DGT - USAGE MANUAL/DOCUMENTATION

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This document is a work in progress and currently only contains a minimum needed to use DGT.

1. About

***D**iagrams to **G**raphs and **T**riangulations* (DGT)[1] is software for generating triangulations and coloured graphs of 3- and 4-manifolds associated to framed links.

2. Usage

Diagrams should first be drawn in SnapPy's PLink Editor utility (available via `https://snappy.math.uic.edu/`). In SnapPy, start the PLink Editor by running the `Manifold()` command. Once the diagram has been drawn, copy the planar diagram code (PD code) via the Info → PD Code menu option in the PLink Editor.

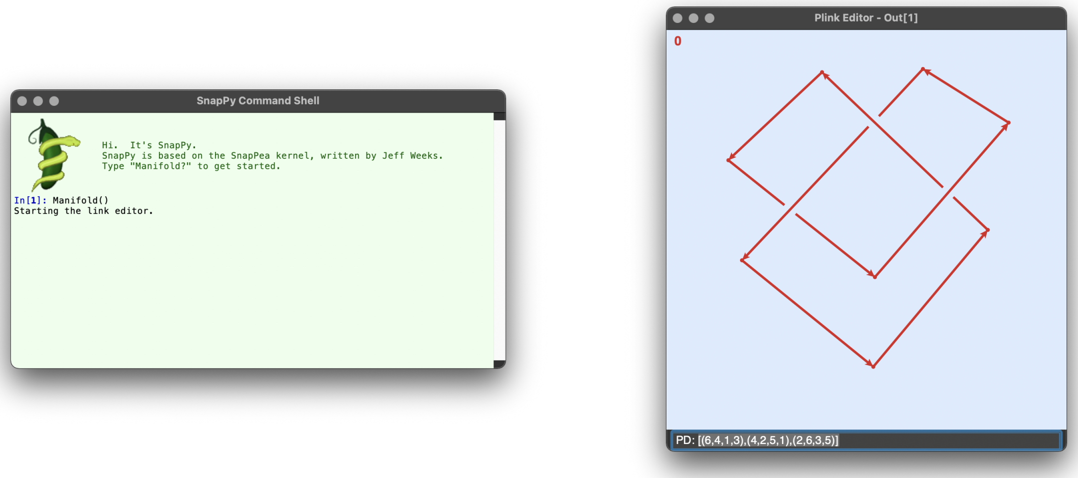


Figure 1. SnapPy and the PLink Editor

In a terminal session, from the DGT directory, DGT is used as follows.

```
> ./DGT (-3 | -4) [OUTPUT_TYPE] [BOUNDARY_TYPE]
```

Precisely one dimension flag must be provided at runtime, either `-3` (`--dim3`) (for a 3-manifold) or `-4` (`--dim4`) (for a 4-manifold). The 3-manifold produced is the result of performing integer Dehn surgery on the given framed link. The 4-manifold produced is the result of attaching

[1]Name subject to change, open to better names.

4-dimensional 2-handles (i.e. copies of $D^2 \times D^2$) to D^4 along the given framed link. Specifically, let (L, c) be a framed link, with l components, and $c = (c_1, \dots, c_l)$, $c_i \in \mathbb{Z}$. Then

$$M^4(L, c) = D^4 \cup_\varphi (H_1 \cup \cdots \cup H_l),$$

where $H_i = D^2 \times D^2$, and $\varphi_i : \partial D^2 \times D^2 \to \partial D^4$ satisfies:

- $\varphi_i(\mathbb{S}^1 \times \{0\}) = L_i$;
- $lk\left(L_i, \varphi_i(\mathbb{S}^1 \times \{x\})\right) = c_i$, for all $x \in D^2 \backslash \{0\}$ (where $lk(A, B)$ is the linking number of two link components A and B) — in other words, L_i has framing c_i.

The boundary of $M^4(L, c)$ is the 3-manifold which is obtained by performing integer Dehn surgery on (L, c).

