Simplifying 4-Manifold Triangulations

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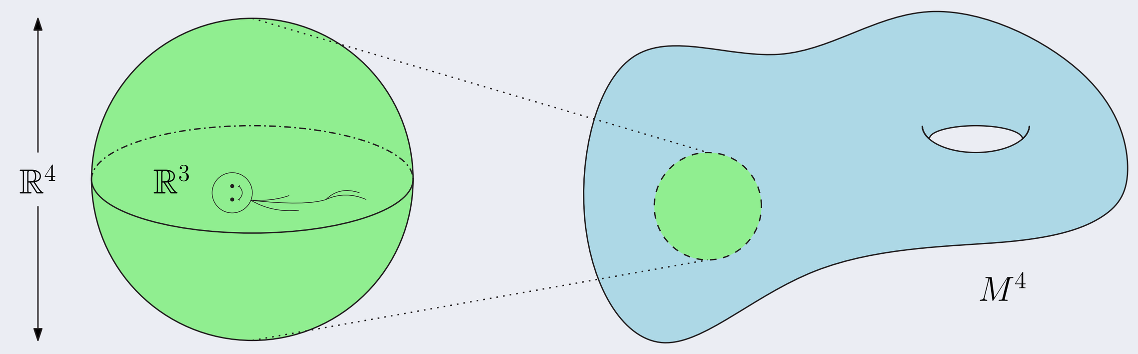
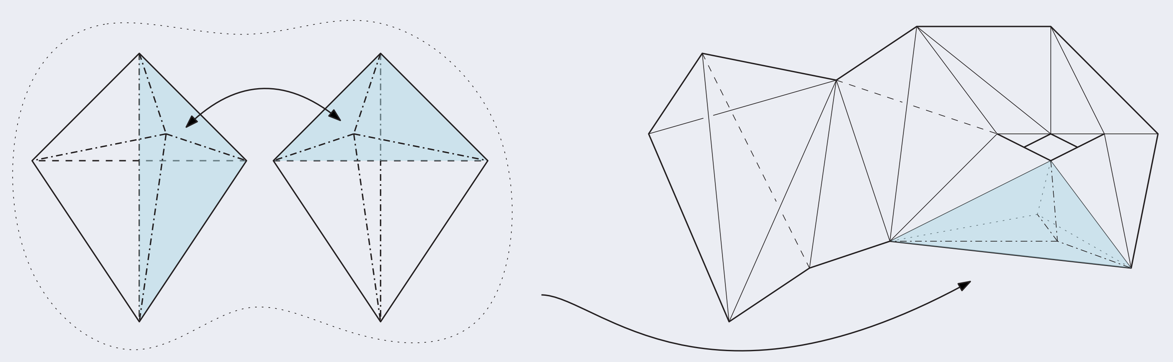
Introduction

- 4-manifolds are spaces which locally "look like" $\mathbb{R}^4$.



- We encode 4-manifolds via **generalised triangulations**: collections of 4-dimensional simplices (**pentachora**), some or all of whose tetrahedral facets are affinely identified in pairs, such that the underlying topological space represents the manifold in question.



- In contrast to three dimensions, where algorithms may exist (even with theoretically poor running times), in four dimensions many problems are provably **undecidable**, meaning no algorithm even exists in the first place.
- Consequently, we can often only hope for **heuristics** which, for as many cases as possible, give the correct answer, in as short as time as possible.
- Such algorithms illustrate the **difference** between what is possible in **theory** versus what is possible in **practice**.

Motivation

- The feasibility of analysing the combinatorial structure of a triangulation is often dependent on having a sufficiently small triangulation to work with.
- As such, it is desirable to have an efficient, and effective tool which can simplify large triangulations.
- Existing simplification heuristics are predominantly tailored towards 3-manifold triangulations, and have limited effectiveness in four dimensions.

