Demographic Change and Secular Trends in Interest Rates and Risk Premia

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Abstract

Demographic change lowers raw labor supply and leads to a relative abundance of physical capital, resulting in an increase of physical capital intensity in production. This leads to lower asset returns. However risk-free and risky returns are affected to a different extent. Both rates fall, but the decrease in the risk-free rate exceeds the decrease in the risky rate. The reasoning is that demographic change increases the share of old-age households with older households holding a larger share of their wealth in risk-free assets, as compared to younger households. The increase in demand for risk-free assets compared to risky assets puts downward pressure on risk-free asset prices and leads to an increase in the risk premium. I develop an overlapping generations model and quantify these effects for the US economy. Until 2050 I find that risky returns and the risk-free rate decrease by roughly 0.85 percentage points, and the risk premium increases by a mild 5.4 basis points. Allowing households to endogenously increase their human capital holdings, these figures reduce to 0.25 percentage points and 1.5 basis points, respectively.

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1 Introduction

In his famous keynote address at the IMF in 2013 and at the NABE Policy conference in 2014, Lawrence Summers (2013, 2014) revived a term originally coined by Hansen (1938): "secular stagnation". By secular stagnation Summers referred to (i) a long-run or trend reduction of growth, (ii) accompanied by decline in the natural rate of interest and (iii) that these long-run changes would require a rethinking of policy approaches.

This paper analyzes the role demographic change plays for a potential secular stagnation, particularly for point (ii), the evolution of the natural rate. But what is the natural rate in a world with multiple assets? According to Wicksell (1898), the natural rate of interest is the interest rate which is compatible with a stable price level. In monetary theory the workhorse model of central bank behavior is the Tayor rule (Taylor 1993) which models the nominal interest rate as the natural rate plus some deviation terms. Within conventional monetary policy central banks set short-term safe rates. Hence, the natural rate of interest could be understood as such. On the other hands, investment decisions of firms and households are taken with respect to other interest rates which are subject to risk. Therefore it is worthwhile studying how various interest rates are affected by demographic change. In this study I differentiate between a safe real short-term rate and a generic real risky rate, which allows me to investigate the long-term trends of risk-premia.

I investigate this question by developing a structural model of an economy with a production and a household sector. Households, which are modeled as overlapping generations, hold government bonds at a safe rate and productive capital which generates risky returns. Intra-generational heterogeneity - due to stochastic labor income and asset returns - and inter-generational heterogeneity - due to age - give rise to trade across households in both assets. In order to realistically capture the effects of demographic change the model features a detailed description of the aging process. Demographic change itself is introduced into the model through an exogenous change in the aging process and fertility.

As it is not just the raw size of the labor force that determines economic outcomes, but also the "quality" of labor, I model human capital. I consider two polar human capital scenarios. In the first one I restrict human capital adjustments, while in the second one households freely adjust their human capital accumulation to changing wages and returns. In the first scenario - with restricted human capital adjustment - both the safe and risky rates decrease by roughly 0.85 percentage points until 2050. But the drop in the safe rate is slightly larger, leading to an increase in the risk premium of 5.4 basis points. Allowing for human capital adjustments the safe and risky rate drop by approximately 0.25 percentage points, while the risk premium increases by only 1.5 basis points. Hence, human capital adjustments mitigate the effects of demographic change on interest rates. These effects are rather mild. In light of real world frictions on markets for human capital (which I do not model) I argue that the two scenarios bracket the evolution of future asset returns.

These results are driven by demographic change which leads to a relative scarcity of raw labor supply and a relative abundance of physical capital. This decreases productivity of capital, which suppresses risky returns. As the supply of safe bonds does not change, safe returns follow suit. The differential effect on safe and risky rates follows from the change in the composition of savers. The portfolio share of the risky asset decreases over the life-cycle. Aging leads to a relative increase of the population of older households relative to young ones and thus to a decline in the demand for risky assets and an increase in demand for the safe assets. This leads to a widening of the risk premium. The scarcity of raw labor also leads to an increase in wages. This increase in wages, paired with the longer life-expectancy lead to an increase in human capital in scenario two - with free human capital adjustments. The decrease in raw labor supply, as measured by the population size of households in working-age, is thus partly mitigated by this increase in human capital. This explains the moderate asset return changes in scenario two.

This paper builds on Geppert, Ludwig, and Abiry (2016), who employ a quantitative overlapping generations model studying the transitional effects of demographic change. The authors find that demographic change lowers economic growth and interest rates, and leads to an increase in the risk premium. My paper extends their study by including a pay-as-you-go pension system. It also differs by using the risky investment framework of Angeletos (2007a) which allows me to disregard from aggregate risk, thus leading to a more parsimonious model and reducing computational complexity significantly.

Related Literature. I contribute to a large and growing literature on the secular decline of interest rates in the last decades. This decline is for example documented in Rachel and Summers (2019) and Rachel and Smith (2015). Rachel and Summers (2019) report a decline of the natural real rate of about 300 basis points during the period from 1970 to 2017. The secular decline in interest rates is one of the topics subsumed under the general term of secular stagnation, that studies also secular trends in growth potential, deficient demand and labor market hysteresis. A collection of overview articles on these topics can be found in Teulings and Baldwin (2014).

Quantitative studies investigating one single real rate or natural rate and the role that demographic change plays in its decline include among others Carvalho, Ferrero, and Nechio (2016) and Rachel and Summers (2019) using perpetual youth models and Krueger and Ludwig (2007) and Papetti (2020) within the context of quantitative overlapping generations

(OLG) models. Looking at advanced economies as one block, Rachel and Summers (2019) attribute a decline of 1.8% in the natural rate from 1970 to 2017 to demographic change. Krueger and Ludwig (2007) study the US case within the context of an open economy and predict the real rate to decline by 0.86% between 2005 and 2080. They find that faster ageing of other regions of the world is "imported" into the US, leading to a larger decline in the real rate. Studying the case of the European Union, Carvalho, Ferrero, and Nechio (2016) and Papetti (2020) attribute a decrease of about 1.5% in the period 1990-2014, respectively 1.4% in the period 1987-2030 of the real rate to demographic change. In both studies most of the decline in the real rate is due to an increase in life-expectancy as compared to a decline in fertility, similar to Krueger and Ludwig (2007).

Turning to the risk premium, assessing historical levels and dynamics of (equity) risk premia is subject to imprecision as unobserved expected returns have to be estimated. Kopecky and Taylor (2020) survey several studies for the case of equity risk premia with various methods. Although estimates vary widely they all tend to have a U-shape pattern in common: a decline in the risk premium from the 1970s to the 1990s followed by an increase since then. Performing a descriptive analysis on data of various sources Rachel and Smith (2015) conclude similarly that the risk premium has increased by 100 basis points since the 1980s.

In contrast to studies on the natural real rate, quantitative investigations on secular trends of risk premia due to ongoing demographic shifts are scarcer. A related literature on aging and the equity premium (Bakshi and Chen 1994; Brooks 2004; Börsch-Supan, Ludwig, and Sommer 2003; Geanakoplos, Magill, and Quinzii 2004; Kuhle 2008) employing stylized overlapping generations models with few generations has not reached a consensus on the quantitative effects of demographic change on differential asset returns. However, such a periodicity severely restricts households to re-balance their portfolios. I avoid such restrictions by employing a large scale overlapping generations model that runs at an annual frequency. The closest papers to my agenda, which are also employing OLG models at an annual frequency are Kopecky and Taylor (2020) and Geppert, Ludwig, and Abiry (2016), both calibrated to US data. Kopecky and Taylor (2020) attribute a decline of 2.7% and 3.5% in the equity and risk-free rate, respectively, to demographic change in the period 1970-2017. This means an increase in the equity premium of around 80 basis points. Geppert, Ludwig, and Abiry (2016) predict a decrease in the equity and risk-free rate of 0.7% and 1% for 2010-2030, implying an increase in the equity premium of 27 basis points.

As described above Geppert, Ludwig, and Abiry (2016) use a similar model to mine, albeit with aggregate risk. Kopecky and Taylor (2020) also model aggregate risk and also feature a social security system, but abstract from endogenous labor supply and solve the model for a stationary equilibrium but not for a dynamic path. I on the other hand refrain from modeling aggregate risk. This allows me to introduce social security into the model. Similarly to Geppert, Ludwig, and Abiry (2016) I also solve the model along the transition.

Some of the before mentioned studies also take social security systems into consideration. While Bonchi and Caracciolo (2021), Papetti (2020), Rachel and Summers (2019) and Krueger and Ludwig (2007) (in the defined benefit case) find that accounting for social security dampens the effects of demographic change, Carvalho, Ferrero, and Nechio (2016) find stronger effects when social security is included. Projecting forward, Bonchi and Caracciolo (2021) compare different social security reforms and find that different reforms can lead to large differences in predicted natural rates. The setup I employ allows me to study the effect that social security systems and their reforms, paired with demographic change, have on interest rates and risk premia.[1]

This paper proceeds as follows. Section 2 illustrates and discusses the mechanisms at play within a simplified three-period model. Section 3 develops the large scale quantitative overlapping generations model. Section 4 describes the approach to numerically solve this model as well as the model's calibration. Section 5 presents the results and, finally, section 6 concludes. Separate appendices provide derivations, proofs of propositions, further details of the calibration and additional results.

2 Three-Period Model

I extend the classical Diamond (1965) economy by introducing idiosyncratic risk and allowing households to save in a risk-free and risky financial asset. Furthermore, labor supply is endogenous by adopting a simple human capital framework. The household's decision problem can be represented by a portfolio choice problem in three assets, risk-free bonds, risky physical capital and risky human capital. Portfolio decisions change over the life cycle as human capital decreases. Changing population and wealth distributions across cohorts - through demographic change - thus alter aggregate average portfolio shares and affect economic outcomes. The mechanics can be boiled down to a three-period model in which working-age and old households hold financial assets, but human capital is only supplied by working-age households.

Two modeling choices simplify the characterization of the equilibrium significantly. First, the household problem setup is such that portfolio shares are independent of wealth. Moreover, contingent on portfolio allocation consumption and savings are proportional to wealth and only functions of age and time, cf. Merton (1969) and Samuelson (1969). Therefore, aggregate dynamics are unaffected by idiosyncrasies due to linear policy functions and the lack

[1] These results are not yet contained in the current version of the paper.

of aggregate uncertainty. Second, by approximating the portfolio allocation by its continuous time equivalent I can derive closed-form solutions of the household problem, cf. Campbell and Viceira (2002).

2.1 Model Setup

Households. Households live for three periods, in which they are identified as young, workers and retirees. When young they invest into human capital which they supply to the labor market when they reach working age. Human capital can only be invested in when young, so that there is no labor supply when old. Young and working households also can invest into risky physical capital and safe bonds, for which they receive returns the next period. That means, retired households only receive income from returns on their physical capital and bond holdings, which they accumulated when they were of working age. The young, working and retired cohorts are identified by the age indices $0, 1$ and 2. The size of the young cohort in period t is given by $N_{t,0}$, the one of the working cohort is $N_{t,1}$ and the one of the old cohort is $N_{t,2}$. I am abstracting from survival risk, such that next period's working-age cohort is the the same as this period's young cohort, $N_{t+1,1} = N_{t,0}$, and next period's old cohort size is the same as this period's working cohort size, $N_{t+1,2} = N_{t,1}$. Cohorts grow at the constant rate g^N, i.e. $\frac{N_{t+1,0}}{N_{t,0}} = 1 + g^N$. Old households only live a fraction $\varsigma \in (0, 1)$ of the period's length, allowing me to capture changes in life-expectancy. Households have log-preferences over consumption in all three periods, $c_{t,0}$, $c_{t+1,1}$, $c_{t+2,2}$. The utility function is given by

$$\log(c_{t,0}) + \mathbb{E}_t \left[\beta \log(c_{t+1,1}) + \beta^2 \varsigma \log(c_{t+2,2})\right],$$

where β is the discount factor and discounting of utility at old age is reduced by the factor ς as households only live a fraction of this period.

Households are born with initial wealth w_0, and divide it up between consumption when young, $c_{t,0}$, human capital, $h_{t+1,1}$, physical capital, $k_{t+1,1}$ and bond, $b_{t+1,1}$, investments.[2]

At working age, households receive the returns from their investments at young age. Households supply one fixed unit of labor and their labor income depends on their human capital investment when young. Labor income is $h_{t+1,1} \cdot (r^H_{t+1} + \eta)$, where the return on human capital has two components:[3] the deterministic market rate r^H_{t+1} and the idiosyncratic risky

[2] As an alternative to the wealth drop from sky, households could be born with an initial human capital level h_0 one period earlier prior, let it be called pre-young. In this period, they would earn labor income on this human capital with which they would enter into their young period. As this would not alter the results qualitatively, this is abstracted from here.

[3] Unlike in the quantitative model, the implicit assumption here is of full depreciation of human capital.

component η, which is i.i.d. across time and households. The total human capital return is log-normally distributed as

$$\log(r_t^H + \eta) \sim \mathcal{N}\left(\log(1 + r_t^H) - \frac{1}{2}\sigma_\eta^2, \sigma_\eta^2\right),$$

where σ_η^2 is a parameter pinning down the variance of the log-returns. The distributional assumption implies that expected returns at the household level are equal to the market rate, i.e. $\mathbb{E}[r_t^H + \eta] = r_t^H$. Bonds have a risk-free gross return of $1 + r_{t+1}^f$, where r_t^f is the risk-free rate. Similar to human capital, physical capital is also subject to idiosyncratic risk, with the gross return of physical capital being equal to $(1 + r_{t+1}^K + \zeta)$. r_{t+1}^K is the deterministic market rate and ζ the idiosyncratic risky component, i.i.d. across time and households. The gross return on physical capital is log-normally distributed as

$$\log(1 + r_t^K + \zeta) \sim \mathcal{N}\left(\log(1 + r_t^K) - \frac{1}{2}\sigma_\zeta^2, \sigma_\zeta^2\right),$$

with the variance of the log-gross-returns being equal to the parameter σ_ζ^2. Given above distribution expected returns at the household level coincide with the market rate, i.e. $\mathbb{E}[r_t^K + \zeta] = r_t^K$.

After receiving the returns on human capital, bonds and physical capital (determined when young), working households decide how much of their income to consume, $c_{t+1,1}$, and how much to save in physical capital, $k_{t+2,2}$, and bonds, $b_{t+2,2}$. Working households do not invest in human capital as they do not supply labor next period when they retire.

When old, households receive risk-free, r_{t+2}^f, and risky, $r_{t+2}^K + \zeta$, returns on bonds and physical capital which is their only source of income and consume all of it, $c_{t+2,2}$.

Summarizing above, the budget constraint of households is given by

$$
\begin{aligned}
&w_0 = c_{t,0} + h_{t+1,1} + k_{t+1,1} + b_{t+1,1} \\
&k_{t+2,2} + b_{t+2,2} + c_{t+1,1} = h_{t+1,1} \cdot (r_t^H + \eta) + k_{t+1,1} \cdot (1 + r_{t+1}^K + \zeta) + b_{t+1,1} \cdot (1 + r_{t+1}^f) \\
&c_{t+2,2} = k_{t+2,2} \cdot (1 + r_{t+2}^K + \zeta) + b_{t+2,2} \cdot (1 + r_{t+2}^f)
\end{aligned}
$$

Let us define total wealth of working-age and retired households as

$$
\begin{aligned}
\hat{w}_{t+1,1} &:= h_{t+1,1} + k_{t+1,1} + b_{t+1,1} \\
w_{t+2,2} &:= k_{t+2,2} + b_{t+2,2}
\end{aligned}
$$

Note the different notation of wealth above, with and without a hat. This is in order to distinguish wealth including human capital from purely financial wealth. For middle-aged households total wealth consists of financial wealth plus human capital, for old households total wealth consists solely out of financial wealth. This notation will also be used for other variables in the following. Using the definition of total wealth above, we can define physical capital and human capital shares in total wealth as $\hat{\alpha}^K_{t+1,1} := \frac{k_{t+1,1}}{\hat{w}_{t+1,1}}$ and $\hat{\alpha}^H_{t+1,1} := \frac{h_{t+1,1}}{\hat{w}_{t+1,1}}$. For old households, let us define the physical capital share in wealth - total and financial wealth are the same for this cohort - as $\alpha^K_{t+2,2} := \frac{k_{t+2,2}}{w_{t+2,2}}$.

Furthermore, we can define expected portfolio returns when middle-aged, $\hat{r}_{t+1}$, and when old, r_{t+2}, as

$$\begin{aligned}
\hat{r}_{t+1,1} &= r^f_{t+1} + \hat{\alpha}^K_{t+1,1} \cdot (r^K_{t+1} - r^f_{t+1}) + \hat{\alpha}^H_{t+1,1} \cdot (r^H_{t+1} - 1 - r^f_{t+1}) \\
r_{t+2,2} &= r^f_{t+2} + \alpha^K_{t+2,2} \cdot (r^K_{t+2} - r^f_{t+2}).
\end{aligned}$$

Finally let us define marginal propensity to consume for young and middle aged, $m_{t,0}$ and $m_{t+1,1}$, as the share of wealth, respectively wealth cum interest, that households consume.[4] At old age households consume all their wealth cum interest ($m_{t+2,2} = 1$). Taking this together we can rewrite above budget constraints as

$$\begin{aligned}
c_{t,0} &= m_{t,0} \cdot w_0 \\
c_{t+1,1} &= m_{t+1,1} \cdot (1 + \hat{r}_{t+1,1} + \varepsilon_{t+1,1}) \cdot \hat{w}_{t+1,1} \\
c_{t+2,2} &= (1 + r_{t+2,2} + \varepsilon_{t+2,2}) \cdot w_{t+2,2},
\end{aligned} \tag{1}$$

where $\varepsilon_{t+1,1} = \hat{\alpha}^K_{t+1,1} \cdot \zeta + \hat{\alpha}^H_{t+1,1} \cdot \eta$ and $\varepsilon_{t+2,2} = \hat{\alpha}^K_{t+2,2} \cdot \zeta$ are mean-zero shocks and wealth accumulates according to

$$\begin{aligned}
\hat{w}_{t+1,1} &= (1 - m_{t,0}) \cdot w_0 \\
w_{t+2,2} &= (1 - m_{t+1,1}) \cdot (1 + \hat{r}_{t+1,1}) \cdot \hat{w}_{t+1,1}.
\end{aligned} \tag{2}$$

This alternative specification of the budget constraint is equivalent to the original one. Instead of optimizing over consumption, human capital, physical capital and bonds households maximize utility over marginal propensities ($m_{t,0}$ and $m_{t+1,1}$) and portfolio shares ($\hat{\alpha}^K_{t+1,1}, \hat{\alpha}^H_{t+1,1}$ and $\alpha^K_{t+2,2}$).

[4]The difference in consumption out of wealth and out of wealth cum interest stems from the different timing assumptions when young and working-age. In the quantitative model timing is unified across all periods.

Optimizing life-time utility results in

$$\begin{aligned} m_{t,0} &= m_0 := \frac{1}{1+\beta+\varsigma\beta^2} \\ m_{t+1,1} &= m_1 := \frac{1}{1+\varsigma\beta}h \end{aligned} \tag{3}$$

and

$$\hat{\alpha}^K_{t+1,1} \approx \frac{\log\left(\frac{1+r^K_{t+1}}{1+r^f_{t+1}}\right)}{\sigma^2_\zeta} \qquad \alpha^K_{t+2,2} \approx \frac{\log\left(\frac{1+r^K_{t+2}}{1+r^f_{t+2}}\right)}{\sigma^2_\zeta} \tag{4a}$$

$$\hat{\alpha}^H_{t+1,1} \approx \frac{\log\left(\frac{1+r^H_{t}}{1+r^f_{t+1}}\right)}{\sigma^2_\eta}. \tag{4b}$$

For the complete derivation see section A.1 in the appendix. Three aspects are noteworthy. First, the marginal propensity to consume is independent of wealth (and due to the choice of log-utility also of time). Second, also the portfolio shares are functions independent of wealth. Third, the portfolio shares in physical capital for working and old cohorts coincide in the same period, that is $\hat{\alpha}^K_{t,1} = \alpha^K_{t,2}$. Horizon effects from a finite horizon of the life-cycle do not arise in this model for the portfolio shares in *total wealth*. However, as working-age cohorts total wealth also includes human capital - in contrast to old cohorts - we see horizon effects of portfolio shares in *financial wealth*, that is $\frac{k_{t,1}}{b_{t,1}+k_{t,1}} \neq \frac{k_{t,2}}{b_{t,2}+k_{t,2}}$.

Firms. The production sector is comprised of a representative competitive firm employing aggregate physical capital, K_t and human capital H_t at the market rates r^K_t and r^H_t. Output Y_t is produced according to a Cobb-Douglas production function

$$Y_t = K^\alpha_t H^{1-\alpha}_t,$$

with capital intensity α. Furthermore, I assume full depreciation of capital, such that we can write the firm's profit maximization problem as

$$\max_{K_t,H_t} \quad K^\alpha_t H^{1-\alpha}_t - (1+r^K_t)K_t - r^H_t H_t.$$

Before stating the FOCs, note that human capital is supplied by the working-age cohort, while physical capital is supplied by the working-age and old cohorts, and therefore we have $H_t = N_{t-1,0} \cdot h_{t,1}$ and $K_t = N_{t-1,0} \cdot k_{t,1} + N_{t-2,0} \cdot k_{t,2}$. Let us define κ_t as (physical) capital intensity, $\kappa_t := \frac{K_t}{H_t}$.