By default, DGT automatically performs a capping off procedure on all 4-manifold constructions. Consequently, if $\partial M^4(L, c)$ is non-spherical, the resulting triangulation will have an *ideal* boundary component, consisting of a single ideal vertex; that is, a vertex whose link is a closed 3-manifold but not a sphere. If however, $\partial M^4(L, c) = \mathbb{S}^3$, then the capping off procedure amounts to the addition of a 4-handle in the handlebody decomposition, i.e. $M^4(L, c) = H^0 \cup_\varphi (H_1^2 \cup \cdots \cup H_l^2) \cup H^4$, and so the resulting triangulation will be of this closed 4-manifold.

Alternatively, by providing the `-r` (`--real`) runtime argument, DGT will not perform the capping off procedure, with the resulting triangulation having a *real* boundary component formed from unglued facets of the 4-simplices of the triangulation.

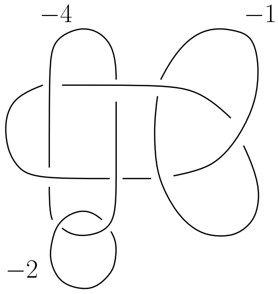
By default, the output of DGT is an *isomorphism signature* associated to the triangulation. An isomorphism signature is a compact sequence of letters, digits and/or punctuation that identifies a triangulation uniquely up to combinatorial isomorphism (i.e., relabelling simplices and their vertices), and can be fed into *Regina* (`https://regina-normal.github.io/`) to reconstruct the full triangualtion.

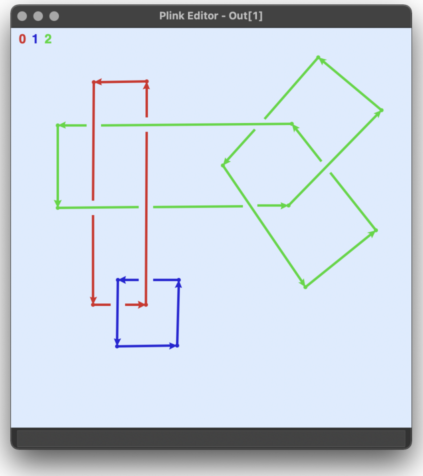
DGT can also produce the edge list of the associated coloured graph. This edge list coincides with the facet identification list, and so can also be used to reconstruct the triangulation. To obtain the edge list, use the `-g` (`--graph`) runtime argument.

On running DGT, paste the PD code copied from SnapPy into the terminal when prompted, and hit return. Then, specify the framing on each component when prompted, and hit return. For multiple component links, the ordering of the framings should be the same as the ordering of components as in SnapPy. For example, consider the following framed link.

If in the PLink Editor we draw the link as shown below,

then when specifying the framings in DGT, we specify them as `-4 -2 -1` (the same order as seen in the upper left corner of the PLink Editor).

