cutoff risk-free rate:* As shown in the proof to proposition 1 and footnote 4, the portfolio choice problem can be regarded separately from the consumption-savings choice problem, i.e. the optimal portfolio share θ solves

$$\theta = \arg\max_{\tilde{\theta}} \mathbb{E}\big[ln(1 + r_t^P(\tilde{\theta}))\big].$$

As

$$\left.\frac{\partial \mathbb{E}\big[ln(1 + r^P(\theta))\big]}{\partial \theta}\right|_{\theta=0} = \mathbb{E}\left[\frac{r^s + \varepsilon - r^f}{1 + r^f}\right] > 0,$$

as $r^s > r^f$. Hence, $\theta > 0$ and thus $\mathbb{E}[R^P] > R^f$.

In general we have

$$\frac{\mathcal{AJ}}{(R^f)^2} = \frac{1 + \beta \mathbb{E}[R^P]}{1 + \beta} \cdot \frac{\mathbb{E}[R^P]}{(R^f)^2}.$$

For $r^f = 0$ we have

$$\left.\frac{\mathcal{AJ}}{(R^f)^2}\right|_{r^f=0} = \frac{1 + \beta \mathbb{E}[R^P]}{1 + \beta} \cdot \mathbb{E}[R^P] > 1,$$

as $\mathbb{E}[R^P] > 1$.

For $r^f = r^s > 0$ we have

$$\left.\frac{\mathcal{AJ}}{(R^f)^2}\right|_{r^f=r^s} = \frac{1 + \beta R^f}{1 + \beta} \frac{1}{R^f} < 1,$$

as now $R^f = \mathbb{E}[R^P] > 1$.

The optimality condition for portfolio choice implies that the risky portfolio share declines with the risk-free rate:

$$- \mathbb{E}\left[\frac{1 + r^s + \varepsilon}{(R^P)^2}\right] \cdot dr^f - \mathbb{E}\left[\left(\frac{r^s + \varepsilon - r^f}{R^P}\right)^2\right] \cdot d\theta = 0$$

$$\Leftrightarrow \frac{d\theta}{dr^f} = -\frac{\mathbb{E}\left[\frac{1+r^s+\varepsilon}{(R^P)^2}\right]}{\mathbb{E}\left[\left(\frac{r^s+\varepsilon-r^f}{R^P}\right)^2\right]} < 0,$$

as $\varepsilon > -(1 + r^s)$. Thus, for the gross portfolio return we have

$$\frac{d\mathbb{E}[R^P]}{dr^f} = 1 - \theta + \frac{d\theta}{dr^f} \cdot \big(r^s - r^f\big) < 1.$$

This together with the fact that $R^f \leq \mathbb{E}[R^P]$ implies that

$$\frac{d\frac{\mathcal{AJ}}{(R^f)^2}}{dr^f} = \frac{1}{1+\beta} \cdot \frac{R^f \cdot \frac{d\mathbb{E}[R^P]}{dr^f} \cdot \left(1 + 2\beta\mathbb{E}[R^P]\right) - 2 \cdot \mathbb{E}[R^P] \cdot \left(1 + \beta\mathbb{E}[R^P]\right)}{(R^f)^3} < 0.$$

Summarizing the findings again, we have $\left.\frac{\mathcal{AJ}}{(R^f)^2}\right|_{r^f=0} > 1$, $\left.\frac{\mathcal{AJ}}{(R^f)^2}\right|_{r^f=r^s} < 1$ and $\frac{d\frac{\mathcal{AJ}}{(R^f)^2}}{dr^f} < 0$. By the intermediate value theorem there must thus exist an $0 < \bar{r}^k < r^s$, such that $\left.\frac{\mathcal{AJ}}{(R^f)^2}\right|_{r^f=\bar{r}^f} = 0$. Furthermore, it holds that $\left.\frac{\mathcal{AJ}}{(R^f)^2}\right|_{r^f<\bar{r}^f} > 1$ and $\left.\frac{\mathcal{AJ}}{(R^f)^2}\right|_{r^f>\bar{r}^f} < 1$ which together with the results from item (iii) give the case differentiation as in corollary 1.