Using this, the firms FOCs yield

$$1 + r_t^K = \alpha\left(\kappa_t\right)^{\alpha-1} \tag{5}$$
$$1 + r_t^H = (1-\alpha)\left(\kappa_t\right)^{\alpha}. \tag{6}$$

Finally, I assume that bonds are in net-zero supply, such that

$$B_t = N_{t-1,0} \cdot b_{t,1} + N_{t-2,0} \cdot b_{t,2} = 0. \tag{7}$$

2.2 Balanced Growth Path Characterization

Population growth at rate g^N constitutes the only growth driver in the economy. Along the *balanced growth path* (BGP), marginal propensity to consume ($m_{t,0} = m_0$ and $m_{t,1} = m_1$), portfolio shares ($\hat{\alpha}_t^K = \alpha^K, \hat{\alpha}_t^H = \alpha^H$), capital intensity ($\kappa_t = \kappa$) and rates of return ($r_t^K = r^K, r_t^H = r^H, r_t^f = r^f$) are constant.

For a derivation of the balanced growth path see section A.2 in the appendix. In steady state capital intensity is given by

$$\kappa = \frac{\alpha^K}{\alpha^H} \cdot (1+\Omega), \tag{8}$$

where

$$\Omega = \frac{(1-m_1)(1+\hat{r}_1)}{1+g^N} = \mathrm{OADR} \cdot \mathrm{OAWR}, \tag{9}$$

with $\mathrm{OADR} = \frac{N_2}{N_1} = \frac{1}{1+g^N}$ and $\mathrm{OAWR} = \frac{w_2}{\hat{w}_1} = (1-m_1)(1+\hat{r}_1)$. The ratio of portfolio shares, $\frac{\alpha^K}{\alpha^H}$, represents the ratio of investment into physical and human capital by young households. This is multiplied by the factor $1+\Omega$ that takes into account that the old cohort only holds physical capital and therefore raises capital intensity. $\Omega = \frac{N_2 \cdot \int w_2\, di}{N_1 \cdot \int \hat{w}_1\, di}$ captures the relative wealth size of the old cohort relative to the working-age cohort. Ω can be further decomposed into an old age dependency ratio (OADR) component and a component that I call old age wealth ratio (OAWR). OADR measures the relative size of the old cohort compared to the young cohort, which is determined by population growth. OAWR captures the relative average wealth of an old household compared to that of a young household, which is driven by wealth accumulation. Wealth accumulation itself depends on the savings rate of the working-age cohort, $(1-m_1)$, and the portfolio return $(1+\hat{r}_1)$ on these savings.

Net-zero bond supply implies in steady state

$$\hat{\alpha}_1^f + \alpha_2^f \cdot \Omega = 0, \tag{10}$$

where $\hat{\alpha}_1^f = \left(1 - \alpha^K - \alpha^H\right)$ and $\alpha_2^f = \left(1 - \alpha^K\right)$ denote the portfolio shares in risk-free bonds of working-age and retired households, respectively. In equilibrium the risk-free share of working-age households is negative, $\hat{\alpha}_1^f < 0$, that is they take up debt to finance human and physical capital investments, and the share of old households is positive, $\alpha_2^f > 0$. Above equation implies that in equilibrium the two risk-free shares of working-age and retired households must sum to zero, where the latter share is corrected by the factor Ω. Ω again captures the relative wealth size of the old cohort compared to the young one.

Combining equations (8) and (10) we get the simple equation

$$\kappa = \frac{\alpha^K}{1 - \alpha^K}, \tag{11}$$

relating capital intensity to the ratio of the portfolio share in physical capital, relative to the share in the other assets $(1-\alpha^K)$. This finding is no surprise, considering that α^K is constant across generations and that in aggregate only physical and human capital are available as assets (due to net-zero bond supply). Therefore α^K denotes the share of aggregate available assets in physical capital and $1 - \alpha^K$ the share in human capital in the economy.

Defining the (physical capital) *relative risk premium*, $\mu^{K,rel}$ as

$$\mu^{K,rel} = \frac{1 + r^K}{1 + r^f}, \tag{12}$$

the ratio of the physical capital gross return and the gross risk-free return, we can write $\alpha^K = \log(\mu^{K,rel})/\sigma_\zeta^2$. Using this in equation (11) we can express the relative risk premium solely as a function of capital intensity:

$$\log(\mu^{K,rel}) = \sigma_\zeta^2 \frac{\kappa}{1+\kappa} \tag{13}$$

Inspecting equations (5), (12) and (13) we can conclude unambiguously that (i) the risky rate (on physical capital) decreases in κ, (ii) the relative risk premium increases in κ and (iii) therefore the risk-free rate decreases in κ. Rephrasing it, a higher capital intensity leads to lower risky and risk-free rates, and also increases the risk-premium. The intuition is straightforward: A higher capital intensity leads to lower risky returns, due to decreasing marginal productivity on physical capital (equation (5)). At the same time, the economy only experiences a higher (physical) capital intensity if households are willing to hold a larger fraction of their wealth in physical capital, and they are willing to do so if the risk premium is higher (equation (4a)). Taken together, a lower risky rate and a higher risk-premium imply a decrease in the risk-free rate.

The relative definition in equation (12) is a rather uncommon definition of a risk premium. A more familiar concept of the risk premium is in absolute terms, that is

$$\mu^{K,abs} = r^K - r^f = \alpha\kappa^{\alpha-1}\left(1 - e^{-\frac{\kappa\sigma_\zeta^2}{1+\kappa}}\right). \tag{14}$$

The absolute risk premium decreases first with capital intensity until a certain threshold of κ, from where on it decreases monotonously. For usual parameter choices in the three period model capital intensity in equilibrium is above this threshold, such that an increase in capital-intensity (e.g. through aging) leads to a decrease in the absolute risk premium unambiguously. However, this need not hold in a more complex model as in the quantitative model in section 3. In fact I find that the increase in capital intensity due to demographic change leads to an increase in the absolute risk premium, as detailed in section 5.4.

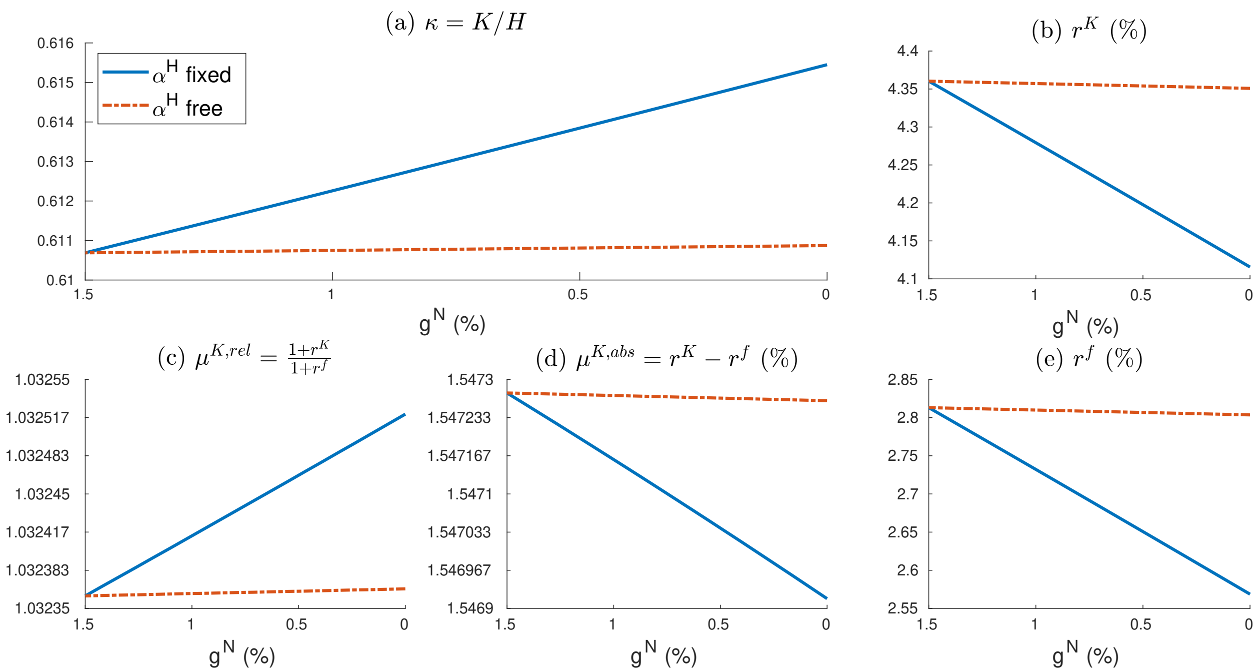
2.3 Demographic Change and the Balanced Growth Path

The three-period model contains two levers through which demographic change affects economic outcomes. First, a decrease in fertility can be simulated through a decline in the population growth rate, g^N. Second, increases in life-expectancy are represented by larger values in ς, the fraction of a period that households live when old. Both changes have qualitatively the same effect. They increase the wealth weight of the old cohort, as measured by Ω, in equations (8) and (10). A lower g^N increases the relative size of the old cohort compared to the young one, leading to an increase in Ω through a larger OADR. A higher ς increases the savings rate, $1 - m_1$, leading old households to hold more wealth in average, thus increasing Ω through a larger OAWR.

This increased wealth weight of the older generation has two effects. First, the old generation holds physical capital, but no human capital. Hence, an increased wealth weight of the old cohort is followed by an increase in capital intensity (see equation (8)). This increase in capital intensity lowers physical capital returns. Second, the old cohort has a higher portfolio share in safe bonds, $\alpha_2^f > 0 > \hat{\alpha}_1^f$. With a constant relative risk premium a higher weight of the old generation would thus lead to an excess demand for bonds. In order to reduce bond demand and equate it back back to zero supply the risk premium, $\mu^{K,rel}$, must increase (see equations (10) and (4a)). Both effects together, a lower risky rate and a higher risk premium, lead to a decrease in the risk-free rate.

Figure 1 shows how capital intensity, relative and absolute risk premia, risky and risk-free rate change when the population growth rate drops. Note that the x-axis order of g^N is reversed (1.5% to 0%) in order to explicitly show the impact of decreasing population growth. As the results are qualitatively the same for decreasing population growth and increasing life-expectancy, the figure of the latter is omitted here, but can be found in section A.2 in the appendix (see figure 8). Two polar scenarios are considered, one in which the human capital share is held constant ($\alpha^H = \overline{\alpha}^H$) and one in which households freely adjust their human capital share according to equation (4b). For comparison reasons the human capital share in the fixed scenario is set equal to the share in the free scenario with $g^N = 1.5\%$. Equilibrium conditions are stated in equations (53) and (54) in appendix A.2.

Figure 1: Capital intensity, interest rates and risk premia as functions of the population growth rate



Notes: The figure displays: (a) capital intensity, κ, (b) risky return (on physical capital), (c) relative risk premium, $\mu^{K,rel}$, (d) absolute risk premium, $\mu^{K,abs}$, and (e) risk-free rate (on bonds) in equilibrium as a function of the population growth rate g^N. The x-axis order is reversed (1.5% to 0%) to explicitly show the impact of decreasing population growth (fertility). κ and $\mu^{K,rel}$ are reported in absolute terms, $r^K, \mu^{K,abs}$ and r^f in percent. The equilibria are shown for (i) keeping the human capital share, α^H, constant and (ii) allowing it to vary according to $\alpha^H = \log((1 + r^H)/(1 + r^f))/\sigma^2_\eta$. The fixed human capital share is set equal to the human capital share in the equilibrium with a free human capital share and $g^N = 0.015$. The parameterization in all equilibria is $\alpha = 0.36, \beta = 0.95, \sigma^2_\eta = 0.15, \sigma^2_\zeta = 0.084$

As discussed earlier, a decrease in population growth leads to an increase in capital intensity, a decrease in the risky rate, an increase in the relative risk premium (decrease in the absolute risk premium) and a decrease in the risk-free rate. Although the directions of these effects coincide across the two scenarios, these effects are much more pronounced in the scenario with fixed human capital shares. Human capital adjustments in the scenario with free human capital shares mitigate much of the effects. The increase in capital intensity induced through lower population growth spurs an increase in the return on human capital (see equation (6)). This in turn incentivizes households to increase their human capital investment by increasing their portfolio share in human capital. This activates two forces. First, by increasing human capital in the economy the increase of capital intensity is attenuated (see equation (8)). Second, the increase in the portfolio share of human capital lowers the share in safe bonds (see equation (10)). This reduces the upward pressure on the relative risk premium.

2.4 Discussion and Link to Quantitative Model

An implication of equation (13) is that aside from σ_ζ^2, κ is sufficient to pin down the (relative) risk premium. Given κ, neither the distribution of characteristics across generations, nor population growth g^N or life-expectancy ς have any bearing on the risk premium. That is, demographic change affects economic outcomes solely through capital intensity in this model.

There are two criteria that are necessary for the above statement to hold. The first one is that the share of physical capital in total wealth is independent of age, i.e. flat across cohorts. The flat structure of capital returns stems from the lack of correlation of physical capital returns with past own and current human capital returns. This is supported by Davis and Willen (2000), who document empirical low correlation between labor and equity returns. In a model similar to mine, but with aggregate risk and thus both auto and cross-return correlation Geppert, Ludwig, and Abiry (2016) find very low horizon effects (see also Barberis (2000)), and a fairly flat risky asset share across age. A flat portfolio share in equities is thus not an unusual finding in the literature.

The second criterion is a net-zero supply of safe assets. This assumption is rather debatable as we can observe for example a large supply of double and triple A rated sovereign and corporate bonds. Furthermore, claims on defined benefits from pay-as-you-go social security systems can also be regarded as constituting a safe asset. Therefore, in a future version of this paper I plan to take the positive supply of safe assets through bonds and social security claims into account.

The introduction of these two safe assets will break the simple relationship between the risk premium and capital intensity in equation (13). The risk premium and the risk-free rate will then also be affected by age-related aspects of demographic change, aside from changing capital intensity. Moreover, the supply of these assets experienced large shifts in the last decades, and may be subject to future large shifts. The increase of public debt in developed nations increased the supply of safe bonds. Also, social security systems are directly and indirectly affected by demographic change. Directly, as aging and decreasing fertility affects the redistribution between young and old generations, and indirectly, as we observe a gradual shift from defined benefit towards defined contribution systems.

Another interesting implication of equation (13) is that the risk premium $\mu^{K,rel}$ is very insensitive to economic changes. Since $\kappa > 0$, the risk premium is located in a narrow interval in equilibrium, namely $1 < \mu^{K,rel} < e^{\sigma_\zeta^2}$. For small values of σ_ζ^2 this interval is approximately $(1, 1 + \sigma_\zeta^2)$.

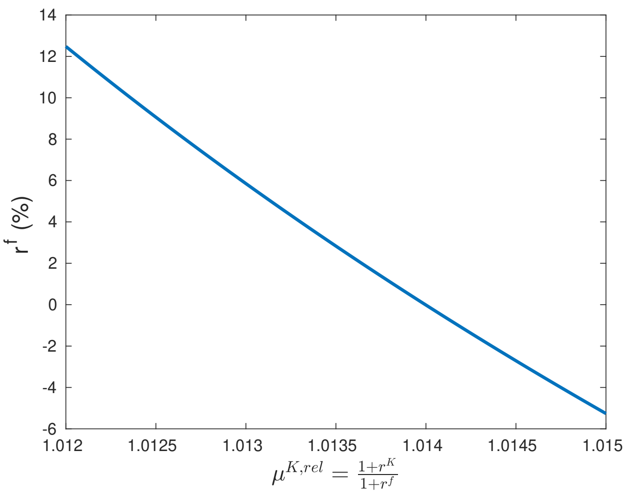
The insensitivity of $\mu^{K,rel}$ can also be illustrated by drawing all possible equilibrium outcomes in the $(\mu^{K,rel}, r^f)$ plane. As the risk premium and the risk-free rate are only dependent on κ we can use equations (5), (12) and (13) to derive

$$1 + r^f = \frac{\alpha}{\mu^{K,rel}} \cdot \left(\frac{\sigma_\zeta^2}{\log(\mu^{K,rel})} - 1 \right)^{1-\alpha} .$$

The $(\mu^{K,rel}, r^f)$-equilibria path for a realistically calibrated model are shown in figure 2. In the narrow interval of $(1.012, 1.015)$ of the risk premium, $\mu^{K,rel}$, the equilibria feature risk-free rates spanning from $+12\%$ to -6%. Hence, while the risk-free rate may change significantly from one steady state equilibrium to another, changes in the risk-premium are very mild. This insensitivity of the risk-premium to economic changes, also hinges on the zero supply of safe assets in the economy. Hence, the analysis of an introduction of positive bond supply and a social security system will also be interesting within this context.

For now the quantitative model does not feature a positive supply of bonds nor a working social security system. However, first it allows for a detailed modeling of the age-structure in the population and thus a realistic picture of past and future demographic trends. Second, it allows for a replication of the life-cycle labor-income distribution in the data. Both these aspects together allow for a quantitative realistic study, of how demographic change affects capital intensity, and how these changes in capital intensity affect interest rates and risk premia. Third, the quantitative model fully models the transitional economic path and thus allows me to follow how demographic change affects interest rates and risk premia along the whole time path. In a future version of this paper these findings will be complemented by a positive bond supply and a social security system, and how demographic change interacted with changes in the bond supply and the social security system affect interest rates and risk premia.

Figure 2: Equilibria in the $(\mu^{K,rel}, r^f)$ plane



Notes: $r^f = \frac{\alpha}{\mu^{K,rel}} \cdot \left(\frac{\sigma_\zeta^2}{\log(\mu^{K,rel})} - 1\right)^{1-\alpha} - 1$ with $\alpha = 0.36$ and $\sigma_\zeta^2 = 0.084$.

3 Quantitative Model

The quantitative model extends the simple model by a higher time-resolution. Instead of dividing the life-cycle into three phases (young, working-age and old), households are modeled at an annual frequency. This is important for the quantification of demographic change for several reasons. First, it allows for a realistic depiction of the unfolding of demographic change. Actual changes in population growth and life-expectancy can be mapped easily into this model. Second, the three-period setup of the simple model severely restricts the ability of households to adjust their portfolios over the life-cycle, only two adjustments are possibly. The quantitative model avoids such a restriction by running at an annual frequency. Third, by matching the life-cycle labor income profile, the quantitative can account for human capital shifts through demographic change.

Besides increasing the frequency, the quantitative model also micro-founds asset return and labor income risk. Adopting the framework developed in Angeletos (2007b) idiosyncratic risky returns on physical capital arise from entrepreneurial activity of households. Labor income risk is implemented by adopting the risky human capital framework developed in Krebs (2003) and Krebs and Wilson (2004) in an overlapping generations setup. By modeling increased human capital depreciation as households age, I can generate hump-shaped human capital holdings and earnings profiles.

As in the simple model, portfolio allocation is independent of wealth and can be approximated in closed-form by continuous time equations, and consumption and savings are proportional to wealth. Hence, like in the simple model, household decisions can be derived in closed form and aggregate dynamics are unaffected by idiosyncrasies. This feature of the model is particularly useful because it enables me to solve a large-scale OLG model with idiosyncratic risk and complex dynamics without incurring large computational costs.

3.1 Demographics

Time is discrete and runs from $t = 0, \ldots, \infty$. In the economy there are overlapping generations. Each generation is populated by a continuum of uniformly distributed households. The dual index j, i denotes the i'th household within the population of age j. Let $N_{t,j}$ denote the population mass of households of age j in time period t. Households born at t live until turning $J_t + 1$. J_t changes over time, capturing changes in life expectancy. The process governing J_t is assumed to be non-stochastic. For a discussion how I deal with sub-period changes see section C.1 in the appendix. I assume $J_{t+1} - J_t \geq -1$. Households enter the model at the age of 20 ($j = 0$). Given the population dynamics, the economy is populated with a changing number of overlapping generations. Population mass is given recursively as

$$N_{t,j} = \begin{cases} N_{t-1,j-1} & \text{for } j \in \{q : 0 \leq q \leq J_{t-q}\}^5 \\ newborns_t & \text{for } j = 0, \\ 0 & \text{for all other } j \end{cases} \tag{15}$$

where $newborns_t$ denotes the mass of the age 0 generation newly entering the economy. The underlying population dynamics is the exogenous driving force of the model.

3.2 Preferences

My solution of the household model builds on the homothetic structure of Epstein-Zin-Weil preferences. I sidestep the issues related to the calibration of risk aversion and inter-temporal substitution in EZW models with survival risk (cf. the discussion in Bommier, Harenberg, Le Grand, and O'Dea (2020), Bommier, Harenberg, and Le Grand (2021) building on the work by Bommier (2006), Hugonnier, Pelgrin, and St-Amour (2013)) by assuming deterministic survival.

[5] $j \in \{q : 0 \leq q \leq J_{t-q}\}$ is equivalent to $j \leq \bar{J}_{t-1}$, where $\bar{J}_{t-1} = \max\{q : 0 \leq q \leq J_{t-q}\}$, i.e. there are no holes in the age set. This follows from assumption $J_{t+1} - J_t \geq -1$.

Let θ be a measure of risk-aversion and ξ denote the elasticity of inter-temporal substitution. Epstein-Zin preferences then write as

$$u_{t,j,i} = \left[c_{t,j,i}^{1-\frac{1}{\xi}} + \beta \cdot \left(\mathbb{E}[u_{t+1,j+1,i}^{1-\theta}] \right)^{\frac{1-\frac{1}{\xi}}{1-\theta}} \right]^{\frac{1}{1-\frac{1}{\xi}}}.$$

$0 < \beta < 1$ is the standard discount factor. For $\theta = 1/\xi$ we have $\gamma = 1$ and are back at standard CRRA preferences. β is the raw time discount factor and $c_{t,j,i}$ is consumption at time t, age j of household i. $\mathbb{E}$ is the expectations operator and expectations are taken with respect to idiosyncratic shocks to human capital and technology. The felicity function in the last period an individual is still alive is given by $u_{t+J_t+1,J_t,i} = c_{t+J_t+1,J_t,i}$. Also observe that $u_{t,j,i} > 0$ for $c_{t,j,i} > 0$. This implies that life-time utility is increasing in J_t.