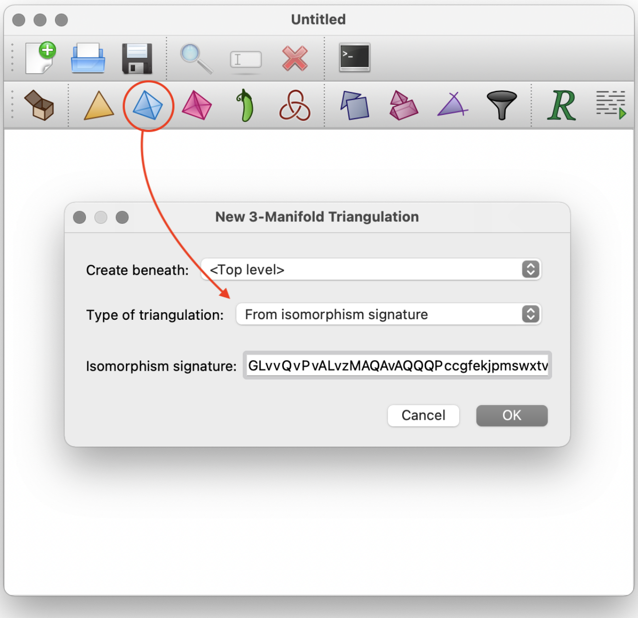

−4
−1
−2

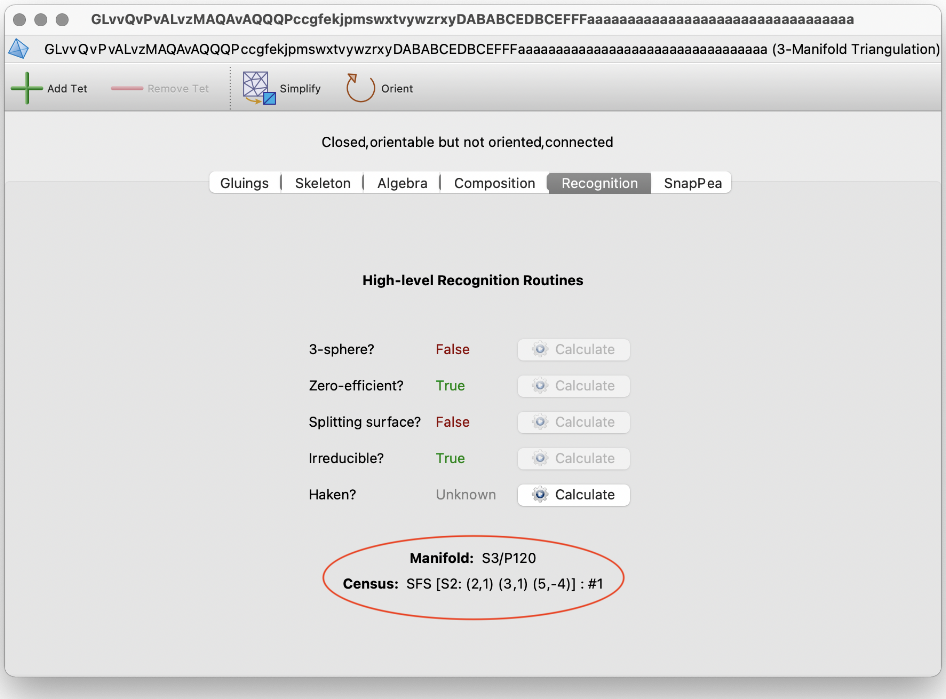

Plink Editor - Out[1]
0 1 2

2.1. **Examples.**

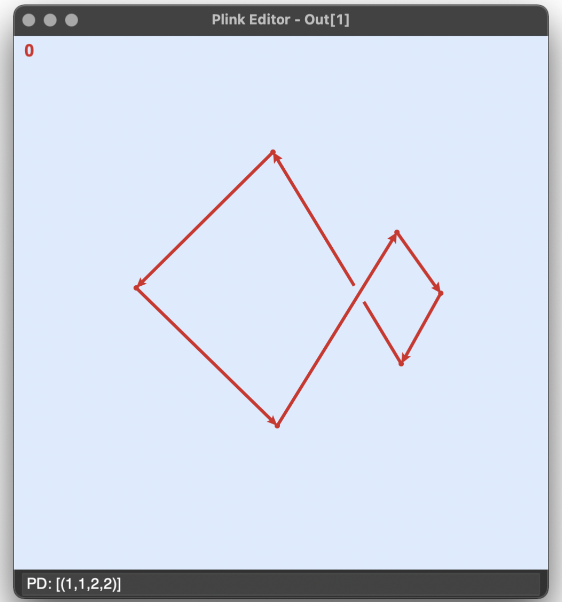
1. The Poincaré Homology 3-Sphere, obtained via +1 framed surgery along the right handed trefoil knot. We draw the trefoil as shown in Figure 1, and copy the associated PD code. The DGT session then runs as follows (where red text is user input).

   todo If we want to then work with the triangulation further, in Regina, we make a new 3-manifold triangulation from the isomorphism signature, as shown below. Using Regina's simplification feature, the manifold is correctly recognised as the Poincaré homology 3-sphere, as desired.