Up Down Simplification

- The new simplification heuristic, called **Up Down Simplification** (*UDS*), uses sequences of **Pachner moves** — moves which alter the triangulation without changing the underlying topology.
- Each run of *UDS* performs an increasing number of **2-4** moves (Fig. 1), in the hopes of "opening up" more **2-0** moves which simplify the triangulation. A number of **3-3** moves (which do not change the size of the triangulation) can also be performed after the 2-4 moves in order to further "shake up" the triangulation in an effort to open up simplification paths. Fig. 2 shows a schematic of the heuristic.

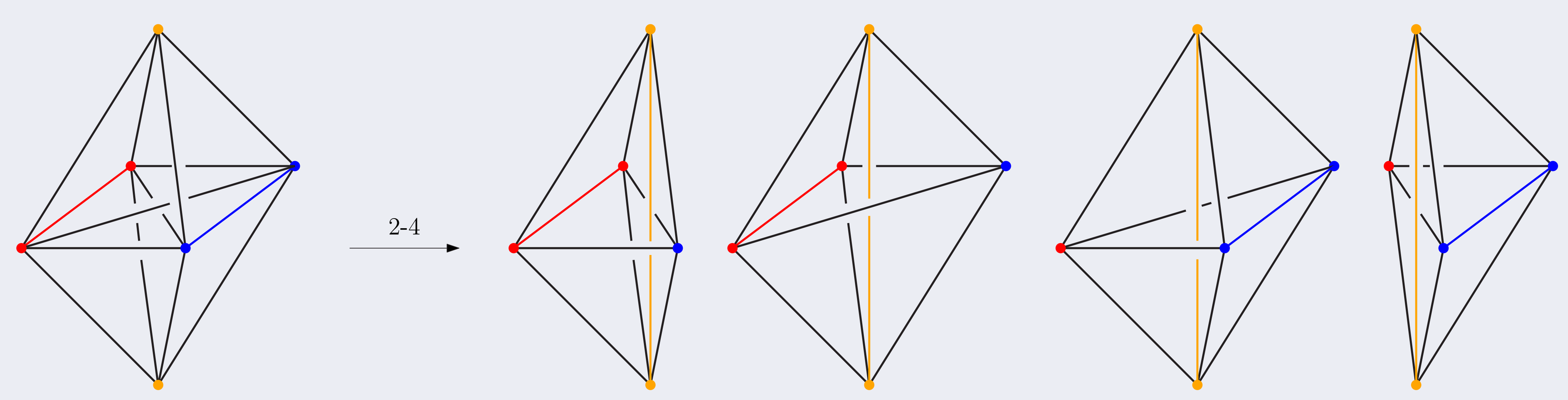


Figure 1: The 2-4 Pachner move.