3.3 Budgets

Households consume $c_{t,j,i}$, hold risky physical capital $k_{t,j,i}$ with capital income $\pi_{t,j,i}(k_{t,j,i})$ (to be specified below), bonds $b_{t,j,i}$ which accrue risk-free interest at the rate r_t^f and human capital $h_{t,j,i}$ earning a gross rate of return of r_t^H. Additionally, the model features a pay as you go social security system. Social security benefits p_t, which are the same across all households and age groups, are lump-sum and are financed by contributions to labor income τ_t. At each age a household works for a fraction $\omega_{t,j} \in [0, 1]$ of her time and is retired with fraction $1 - \omega_{t,j}$. By letting $\omega_{t,j} = 1$ for j considerably before retirement and letting $\omega_{t,j}$ gradually decrease to 0 around the official retirement age I can accommodate the fact that actual age at retirement varies across individuals. Furthermore, by letting $\omega_{t,j}$ increase with t, I can account for an increasing retirement age with time, due to changes in social security eligibility or observed labor force participation patterns. While I assume human capital supply to be unaffected by $\omega_{t,j}$ I account for declining labor income at older age by incorporating $\omega_{t,j}$ into the calibration of the life-cycle labor income profile (see section 4 on the calibration strategy). The budget constraint of household i of age j in period t is given by

$$\begin{aligned} b_{t+1,j+1,i} + k_{t+1,j+1,i} = {} & b_{t,j,i}(1 + r_t^f) + \pi_{t,j,i}(k_{t,j,i}) + \\ & h_{t,j,i}(1 - \tau_t) r_t^H + (1 - \omega_j) p_t - c_{t,j,i} - e_{t,j,i} \end{aligned} \tag{16}$$

When entering the economy at age $j = 0$, any household i is endowed with an initial level of wealth $w_{t,0}^{init}$, which they can distribute across bond holdings, physical capital and human capital (there are no bequest flows to households).[6,7]

3.4 Technology

Every household i owns a firm that produces a final good according to the Cobb-Douglas production function

$$y_{t,j,i} = \psi k_{t,j,i}^{\alpha}(\Upsilon_t h_{ji}^D)^{1-\alpha} + k_{t,j,i} \cdot \zeta_{t,j,i},$$

The household supplies her capital holdings $k_{t,j,i}$ to her firm and hires human capital $h_{t,j,i}^D$ at the market rate r_t^H. $h_{t,j,i}^D$ differs from $h_{t,j,i}$, the human capital supplied by household i. Human capital is both, supplied to and demanded from the labor market by households and their firms. Households do not employ their own human capital directly in their firms. Υ_t is a human capital augmenting productivity parameter which grows at the exogenous constant rate g to capture the observed trend growth of GDP and ψ is a scaling parameter. $\zeta_{t,j,i}$ is an idiosyncratic firm-specific physical capital shock with distribution defined subsequently.

In order to work with stationary variables I de-trend the production function by dividing it by Υ_t, yielding

$$\tilde{y}_{t,j,i} = \psi \tilde{k}_{t,j,i}^{\alpha}(h_{ji}^D)^{1-\alpha} + \tilde{k}_{t,j,i} \cdot \zeta_{t,j,i}, \tag{17}$$

where $\tilde{y}_{t,j,i} = \frac{y_{t,j,i}}{\Upsilon_t}$ and $\tilde{k}_{t,j,i} = \frac{k_{t,j,i}}{\Upsilon_t}$.

Whereas $\tilde{k}_{t,j,i}$ is determined before the firm specific shock realizes, $h_{t,j,i}$ is determined after its realization, allowing for closed form policies as shown in the following. Hence, the firm chooses $h_{t,j,i}^D$ to maximize (de-trended) capital income (net of labor costs and depreciation of physical capital at rate δ^K)

$$\tilde{\pi}_{t,j,i}(\tilde{k}_{t,j,i}) = \max_{\ell} \tilde{y}_{t,j,i} - \tilde{r}_t^H h_{t,j,i}^D + (1 - \delta^K)\tilde{k}_{t,j,i},$$

[6] In order to ensure the existence of a trend-stationary growth path, $w_{t,0}^{init}$ grows with the same rate as Υ_t, that is $w_t^{init} = \Upsilon_t \cdot w_0^{init}$, where w^{init} is a constant parameter. For the definition of Υ_t, see section 3.4.

[7] This choice of initial wealth is a shortcut to the introduction of a pre-initial period in which households are born with an initial human capital level only. See also footnote 2.

given the amount of physical capital, where $\tilde{r}^H_t = \frac{r^H_t}{\Upsilon_t}$. This results in human capital demand

$$h^D_{t,j,i} = \left(\frac{(1-\alpha)\psi}{\tilde{r}^H_t}\right)^{\frac{1}{\alpha}} \tilde{k}_{t,j,i}, \tag{18}$$

output

$$\tilde{y}_{t,j,i} = (1-\alpha)^{\frac{1-\alpha}{\alpha}} \left(\psi(\tilde{r}^H_t)^{\alpha-1}\right)^{\frac{1}{\alpha}} \tilde{k}_{t,j,i} + \zeta_{t,j,i} \cdot \tilde{k}_{t,j,i}, \tag{19}$$

and capital income

$$\tilde{\pi}_{t,j,i} = \left[1 + \alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}} \left(\psi(\tilde{r}^H_t)^{\alpha-1}\right)^{\frac{1}{\alpha}} - \delta^k + \zeta_{t,j,i}\right] \tilde{k}_{t,j,i},$$

which is proportional to $\tilde{k}_{t,j,i}$. Hence we can write capital income as

$$\tilde{\pi}_{t,j,i}(\tilde{k}_{t,j,i}) = (1 + r^K_{t,j,i} + \zeta_{t,j,i})\tilde{k}_t, \tag{20}$$

where the deterministic component of the return on physical capital is defined as

$$r^K_t := \alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}} \left(\psi(\tilde{r}^H_t)^{\alpha-1}\right)^{\frac{1}{\alpha}} - \delta^K. \tag{21}$$

Note that the deterministic component is independent of any household characteristics and her age, it only depends on the economy-wide market rate on human capital. In equilibrium the market-wide rate on physical capital is equal to (21).

The distribution of the idiosyncratic shock is implicitly given by

$$\log(1 + r^K_t + \zeta_{t,j,i}) \sim \mathcal{N}\left(\log(1 + r^K_t) - \frac{1}{2}\log\left(\frac{\sigma^2_\zeta}{(1 + r^K_t)^2} + 1\right), \log\left(\frac{\sigma^2_\zeta}{(1 + r^K_t)^2} + 1\right)\right), \tag{22}$$

where σ^2_ζ is a variance parameter. Note that the expected idiosyncratic return is equal to its deterministic component, i.e. $\mathbb{E}\left[1 + r^K_t + \zeta_{t,j,i}\right] = 1 + r^K_t$, and its variance is equal to the variance parameter, i.e. $Var\left(1 + r^K_t + \zeta_{t,j,i}\right) = Var\left(\zeta_{t,j,i}\right) = \sigma^2_\zeta$.

Dividing household budget (16) by Υ_t and using equation (20) in place of capital income we get

$$\begin{aligned}\tilde{b}_{t+1,j+1,i} + \tilde{k}_{t+1,j+1,i} = \frac{1}{1+g}\Big(&\tilde{b}_{t,j,i}(1 + r^f_t) + \tilde{k}_{t,j,i}(1 + r^K_t + \zeta_{t,j,i}) + \\ &(1-\tau_t)\tilde{r}^H_t h_{t,j,i} + (1-\omega_j)\tilde{p}_t - \tilde{c}_{t,j,i} - \tilde{e}_{t,j,i}\Big),\end{aligned} \tag{23}$$

where $\tilde{b}_{t,j,i} = \frac{b_{t,j,i}}{\Upsilon_t}$, $\tilde{c}_{t,j,i} = \frac{c_{t,j,i}}{\Upsilon_t}$ and $\tilde{p}^H_t = \frac{p^H_t}{\Upsilon_t}$.

3.5 Human Capital Accumulation

I assume that human capital depreciates at the individual level by the age-specific deterministic rate δ_j^h. The age-profile of $\{\delta_j^h\}_{j=1}^{j_r}$ enables me to calibrate the model such that it mimics decreasing returns to human capital accumulation as assumed elsewhere in the literature (e.g., Huggett, Ventura, and Yaron (2011)). I assume the following functional form

$$\delta_j^h = -\chi_0 + \exp(\chi_1 \cdot j), \quad \chi_0 > 0, \chi_1 \geq 0,$$

which is monotonically increasing in j so that $1 - \chi_0 \leq \delta_j^h \leq \delta_{j+1}^h$ for all j. χ_1 is the rate at which the household's human capital depreciation accelerates when getting older.

Aside from deterministic depreciation, human capital accumulation is also subject to an idiosyncratic shock, $\eta_{t,j,i}$, whose distribution is defined further below. Collecting these elements, the human capital accumulation equation in period t, age j, of household i is given by

$$h_{t+1,j+1,i} = h_{t,j,i} \cdot (1 - \delta_j^h + \eta_{t,j,i}) + \tilde{e}_{t,j,i}^h, \qquad h_{t,j,i} \geq 0 \ \ \forall \ t, j, i, \tag{24}$$

where $\tilde{e}_{t,j,i}^h \equiv e_{t,j,i}^h / \Upsilon_t$.[8] Note that all variables in (24) are trend-stationary.[9]

Adding up human capital accumulation and labor income through human capital returns we can define the deterministic part of the gross return on growth rate adjusted human capital holdings, $\frac{h_{t,j,i}}{1+g}$,[10] as

$$1 + \hat{r}_{t,j}^H := (1+g)(1 + (1-\tau_t)\tilde{r}_t^H - \delta_j^H). \tag{25}$$

This deterministic return component depends on time and age of the household, but besides that is independent of any household specific characteristics, like her wealth level. Households' idiosyncratic gross return on their growth adjusted human capital holdings is hence $1 + \hat{r}_t^H + (1+g)\eta_{t,j,i}$.

Following the approach taken with physical capital returns, the distribution of shock $\eta_{t,j,i}$ is implicitly given by

$$\log\left(1 + \hat{r}_{t,j}^H + (1+g)\eta_{t,j,i}\right) \sim \mathcal{N}\left(\log(1 + \hat{r}_{t,j}^H) - \frac{1}{2}\log\left(\frac{\sigma_\eta^2(1+g)^2}{(1+\hat{r}_{t,j}^H)^2} + 1\right), \log\left(\frac{\sigma_\eta^2(1+g)^2}{(1+\hat{r}_{t,j}^H)^2} + 1\right)\right), \tag{26}$$

where σ_η^2 is a variance parameter.

[8] I assume that costs for human capital investment, grow with the same rate as Υ_t.

[9] The return to human capital r_t^H already exhibits a trend growth along with Υ_t. Hence, human capital must be trend stationary in order to assure that gross human capital earnings, $h_{t,j,i} \cdot r_t^H$, grow at the same rate as Υ_t over time.

[10] The scaling by $\frac{1}{1+g}$ is necessary as human capital is trend-stationary, while physical capital and bond holdings exhibit trend growth. See equation (30) for a derivation.

Note that the expected idiosyncratic return is equal to its deterministic component, i.e. $\mathbb{E}\left[1+\hat{r}^H_{t,j}+(1+g)\eta_{t,j,i}\right] = 1+\hat{r}^H_{t,j}$, and its variance is equal to the scaled variance parameter, i.e. $Var\left(1+\hat{r}^H_{t,j}+(1+g)\eta_{t,j,i}\right) = (1+g)^2 \cdot Var\left(\eta_{t,j,i}\right) = (1+g)^2\sigma^2_\eta$.

3.6 Social Security Benefits

Denote by

$$\begin{aligned} ps_{t,j} =& \frac{(1-\omega_{t,j})p_t}{1+r^f_t} + \frac{(1-\omega_{t+1,j+1})p_{t+1}}{(1+r^f_t)(1+r^f_{t+1})} + \cdots + \frac{(1-\omega_{t+J_{t-j}+1-j,J_{t-j}+1})p_{t+J_{t-j}+1-j}}{\prod_{s=0}^{J_{t-j}+1-j}(1+r^f_{t+s})} \\ =& \sum_{s=0}^{J_{t-j}+1-j} (1-\omega_{j+s})p_{t+s} \prod_{q=0}^{s}\left(1+r^f_{t+q}\right)^{-1} \end{aligned} \tag{27}$$

the discounted value of social security benefits for households of age j in period t (before interest accrues). Thereby the sequence of future risk-free rates, which are deterministic and thus known, feature as discount rates. Hereafter $ps_{t,j}$ is referred to as pension stock.

Recursively the pension stock can be represented by

$$ps_{t+1,j+1} = ps_{t,j}(1+r^f_t) - (1-\omega_{t,j})p_t.$$

De-trended (by Υ_t) pension stock thus evolves according to

$$\widetilde{ps}_{t+1,j+1} = \frac{1}{1+g}\left(\widetilde{ps}_{t,j}(1+r^f_t) - (1-\omega_{t,j})\tilde{p}_t\right). \tag{28}$$

with pension stock after death being zero, i.e.

$$\widetilde{ps}_{t+J_{t-j}+2,J_{t-j}+2} = 0. \tag{29}$$

3.7 Transformations and Recursive Household Problem

Transformations. Add $\frac{1}{1+g}$ times equation (24) and equation (28) to the de-trended dynamic budget constraint (23) yielding

$$\begin{aligned} \tilde{b}_{t+1,j+1,i}+&\tilde{k}_{t+1,j+1,i} + \frac{h_{t+1,j+1,i}}{1+g} + \widetilde{ps}_{t+1,j+1,i} = \\ &\frac{1}{1+g}\Bigg(\tilde{b}_{t,j,i}(1+r^f_t) + \tilde{k}_{t,j,i}(1+r^K_t+\zeta_{t,j,i}) + \\ &\qquad \frac{h_{t,j,i}}{1+g}\left(1+\hat{r}^H_{t,j}+(1+g)\eta_{t,j,i}\right) + \widetilde{ps}_{t,j}(1+r^f_t) - \tilde{c}_{t,j,i}\Bigg), \end{aligned} \tag{30}$$

where $1+\hat{r}^H_{t,j}$ is given in equation (25).

Now, we can define total household wealth as the sum of holdings of physical capital, bond holdings, (growth rate adjusted) human capital and pension stock,

$$\tilde{w}_{t,j,i} = \tilde{b}_{t,j,i} + \tilde{k}_{t,j,i} + \frac{h_{t,j,i}}{1+g} + \widetilde{ps}_{t,j,i}.$$

Let $\hat{\alpha}^B_{t,j,i} = \frac{b_{t,j,i}}{w_{t,j,i}}$, $\hat{\alpha}^K_{t,j,i} = \frac{k_{t,j,i}}{w_{t,j,i}}$, $\hat{\alpha}^H_{t,j,i} = \frac{h_{t,j,i}/(1+g)}{w_{t,j,i}}$, $\hat{\alpha}^{PS}_{t,j,i} = \frac{ps_{t,j,i}}{w_{t,j,i}}$ denote the household i, period t, age j holdings of bonds, physical capital, human capital and pension stock relative to total wealth, respectively. Furthermore, let $\hat{\alpha}^f_{t,j,i} = \hat{\alpha}^B_{t,j,i} + \hat{\alpha}^{PS}_{t,j,i}$ denote the fraction of risk-free holdings in total wealth. Then $\hat{\alpha}^f_{t,j,i} = 1 - \hat{\alpha}^K_{t,j,i} - \hat{\alpha}^H_{t,j,i}$. Finally, define the gross portfolio return as $1+\hat{r}_{t,j}+\varepsilon_{t,j,i}$, where the deterministic component is given by

$$1+\hat{r}_{t,j,i} := 1 + r^f_t + \hat{\alpha}^K_{t,j,i}\cdot(r^K_t - r^f_t) + \hat{\alpha}^H_{t,j,i}\cdot\left(\hat{r}^H_{t,j} - r^f_t\right), \tag{31}$$

and the stochastic component by $\varepsilon_{t,j,i} = \hat{\alpha}^K_{t,j,i}\cdot\zeta_{t,j,i} + \hat{\alpha}^H_{t,j,i}\cdot(1+g)\eta_{t,j,i}$. Note that the expected portfolio return equals its deterministic component, $\mathbb{E}\left[1+\hat{r}_{t,j,i}+\varepsilon_{t,j,i}\right] = 1+\hat{r}_{t,j,i}$. With this definition at hand we can rewrite equation (30) as

$$\tilde{w}_{t+1,j+1,i} = \frac{\tilde{w}_{t,j,i}}{1+g}\cdot(1+\hat{r}_{t,j}+\varepsilon_{t,j,i}) - \frac{\tilde{c}_{t,j,i}}{1+g} \tag{32}$$

As a last step let cash-on-hand be wealth cum interest $\tilde{x}_{t,j,i} := \tilde{w}_{t,j,i}(1+\hat{r}_{t,j,i})$. With this definition rewrite above equation as

$$\tilde{x}_{t+1,j+1,i} = (\tilde{x}_{t,j,i} - \tilde{c}_{t,j,i})\cdot\frac{1}{1+g}\cdot(1+\hat{r}_{t,j}+\varepsilon_{t,j,i}). \tag{33}$$

Arriving at (33) is a key transformation as all additive terms have vanished. In combination with homothetic preferences, this will give rise to the closed form solutions stated in the subsequent proposition. Note that within the transformed problem, period t age j choice variables are $\tilde{c}_{t,j,i}, \tilde{x}_{t+1,j+1,i}$ as well as the portfolio shares $\hat{\alpha}^K_{t+1,j+1,i}, \hat{\alpha}^H_{t+1,j+1,i}$ which together determine the period $t+1$, age $j+1$ holdings of physical capital, human capital and risk-free holdings. Social security benefits p_t are either exogenously given or are indirectly determined by the social contribution rate τ_t. Given the sequence of p_t and r^f_t, pension stock holdings and shares in total wealth are pinned down. This allows me to determine bond holdings by $\hat{\alpha}^B_{t+1,j+1,i} = \hat{\alpha}^f_{t+1,j+1,i} - \hat{\alpha}^{PS}_{t+1,j+1,i}$.

Recursive Household Problem. In the following I define the household problem recursively. It is convenient to express next period's values with symbol $'$, irrespective of whether they are only time-dependent or both, age- and time-dependent. The household problem is solved contingent on age j, the endogenous idiosyncratic state of (de-trended) cash-on-hand, $\tilde{x}$ and prices next period $\phi_t' = \{r_t^{f\prime}, \tilde{r}_t^{H\prime}, p_t', \tau_t'\}$.

$$
\begin{aligned}
v(t,j,\tilde{x}) = \max_{\tilde{c},\tilde{x}',\widehat{\alpha}^{k\prime},\widehat{\alpha}^{h\prime}} & \left\{ \tilde{c}^{1-\frac{1}{\xi}} + \widehat{\beta} \cdot \left(\mathbb{E}[v(t+1,j+1,\tilde{x}')^{1-\theta}]\right)^{\frac{1-\frac{1}{\xi}}{1-\theta}} \right\}^{\frac{1}{1-\frac{1}{\xi}}} \\
\text{s.t. } \tilde{x}' = & \frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}) \cdot \left(1 + \widehat{r}(\phi_t', j+1) + \varepsilon'\right) \\
& \widehat{r}(\phi_t', j+1) = r^{f\prime} + \hat{\alpha}^{K\prime} \cdot \left(r^K(\tilde{r}^{H\prime}) - r^{f\prime}\right) + \hat{\alpha}^{H\prime} \cdot \left(\hat{r}^H(\tilde{r}^{H\prime}, \tau', j+1) - r^{f\prime}\right) \\
& \varepsilon' = \hat{\alpha}^{K\prime} \cdot \zeta' + \hat{\alpha}^{H\prime} \cdot (1+g)\eta' \\
& t \mapsto \phi_t \\
& \zeta' \sim \text{ equation (22)} \\
& \eta' \sim \text{ equation (26)},
\end{aligned}
$$

where $\widehat{\beta} \equiv \beta \cdot (1+g)^{1-\frac{1}{\xi}}$.

Proposition 1. *Conditional on the portfolio shares $\hat{\alpha}$, policy functions for consumption are linear in cash-on-hand*

$$
\tilde{c}(t,j,\tilde{x}) = m(t,j) \cdot \tilde{x}. \tag{35}
$$

Furthermore portfolio shares $\hat{\alpha}^K(t,j,\tilde{x}) = \hat{\alpha}^K(t)$ and $\hat{\alpha}^H(t,j,\tilde{x}) = \hat{\alpha}^H(t,j)$ are approximately

$$
\hat{\alpha}^K(t+1) \approx \frac{\log\left(\frac{1+r^{K\prime}}{1+r^{f\prime}}\right)}{\theta \cdot Var(\log(1 + r^{K\prime} + \zeta'))} \tag{36a}
$$

$$
\hat{\alpha}^H(t+1,j+1) \approx \frac{\log\left(\frac{1+\hat{r}^{H\prime}}{1+r^{f\prime}}\right)}{\theta \cdot Var(\log(1 + \hat{r}^{H\prime} + \eta'))} \tag{36b}
$$

and the marginal propensity to consume out of cash-on-hand is given by

$$
m = \frac{(\beta^{\frac{1-\theta}{1-\frac{1}{\xi}}} \cdot \mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1-\xi}{1-\theta}} \cdot m'}{1 + (\beta^{\frac{1-\theta}{1-\frac{1}{\xi}}} \cdot \mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1-\xi}{1-\theta}} \cdot m'}, \tag{37}
$$

where $\mathbb{E}[(1+\widehat{r}')^{1-\theta}$ can be approximated by equation (57) in the appendix.