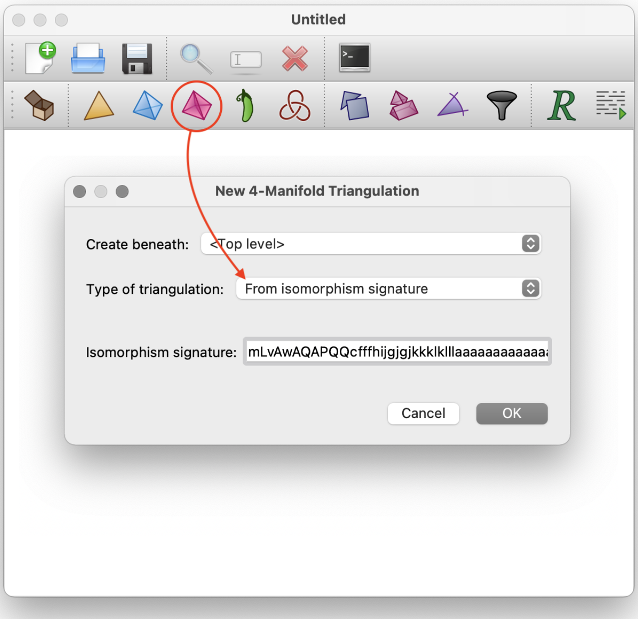

GLvvQvPvALvzMAQAvAQQQPccgfekjpmswxtvywzrxyDABABCEDBCEFFFaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaa
GLvvQvPvALvzMAQAvAQQQPccgfekjpmswxtvywzrxyDABABCEDBCEFFFaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaa (3-Manifold Triangulation)
Add Tet
Remove Tet
Simplify
Orient
Closed,orientable but not oriented,connected
Gluings
Skeleton
Algebra
Composition
Recognition
SnapPea
High-level Recognition Routines
3-sphere? False Calculate
Zero-efficient? True Calculate
Splitting surface? False Calculate
Irreducible? True Calculate
Haken? Unknown Calculate
Manifold: S3/P120
Census: SFS [S2: (2,1) (3,1) (5,-4)] : #1

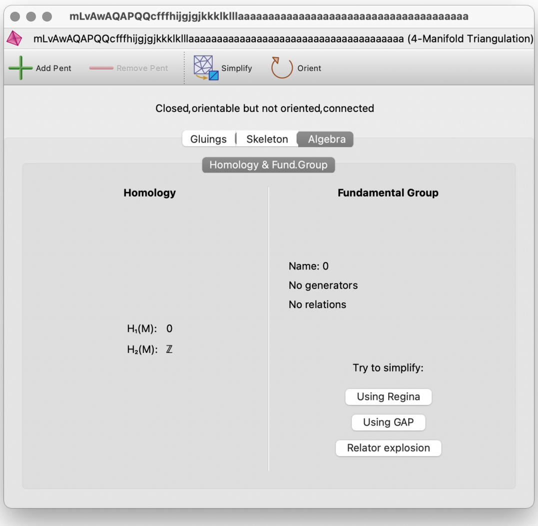
2. $\mathbb{CP}^2$ has a handle decomposition of the form $H^0 \cup H^2 \cup H^4$, where the 2-handle is attached along an unknot with framing $+1$. Due to the nature of PD codes, we require at least a single crossing in our diagram, and so we draw the unknot with a single trivial twist as shown below.

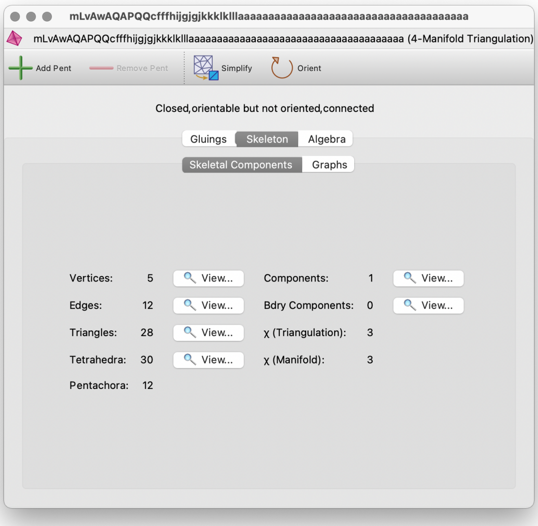


The DGT sessions then runs as follows.