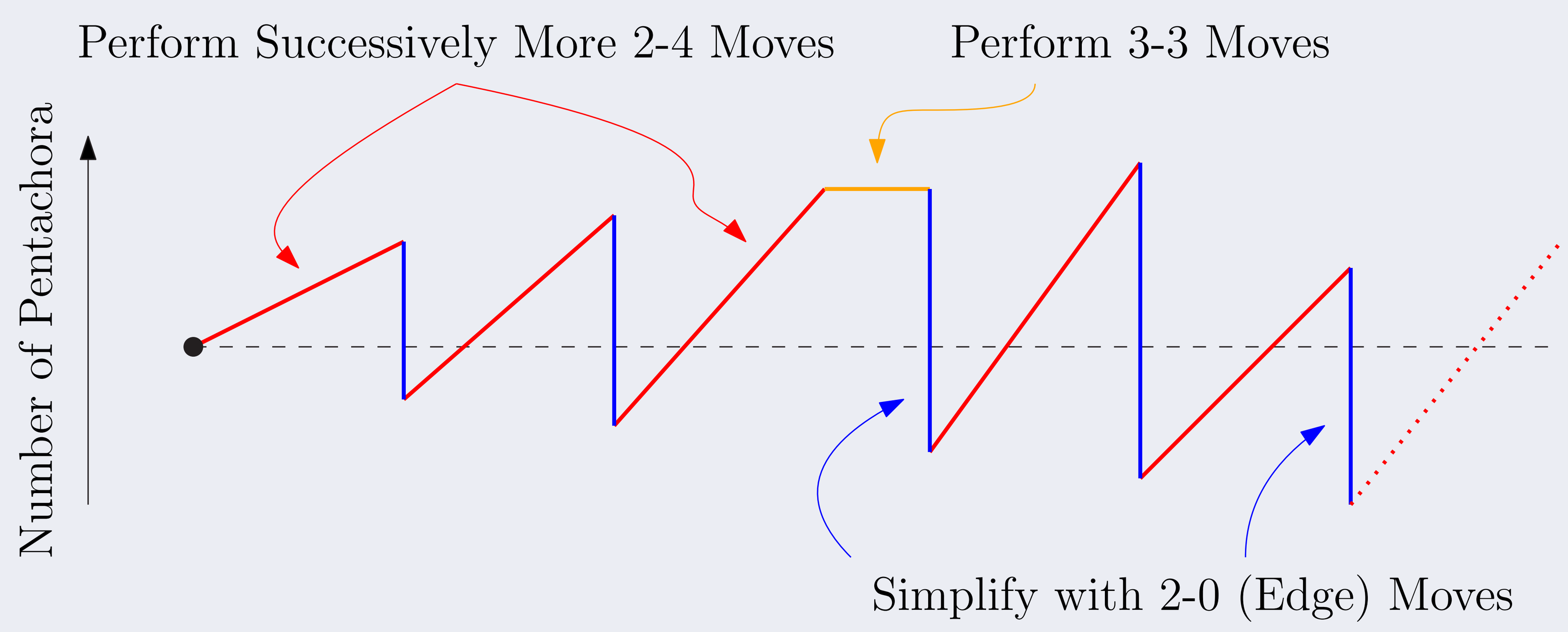


Figure 2: The UDS heuristic.

Example: The $K3$ Surface

- The $K3$ surface is one of four "fundamental" closed simply-connected 4-manifolds.
- $K3_{16}$ [1] and $K3_{17}$ [2] are two triangulations of the $K3$ surface.
- $K3_{17}$ is known to have a standard PL (piecewise linear) structure (this is equivalent to a smooth structure), but the PL type of $K3_{16}$ is unknown (conjectured to be standard).
- $K3_{16}$ and $K3_{17}$ have **288** and **312** pentachora respectively.
- Previous best efforts to simplify these triangulations resulted in triangulations on the order of $>$ **100** pentachora.
- Starting from a 134 pentachora triangulation derived from $K3_{17}$, *UDS* yielded a **60** pentachora triangulation.
- Additionally, starting from a **2048** pentachora triangulation of the $K3$ surface based on the handle decomposition shown in Fig. 3 , *UDS* produced a triangulation with just **54** pentachora.
- This is currently the smallest known triangulation of a K3 surface, and the first with $<$ **100** pentachora.
- Having such a small triangulation represents new potential in being able to settle Conjecture 1.1 of [3].