Proof. See Section C.2 in the appendix. □

Note that both, marginal propensity to consume m as well as portfolio shares $\hat{\alpha}^K$ and $\hat{\alpha}^H$ are independent of the household specific state realizations in η, ζ and $\tilde{x}$. This allows me to write the deterministic part of the household return as only a function of time and age, that is

$$1 + \hat{r}_{t,j} = 1 + r_t^f + \hat{\alpha}_t^K \cdot (r_t^K - r_t^f) + \hat{\alpha}_{t,j}^H \cdot \left(\hat{r}_{t,j}^H - r_t^f\right).$$

Moreover, the portfolio shares in physical capital are independent of age and thus do not experience any horizon effects.

3.8 Bond Supply and Social Security System

I assume a net-zero-supply of bonds,

$$B_t = 0, \tag{38}$$

where B_t are the aggregate bonds supplied in period t.

The social security system is exclusively financed by labor income contributions. Benefits are linked to labor income through $\tilde{p}_t = \rho_t \cdot \frac{(1-\tau_t)\cdot \tilde{r}_t^H \cdot L_t}{\sum_j N_{t,j}\cdot \omega_{t,j}}$, with ρ_t being the replacement rate and H_t the aggregate human capital employed in the economy. That is, benefits are equal to the replacement rate times the average net labor income. Closed financing of the pension system implies

$$\left(\sum_j N_{t,j} \cdot \omega_{t,j}\right) \tau_t \tilde{r}_t^H H_t = \left(\sum_j N_{t,j} \cdot (1 - \omega_{t,j})\right) \cdot \rho_t \cdot (1 - \tau_t) \cdot \tilde{r}_t^H \cdot H_t. \tag{39}$$

After simplification it implies that in equilibrium the labor contribution rate is

$$\tau_t = \frac{\sum_j N_{t,j}(1 - \omega_{t,j})\rho_t}{\sum_j N_{t,j}\omega_{t,j} + \sum_j N_{t,j}(1 - \omega_{t,j})\rho_t} \tag{40}$$

or equivalently the replacement rate is

$$\rho_t = \frac{\tau_t}{1 - \tau_t} \cdot \frac{\sum_j N_{t,j}\omega_{t,j}}{\sum_j N_{t,j}(1 - \omega_{t,j})}. \tag{41}$$

Note that the labor contribution rate (τ_t) or the replacement rate (ρ_t) - given the other - is only determined by demographics, independent of any prices or aggregates. It is the population sizes of workers and retirees which is determining τ_t and ρ_t.

3.9 Equilibrium

Aggregate quantities are stated in capital letters. Before I give a definition of the equilibrium I determine aggregate human capital demand by aggregating equation (18) across ages j and firms i:

$$\begin{aligned} H_t^D &= \sum_j N_{t,j} \cdot \int \left(\frac{(1-\alpha)\psi}{\tilde{r}_t^H} \right)^{\frac{1}{\alpha}} \tilde{k}_{t,j,i} \, di \\ &= \left(\frac{(1-\alpha)\psi}{\tilde{r}_t^H} \right)^{\frac{1}{\alpha}} \cdot \widetilde{K}_t. \end{aligned} \tag{42}$$

Aggregating equation (19) gives aggregate output as

$$\widetilde{Y}_t = \left(\frac{1-\alpha}{\tilde{r}_t^H} \right)^{\frac{1-\alpha}{\alpha}} \psi^{\frac{1}{\alpha}} \cdot \widetilde{K}_t, \tag{43}$$

which holds by the law of large numbers as $\zeta_{t,j,i}$ and $\tilde{k}_{t,j,i}$ are independently distributed. Note that like in the case with a representative firm with Cobb-Douglas technology, the labor income share (here the share of human capital income) is $1-\alpha$ and the physical capital income share (including depreciated capital) is α.

Similarly, the aggregate pension stock is determined by aggregating the individual pension stocks across ages j:

$$\widetilde{PS}_t = \sum_j N_{t,j} \widetilde{ps}_{t,j}, \tag{44}$$

where $\widetilde{ps}_{t,j}$ is the same across households of the same cohort and is determined by equation (27) or equivalently equation (28).

Age dependent accumulation of cash-on-hand determines the aggregate law of motion. Average cash-on-hand holdings of cohort t,j are given by aggregating the individual cash-on-hand accumulation equation (33) across households, i.e.

$$\begin{aligned} \widetilde{X}_{t+1,j+1} =& \frac{1}{1+g} \cdot \int \tilde{x}_{t,j,i} \cdot (1-m_{t,j}) \cdot (1 + \hat{r}_{t+1,j+1} + \varepsilon_{t+1,j+1,i}) \, di \\ =& \frac{1}{1+g} \cdot \widetilde{X}_{t,j} \cdot (1-m_{t,j}) \cdot (1 + \hat{r}_{t+1,j+1}), \end{aligned} \tag{45}$$

which follows from $\hat{\alpha}^f, \hat{\alpha}^K$ and m not being functions of cash on hand and $(\eta_{t,j,i}, \zeta_{t,j,i})$ being independently distributed from $\tilde{x}_{t,j,i}$.

The equilibrium in the economy is defined recursively and presented in de-trended form, cf. Section 3.7. Market clearing is required in all periods while household optimality and aggregation conditions have to hold. In the following, symbol $'$ indicates next period's variables.

Definition 1. *A recursive competitive equilibrium is a value function $v(t, j, \tilde{x})$ and policy functions, $\widehat{\alpha}^{K\prime}(t, j)$, $\widehat{\alpha}^{H\prime}(t, j)$, $m(t, j)$ for the household, functions for aggregate human capital demand $H^D(\tilde{r}^H, \widetilde{K})$, bond supply $B'(t)$, physical capital holdings $K'(t)$, aggregate pension stock $\widetilde{PS}(t)$, aggregate cash-on-hand holdings of generation j, $\widetilde{X}'_j(\tilde{X}_j, \phi')$, pricing functions $\phi'(t) = \{r^{f\prime}(t), \tilde{r}^{H\prime}(t), p'(t), \tau'(t)\}$ and the demographic distribution, $N(t, j)$, such that for all t*

1. *$v(\cdot)$ satisfies the household's recursive problem, and $\widehat{\alpha}^{K\prime}(\cdot)$, $\widehat{\alpha}^{H\prime}(\cdot)$, $m(\cdot)$ are the associated policy functions following from the conditions in Proposition 1, given $\phi'(t)$*

2. *aggregate human capital demand $H^D(\tilde{r}^H, \widetilde{K})$ is given by (42),*

3. *aggregate pension stock is given by (44) and thus a function of future benefits and risk-free rates, $\widetilde{PS}(\{\rho(s), \tau(s), \tilde{r}^H(s), r^f(s)\}_t^{t+J_t})$,*

4. *the aggregate bond supply is given by (38).*

5. *Pension payments are financed through labor income contributions as given in equation (39).*

6. *Risk free assets and labor markets clear:*

$$\begin{aligned}
&\widetilde{B}'(r^f(t), Y(t), B(t)) + \widetilde{PS}'(\{\rho(s), \tau(s), \tilde{r}^H(s), r^f(s)\}_t) = \\
&\frac{1}{1+g} \cdot \sum_j N(t, j) \cdot \widetilde{X}_j(\widetilde{X}_{j-1}(t-1), \phi(t)) \cdot (1 - m(t, j)) \cdot (1 - \widehat{\alpha}^{K\prime}(t, j) - \widehat{\alpha}^{H\prime}(t, j))^{10}
\end{aligned} \tag{46a}$$

$$\begin{aligned}
&H^{D\prime}(\tilde{r}^{H\prime}(t), \widetilde{K}'(t)) = \\
&H^{S\prime}(t) \equiv \sum_j N(t, j) \cdot \widetilde{X}_j(\widetilde{X}_{j-1}(t-1), \phi(t)) \cdot (1 - m(t, j)) \cdot \hat{\alpha}^{H\prime}(t, j) \cdot \omega_{t+1, j+1},^{10}
\end{aligned} \tag{46b}$$

7. *aggregate law of motion $\widetilde{X}'_j(\widetilde{X}_j, \phi')$ is given by equation (45), where*[11]

$$\widetilde{X}_0(t) = w^{init} \cdot (1 + \hat{r}(t, 0)) + \widetilde{ps}_{t,0} \cdot (1 + r^f(t)). \tag{47}$$

[10] Aggregation results are based on the fact that $m(j, \cdot), \hat{\alpha}^{f\prime}(\cdot)$ and $\hat{\alpha}^{K\prime}(\cdot)$ policies are independent of individual cash-on-hand levels.

[11] Initial cash-on-hand level is equal to the given initial wealth distributed across bonds, physical and human capital, according to the portfolio shares given in (36). See also footnote 6.

8. *and either $\rho(t)$ or $\tau(t)$ is given.*[12]

Lemma 1 (Walras' Law). *The definition of a recursive equilibrium above ensures that the aggregate resource constraint holds:*

$$\widetilde{Y}(t) = \widetilde{C}(t) - N(t,0) \cdot \widetilde{B}(t+1,0) + \widetilde{I}^K(t) + I^H(t), \tag{48}$$

where $\tilde{I}^K(t)$ and $I^H(t)$ are aggregate (de-trended) investments into physical capital and human capital, respectively.

Proof. See Section C.3 in the Appendix. □

Definition 2. *A stationary recursive competitive equilibrium is a special case of the equilibrium described above. It is characterized by time-constant prices $\phi(t)$. This requires a time-constant demographic distribution, $N(t)$.*

4 Solution Method and Calibration

Solution Method. As shown in proposition 1 households' portfolio shares and marginal propensities to consume are independent of individual variables (shock realizations and cash-on-hand). Instead they only depend on time t and age j. By the law of large numbers the idiosyncratic shocks cancel each other out on the cohort level. This allows me to to treat the model at the aggregate level as a deterministic OLG model with one representative agent per cohort, where as sole state variable I have time and cohort specific (de-trended) cash-on-hand, $\widetilde{X}_{t,j}$. I solve for an initial stationary equilibrium in the year 1960 and a final one in the year 2500. Starting with the initial stationary equilibrium in 1960 I solve the model for the transitional path until 2500. For more details on the computational approach see appendix D.

Calibration. The model is calibrated in part by reference to other studies and in part by informal matching of moments procedures. Period length is one year. Table 1 summarizes structural model parameters where target values refer to the referenced study or in the other cases to the average of years $1960-2010$. The additional parameters governing demographic and stochastic processes are described below.

Time-specific cohort life-expectancies, J_t, are computed using data from Noymer and Garenne (2000) and from mortality rates data taken from the Human Mortality Database

[12] Additionally, the pension stocks across generations in the initial period, $\{\widetilde{ps}_{0,j}\}_{j=0}^{J_0}$, and in the final period, $\{\widetilde{ps}_{T,j}\}_{j=T}^{J_T}$ are given. See appendix D.1.1, and there especially equation (62) (combined with equation (28)).

Table 1: First and Second Stage Parameters

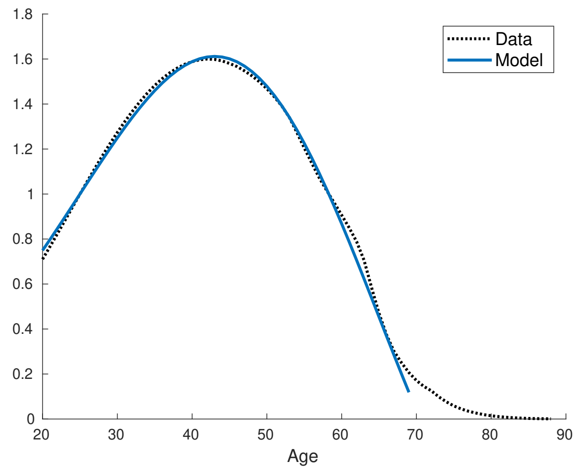
Parameter	Value	Target	Target Source, Comment
Preferences			
Elasticity inter-temp. substit.: ξ	0.5	1^{st} stage	Havránek (2015)
Time discount factor: β	0.98	$K/Y = 2.86$	NIPA
Relative risk aversion: θ	3.99	$\mathbb{E}[r]^s - r^f = 0.0223$	PST
Budgets			
Endowment: w_0	1.0	1^{st} stage	normalization
Technology			
Capital share: α	0.36	1^{st} stage	wage share (NIPA)
Technological progress: g	0.018	1^{st} stage	TFP growth (NIPA)
Depreciation rate K: δ_0^K	0.0786	$\mathbb{E}[r^K] = 0.0473$	PST
TFP Scaling factor: ψ	1	–	
TFP shock std.: σ_ζ	0.0841	$std(r^k) = 0.0841$ (std. of stock returns)	PST
Human Capital			
Depreciation rate h intercept: χ_0	0.289	$\frac{r^H \cdot h_{40}}{r^H \cdot h_{25}} = 1.59$	PSID
Depreciation rate h slope: χ_1	0.00468	$\frac{r^H \cdot h_{52}}{r^H \cdot h_{25}} x = 1.39$	PSID
Hum cap. shock std.: σ_η	0.958	$std(\Delta \log \widetilde{inc}) = 0.18$	Krebs (2003)

Notes: PST $\hat{=}$ Piazzesi, Schneider, and Tuzel (2007). PSID $\hat{=}$ Panel Study of Income Dynamics.

(HMD). Newborn popoulation dynamics, $newborns_t$, are based on two sources. First, a population model with survivial rates taken from the HMD and fertility rates taken from the United Nations' population projections (United Nations 2007) is used. Second, differences in these projections and actual population data is also attributed to newborns. The sum of these two is normalized by the US population size in 1965. Time-age-specific working time fractions, $\omega_{t,j}$ consist of two components, (i) survival rates conditional on reaching working age 20 and (ii) participation rates. Data from the HMD and OECD is used to set these. For details refer to appendix D.2.1.

I calibrate the human capital depreciation rate, δ^h, by setting the corresponding parameters, χ_0 and χ_1, such that the model matches an observed labor income profile based on PSID data. The profiles is based on smoothed post government labor income, net of all taxes and transfers, where averages are taken over the years from 1960 to 2010. This profile is computed for households that have positive labor income. The model on the other side, subsumes also households in every cohort that do not participate in the labor market. Therefore I scale the labor income profile with the age dependent working time fractions of the year 1965, $\omega_{1965,j}$, and match this profile. Figure 3 compares the scaled labor income profiles from data (computed as mentioned above) and of the calibrated steady state in the model.

Figure 3: Labor Income Profile in 1965



Notes: Labor income profiles in the data and the model are normalized such that age 25 income equals one. The data profile is based on PSID data averaged from 1960-2010, scaled by age-dependent factors $\omega_{1965,j}$ of the year 1965. These factors account for age-contingent participation and survival probabilities. See appendix D.2.1 for more details.

The standard deviation parameter of human capital returns, σ_η, is pinned down such that I match observed labor income risk by requiring labor income growth to have an average standard deviation of 0.18, slightly higher than in Krebs (2003), who sets it to 0.15. Section D.2.2 in the appendix gives details on how the variance of income growth is computed.

I set the elasticity of inter-temporal substitution $\xi = 0.5$, and thus well below the value of one. This is in line with empirical estimates, c.f. Havránek (2015) for a meta-analysis of the literature on the value of the elasticity of intertemporal substitution. The value of households' raw time discount factor, β, is calibrated to yield a capital-output ratio of 2.86, as measured in NIPA data, cf., e.g., Ludwig, Schelkle, and Vogel (2012), yielding $\beta = 0.98$.

Due to homothetic preferences, the initial level of de-trended wealth w_0 is irrelevant and I normalize it to one.

The value of the capital share parameter $\alpha = 0.36$, is based on an estimation of the aggregate production function for the US, cf. Krueger and Ludwig (2007). It lies in the usual range considered in the literature. I set the average risk-free rate to 2.5%, the approximate average US bank lending rates in the 1960s according to Worldbank data. Piazzesi, Schneider, and Tuzel (2007) report average risky stock and housing returns of 6.94% and 2.52%, respectively. Following Glover, Heathcote, Krueger, and Rios-Rull (2014), I set the risky rate on physical capital as the return of an equally weighted portfolio of stocks and housing, yielding roughly 4.7%. This implies a risk premium, measured as the difference between the risky and risk-free rate of about 2.2%. The risky and risk-free rate, or equivalently the risky rate and the risk-premium are used as targets in the calibration of the depreciation rate parameter of physical capital, δ_0^K, and the risk aversion parameter, θ, yielding parameter values within the range considered in the literature. The standard deviation of the equally weighted risky portfolio is 8.41% which acts as a target in the calibration of the standard deviation parameter of the TFP shock, σ_ς. The TFP scaling factor is set to one.

The social security system is switched off for now, such that contribution rate $\tau_t = 0$ and replacement rate $\rho_t = 0$ for all years. It will be introduced in a later version of this paper.

5 Results

Given the calibration, as described in section 4, I simulate the model for 500 years (= periods), from 1965 to 2465, where the initial wealth distribution corresponds to the distribution in the calibrated steady state. Demographic variables (life-expectancy, size of new cohorts entering the economy and work time fractions) change until the year 2100, and are constant from then on. This guarantees convergence to a new steady state. In the following I depict and discuss macroeconomic dynamics for the horizon 2010 to 2050. Two scenarios are investigated within this time horizon, a *fixed* and a *free* human capital scenario, which correspond to the fixed and free scenario discussed within the context of the simple model. Both scenarios start in 2010 from the same realization of the dynamic path, that is have equal aggregate levels and wealth distributions.

Before I get to the discussion of how demographic change affects the dynamics of interest rates and risk premia I first show cross-sectional profiles in the year 2010.

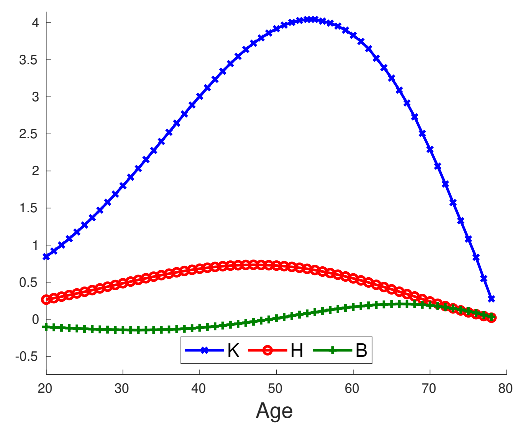
5.1 Cross-Sectional Profiles in 2010

Figure 4 shows cross-sectional age profiles of the model economy in the year 2010. The left panel depicts the portfolio allocation of households by age. Households enter their economically relevant lifetime with zero financial assets but positive human capital. Human capital follows a hump-shaped pattern over the working life which results in a corresponding pattern in the age-earnings profile (as shown earlier in figure 3). This is a target in the calibration.

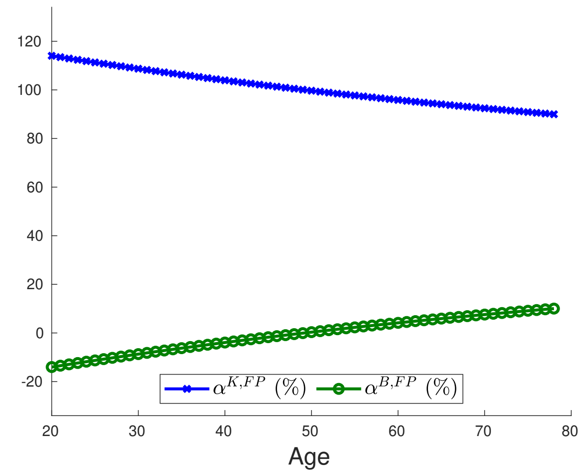
The share of financial asset holdings in physical capital, $\alpha_{t,j}^K$, is shown in the right panel. This share is defined as $\alpha_{t,j}^{K,FP} = \frac{K_{t,j}}{B_{t,j}+K_{t,j}} = \frac{\hat{\alpha}_{t,j}^K W_{t,j}}{B_{t,j}+K_{t,j}}$. The portfolio share in total wealth (including human capital), $\hat{\alpha}_{t,j}^K$, is constant over the life-cycle. There are no horizon effects arising from the finite horizon of the life-cycle model. Small horizon effects are a well-known feature of portfolio models such as mine (cf. Campbell and Viceira (2002, ch. 6), Barberis

(2000) and Geppert, Ludwig, and Abiry (2016)). As a consequence, the dynamics of $\alpha^K_{t,j}$ over the life-cycle are driven by the dynamics of the share of financial wealth in total wealth, $\frac{B_{t,j}+\tilde{K}_{t,j}}{\tilde{W}_{t,j}}$. As households decrease their share of human capital in total wealth over their life-cycle (SEE APPENIX FIGURE), the share of financial wealth in total wealth is increasing at the same time. Intuitively, the decrease in human capital reduces the diversification of the portfolio held by any household. Therefore, in order to keep the overall risk of their portfolio down, households reduce their share risky physical capital.