□

Proof of proposition 3. From corollary 1 we know that $\frac{dp}{d\tau} > 0$. Hence, it is sufficient to look at the quantities $\frac{dr_j^{FP}}{dp}$ and $\frac{d(1-m_0^{FP})}{dp}$

By equations (7) and (3) we have

$$\begin{aligned} x_1 &= (1-m_0)(1+r^P)x_0 \\ &= (1-m_0)(1+r^P) \cdot \left(h + \frac{p}{(R^f)^2}\right) \end{aligned}$$

and thus (again using (3))

$$\begin{aligned} \frac{dx_1}{dp} &= \frac{(1-m_0)(1+r^P)}{(R^f)^2} \\ \frac{d(b_1+s_1)}{dp} &= \frac{(1-m_0)(1+r^P) - R^f}{(R^f)^2} \end{aligned}$$

Note, that

$$\frac{d(b_0+s_0)}{dp} = 0.$$

Before we start, here again the quantities of interest:

$$\begin{aligned} \theta_j^{FP} &= \frac{s_j}{b_j+s_j} = \theta \cdot \left(1 + \frac{\frac{p}{(R^f)^{2-j}}}{b_j+s_j}\right) \\ r_j^{FP} &= r^f + \theta_j^{FP} \cdot (r^s + \varepsilon_j - r^f) \\ m_0^{FP} &= m_0 \cdot \frac{1+r^P}{1+r_0^{FP}} \cdot \left(1 + \frac{\frac{p}{(R^f)^2}}{b_0+s_0}\right). \end{aligned}$$

Hence

$$\frac{d\theta_0^{FP}}{dp} = \theta \cdot \frac{1}{h \cdot (R^f)^2} > 0$$

and

$$\begin{aligned}
\frac{d\theta_1^{FP}}{dp} &= \theta \cdot \frac{\frac{b_1+s_1}{R^f} - \frac{d(b_1+s_1)}{dp} \cdot \frac{p}{R^f}}{(b_1+s_1)^2} \\
&= \theta \cdot \frac{\left[(1-m_0)(1+r^P) \cdot \left(h + \frac{p}{(R^f)^2}\right) - \frac{p}{R^f}\right] - \left[\frac{(1-m_0)(1+r^P)-R^f}{(R^f)^2} \cdot p\right]}{R^f \cdot (b_1+s_1)^2} \\
&= \theta \cdot \frac{(1-m_0)(1+r^P) \cdot h}{R^f \cdot (b_1+s_1)^2} > 0,
\end{aligned}$$

given that $1 + r^P > 0 \;\; \forall \varepsilon$ with positive probability mass/density.

For the rate of return we get

$$\frac{dr_j^{FP}}{dp} = \frac{d\theta_j^{FP}}{dp} \cdot (r^s + \varepsilon_j - r^f)$$

and thus

$$\mathbb{E}\left[\frac{dr_j^{FP}}{dp}\right] > 0.$$

The marginal propensity to consume above can be rewritten as

$$m_0^{FP} = \frac{m_0(1+r^P)}{h(R^f)^2} \cdot \frac{h(R^f)^2 + p}{1 + r_0^{FP}}.$$

Thus, for the marginal propensity to consume we get

$$\begin{aligned}
\frac{dm_0^{FP}}{dp} &= \frac{m_0(1+r^P)}{h(R^f)^2} \cdot \frac{1 + r_0^{FP} - \frac{dr_0^{FP}}{dp}\left(h(R^f)^2 + p\right)}{\left(1 + r_0^{FP}\right)^2} \\
&= \frac{m_0(1+r^P)}{h(R^f)^2} \cdot \frac{1 + r_0^{FP} - (r^s + \varepsilon - r^f)\frac{\theta}{h(R^f)^2}\left(h(R^f)^2 + p\right)}{\left(1 + r_0^{FP}\right)^2} \\
&= \frac{m_0(1+r^P)}{h(R^f)^2} \cdot \frac{1 + r_0^{FP} - \theta_0^{FP}(r^s + \varepsilon - r^f)}{\left(1 + r_0^{FP}\right)^2} \\
&= \frac{m_0(1+r^P)}{hR^f} \cdot \frac{1}{\left(1 + r_0^{FP}\right)^2} > 0,
\end{aligned}$$

given that $1 + r^P > 0 \;\; \forall \varepsilon$ with positive probability mass/density.

$\square$

B General Equilibrium

B.1 Variance Calibration Through Portfolio Choice

Due to the properties of the Log-Normal distribution we have that

$$\begin{aligned} Var(1 + r^s + \varepsilon) = Var(\varepsilon) &= \left[\exp(\sigma^2) - 1\right] \exp(\ln(1 + r^s)) \\ &= \left[\exp(\sigma^2) - 1\right] (1 + r^s) \end{aligned}$$

Setting $Var(1 + r^s + \varepsilon) = z$ for some target value z we obtain

$$\sigma^2 = \log\left(1 + \frac{z}{1 + r^s}\right)$$

B.2 Cobb Douglas Production With Full Depreciation

Calibration. The parameters A and α need to be calibrated.