todo In Regina, we then reconstruct the triangulation by building a new 4-manifold triangulation from the isomorphism signature:


Untitled
New 4-Manifold Triangulation
Create beneath:
<Top level>
Type of triangulation:
From isomorphism signature
Isomorphism signature:
mLvAwAQAPQQcfffhijgjgjkkklklllaaaaaaaaaaaaa
Cancel
OK

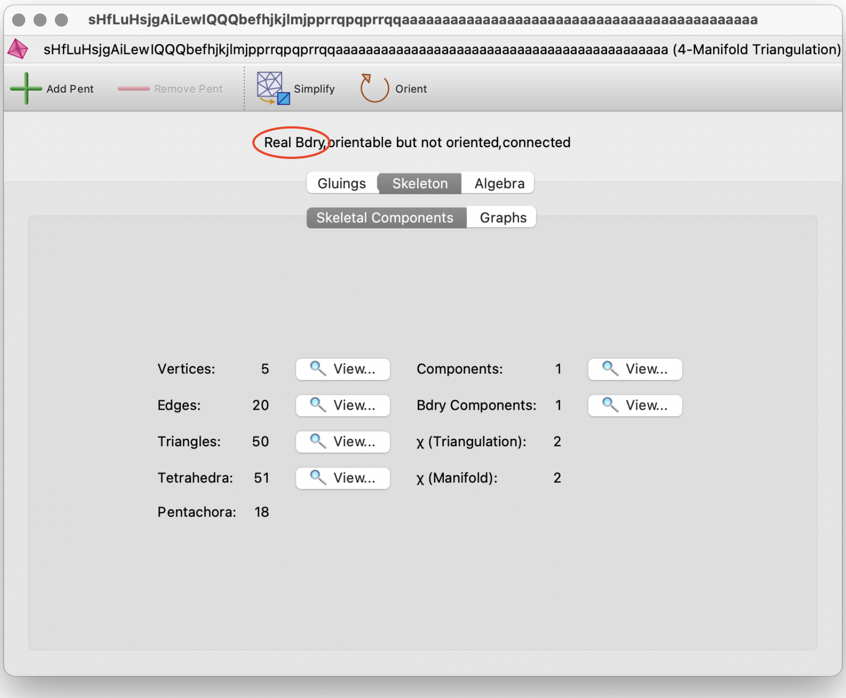

Closed,orientable but not oriented,connected
Gluings
Skeleton
Algebra
Homology & Fund.Group
Homology
Fundamental Group
Name: 0
No generators
No relations
Try to simplify:
Using Regina
Using GAP
Relator explosion