Figure 4: Cross-Sectional Profiles in 2010



Assets



Financial Portfolio Shares

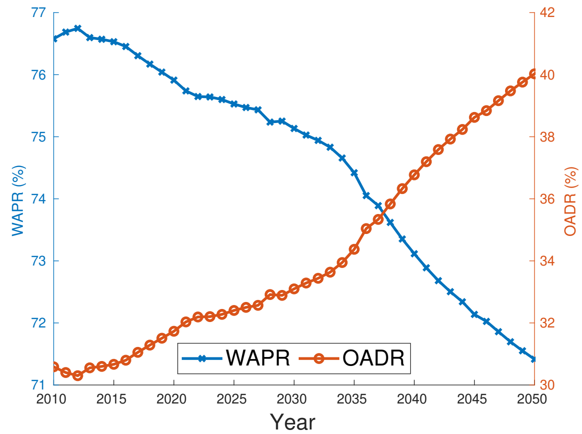
Notes: Selected average cross-sectional age profiles in the year 2010. The left panel depicts average physical capital ($K_{2010,j}$), human capital ($K_{2010,j}$) and bond ($B_{2010,j}$) holdings over the life-cycle. The right panel shows the average financial portfolio shares in risky physical capital $\left(\alpha^{K,FP}_{2010,j} = \frac{K_{2010,j}}{K_{2010,j}+B_{2010,j}}\right)$ and risk-free bonds $\left(\alpha^{B,FP}_{2010,j} = \frac{B_{2010,j}}{K_{2010,j}+B_{2010,j}}\right)$.

5.2 Asset Returns

I now turn to the key question, how differential asset returns are affected by the demographic transition. The demographic transition is taken as exogenous driving force. Key summary statistics are depicted in figure 5, namely the working age to population ratio (WAPR) and the old age dependency ratio (OADR). WAPR measures the ratio of the working population to the total adult population of age 20 and above and OADR the ratio of the old-age population to the working-age population. Following the model setup I define the working(-age) population size as $\sum_j N_{t,j} \cdot \omega_{t,j}$ and the old-age population size as $\sum_j N_{t,j} \cdot (1 - \omega_{t,j})$.

WAPR is projected to decrease by about 5 percentage points until 2050 and as a mirror image OADR is projected to increase by almost 10 percentage points over the same period.[13]

Figure 5: Working Age to Population Ratio and the Old Age Dependency Ratio



Notes: The working age-to-population ratio equals the working-age population ($\sum_j N_{t,j} \cdot \omega_{t,j}$) over the total population of the model economy. The old-age-dependency-ratio equals the old-age population ($\sum_j N_{t,j} \cdot (1 - \omega_{t,j})$) over the working-age population.

I analyze the macroeconomic consequences of demographic change in two distinct human capital scenarios, a *fixed* and a *free* scenario. Within the fixed scenario human capital shares in total wealth are held constant at 2010 levels, $\hat{\alpha}^H_{t,j} = \hat{\alpha}^H_{2010,j}$. I thereby approximate a model without human capital adjustments.[14] In the free scenario, households can freely adjust human capital shares, as given in equation (36b). The comparison of both scenarios allows me to illustrate the mitigating effects that endogenous human capital adjustments have on the macroeconomic dynamics triggered through demographic change. I regard the two scenarios as polar extremes. On the one hand, the fixed scenario is likely too restrictive in modeling human capital adjustment, while on the other hand the free scenario overstates these adjustments due to the lack of any human capital adjustment frictions. These two scenarios allow me to bracket the likely dynamics in asset returns in the coming decades.

[13]Differences in WAPR and OADR in levels and trends from comparable literature (e.g. Geppert, Ludwig, and Abiry (2016)) are for two reasons. First, I model survival deterministically (for reasons of the value of life, see section 3.2), while other studies choose stochastic survival. Second, I do not impose a retirement age at which households have to retire. Instead, transition into retirement is smooth, that is as households age a larger share of any cohort is characterized as retirees, as measured by the share $1 - \omega_{t,j}$.

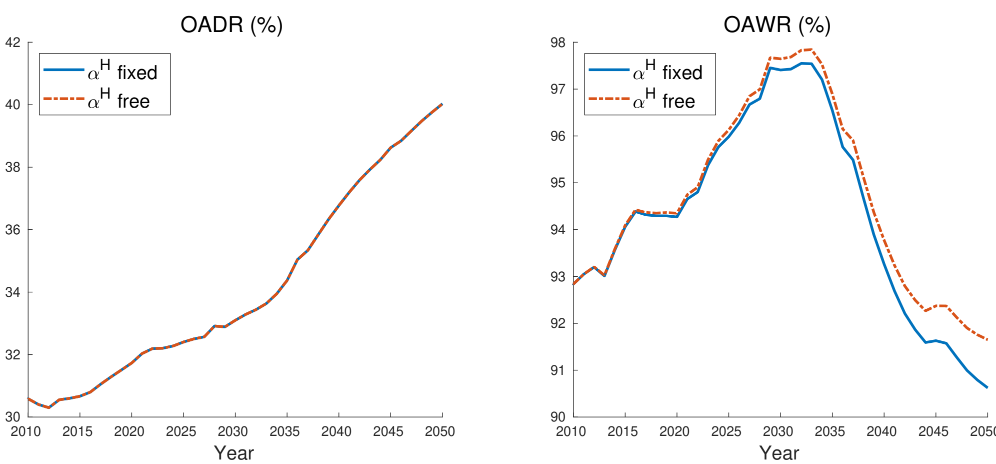
[14]In order to preserve closed form solutions in the model, I cannot directly hold constant $h_{t,j}$, but instead fix $\hat{\alpha}^H_{t,j}$. Therefore, I speak of an approximation to a model with constant human capital.

5.3 Disentangling the Old-Age Factor

As pointed out in the discussion of the simple model following equation (9), demographic change affects interest rates through two channels, (i) increasing the size of older cohort relative to younger ones and (ii) through changing the relative average wealth level of older households relative to younger ones.

Figure 6 shows how these two channels play out in this model along the transition. On the left panel is the same old age dependency ratio (OADR) depicted as in figure 5. This represents a summary statistics for the relative growth of the size of the old cohorts. The right panel shows what I define as old age wealth ratio (OAWR). This is the ratio of the average total wealth of old households to the average wealth of young households. Average young household wealth is defined as $\frac{\sum_j N_{t,j}\omega_{t,j} \int w_{t,j,i}\ di}{\sum_j N_{t,j}\omega_{t,j}}$ and average old household wealth as $\frac{\sum_j N_{t,j}(1-\omega_{t,j}) \int w_{t,j,i}\ di}{\sum_j N_{t,j}(1-\omega_{t,j})}$. The figure shows the dynamics of OADR and OAWR for both human capital scenarios. As the population dynamics are purely exogenous, both scenarios exhibit the same OADR evolution. However, savings and portfolio allocation patterns are endogenous quantities in the model and hence differ between the two scenarios, and so do also OAWR dynamics.

Figure 6: Old Age Dependency Ratio and Old Age Wealth Ratio



Notes: The left panel depicts the old age dependency ratio (OADR), the right panel the old age wealth ratio (OAWR), both reported in percent. OADR is equal to OADR in figure 5. OAWR is the ratio of average wealth of old households to average wealth of young households, where the latter is $(\sum_j N_{t,j}\omega_{t,j} \int w_{t,j,i}\ di)/(\sum_j N_{t,j}\omega_{t,j})$ and the former is $(\sum_j N_{t,j}(1-\omega_{t,j}) \int w_{t,j,i}\ di)/(\sum_j N_{t,j}(1-\omega_{t,j}))$.

The left panel shows that the relative size of the older cohorts (compared to the younger ones) increases from around 30% to above 40% from 2010 to 2050, thus imposing downward pressure on interest rates and upward pressure on the (relative) risk premium, see the discussion in section 2.3. In the right panel we can see that initially average old wealth equals 93% of young wealth in 2010 and rises to around 97-98% in 2030. However, from 2035 onward it reverts back to 91-92%. These dynamics of the wealth ratio impose initially a downward (upward) pressure on interest rates (risk premium) and subsequently an upward (downward) pressure. This means that from 2035 onward OADR and OAWR exert pressure in opposite directions. However the changes in OADR (34% to 40% from 2035 to 2050 corresponds to a factor increase of 1.18) are of a bigger magnitude than the changes in OAWR (97% to 91% correspond to a factor decrease of 1.066)and thus dominate them. We hence expect a persistent decrease in interest rates and increase in the relative risk premium, as discussed in the next section.

5.4 Interest Rates and the Risk Premium

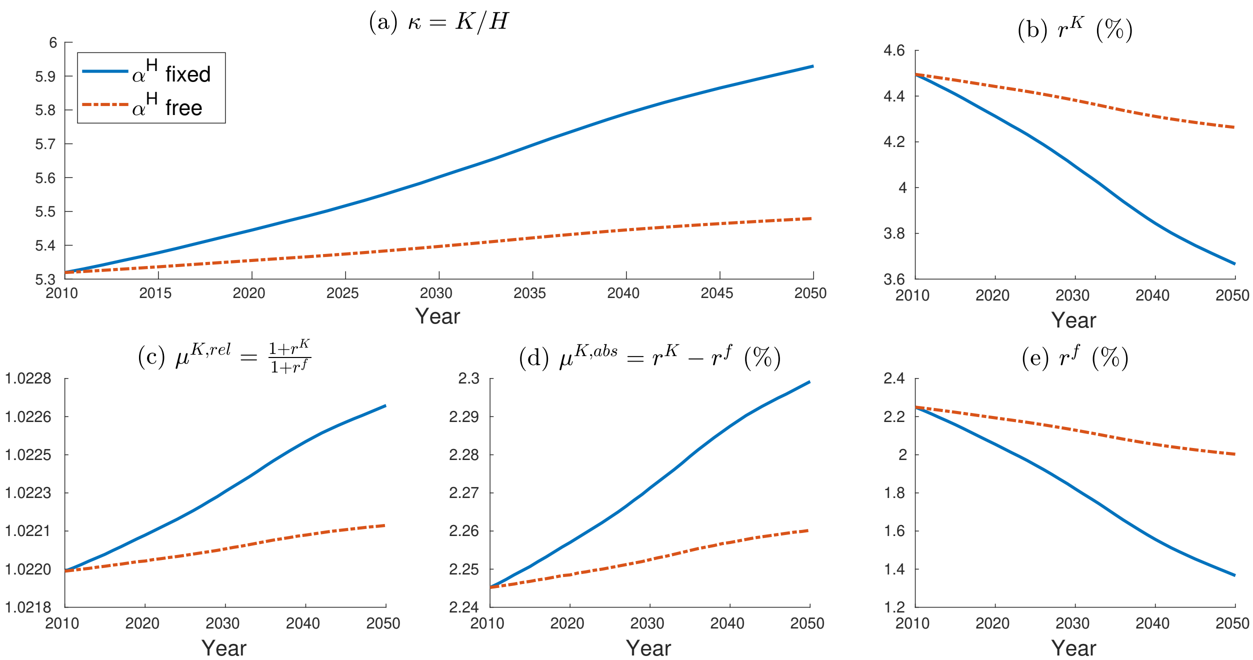
Following the discussion of the simple model I define also here capital intensity, $\kappa_t = \frac{K_t}{H_t}$, relative risk premium, $\mu_t^{K,rel} = \frac{1+r_t^K}{1+r^f}$ and absolute risk premium $\mu_t^{K,abs} = r_t^K - r_t^f$.

Figure 7 shows how capital intensity, relative and absolute risk premia, risky and risk-free rates evolve along the transition, from 2010 to 2050. It is the equivalent of figures 1 and 8 of the simple model. As in the simple model dynamics for the two scenarios, fixed and free, are shown.

As discussed in section 5.3, demographic change puts upward pressure on capital intensity, which leads to a decline in the risky and risk-free rate as well as an increase in the relative risk-premium. This is in line with the predictions of the simple model. However, in contrast to the simple model the quantitative model predicts an increase in the absolute risk premium. In the discussion of the simple model, I pointed out that the directional effect of demographic change on the absolute risk premium is not definite, it can be decreasing or increasing. In the quantitative model we observe an increase.

Comparing the two scenarios, fixed and free, we observe the same two characteristics as in the simple model. First, qualitatively the effects are the same - rising capital intensity, falling interest rates and increasing risk premium (relative and absolute). Second, the magnitude of the changes in the fixed scenario are significantly larger than in the free scenario. Human capital adjustments mitigate a large share of the effects from demographic change. In the examined horizon of 2010 to 2050, the fixed scenario predicts the risky rate to decline by 0.83 percentage points (from 4.50% to 3.67%) and the risk-free rate by 0.88 percentage points (from 2.25% to 1.37%). The respective declines in the free scenarios only account to 0.23% and 0.25%. Increases in the risk premium are modest in both scenarios. Absolute risk premium increases by 5.4 basis points (from 2.25% to 2.30%) in the fixed scenario and only

Figure 7: Capital Intensity, Interest Rates and Risk Premia



Notes: The figure displays: (a) capital intensity, κ, (b) risky return (on physical capital), (c) relative risk premium, $\mu^{K,rel}$, (d) absolute risk premium, $\mu^{K,abs}$, and (e) risk-free rate (on bonds) in equilibrium as predicted by the quantitative model over the horizon 2010 to 2050. κ and $\mu^{K,rel}$ are reported in absolute terms, r^K, $\mu^{K,abs}$ and r^f in percent. The quantities are shown for two scenarios (i) keeping the human capital share, α^H, constant and (ii) allowing it to vary according to equation (36b). The fixed human capital share is set equal to the human capital share in the scenario with a free human capital share of the year 2010.

1.5 basis points in the free scenario. As discussed within the simple model context in section 2.2, the finding of mild changes in the risk premium is no surprise.

6 Conclusion

In this paper I explore how demographic change affects the dynamics of interest rates and risk premia. I develop a theoretical and quantitative overlapping generations model with a production economy that features portfolio choice in a risk-free (bonds) and risky financial asset (physical capital). Furthermore, households can endogenously alter their labor supply through the adjustment of risky human capital accumulation. The modeling choices allow for closed form solutions at the household level and thus clear identification of mechanisms as well as complex dynamics without incurring large computational costs.

Demographic change leads to a decrease in population growth and an increase in life-expectancy. Both dynamics lead to an increase in the population size of older cohorts relative to younger ones in the economy. I show that this leads to an unambiguous increase in (physical) capital intensity in the economy, which in turn leads to lower interest rates, both risky and risk-free. Within this setup the relative risk-premium - the ratio of risky to risk-free gross returns - also increases unambiguously, implying a stronger decrease in the risk-free rate compared to the risky one. The dynamics for the more conventional absolute risk premium - the difference between risky and risk-free rates - are not that explicit. Theory shows that they can increase or decrease in the face of demographic change. However, the quantitative model predicts that also the absolute risk premium will increase in the coming decades.

I identify two driving forces through which demographic change affects interest rates and risk premia, (i) shifts in the relative population size of older cohorts relative to younger ones - measured through the old age dependency ratio (OADR), and (ii) shifts in the relative average wealth of older households relative to younger ones - measured through what I call old age wealth ratio (OAWR). Either of the two types of shifts increases the overall wealth that older generations hold in the economy relative to younger ones. Older household hold relatively more financial assets (= physical capital) than human capital compared to younger ones, and this puts a downward pressure on interest rates. In a similar fashion, older households hold relatively more risk-free assets compared to younger ones, and thus put an upward pressure on the (relative) risk premium. In the quantitative model I find that both OADR and OAWR increase until around 2030. However, from 2035 onward OAWR starts decreasing. As the increase in OADR outpaces the decrease in OAWR, the overall effect is still a decrease in interest rates and increase in risk premia. In the period from

2010 to 2050 the model predicts a decrease in the risk-free rate and risky rate of 0.88% and 0.83%, respectively. The (absolute) risk premium, measured as the difference between expected risky returns and risk-free returns, is predicted to increase by a mild 5.4 basis points. Taking human capital accumulation reactions into account mitigates a large fraction of interest rate swings, and I find that absolute changes in above figures (risk-free rate, risky rate and risk premium) reduce to 0.25%, 0.23% and 1.5 basis points.

My findings are within an economy with a net-zero supply of risk-free assets and without a social security system in place. As discussed in the theoretical section of this paper I identify two reasons why this may play an important role for both the qualitative and quantitative results. First, positive bond supply and social security benefits act as surrogate bonds and allow for demographic change to affect interest rates directly, and not just through altering capital intensity. Through these surrogate bonds, demographic change may have in particular a stronger impact on risk premia. Second, throughout industrialized countries we observe a shift in social security systems from defined benefit to defined contribution systems as well as an increase in sovereign debt. These shifts themselves affect interest rate and risk premium dynamics and should be investigated together with demographic dynamics.

As laid out in the modeling section of the quantitative model, a social security setup is already in place. However, due to the structure of the model, it features certain linear dynamics, which complicate calibration and equilibrium finding in a setup with a realistic social security system (see section D.1.1 in the appendix). Getting the model up and running with a sizable social security system is part of my ongoing research.

A Three-Period Model

A.1 Optimal Household Policy

Household problem. Combining equations (1) and (2) the households optimization problem can be stated as

$$\max_{m_{t,0}, m_{t+1,1}, \hat{\alpha}^K_{t+1,1}, \hat{\alpha}^H_{t+1,1}, \alpha^K_{t+2,2}} \log(c_{t,0}) + \mathbb{E}_t \left[\beta \log(c_{t+1,1}) + \beta^2 \varsigma \log(c_{t+2,2}) \right],$$

where

$$\begin{aligned}
c_{t,0} &= m_{t,0} \cdot w_0 \\
c_{t+1,1} &= m_{t+1,1} \cdot (1 + \hat{r}_{t+1,1}) \cdot \underbrace{(1 - m_{t,0}) w_0}_{\hat{w}_{t+1,1}} \\
c_{t+2,2} &= (1 + r_{t+2,2}) \cdot \underbrace{(1 - m_{t+1,1})(1 + \hat{r}_{t+1,1})(1 - m_{t,0}) w_0}_{w_{t+2,2}},
\end{aligned}$$

and

$$\begin{aligned}
\hat{r}_{t+1,1} + \varepsilon_{t+1,1} &= r^f_{t+1} + \hat{\alpha}^K_{t+1,1} \cdot (r^K_{t+1} + \zeta - r^f_{t+1}) + \hat{\alpha}^H_{t+1,1} \cdot (r^H_{t+1} + \eta - 1 - r^f_{t+1}) \\
r_{t+2,2} + \varepsilon_{t+2,2} &= r^f_{t+2} + \alpha^K_{t+2,2} \cdot (r^K_{t+2} + \zeta - r^f_{t+2}).
\end{aligned}$$

Marginal propensity to consume. The FOCs with respect to $m_{t,0}$ and $m_{t+1,1}$ yield

$$\begin{aligned}
\frac{1}{m_{t,0}} - \mathbb{E}_t \left[\frac{\beta}{1 - m_{t,0}} \right] - \mathbb{E}_{t+1} \left[\frac{\beta^2 \varsigma}{1 - m_{t,0}} \right] &= 0 \\
\frac{\beta}{m_{t+1,1}} - \mathbb{E}_{t+1} \left[\frac{\beta^2 \varsigma}{1 - m_{t+1,1}} \right] &= 0.
\end{aligned}$$

Noting that deterministic and time independent $m_{t,0} = m_0$ and $m_{t+1,1} = m_1$ solve above equations, we get optimal m_0 and m_1 as in equation (3).

Portfolio shares. The FOC with respect to $\hat{\alpha}^K_{t+1,1}$ yields

$$\frac{\partial \, \mathbb{E}_t [\log(1 + \hat{r}_{t+1,1} + \varepsilon_{t+1,1})]}{\partial \, \hat{\alpha}^K_{t+1,1}} = 0,$$

As discussed in Campbell and Viceira (2002, ch. 2) the portfolio return - which is the weighted average of multiple log-normally distributed returns - can be approximated by its continues time equivalent, namely by

$$
\begin{aligned}
\log(1+\hat{r}_{t+1,1}+\varepsilon_{t+1,1}) \approx & \log(1+r^f_{t+1}) + \hat{\alpha}^K_{t+1,1}\cdot\log\left(\frac{1+r^K_{t+1}+\zeta}{1+r^f_{t+1}}\right) + \hat{\alpha}^H_{t+1,1}\cdot\log\left(\frac{1+r^H_{t+1}+\eta}{1+r^f_{t+1}}\right)\\
& + \frac{1}{2}\hat{\alpha}^K_{t+1,1}(1-\hat{\alpha}^K_{t+1,1})Var(\log(1+r^K_{t+1}))\\
& + \frac{1}{2}\hat{\alpha}^H_{t+1,1}(1-\hat{\alpha}^H_{t+1,1})Var(\log(1+r^H_{t+1})).\\
= & \log(1+r^f_{t+1}) + \hat{\alpha}^K_{t+1,1}\cdot\log\left(\frac{1+r^K_{t+1}+\zeta}{1+r^f_{t+1}}\right) + \hat{\alpha}^H_{t+1,1}\cdot\log\left(\frac{1+r^H_{t+1}+\eta}{1+r^f_{t+1}}\right)\\
& + \frac{1}{2}\hat{\alpha}^K_{t+1,1}(1-\hat{\alpha}^K_{t+1,1})\sigma^2_\zeta + \frac{1}{2}\hat{\alpha}^H_{t+1,1}(1-\hat{\alpha}^H_{t+1,1})\sigma^2_\eta.
\end{aligned}
$$

As the time interval shrinks, the approximation error shrinks, and in continuous time above approximation is exact. Above equation implies that the expectation of the log gross portfolio return can be approximated by

$$
\begin{aligned}
\mathbb{E}_t[\log(1+\hat{r}_{t+1,1}+\varepsilon_{t+1,1})] \approx & \log(1+r^f_{t+1}) + \hat{\alpha}^K_{t+1,1}\cdot\mathbb{E}\left[\log\left(\frac{1+r^K_{t+1}+\zeta}{1+r^f_{t+1}}\right)\right] + \hat{\alpha}^H_{t+1,1}\cdot\mathbb{E}\left[\log\left(\frac{1+r^H_{t+1}+\eta}{1+r^f_{t+1}}\right)\right]\\
& + \frac{1}{2}\hat{\alpha}^K_{t+1,1}(1-\hat{\alpha}^K_{t+1,1})\sigma^2_\zeta + \frac{1}{2}\hat{\alpha}^H_{t+1,1}(1-\hat{\alpha}^H_{t+1,1})\sigma^2_\eta\\
= & \log(1+r^f_{t+1}) + \hat{\alpha}^K_{t+1,1}\cdot\mathbb{E}\left[\log\left(\frac{1+r^K_{t+1}}{1+r^f_{t+1}}\right)\right] + \hat{\alpha}^H_{t+1,1}\cdot\mathbb{E}\left[\log\left(\frac{1+r^H_{t+1}}{1+r^f_{t+1}}\right)\right]\\
& - \frac{1}{2}\left(\hat{\alpha}^K_{t+1,1}\right)^2\sigma^2_\zeta - \frac{1}{2}\left(\hat{\alpha}^H_{t+1,1}\right)^2\sigma^2_\eta,
\end{aligned}
\tag{49}
$$

where I used the fact that the distributional assumption on physical and human capital returns imply that $\mathbb{E}_t\left[\log(1+r^K_{t+1}+\zeta)\right] = \log(1+r^K_{t+1}) - \sigma^2_\zeta/2$ and $\mathbb{E}_t\left[\log(1+r^H_{t+1}+\eta)\right] = \log(1+r^H_{t+1}) - \sigma^2_\eta/2$.