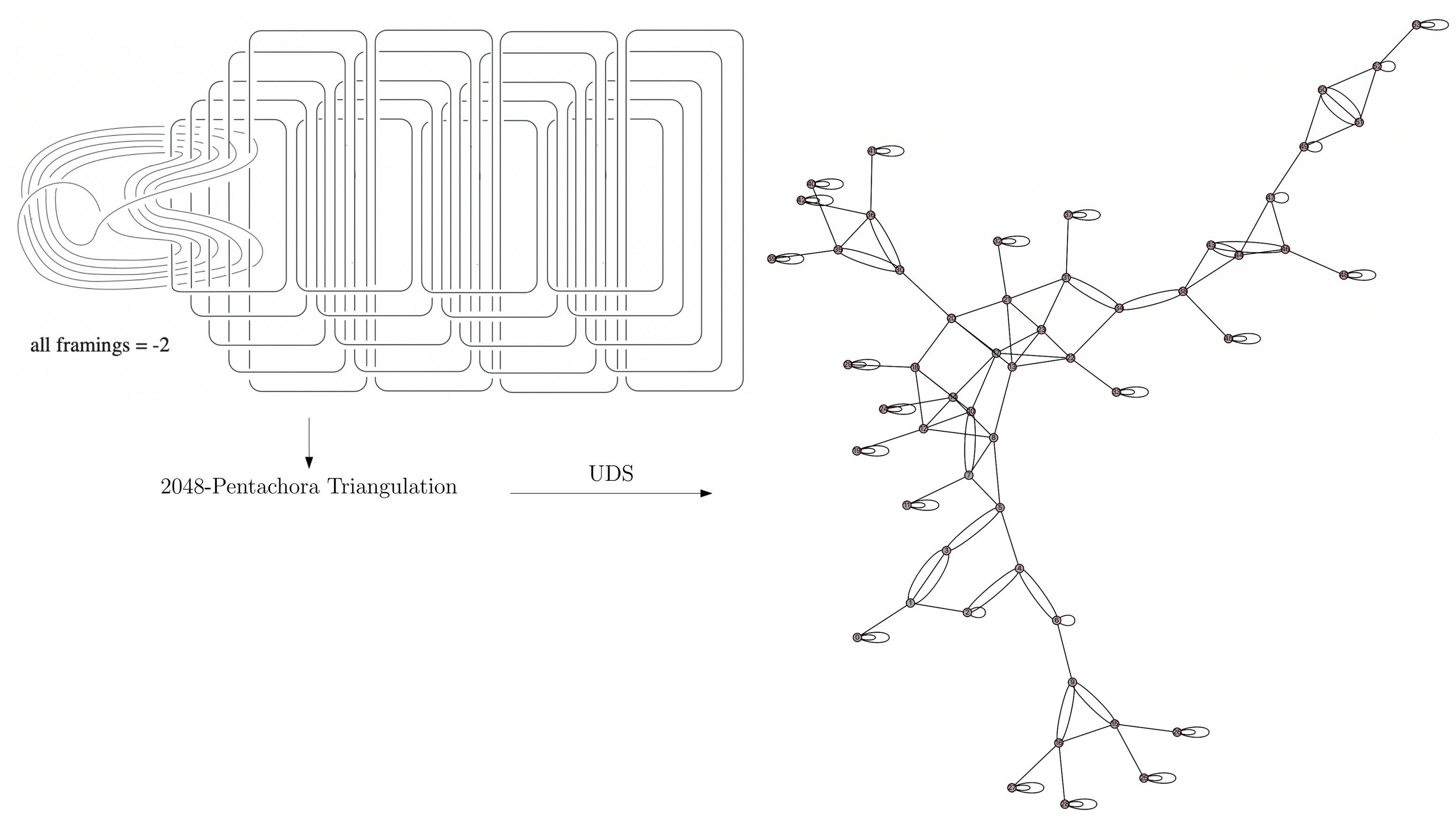


Figure 3: A 54 pentachora triangulation of the K3 surface.

Example: Exotic Pairs

- Dimension four is the first dimension where **exotic** structures appear: manifolds which are homeomorphic, but not diffeomorphic (i.e. they are **topologically the same**, but for which there is **no smooth deformation** of one into the other).
- Recent work of the author has been the production of triangulations of exotic pairs, however the initial triangulations were very large.
- We wish to determine how these exotic structures are reflected in the combinatorics of the triangulations, and so desire smaller triangulations. *UDS* has been highly effective at simplifying these large triangulations as illustrated in Fig. 4.