The production function implies that gross returns are

$$R^s = \alpha A \left(\frac{S}{B}\right)^{\alpha - 1} \tag{22}$$

$$R^f = (1 - \alpha) A \left(\frac{S}{B}\right)^{\alpha} \tag{23}$$

Dividing the gross returns by each other yields

$$\frac{R^s}{R^f} = \frac{\alpha}{1 - \alpha} \frac{B}{S} \tag{24}$$

$$\Leftrightarrow \alpha = \frac{\frac{R^s}{R^f}}{\frac{B}{S} + \frac{R^s}{R^f}}$$

Plugging this back into the equation for the gross free rate we get

$$A = R^f \frac{\frac{B}{S} + \frac{R^s}{R^f}}{\frac{R^s}{R^f}} \left(\frac{B}{S}\right)^{\alpha - 1}.$$

Given R^f and R^s as targets B and S are determined by the demand of households. As we calibrate the model with $\tau = 0$ we have

$$\frac{B}{S} = \frac{1 - \theta\left(\frac{R^s}{R^f}\right)}{\theta\left(\frac{R^s}{R^f}\right)},$$

with

$$\theta\left(\frac{R^s}{R^f}\right) = \frac{\log\left(\frac{R^s}{R^f}\right)}{\sigma^2}$$

This does then pin down α and A by the equations above.

References

Abiry, R. (2020). Demographic Change and the Secular Decline of Risk-Free and Risky Rates. *Unpublished manuscript*.

Bach, L., L. E. Calvet, and P. Sodini (2020). Rich pickings? risk, return, and skill in household wealth. *American Economic Review 110*(9), 2703–47.

Blanchard, O. (2019). Public debt and low interest rates. *American Economic Review 109*(4), 1197–1229.

Campbell, J. Y. and L. M. Viceira (2002). *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*. Oxford University Press.

Diamond, P. A. (1965). National debt in a neoclassical growth model. *The American Economic Review 55*(5), 1126–1150.

Fagereng, A., L. Guiso, D. Malacrino, and L. Pistaferri (2020). Heterogeneity and persistence in returns to wealth. *Econometrica 88*(1), 115–170.

Geppert, C., A. Ludwig, and R. Abiry (2016). Secular Stagnation? Growth, Asset Returns and Welfare in the Next Decades: First Results. SAFE Working Paper No. 145.

Glover, A., J. Heathcote, D. Krueger, and J. V. Rios-Rull (2014). Intergenerational Redistribution in the Great Recession.

Harenberg, D. and A. Ludwig (2019). Idiosyncratic risk, aggregate risk, and the welfare effects of social security. *International Economic Review 60*(2), 661–692.

İmrohoroglu, A., S. Imrohoroglu, and D. H. Joines (1995). A life cycle analysis of social security. *Economic theory 6*(1), 83–114.

Krebs, T. (2003). Human Capital Risk and Economic Growth. *Quarterly Journal of Economics 118*, 709–745.

Krebs, T. and B. Wilson (2004). Asset Returns in an Endogenous Growth Model with Incomplete Markets. *Journal of Economic Dynamics and Control*, 817–839.

Krueger, D. and F. Kubler (2006). Pareto-improving social security reform when financial markets are incomplete!? *American Economic Review 96*(3), 737–755.

Krueger, D. and A. Ludwig (2019). Optimal Taxes on Capital in the OLG Model with Uninsurable Idiosyncratic Income Risk. *MEA Discussion Paper* (No. 02-2018).

Ludwig, A. and E. Vogel (2010). Mortality, fertility, education and capital accumulation in a simple olg economy. *Journal of Population Economics 23*(2), 703–735.

Merton, R. C. (1969). Litetime Portfoliio Selection under Uncertainty: The Continuous-Time Case. *The Review of Economics and Statistics 51(3)*, 247–257.

Merton, R. C. (1981). On the role of social security as a means for efficient risk-bearing in an economy where human capital is not tradeable.

Obstfeld, M. (1992). Risk-taking, global diversification, and growth. Technical report, National bureau of economic research.

Piazzesi, M., M. Schneider, and S. Tuzel (2007). Housing, Consumption, and Asset Pricing. *Journal of Financial Economics 83(3)*, 531–569.

Samuelson, P. A. (1969). Lifetime Portfolio Selection by Dynamic Stochastic Programming. *The Review of Economics and Statistics 51(3)*, 239–246.

Sargent, T. J. (1987). *Dynamic macroeconomic theory.* Harvard University Press.

Storesletten, K., C. I. Telmer, and A. Yaron (1999). The risk-sharing implications of alternative social security arrangements. In *Carnegie-Rochester Conference Series on Public Policy*, Volume 50, pp. 213–259. Elsevier.