Hence

$$
\frac{\partial\,\mathbb{E}_t[\log(1+\hat{r}_{t+1,1}+\varepsilon_{t+1,1})]}{\partial\,\hat{\alpha}^K_{t+1,1}} = \mathbb{E}\left[\log\left(\frac{1+r^K_{t+1}}{1+r^f_{t+1}}\right)\right] - \hat{\alpha}^K_{t+1,1}\cdot\sigma^2_\zeta.
$$

Following the FOC and setting this to zero yields the optimal physical capital share as in equation (4a). Note that above equation does not contain any cohort specific variables besides the portfolio share, such that $\hat{\alpha}^K_{t+1,1} = \hat{\alpha}^K_{t+2,2}$. Taking the derivative of equation (49) with respect to $\hat{\alpha}^H_{t+1,1}$ and setting it to zero, yields the optimal human capital share as in equation (4b).

A.2 Balanced Growth Path

Physical capital intensity is given as

$$\begin{aligned}
\kappa_t &= \frac{K_t}{H_t} \\
&= \frac{N_{t-1,0} \cdot k_{t,1} + N_{t-2,0} \cdot k_{t,2}}{N_{t-1,0} \cdot h_{t,1}} \\
&= \frac{k_{t,1}}{h_{t,1}} + \frac{1}{1+g^N} \frac{k_{t,2}}{h_{t,1}} \\
&= \frac{\hat{\alpha}^K_{t,1}}{\alpha^H_{t,1}} + \frac{1}{1+g^N} \cdot \frac{\alpha^K_{t,2} \cdot (1-m_{t-1,1})(1+\hat{r}_{t-1,1}) \cdot \hat{w}_{t-1,1}}{\alpha^H_{t,1} \cdot \hat{w}_{t,1}}
\end{aligned}$$

In steady state this simplifies to equation (8):

$$\kappa = \frac{\alpha^K}{\alpha^H} \cdot \left(1 + \frac{(1-m_1)(1+\hat{r}_1)}{1+g^N}\right).$$

Net-zero bond condition (7) implies that

$$\begin{aligned}
& N_{t-1,0} \cdot b_{t,1} + N_{t-2,0} \cdot b_{t,2} = 0 \\
\Leftrightarrow\ & (1+g^N) \cdot b_{t,1} + b_{t,2} = 0 \\
\Leftrightarrow\ & (1+g^N) \cdot \left(1 - \hat{\alpha}^K_{t,1} - \hat{\alpha}^H_{t,1}\right) + \left(1 - \alpha^K_{t,2}\right) \cdot (1-m_{t-1,1})(1+\hat{r}_{t-1,1}) = 0.
\end{aligned}$$

In steady state this simplifies to

$$(1+g^N) \cdot \left(1 - \alpha^K - \alpha^H\right) + \left(1 - \alpha^K\right) \cdot (1-m_1)(1+\hat{r}_1) = 0.$$

Rearranging this equation yields equation (7).

Let

$$\mu^{H,rel} = \frac{1+r^H}{1+r^f}$$

denote the relative human capital risk premium. Then, by dividing the gross market rate on human capital by the rate on physical capital we get

$$\begin{aligned}
& \frac{1+r^H}{1+r^K} = \frac{\mu^{H,rel}}{\mu^{K,rel}} = \frac{1-\alpha}{\alpha} \cdot \kappa \\
\Leftrightarrow\ & \mu^{H,rel} = \mu^{K,rel} \cdot \frac{1-\alpha}{\alpha} \cdot \kappa \qquad (50)
\end{aligned}$$

Using equations (13) and (50) the portfolio shares can be expressed as

$$\alpha^K = \frac{\log(\mu^K)}{\sigma_\zeta^2} = \frac{\kappa}{1+\kappa} \tag{51}$$

$$\alpha^H = \frac{\log(\mu^H)}{\sigma_\eta^2} = \log\left(\frac{1-\alpha}{\alpha}\kappa\right) \cdot \frac{1}{\sigma_\eta^2} + \frac{\kappa}{1+\kappa} \cdot \frac{\sigma_\zeta^2}{\sigma_\eta^2}. \tag{52}$$

Using $1 + r^f = \frac{1+r^K}{\mu^K}$ the deterministic part of the steady state gross portfolio return of working age households can be written as

$$1 + \hat{r}_1 = \frac{1+r^K}{\mu^K} \cdot \left(1 - \alpha^K - \alpha^H\right) + (1+r^K)\alpha^K + (1+r^H)\alpha^H.$$

Plugging this into equation (8) and replacing $\alpha^K, (1+r^K)$ and $1+r^H$ by equations (51), (5) and (6) we get an equilibrium condition in κ and α^H:

$$\kappa \left(\frac{\frac{(m_1-1)\kappa^{\alpha-1} e^{-\frac{\kappa\sigma^\zeta}{\kappa+1}} \left(\alpha\left(\kappa e^{\frac{\kappa\sigma^\zeta}{\kappa+1}}+1\right)\left(\kappa\alpha^H+\alpha^H-1\right)-\kappa(\kappa+1)\alpha^H e^{\frac{\kappa\sigma^\zeta}{\kappa+1}}\right)}{(\kappa+1)(g^N+1)} + 1}{(\kappa+1)\alpha^H} - 1 \right) \stackrel{!}{=} 0 \tag{53}$$

This constitutes the equilibrium condition in the scenario with a fixed human capital share. To arrive at the equilibrium condition in the scenario with a free human capital share, replace α^H in above equation by its optimal policy, equation (52), yielding

$$\begin{aligned}
&\frac{\frac{(m_1-1)\kappa^{\alpha} e^{-\frac{\kappa\sigma_\zeta^2}{\kappa+1}} \left(\kappa e^{\frac{\kappa\sigma_\zeta^2}{\kappa+1}}\left((\alpha-1)\Gamma(\kappa)-\alpha\sigma_\eta^2\right)+\alpha\left(\Gamma(\kappa)-\sigma_\eta^2\right)\right)}{(\kappa+1)(g^N+1)} + \kappa\sigma_\eta^2}{\Gamma(\kappa)} \stackrel{!}{=} \kappa \\
\Leftrightarrow\ &\frac{\frac{(m_1-1)\kappa^{1+\alpha} \left(\alpha\left(\Gamma(\kappa)-\sigma_\eta^2\right)\cdot\left(1+\kappa^{-1}e^{-\frac{\kappa\sigma_\zeta^2}{\kappa+1}}\right)-\Gamma(\kappa)\right)}{(\kappa+1)(g^N+1)} + \kappa\sigma_\eta^2}{\Gamma(\kappa)} \stackrel{!}{=} \kappa \\
\Leftrightarrow\ &\frac{(1-m_1)\,\kappa^{1+\alpha}\left(1+\alpha\left(\frac{\sigma_\eta^2}{\Gamma(\kappa)}-1\right)\cdot\left(1+\kappa^{-1}e^{-\frac{\kappa\sigma_\zeta^2}{\kappa+1}}\right)\right)}{(\kappa+1)\left(g^N+1\right)} + \frac{\kappa}{\Gamma(\kappa)}\sigma_\eta^2 \stackrel{!}{=} \kappa,
\end{aligned} \tag{54}$$

where $\Gamma(\kappa)$ is defined as

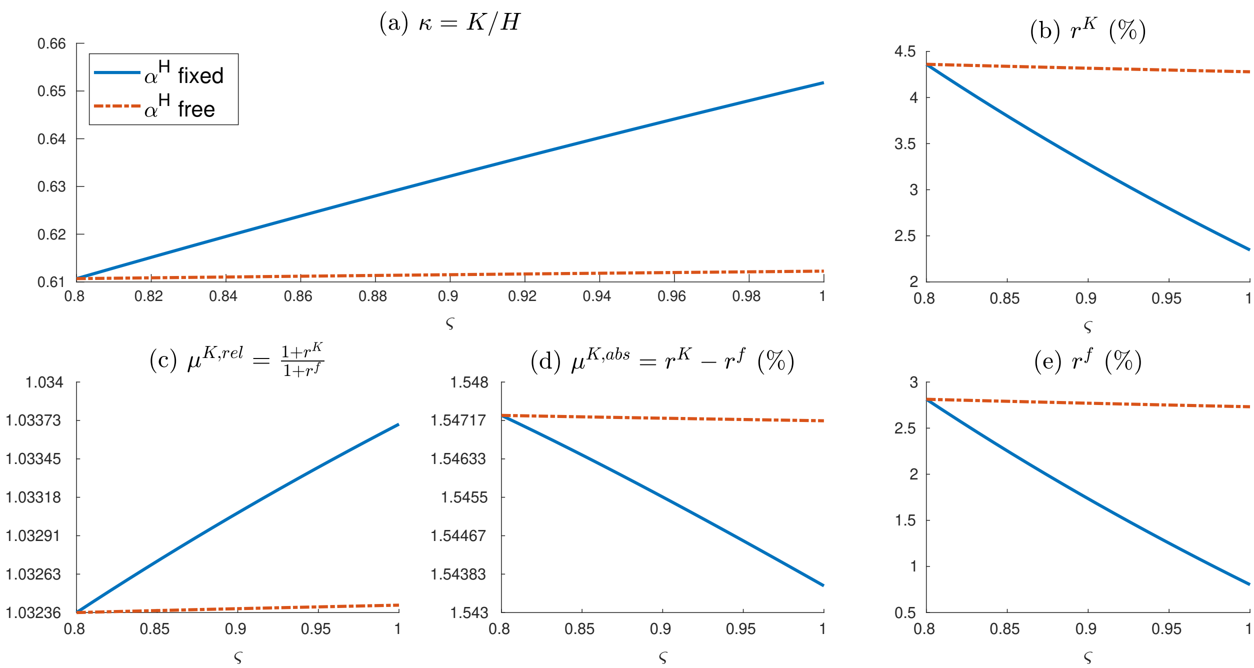
$$\Gamma(\kappa) = (1+\kappa)\cdot\log\left(\frac{1-\alpha}{\alpha}\kappa\right) + \kappa\sigma_\zeta^2. \tag{55}$$

Conjecture 1 (Existence, uniqueness and comparative statics)**.** *Let κ^{Fix} and κ^{Free} denote an equilibrium with a fixed and a free human capital share, respectively. That is κ^{Fix} solves equation* (53) *and κ^{Free} solves equation* (54)*. It holds that*

(a) $\kappa^{Fix} \in \mathbb{R}^+$ and $\kappa^{Free} \in \mathbb{R}^+$ exist and are unique.

(b) κ^{Fix} and κ^{Free} decrease in g^N and m_1.

Figure 8: Capital intensity, interest rates and risk premia as functions of life-expectancy

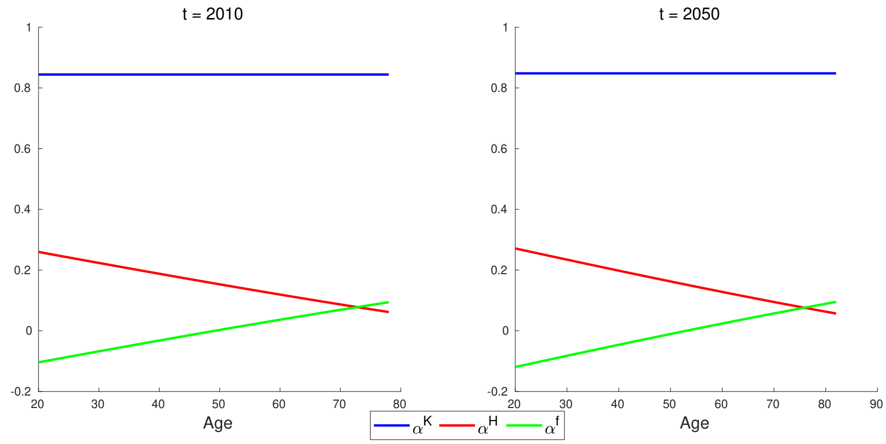


Notes: The figure displays: (a) capital intensity, κ, (b) risky return (on physical capital), (c) relative risk premium, $\mu^{K,rel}$, (d) absolute risk premium, $\mu^{K,abs}$, and (e) risk-free rate (on bonds) in equilibrium as a function of life-expectancy ς. κ and $\mu^{K,rel}$ are reported in absolute terms, $r^K, \mu^{K,abs}$ and r^f in percent. The equilibria are shown for (i) keeping the human capital share, α^H, constant and (ii) allowing it to vary according to $\alpha^H = \log((1 + r^H)/(1 + r^f))/\sigma^2_\eta$. The fixed human capital share is set equal to the human capital share in the equilibrium with a free human capital share and $\varsigma = 0.8$. The parameterization in all equilibria is $\alpha = 0.36, \beta = 0.95, \sigma^2_\eta = 0.15, \sigma^2_\varsigma = 0.084$.

B Quantitative Model - Further Figures

Figure 9 shows the cross-sectional life-cycle profiles of portfolio shares in total wealth (including human capital) in 2010 and 2050. It is hard to see with the naked eye, but the human and physical capital share profiles moves up, and the profile in risk-free bonds down from 2010 to 2050.

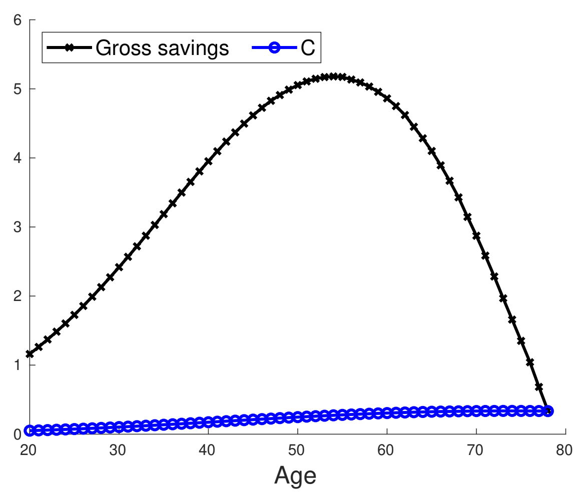
Figure 9: Cross-Sectional Profiles of portfolio shares in total wealth in 2010 and 2050



Notes: The figure shows the age profiles of portfolio shares in total wealth in the years 2010, and 2050 in the quantitative model.

Figure 10 shows the cross-sectional life-cycle profiles of wealth cum interest (gross savings) and of consumption in the year 2010.

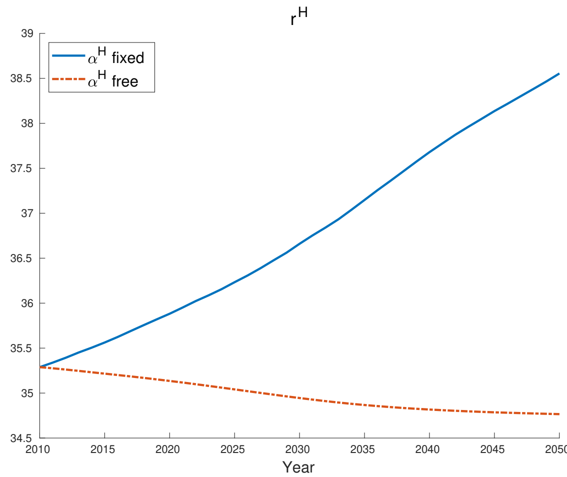
Figure 10: Cross-Sectional gross savings and consumption profiles in 2010



Notes: The figure shows the age profiles of wealth cum interest ($X_{2010,j}$) and consumption ($C_{2010,j}$) in the year 2010 in the quantitative model.

Figure 11 shows the dynamics of average returns on human capital at the household level. That is including age-specific depreciation rates. In the fixed scenario average individual returns decrease with time. This is due to the increasing average age of the work force, which thus exhibits a stronger depreciation of human capital. The market rate on human capital (25) is slightly increasing within the same time horizon (not reported here). The higher depreciation, however, more than counterbalances this slight increase.

Figure 11: Return on human capital at the household level from 2010 to 2050



Notes: The figure displays return on human capital in equilibrium as predicted by the quantitative model over the horizon 2010 to 2050. Return on human capital is reported as average human capital at the household level. That is it takes into account age-dependent human capital depreciation. The average is computed by weighting age-specific returns by human capital holdings of the respective cohort. Returns are reported in percent. The quantities are shown for two scenarios (i) keeping the human capital share, α^H, constant and (ii) allowing it to vary according to equation (36b). The fixed human capital share is set equal to the human capital share in the scenario with a free human capital share of the year 2010.

C Quantitative Model - Theoretical Appendix

C.1 Sub-period life expectancy

As sub-period life expectancy I understand fractional final years. E.g. a life expectancy of 77.4 has a 0.4 sub-period in the final period. In order to capture that policy rules of an individual born at period t are constructed as a weighted average of the policy rules of an individual with maximum age $77 - 20 = 57$ and another with maximum age $78 - 20 = 58$. The weight ψ_t in this example is 0.4. Hence $m_{j,t} = \psi_{t-j} \cdot m^1_{t,j} + (1 - \psi_{t-j})m^2_{t,j}$, $\hat{\alpha}^{K\prime}_{j,t} = \psi_{t-j} \cdot \hat{\alpha}^{K\prime,1}_{t,j} + (1 - \psi_{t-j})\hat{\alpha}^{K\prime,2}_{t,j}$ and $\hat{\alpha}^{H\prime}_{j,t} = \psi_{t-j} \cdot \hat{\alpha}^{H\prime,1}_{t,j} + (1 - \psi_{t-j})\hat{\alpha}^{H\prime,2}_{t,j}$, where $m^1, \hat{\alpha}^{K\prime,1}, \hat{\alpha}^{H\prime,1}$ are the policy functions of an individual born at t with maximum age J_t and $m^2, \hat{\alpha}^{K\prime,2}, \hat{\alpha}^{H\prime,2}$ the policy functions of an individual born at t with maximum age $J_t + 1$.

C.2 Solution of the Household Problem

Proof of Proposition 1. In the following let $\gamma := \frac{1-\theta}{1-\frac{1}{\xi}}$.