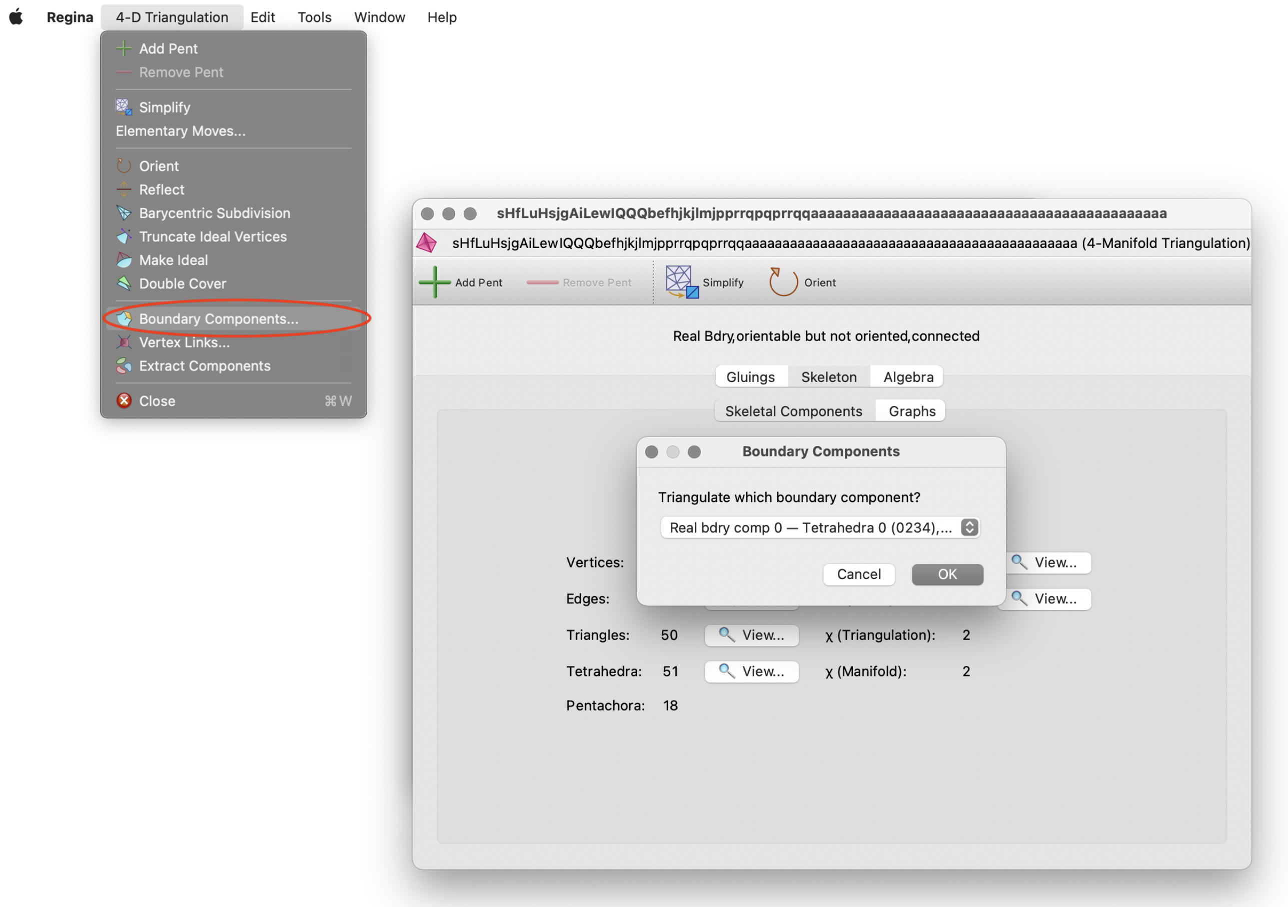

Closed,orientable but not oriented,connected
Gluings
Skeleton
Algebra
Skeletal Components
Graphs
Vertices: 5
Edges: 12
Triangles: 28
Tetrahedra: 30
Pentachora: 12
Components: 1
Bdry Components: 0
χ (Triangulation): 3
χ (Manifold): 3

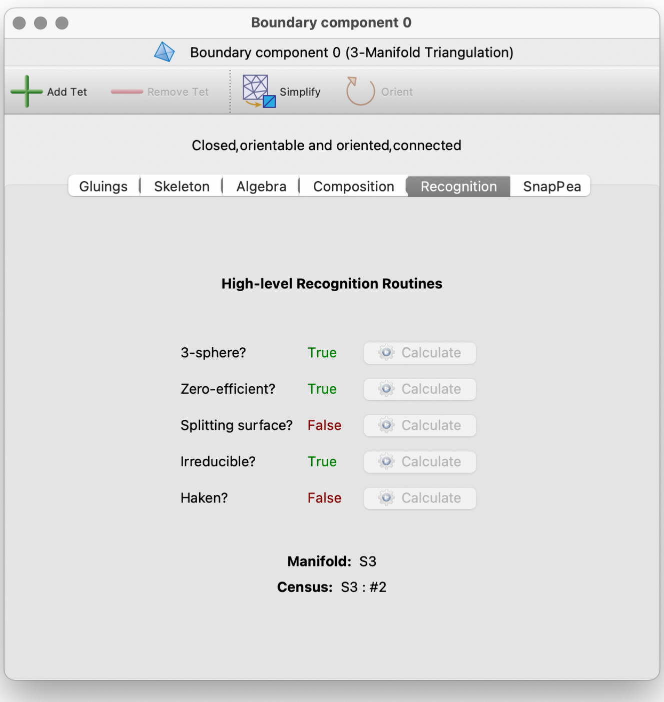
3. In this example, we construct the edge list of the coloured graph associated to $\mathbb{CP}^2 - D^4$. The DGT session this time runs as follows, where we specify both the `-g` and `-r` runtime arguments, in order to produce the edge list (`-g`) of the graph, without performing the capping off procedure in order to obtain a manifold with real boundary component (`-r`).