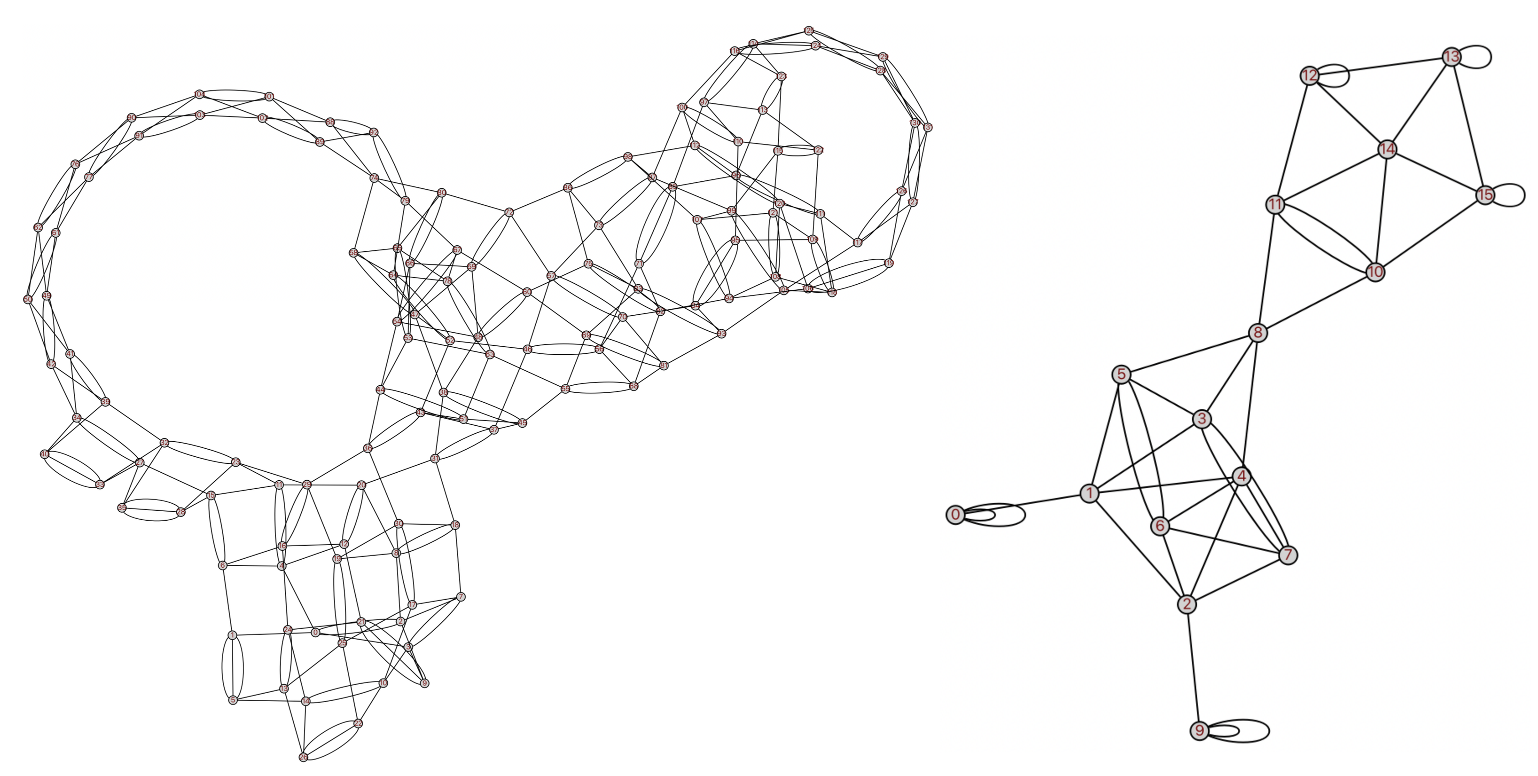


Figure 4: One manifold from an exotic pair, before and after UDS.

References

[1] Mario Casella and Wolfgang Kühnel.
A triangulated k3 surface with the minimum number of vertices, July 2001.

[2] Jonathan Spreer and Wolfgang Kühnel.
Combinatorial properties of the k3 surface: Simplicial blowups and slicings, July 2009.

[3] Benjamin A. Burton and Jonathan Spreer.
Computationally proving triangulated 4-manifolds to be diffeomorphic, 2014.

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