Modified household problem. I guess that $v = m^l \cdot \tilde{x}$ where l is some parameter to be determined below and m is the deterministic marginal propensity to consume out of $\tilde{x}$ and show below that this is indeed true. From the guess it follows that

$$
\begin{aligned}
v = & \max_{\tilde{c},\tilde{x}',\widehat{\alpha}^{s\prime},\widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta} \cdot (\mathbb{E}[(m'^{l} \cdot \tilde{x}')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \text{ s.t. } \tilde{x}' = \frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}) \cdot (1 + \widehat{r}') \\
= & \max_{\tilde{c},\widehat{\alpha}^{s\prime},\widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot m'^{l \cdot \frac{1-\theta}{\gamma}} \cdot (\mathbb{E}[(1 + \widehat{r}' + \varepsilon')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \qquad (56)
\end{aligned}
$$

Next, we compute the first-order conditions (FOCs) with respect to $\tilde{c}, \widehat{\alpha}^{k\prime}, \widehat{\alpha}^{h\prime}$:

Consumption. The FOC of (56) with respect to consumption yields

$$
\begin{aligned}
0 = & \frac{\gamma}{1-\theta} \cdot \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot m'^{l \cdot \frac{1-\theta}{\gamma}} \cdot (\mathbb{E}[(1 + \widehat{r}' + \varepsilon')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}-1} \\
& \cdot \{\frac{1-\theta}{\gamma} \cdot \tilde{c}^{\frac{1-\theta-\gamma}{\gamma}} - \frac{1-\theta}{\gamma \cdot (1+g)} \cdot (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta-\gamma}{\gamma}} \cdot \widehat{\beta} \cdot m'^{l \cdot \frac{1-\theta}{\gamma}} \cdot (\mathbb{E}[(1 + \widehat{r}' + \varepsilon')^{1-\theta}])^{\frac{1}{\gamma}}\} \\
\tilde{c} = & (\tilde{x} - \tilde{c}) \cdot (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m'^{l \cdot \frac{1-\theta}{1-\theta-\gamma}} \cdot (\mathbb{E}[(1 + \widehat{r}' + \varepsilon')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}
\end{aligned}
$$

Defining $n := \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m'^{l \cdot \frac{1-\theta}{1-\theta-\gamma}} \cdot (\mathbb{E}[(1 + \widehat{r}' + \varepsilon')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}$, $o := (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}}$, and $m := \frac{o \cdot n}{1+o \cdot n}$, where m is a deterministic function of age and time only, we get

$$
\tilde{c} = m \cdot \tilde{x}.
$$

Portfolio shares. As discussed in Campbell and Viceira (2002, ch. 2) the portfolio return - which is the weighted average of multiple log-normally distributed returns - can be approximated itself by a log-normal distribution, where

$$
\begin{aligned}
\log(1 + \hat{r}) \approx & \log(1 + r^f) + \hat{\alpha}^K(\log(1 + r^K + \zeta) - \log(1 + r^f)) + \hat{\alpha}^H(\log(1 + \hat{r}^H + \eta) - \log(1 + r^f)) \\
& + \frac{1}{2}\hat{\alpha}^K(1 - \hat{\alpha}^K)Var(\log(1 + r^K + \zeta)) + \frac{1}{2}\hat{\alpha}^H(1 - \hat{\alpha}^H)Var(\log(1 + \hat{r}^H + \eta)).
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
\mathbb{E}[\log(1 + \widehat{r})] \approx & \ (1 - \hat{\alpha}^K - \hat{\alpha}^H)\log(1 + r^f) + \hat{\alpha}^K\mathbb{E}[\log(1 + r^K + \zeta)] + \hat{\alpha}^H\mathbb{E}[\log(1 + \hat{r}^H + \eta)] \\
& + \frac{1}{2}\hat{\alpha}^K(1 - \hat{\alpha}^K)Var(\log(1 + r^K + \zeta)) + \frac{1}{2}\hat{\alpha}^H(1 - \hat{\alpha}^H)Var(\log(1 + \hat{r}^H + \eta)) \\
= & \ (1 - \hat{\alpha}^K - \hat{\alpha}^H)\log(1 + r^f) + \hat{\alpha}^K(\log(1 + r^K)) + \hat{\alpha}^H(\log(1 + \hat{r}^H)) \\
& - \frac{1}{2}\left(\hat{\alpha}^K\right)^2 Var(\log(1 + r^K + \zeta)) - \frac{1}{2}\left(\hat{\alpha}^H\right)^2 Var(\log(1 + \hat{r}^H + \eta)),
\end{aligned}
$$

where I used the fact that the distributional assumption on physical and human capital returns imply that $\mathbb{E}\left[\log(1 + r^K + \zeta)\right] = \log(1 + r^K) - \frac{1}{2}Var(\log(1 + r^K + \zeta))$ and

$\mathbb{E}\left[\log(1+\hat{r}^H+\eta)\right] = \log(1+\hat{r}^H) - \frac{1}{2}Var(\log(1+\hat{r}^H+\eta))$. It also follows that

$$Var(\log(1+\widehat{r})) \approx (\hat{\alpha}^K)^2 Var(\log(1+r^K+\zeta)) + (\hat{\alpha}^H)^2 Var(\log(1+\hat{r}^H+\eta)).$$

Using the equations for expectation and variance of the log-portfolio-return we can derive

$$\begin{aligned}
\mathbb{E}[(1+\widehat{r})^{1-\theta}] &\approx \mathbb{E}[\exp((1-\theta)\log(1+\widehat{r}))] \\
&= \exp\left\{(1-\theta)\mathbb{E}[\log(1+\widehat{r})] + (1-\theta)^2\frac{Var(\log(1+\widehat{r}))}{2}\right\} \\
&= \exp\left\{(1-\theta)\cdot\Big((1-\hat{\alpha}^K-\hat{\alpha}^H)\log(1+r^f) + \hat{\alpha}^K(\log(1+r^K)) + \hat{\alpha}^H(\log(1+\hat{r}^H))\Big)\right. \\
&\qquad - (1-\theta)\cdot\left(\frac{\theta}{2}\left(\hat{\alpha}^K\right)^2 Var(\log(1+r^K+\zeta))\right. \\
&\qquad\qquad \left.\left. + \frac{\theta}{2}\left(\hat{\alpha}^H\right)^2 Var(\log(1+\hat{r}^H+\eta))\right)\right\}.
\end{aligned} \tag{57}$$

The FOC of (56) with respect to the physical capital portfolio share yields

$$\begin{aligned}
0 =& \frac{\gamma}{1-\theta}\cdot\{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g}\cdot(\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}}\cdot\widehat{\beta}\cdot m'^{l\cdot\frac{1-\theta}{\gamma}}\cdot(\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}-1} \\
&\cdot(\frac{1}{1+g}\cdot(\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}}\cdot\widehat{\beta}\cdot\frac{1}{\gamma}\cdot m'^{l\cdot\frac{1-\theta}{\gamma}}\cdot(\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{\gamma}-1} \\
&\cdot\frac{\partial\,\mathbb{E}[(1+\widehat{r}'+\varepsilon)^{1-\theta}]}{\partial\,\hat{\alpha}^{K\prime}} \\
\Leftrightarrow 0 =& \frac{\partial\,\mathbb{E}[(1+\widehat{r}'+\varepsilon)^{1-\theta}]}{\partial\,\hat{\alpha}^{K\prime}}.
\end{aligned}$$

Taking the derivative of equation (57) with respect to the physical capital portfolio share we get

$$0 = \Big(\log(1+r^K) - \log(1+r^f) - \theta\cdot\hat{\alpha}^K\cdot Var(\log(1+r^K+\zeta))\Big)\cdot(1-\theta)\cdot\exp\Big\{\cdot\Big\}.$$

Solving this equation for $\hat{\alpha}^K$ yields the portfolio share as in equation (36a). Following the same steps with the human capital share yields equation (36b).

Verify guess. What is left is to show that indeed $v = m^l \cdot \tilde{x}$. Using $\tilde{c} = m \cdot \tilde{x}$, $n = \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m'^{l \cdot \frac{1-\theta}{1-\theta-\gamma}} \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}$, $m = \frac{o \cdot n}{1+o \cdot n}$, and $o = (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}}$ in u we get:

$$
\begin{aligned}
v &= \left\{ (m \cdot \tilde{x})^{\frac{1-\theta}{\gamma}} + \left(\frac{1}{1+g} \cdot (\tilde{x} - m \cdot \tilde{x}) \right)^{\frac{1-\theta}{\gamma}} \cdot n^{\frac{1-\theta-\gamma}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}} \\
&= \tilde{x} \cdot \left\{ \frac{(o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}}{(1+o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}} \right\}^{\frac{\gamma}{1-\theta}} = \tilde{x} \cdot m^{\frac{1-\theta-\gamma}{1-\theta}}
\end{aligned}
$$

Hence, $v = m^l \cdot \tilde{x}$ where $l = \frac{1-\theta-\gamma}{1-\theta}$.

Defining $\wp := \mathbb{E}[(m'^{\frac{1-\theta-\gamma}{1-\theta}} \cdot (1+\widehat{r}'))^{1-\theta}]$, and noting that $n = \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m' \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}$ the marginal propensity to consume equals:

$$
\begin{aligned}
m = \frac{o \cdot n}{1 + o \cdot n} &= \frac{(\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m' \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}}{1 + (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m' \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}} \\
&= \frac{\beta^{\frac{\gamma}{1-\theta-\gamma}} \cdot m' \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}}{1 + \beta^{\frac{\gamma}{1-\theta-\gamma}} \cdot m' \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}} = \frac{(\beta^{\gamma} \cdot \mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}} \cdot m'}{1 + (\beta^{\gamma} \cdot \mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}} \cdot m'}
\end{aligned}
$$

□

C.3 Derivation of the Aggregate Resource Constraint

Proof of Lemma 1. Using the recursive pension equation (28) in the dynamic budget constraint (30) we get

$$
\begin{aligned}
&\tilde{b}_{t+1,j+1,i} + \tilde{k}_{t+1,j+1,i} + \frac{h_{t+1,j+1,i}}{1+g} = \\
&\frac{1}{1+g} \Big(\tilde{b}_{t,j,i}(1+r_t^f) + \tilde{k}_{t,j,i}(1+r_{t,j,i}^K) + h_{t,j,i} \left(1 + (1-\tau_t)\tilde{r}_t^H - \delta_j^H + \eta_{t,j,i}\right) + (1-\omega_{t,j})\tilde{p}_t - \tilde{c}_{t,j,i} \Big).
\end{aligned}
$$

Now, in order to derive the aggregate resource constraint, we take the population weighted sum of above individual budget constraints (pre-multiplied by $(1+g)$). Note that it is understood that we sum over all individuals of each age bin characterized by the idiosyncratic mean-zero shock η and idiosyncratic mean-one shock ζ without making this explicit.[15] We

[15] This is possible as the idiosyncratic shocks $\eta_{t,j,i}$ and $\zeta_{t,j,i}$ are independent of $\tilde{k}_{t,j,i}$ and $h_{t,j,i}$.

then get

$$
\begin{aligned}
(1+g)\cdot \sum_j N_{t,j}\cdot \widetilde{B}_{t+1,j+1} &+ (1+g)\cdot \sum_j N_{t,j}\cdot \widetilde{K}_{t+1,j+1} + \sum_j N_{t,j}\cdot H_{t+1,j+1} \\
&= \sum_j N_{t,j}\cdot \widetilde{B}_j(1+r^f_t) + \sum_j N_{t,j}\cdot \widetilde{K}_j(1+r^K_t) \\
&\quad + \sum_j N_{t,j}\cdot H_{t,j}\cdot \left(1+(1-\tau_t)\tilde{r}^h_t - \delta^h_j + \int \eta_{t,j,i}\ di\right) \\
&\quad + \sum_j N_{t,j}\cdot(1-\omega_{t,j})\tilde{p}_j - \sum_j N_{t,j}\cdot \widetilde{C}_j,
\end{aligned}
$$

where $\widetilde{B}_{t,j}, \widetilde{K}_{t,j}, H_{t,j}, \widetilde{C}_{t,j}$ are the generation j averages of $\tilde{b}_{t,j,i}, \tilde{k}_{t,j,i}, h_{t,j,i}, \widetilde{c}_{t,j,i}$, $r^K_t = \int r^K_{t,j,i}di$ and $\int \eta_{t,j,i}\ di = 0$ due to the law of large numbers. Noting that $H_t = \sum_j N_{t,j}\cdot H_{t,j}\cdot \omega_{t,j}$ we can rewrite above equation in terms of aggregates as

$$
\begin{aligned}
(1+g)\cdot \widetilde{B}_{t+1} &+ (1+g)\cdot \widetilde{K}_{t+1} + H_{t+1} - N_{t,0}\cdot(1+g)\cdot\left(\widetilde{B}_{t+1,0} + \widetilde{K}_{t+1,0} + \frac{H_{t+1,0}}{1+g}\right) \\
&= \ \widetilde{B}_t(1+r^f_t) + \widetilde{K}_t(1+r^k_t) + \sum_j N_{t,j}\cdot H_{t,j}(1-\delta^h_j) + L_t(1-\tau_t)\tilde{r}^h_t \\
&\quad + \sum_j N_{t,j}\cdot(1-\omega_{t,j})\tilde{p}_j - \widetilde{C}_t \\
\Leftrightarrow\ &\left[(1+g)\cdot \widetilde{B}_{t+1} - \widetilde{B}_t(1+r^f_t)\right] + \left[(1+g)\cdot\left(\widetilde{K}_{t+1} - N_{t,0}\cdot \widetilde{K}_{t+1,0}\right) - (1-\delta^k)\widetilde{K}_t\right] \\
&\quad + \left[H_{t+1} - N_{t,0}\cdot H_{t+1,0} - \sum_j N_{t,j}\cdot H_{t,j}(1-\delta^h_j)\right] - (1+g)\cdot N_{t,0}\cdot \widetilde{B}_{t+1,0} + \widetilde{C}_t \\
&= \ \left[\widetilde{K}_t(r^k_t+\delta^k) + L_t\tilde{r}^h_t\right] + \left[\sum_j N_{t,j}\cdot(1-\omega_{t,j})\tilde{p}_t - L_t\tilde{r}^h_t\tau_t\right]
\end{aligned}
\tag{58}
$$

The definition of the capital income function and equation (20) imply that $\tilde{k}_{t,j,i}(r^K_{t,j,i} + \delta^k) + h_{t,j,i}\tilde{r}^H_t = \tilde{y}_{t,j,i}$ and thus in the aggregate $\widetilde{K}_t(r^K_t + \delta^k) + H_t\tilde{r}^H_t = \widetilde{Y}_t$. Note, that $\sum_j N_{t,j}\cdot \widetilde{B}_{t+1,j+1} = \widetilde{B}_{t+1} - N_{t,0}\cdot \widetilde{B}_{t+1,0}$, $\sum_j N_{t,j}\cdot \widetilde{K}_{t+1,j+1} = \widetilde{K}_{t+1} - N_{t,0}\cdot \widetilde{K}_{t+1,0}$ and $\sum_j N_{t,j}\cdot H_{t+1,j+1} = H_{t+1} - N_{t,0}\cdot H_{t+1,0}$ as $\tilde{w}_0 > 0$. Using this together with the bond supply equation (38), pension system financing (39) and definitions for physical capital investment, $\widetilde{I}^K_t = (1+g)\cdot\left(\widetilde{K}_{t+1} - N_{t,0}\cdot \widetilde{K}_{t+1,0}\right) - (1-\delta^k)\widetilde{K}_t$, and human capital investment, $I^H_t = H_{t+1} - N_{t,0}\cdot H_{t+1,0} - \sum_j N_{t,j}\cdot H_{t,j}(1-\delta^h_j)$ in equation (58) yields the aggregate resource constraint

$$\widetilde{I}^K_t + I^H_t - (1+g)\cdot N_{t,0}\cdot \widetilde{B}_{t+1,0} + \widetilde{C}_t = \widetilde{Y}_t.$$

$\square$

D Quantitative Model - Computational Appendix

D.1 Solving for the equilibrium

Let $\Phi = \{\phi_t\}_{t=1}^T$ with $\phi_t = \{\tilde{r}_t^h, r_t^f\}$, $Q = \{Q_t\}_{t=1}^T$ with $Q_t = \{\widetilde{B}_t, \widetilde{K}_t, H_t\}$. Collect equations (38) and (42) and rearrange them so that the risk-free rate and the return on human capital are on the left-hand side. This gives us a two-dimensional equation in prices that I call institutional side, $\Phi = IS(Q) = \{IS_t(Q)\}_t$, where

$$\phi_t = IS_t(Q) = \begin{cases} r_t^f &= \frac{(1+g)\widetilde{B}_{t+1}}{\widetilde{B}_t} - 1 \\ \tilde{r}_t^h &= (1-\alpha)\psi\left(\frac{\widetilde{K}_t}{H_t}\right)^\alpha \mathbb{E}\left[\zeta^{\frac{1}{\alpha}}\right]^\alpha \end{cases} \tag{59}$$

The other side is the household side $Q = HS(P) = \{HS_t(P)\}_t$, where I collect equations (44), (46) and additionally the equation for aggregate physical capital holdings, yielding

$$Q_t = HS_t(\Phi) = \begin{cases} \widetilde{B}_{t+1} = \frac{1}{1+g}\sum_j N_{t,j} \cdot \tilde{X}_{t,j} \cdot (1-m_{t,j}) \cdot (1-\hat{\alpha}_{t,j}^{K\prime} - \hat{\alpha}_{t,j}^{H\prime}) - \sum_j N_{t+1,j}\widetilde{ps}_{t+1,j+1} \\ \widetilde{K}_{t+1} = \frac{1}{1+g}\sum_j N_{t,j} \cdot \tilde{X}_{t,j} \cdot (1-m_{t,j}) \cdot \hat{\alpha}_{t,j}^{K\prime} \\ H_{t+1} = \sum_j N_{t,j} \cdot \tilde{X}_{t,j} \cdot (1-m_{t,j}) \cdot \hat{\alpha}_{jt}^{H\prime} \cdot \omega_{t+1,j+1}, \end{cases} \tag{60}$$

where $\{m_{t,j}, \hat{\alpha}_{t,j}^K, \hat{\alpha}_{t,j}^H\}_{t,j}$ are given by the policy functions (35) and (36) and $\widetilde{ps}_{t,j}$ is recursively given by equations (28) and (29).

$\{\tilde{X}_{t,j}\}_{t,j}$ is determined by the aggregate law of motion (45) with starting values given by equation (47). In order to determine the starting values $\{\tilde{X}_{t,0}\}_t$, the initial pension stock $\widetilde{ps}_{t,0}$ needs to be determined, which is done using equation (27), which in its detrended form and with pension payments substituted, i.e. $\tilde{p}_t = \rho_t \cdot \frac{(1-\tau_t)\cdot\tilde{r}_t^H \cdot H_t}{\sum_j N_{t,j}\cdot\omega_{t,j}}$, is

$$\begin{aligned} \widetilde{ps}_{t,0} &= \sum_{s=0}^{J_{t-j}+1} (1-\omega_s)\tilde{p}_{t+s} \prod_{q=0}^{s}\left(1+r_{t+q}^f\right)^{-1} \\ &= \sum_{s=0}^{J_{t-j}+1} (1-\omega_s)\rho_{t+s} \cdot \frac{(1-\tau_{t+s})\cdot\tilde{r}_{t+s}^H \cdot H_{t+s}}{\sum_j N_{t,j}\cdot\omega_{t,j}} \prod_{q=0}^{s}\left(1+r_{t+q}^f\right)^{-1}. \end{aligned} \tag{61}$$

But $\{H_t\}_t$ is itself a function of $\widetilde{ps}_{t,0}$ as it depends on $\{\tilde{X}_{t,j}\}_{t,j}$, which itself depends on the initial pension stock. Hence we have to solve jointly for $\{H_t\}_t$ and $\{\widetilde{ps}_{t,0}\}_t$. We do this by a fixed point iteration iterating over equation (61) for the initial pension stock and the cash-on-hand accumulation equation (45), computing $\{H_t\}_t$ using equation (46b). Note that the household problem, i.e. determining $\{m_{t,j}\}_{t,j}$ and the portfolio shares, is independent of $\{\widetilde{ps}_{t,0}\}_t$. Thus, in the above fixed point iteration we do not have to recompute the household problem in every iteration. I only iterate over equations (61) and (45), given the values for $\{m_{t,j}\}_{t,j}$ and the portfolio shares. Instead of using equation (61) we can equivalently determine $\widetilde{ps}_{t,0}$ by using equation (28) recursively backwards together with the final condition (29).

The equilibrium is then given by the fixed point $\Phi^\star$ in

$$\Phi^\star = IS(HS(\Phi^\star)).$$

I determine $\Phi^\star$ by using the Gauss-Seidel algorithm with one-parameter dampening where each iteration n is given by

$$\Phi^{n+1} = \Phi^n - w(\Phi^n - IS(HS(\Phi^n))),$$

with w being the dampening factor.

Let steady state values be denoted by a bar. Furthermore, initial (final) steady state values are denoted by a superscript 1 (2). See below for a detailed description of the steady states. The economy is assumed to start form the initial steady state, i.e. $N_{0,j} = \bar{N}^1_j, \widetilde{X}_{0,j} = \bar{\widetilde{X}}^1_j$ and $\widetilde{B}_0 = \bar{\widetilde{B}}^1$. Additionally I assume prices and human capital after period T to be according to the final steady state, i.e. $\phi_t = \bar{\phi}^2$ for $t \in \{T+1, \ldots, T+J_T+1\}$.

D.1.1 Initial Pension Stock

The pension payments depend on the aggregate labor supply. But aggregate labor supply also depends on the pension payments, on the initial pension stock. Human capital holdings depend on the initial wealth, which depends on the initial pension stock. Hence, I have to jointly solve for the aggregate labor supply and the initial pension stock. As shown here, I get a closed form solution for this.

Iterating over the average cash-on-hand accumulation equation (33) we get cash-on-hand as a function of initial wealth:

$$\widetilde{W}_{t,j} = \frac{\prod_{s=1}^{j}(1 - m_{t-s,j-s}) \cdot (1 + \mathbb{E}_{\eta,\zeta}\left[\hat{r}_{t-s,j-s,i}\right])}{(1+g)^j} \widetilde{W}_{t-j,0},$$

where

$$\widetilde{W}_{t,0} = \frac{h_0}{1+g} + \widetilde{ps}_{t,0}.$$

is the average initial wealth level. Plugging this into the aggregate labor supply as given in equation (46b) we get

$$H_t = \sum_{j=0}^{J+1} N_{t,j} \cdot \frac{\prod_{s=1}^{j}(1-m_{t-s,j-s}) \cdot (1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{t-s,j-s,i}\right])}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^H \cdot \omega_{t,j} \cdot \left(\frac{h_0}{1+g} + \widetilde{ps}_{t-j,0}\right), \quad (62)$$

where I define $\prod_{s=1}^{j}(1-m_{t-s,j-s}) \cdot (1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{t-s,j-s,i}\right]) \equiv 1$. This gives us aggregate labor supply H_t as a function of the initial pension stock $\widetilde{ps}_{t-(J+1)}$ to $\widetilde{ps}_t$. Note that H_t is linear in the initial pension stocks.