   todo

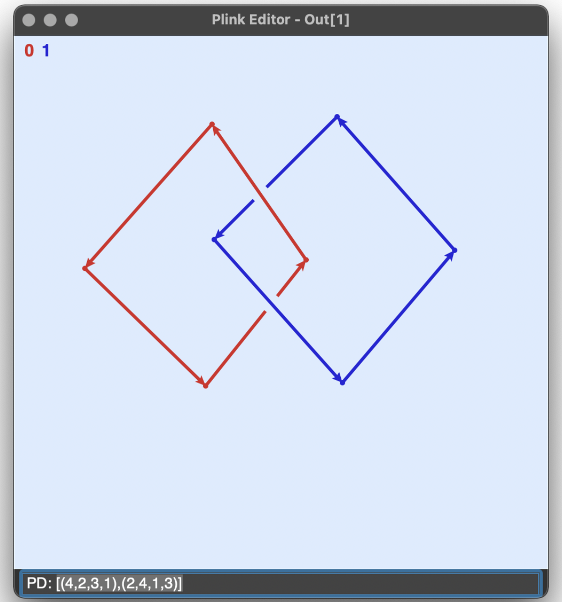
   Using a simple script in regina, we can reconstruct the triangulation from the above edge list. As can be seen in the screenshots below, the resulting triangulation has a real boundary component, and by using Regina's built-in functionality to build a triangulation of the boundary itself, we find that the boundary is $\mathbb{S}^3$ (as expected).




Regina
4-D Triangulation
Edit
Tools
Window
Help
Add Pent
Remove Pent
Simplify
Elementary Moves...
Orient
Reflect
Barycentric Subdivision
Truncate Ideal Vertices
Make Ideal
Double Cover
Boundary Components...
Vertex Links...
Extract Components
Close
sHfLuHsjgAiLewIQQQbefhjkjlmjpprrqpqprrqqaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaa
sHfLuHsjgAiLewIQQQbefhjkjlmjpprrqpqprrqqaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaa (4-Manifold Triangulation)
Add Pent
Remove Pent
Simplify
Orient
Real Bdry,orientable but not oriented,connected
Gluings
Skeleton
Algebra
Skeletal Components
Graphs
Boundary Components
Triangulate which boundary component?
Real bdry comp 0 — Tetrahedra 0 (0234),...
Cancel
OK
Vertices:
Edges:
Triangles: 50
Tetrahedra: 51
Pentachora: 18
View...
χ (Triangulation): 2
χ (Manifold): 2


Boundary component 0
Boundary component 0 (3-Manifold Triangulation)
Add Tet
Remove Tet
Simplify
Orient
Closed,orientable and oriented,connected
Gluings
Skeleton
Algebra
Composition
Recognition
SnapPea
High-level Recognition Routines
3-sphere? True
Zero-efficient? True
Splitting surface? False
Irreducible? True
Haken? False
Calculate
Manifold: S3
Census: S3 : #2

4. $\mathbb{S}^2 \times \mathbb{S}^2$ is obtained by attaching two 2-handles to D^4 along a Hopf link with 0-framings on each component (and capping off with a 4-handle). In PLink we draw the Hopf link as shown below, and copy the generated PD code.



The DGT session then runs as follows.

todo As before, we can then use the isomorphism signature to reconstruct the triangulation in Regina for further analysis.

2.2. **Exotic 4-Manifolds.** DGT has been used to successfully generate triangulations of exotic pairs of 4-manifolds. For more information on this and the isomorphism signatures of the exotic pairs, see `https://raburke.github.io/`.