Equation (27) gives us the initial pension stock as

$$\widetilde{ps}_{t,0} = \sum_{j=0}^{J_t+1} (1+g)^j \cdot (1-\omega_{t+j,j}) \tilde{p}_{t+j} \cdot \prod_{s=0}^{j} \left(1+r_{t+s}^f\right)^{-1}.$$

Pension market clearing implies that pension benefits are given by

$$\tilde{p}_t = \frac{\tau_t \cdot \tilde{r}_t^H \cdot H_t}{\sum_j N_{t,j}(1-\omega_{t,j})},$$

yielding

$$\widetilde{ps}_{t,0} = \sum_{j=0}^{J_t+1} (1+g)^j \cdot (1-\omega_{t+j,j}) \cdot \frac{\tau_{t+j}\tilde{r}_{t+j}^H}{\sum_s N_{t+j,s}(1-\omega_{t+j,s})} \cdot H_{t+j} \cdot \prod_{s=0}^{j} \left(1+r_{t+s}^f\right)^{-1} \quad (63)$$

Combining equations (62) and (63) we get the equilibrium quantities of H_t and $\widetilde{ps}_{t,0}$. Defining

$$\mathcal{A}_{t,j} = N_{t,j} \cdot \frac{\prod_{s=1}^{j}(1-m_{t-s,j-s}) \cdot (1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{t-s,j-s,i}\right])}{(1+g)^{j-1}} \cdot \hat{\alpha}_j^H \cdot \omega_{t,j}$$

$$\mathcal{B}_{t,j} = (1+g)^j \cdot (1-\omega_{t+j,j}) \cdot \frac{\tau_{t+j}\tilde{r}_{t+j}^H}{\sum_s N_{t+j,s}(1-\omega_{t+j,s})} \cdot \prod_{s=0}^{j} \left(1+r_{t+s}^f\right)^{-1}$$

the condition for labor supply is

$$L_t = \sum_{j=0}^{J+1} \mathcal{A}_{t,j} \cdot \frac{h_0}{1+g} + \sum_{j=0}^{J+1} \mathcal{A}_{t,j} \sum_{q=0}^{J_{t-j}} \mathcal{B}_{t-j,q} \cdot L_{t-j+q}. \tag{64}$$

Above equation gives us labor supply as a difference equation, where in general current labor supply depends both on past and future labor supply. If, once retired, households do not work anymore, current labor supply only depends on future labor supply.

Steady State: In steady state equations (62) and (63) simplify to

$$H = N \cdot \sum_{j=0}^{J+1} \frac{\prod_{s=1}^{j}(1-m_{j-s}) \cdot (1+\mathbb{E}_{\eta,\zeta}[\hat{r}_{j-s,i}])}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^H \cdot \omega_j \cdot \left(\frac{h_0}{1+g} + \widetilde{ps}_0\right) \tag{65}$$

$$\widetilde{ps}_0 = \frac{\tau \cdot \tilde{r}^H}{1+r^f} \cdot \frac{1}{N} \cdot \frac{\sum_{j=0}^{J+1}\left(\frac{1+g}{1+r^f}\right)^j (1-\omega_j)}{\sum_{j=0}^{J+1}(1-\omega_j)} \cdot H, \tag{66}$$

resulting in the equivalent of equation (64) giving us aggregate labor supply as

$$H = \frac{N \cdot \left(\sum_{j=0}^{J+1} \frac{\prod_{s=1}^{j}(1-m_{j-s})\cdot\left(1+\mathbb{E}_{\eta,\zeta}[\hat{r}_{j-s,i}]\right)}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^H \cdot \omega_j\right)}{1 - \left(\sum_{j=0}^{J+1} \frac{\prod_{s=1}^{j}(1-m_{j-s})\cdot\left(1+\mathbb{E}_{\eta,\zeta}[\hat{r}_{j-s,i}]\right)}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^H \cdot \omega_j\right) \cdot \frac{\tau \cdot \tilde{r}^H}{1+r^f} \cdot \frac{\sum_{j=0}^{J+1}\left(\frac{1+g}{1+r^f}\right)^j (1-\omega_j)}{\sum_{j=0}^{J+1}(1-\omega_j)}} \cdot \frac{h_0}{1+g}. \tag{67}$$

If the denominator is not positive, no steady state equilibrium exists.

With subperiod life-expectancy: As discussed in the appendix C.1 I model subperiod life-expectancy by having two types of households in the model with different life-expectancies. In the implementation of the model I have to take account of that, resulting in slightly different equations than above. Here only the equations for the steady state are reported. Let $\ell \in 1, 2$ denote the type of household.

The equivalent of equations (65) and (66) are

$$H = \sum_{\ell=1}^{2} \left\{ N^\ell \cdot \sum_{j=0}^{J^\ell+1} \frac{\prod_{s=1}^{j}(1-m_{j-s}^\ell) \cdot \left(1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{j-s,i}^\ell\right]\right)}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^{H,\ell} \cdot \omega_j \cdot \left(\frac{h_0}{1+g} + \widetilde{ps}_0^\ell\right) \right\}$$

$$\widetilde{ps}_0^\ell = \frac{\tau \cdot \tilde{r}^H}{1+r^f} \cdot \frac{\sum_{j=0}^{J^\ell+1}\left(\frac{1+g}{1+r^f}\right)^j (1-\omega_j)}{N^1 \sum_j^{J^1+1}(1-\omega_j) + N^2 \sum_j^{J^2+1}(1-\omega_j)} \cdot H,$$

where the second equation results from pension market clearing implying

$$\tilde{p}_t = \frac{\tau_t \cdot \tilde{r}_t^H \cdot H_t}{N^1 \sum_j^{J^1+1}(1-\omega_j) + N^2 \sum_j^{J^2+1}(1-\omega_j)}.$$

The equivalent of (67) is thus

$$H = \frac{\sum_{\ell=1}^2 \left(N^\ell \cdot \sum_{j=0}^{J^\ell+1} \frac{\prod_{s=1}^j (1-m_{j-s}^\ell)\cdot\left(1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{j-s,i}^\ell\right]\right)}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^{H,\ell} \cdot \omega_j \right)}{1 - \sum_{\ell=1}^2 \left\{ \left(N^\ell \cdot \sum_{j=0}^{J^\ell+1} \frac{\prod_{s=1}^j (1-m_{j-s}^\ell)\cdot\left(1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{j-s,i}^\ell\right]\right)}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^{H,\ell} \cdot \omega_j \right) \cdot \frac{\tau \cdot \tilde{r}^H}{1+r^f} \cdot \frac{\sum_{j=0}^{J^\ell+1}\left(\frac{1+g}{1+r^f}\right)^j (1-\omega_j)}{N^1 \sum_j^{J^1+1}(1-\omega_j)+N^2\sum_j^{J^2+1}(1-\omega_j)} \right\}} \cdot \frac{h_0}{1+g}.$$

D.1.2 Balanced Growth Path

A trend-stationary equilibrium is characterized by $N_{t,j} = \bar{N}_j$ $\phi_t = \bar{\phi} = \{\bar{\tilde{r}}^h, \bar{r}^f\}$, $Q_t = \bar{Q} = \{\bar{\tilde{B}}, \bar{\tilde{K}}, \bar{H}\}$, $\widetilde{X}_{t,j} = \bar{X}_j$, $m_{jt} = \bar{m}_j$, $\hat{\alpha}_{t,j}^K = \bar{\hat{\alpha}}_j^K$ and $\hat{\alpha}_{t,j}^H = \bar{\hat{\alpha}}_j^H$ for all t.[16] Trend-stationary analogues of equations (59) and (60) are

$$\bar{\phi} = IS(\bar{Q}) = \begin{cases} \bar{r}^f = g \\ \bar{\tilde{r}}^h = (1-\alpha)\psi \left(\frac{\bar{\tilde{K}}}{\bar{H}}\right)^\alpha \mathbb{E}\left[\zeta^{\frac{1}{\alpha}}\right]^\alpha \end{cases} \tag{68}$$

and

$$\bar{Q} = HS(\bar{\phi}) = \begin{cases} \bar{\tilde{B}} = \frac{1}{1+g} \sum_j \bar{N}_j \cdot \bar{\tilde{X}}_j \cdot (1-\bar{m}_j) \cdot (1 - \bar{\hat{\alpha}}_j^{K\prime} - \bar{\hat{\alpha}}_j^{H\prime}) - \sum_j N_j \widetilde{ps}_{j+1} \\ \bar{\tilde{K}} = \frac{1}{1+g} \sum_j \bar{N}_j \cdot \bar{\tilde{X}}_j \cdot (1-\bar{m}_j) \cdot \bar{\hat{\alpha}}_j^{K\prime} \\ \bar{H} = \sum_j \bar{N}_j \cdot \bar{\tilde{X}}_j \cdot (1-\bar{m}_j) \cdot \bar{\hat{\alpha}}_j^{H\prime} \cdot \bar{\omega}_{j+1}. \end{cases}$$

The steady state equilibrium is then given by

$$\bar{\phi}^\star = IS(HS(\bar{\phi}^\star)),$$

which is also solved by using the Gauss-Seidel algorithm.

The Initial steady state corresponds to constant fertility rates, $\bar{f}_j^1 = f_{0,j}$, life expectancy $\bar{J}^1 = J_0$, working time shares $\bar{\omega}_j^1 = \omega_{0,j}$ as of the initial period. The final steady state corresponds to these quantities as of the final period, i.e. $\bar{f}_j^2 = f_{T,j}$, $\bar{J}^2 = J_T$ and $\bar{\omega}_j^2 = \omega_{T,j}$.

[16] Furthermore the distribution of cash-on-hand holdings across households, $\{\tilde{x}_{t,j,i}\}$, is constant across time. But this distributions does not have any effect on aggregate prices and quantities.

D.2 Calibration

I calibrate the trend-stationary equilibrium. Along the balanced growth path the risk-free rate is pinned down by the growth rate and the moment condition on the capital-output ratio pins down the capital-labor ratio and thus the return on human capital. This can be seen by referring to equation (68) and by combining equations (42) and (43) yielding

$$\frac{\widetilde{K}_t}{H_t} = \left(\psi \mathbb{E}\left[\zeta^{\frac{1}{\alpha}}\right]^{\alpha} \frac{\widetilde{K}_t}{\widetilde{Y}_t}\right)^{\frac{1}{1-\alpha}}.$$

Hence, the moment conditions pin down the equilibrium prices as given in equation (68).

Given these prices I search for the parameter values such that the moment conditions are met. This entails computing the household model and aggregating resulting quantities. But I do not need to solve for the equilibrium, as I already set prices to equilibrium prices.

Two parameters, ψ and σ^2_{ζ}, can even be set without having to compute the household model. To see that we plug the market clearing condition for the return on human capital from equation (68) into the return on physical capital as given in equation (21) yielding

$$r^K_{t,j,i} = \alpha\psi\zeta^{\frac{1}{\alpha}}_{t,j,i}\mathbb{E}\left[\zeta^{\frac{1}{\alpha}}_{t,j,i}\right]^{\alpha-1}\left(\frac{\widetilde{K}}{\widetilde{H}}\right)^{\alpha-1} - \delta^K$$

Hence

$$\begin{aligned}\mathbb{E}\left[r^K_{t,j,i}\right] &= \alpha\psi\mathbb{E}\left[\zeta^{\frac{1}{\alpha}}_{t,j,i}\right]^{\alpha}\left(\frac{\widetilde{K}}{\widetilde{H}}\right)^{\alpha-1} - \delta^K \\ &= \alpha\frac{\widetilde{Y}}{\widetilde{K}} - \delta^K.\end{aligned}$$

and

$$\begin{aligned}Var\left(r^K_{t,j,i}\right) &= \left(\alpha\psi\mathbb{E}\left[\zeta^{\frac{1}{\alpha}}_{t,j,i}\right]^{\alpha-1}\left(\frac{\widetilde{K}}{\widetilde{H}}\right)^{\alpha-1}\right)^2 \cdot Var\left(\zeta^{\frac{1}{\alpha}}_{t,j,i}\right) \\ &= \left(\alpha\mathbb{E}\left[\zeta^{\frac{1}{\alpha}}_{t,j,i}\right]^{-1}\frac{\widetilde{Y}}{\widetilde{K}}\right)^2 \cdot Var\left(\zeta^{\frac{1}{\alpha}}_{t,j,i}\right).\end{aligned}$$

Thus, the moment condition on the capital-output ratio plus the ones on the expectation and variance of the physical capital return pin down ψ and σ^2_{ζ}.

D.2.1 Demographic Model Parameters

Life-expectancy. Before getting to the calibration of the life-expecancy dynamics in the model, I want to point out that life-expectancy in the model is cohort-life-expectancy, that is the expected length of life at birth. On the other hand the life-expectancy taken from data follows the cross-secional principle, giving the average life-expectancy across all cohorts at a specific point in time. Let $cross_t$ denote cross-section-life-expectancy and J_t cohort-life-expectancy (as in the model), in time period (year) t. Iterating backwards I translate cross-section-life-expectancy into cohort-life-expectancy by, $J_{t-cross_t} = cross_t$. To give an example, let $cross_{2000} = 71$, then I would set $J_{1929} = 71$, households born in 1929 would live until age 71 in the year 2000 . For some periods this algorithm would assign two distinct values to J_t. In such a case I choose the larger value.

As mentioned above, cross-section-life-expectancy is taken or constructed from data. In the following I will refer to it as life-expectancy only. For the years 1900 to 1949 I take life-expectancy data from Noymer and Garenne (2000) and approximate it by a linear function. For the years 1950 onward I draw on mortality rate data from the Human Mortality Database by sex and fit a Lee-Carter model to this data. I use the fitted Lee-Carter model to project sex-independent mortality rates for the years 1950 to 2100.[17] For the years 2101 to 2465 I keep mortality rates fixed at the levels of 2100. Given the mortality rates I compute the (cross-section-)life-expectancies as

$$cross_t = \sum_{j=0}^{100} \prod_{i=0}^{j} \psi_{i,t},$$

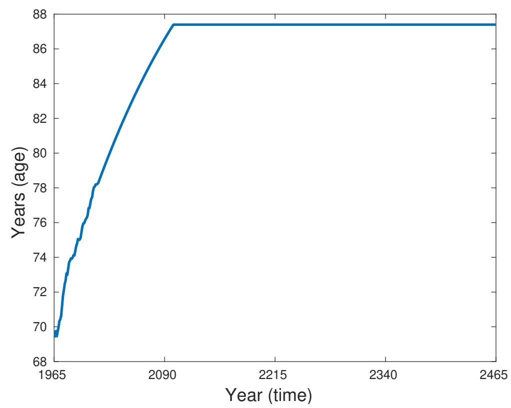
where $\psi_{i,t}$ is the survival rate of an individual of age i in the year t.

From the year 2100 onward I keep life-expectancy constant. A figure with the (cross-section-)life-expactancies supplied to the model is shown in figure 12.

Newborns. Newborns in the model,$newborns_t$, stem from two sources. First, by simulating a population model with the survival rates estimated above and fertility rates taken from the United Nations' population projections (United Nations 2007), I obtain the first measure of newborns in every year. Second this population simulation leads to population sizes that differ from actual data. These differences I also consider as newborns and add to the first measure. This I can do for the years up to and including 2015. For the years thereafter I add the average of the years 1950-2015.

[17] I am grateful to Alexander Ludwig for providing me with the Matlab codes for estimating the Lee-Carter model and performing projections, and also to Leon Hütsch and Leon Stolle for implementing these steps.

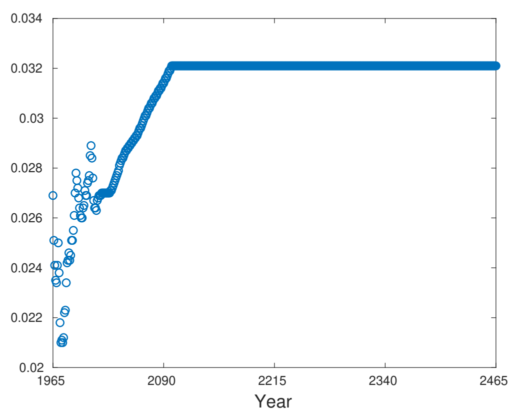
Figure 12: Life-expectancy dynamics



Notes: Cross-sectional life expectancy supplied to the quantitative model, from 1965 to 2465.

From the year 2100 onward, I keep the newborn population constant. The dynamics of the newborns supplied to the model are shown in figure 13.

Figure 13: Newborns dynamics



Notes: Total newborns (due to population model with survival risk and fertility plus differences from actual population data) as a share of total 1965 population, from 1965 to 2465.

Work time fraction. Time and age dependent work time fractions, $\omega_{t,j}$ capture two aspects, contingent survival during working-life and non-participation in the labor market. Let $surv_{t,h}$ capture the former and $part_{t,j}$ the latter. Work time fractions are computed as the product of the two, that is

$$\omega_{t,j} = surv_{t,j} \cdot part_{t,j}.$$

As survival is deterministic in the model all households live until social security's full-benefit retirement age. However, we observe that due to survival risk not all households make it to this age and hence do not supply labor till this age. This shortened working-life span is accounted for through the cumulative survival probability conditional on reaching working age of 20, that is

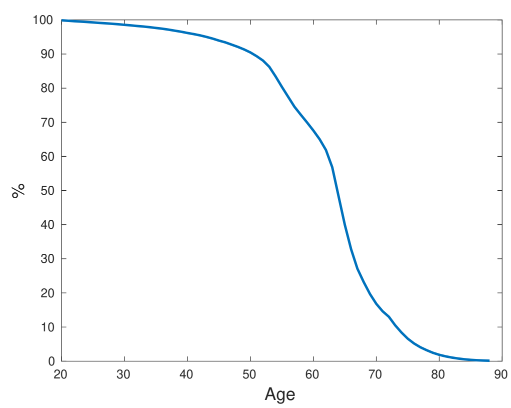
$$surv_{t,j} = \prod_{i=0}^{j} \psi_{i,t},$$

where $\psi_{i,t}$ is again the survival rate of an individual of age i in the year t.

As a second aspect, not all households participate in the labor market. Since the focus of this study is on demographic change, I only want to take into account non-participation due to early retirement and health related incapacity to work. In order to do so I draw on data on participation rates of US men from the Labor Force Statistics of the OECD. Until the age of 50 I set $part_{t,j} = 1$, that is all households participate in the labor market (abstracted from $surv_{t,j}$). From age 51 on participation is equal to participation data normalized by participation at age 50, thus guaranteeing a smooth decline.

Figure 14 shows the resulting working time fractions in the initial steady state (year 1965).

Figure 14: Working time fraction life cycle profile



Notes: The life-cycle profile of working time fractions in the year 1965, where $\omega_{1965,j} = surv_{1965,j} \cdot part_{1965,j}$. Working time fractions are depicted in percent.

D.2.2 Average Variance of Income Growth

Using the definition of cash-on-hand, its transitional equation (33), $\frac{h_{t,j,i}}{1+g} = \hat{\alpha}^H_{t,j}$ and $c_{t,j,i} = m_{t,j} x_{t,j,i}$ we have

$$h_{t+1,j+1,i} = \frac{1}{1+g} \frac{\hat{\alpha}^H_{t+1,j+1}}{\hat{\alpha}^H_{t,j}} (1 - m_{t,j})(1 + \hat{r}_{t,j,i}(\eta_{t,j,i})) h_{t,j,i},$$

where $\hat{r}_{t,j,i}(\eta_{t,j,i})$ is the portfolio return as given in equation (31), explicitly denoted as a function of the human capital shock $\eta_{t,j,i}$. Let $\widetilde{inc}_{t,j,i} = \omega_j \tilde{r}^H_t h_{t,j,i}$ denote the (de-trended) gross labor income. Hence, the growth rate of labor income, as approximated by the difference in logs is

$$\begin{aligned}
\Delta \log \widetilde{inc}_{t+1,j+1,i} &= \Delta \log(\omega_{j+1} \tilde{r}^H_{t+1}) + \Delta \log h_{t+1,j+1,i} \\
&= \Delta \log(\omega_{j+1} \tilde{r}^H_{t+1}) - \log(1+g) + \log \frac{\hat{\alpha}^H_{t+1,j+1}}{\hat{\alpha}^H_{t,j}} + \log(1 - m_{t,j}) + \log(1 + \hat{r}_{t,j,i}(\eta_{t,j,i}, \zeta_{t,j,i})) \\
&\approx D_{t,j} + E_{t,j}(\zeta_{t,j,i})^{\frac{1}{\alpha}} + F_{t,j} \eta_{t,j,i},
\end{aligned}$$

where $D_t = \Delta \log(\omega_{j+1} \tilde{r}^H_{t+1}) - \log(1+g) + \log \frac{\hat{\alpha}^H_{t+1,j+1}}{\hat{\alpha}^H_{t,j}} + \log(1 - m_{t,j}) + (1 - \hat{\alpha}^K_{t,j} - \hat{\alpha}^H_{t,j}) r^f_t - \hat{\alpha}^K_{t,j} \delta^K + (1+g)(1 + (1-\tau_t)\omega_j \tilde{r}^H_t - \delta^H_j) - 1$, $E_{t,j} = \hat{\alpha}^K_{t,j} \alpha (1-\alpha)^{\frac{1-\alpha}{\alpha}} \left(\psi(\tilde{r}^H_t)^{\alpha-1}\right)^{\frac{1}{\alpha}}$ and $F_{t,j} = (1+g)\hat{\alpha}^H_{t,j}$. The last line results from the approximation $\log(1+x) \approx x$ and the equations for portfolio, physical capital and human capital returns. Hence, variance of income growth of an individual aged j in period t is

$$Var(\Delta \log \widetilde{inc}_{t+1,j+1,i}) \approx E^2_{t,j} \sigma^2_{\zeta^{\frac{1}{\alpha}}} + F^2_{t,j} \sigma^2_\eta + 2 E_{t,j} F_{t,j} \sigma_{\zeta\eta}.$$

The average variance at time t is thus

$$\begin{aligned}
\overline{Var(\Delta \log \widetilde{inc}_{t+1,j+1,i})} &= \frac{1}{N} \sum_j N_{t,j} Var\left(\Delta \log \widetilde{inc}_{t+1,j+1,i}\right) \\
&\approx \frac{1}{N} \sum_j N_{t,j} \left(E^2_{t,j} \sigma^2_{\zeta^{\frac{1}{\alpha}}} + F^2_{t,j} \sigma^2_\eta + 2 E_{t,j} F_{t,j} \sigma_{\zeta\eta}\right)
\end{aligned}$$

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