Triangulating Exotic 4-Manifolds

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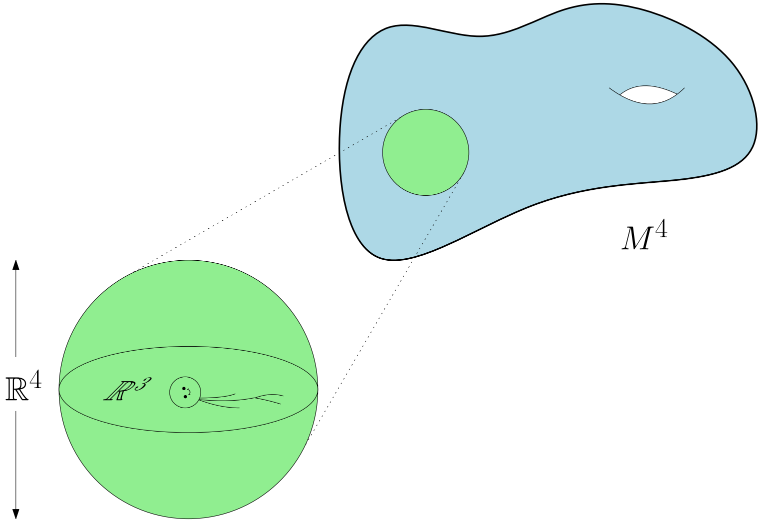
Outline

4-Manifolds are spaces which are locally modelled on $\mathbb{R}^4$:



The Wild World of 4-Manifolds

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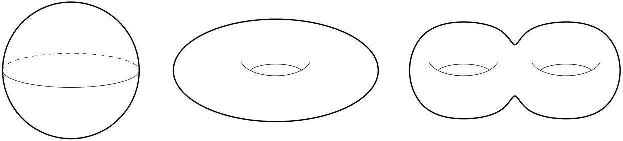

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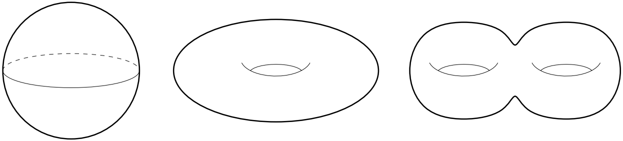

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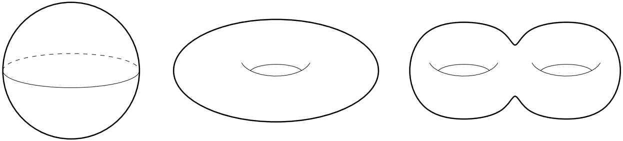

- In dimensions ≥ 5, algebra and surgery theory take over.
- The situations is quite different in four dimensions...

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- Simply-connected *topological* 4-manifolds have been classified by **Freedman** (1982).
- Dimension 4 is the first dimension in which the distinction between *topological* and *smooth* manifolds appear.
- Smooth 4-manifolds remain something of a mystery.

Exotic Structures

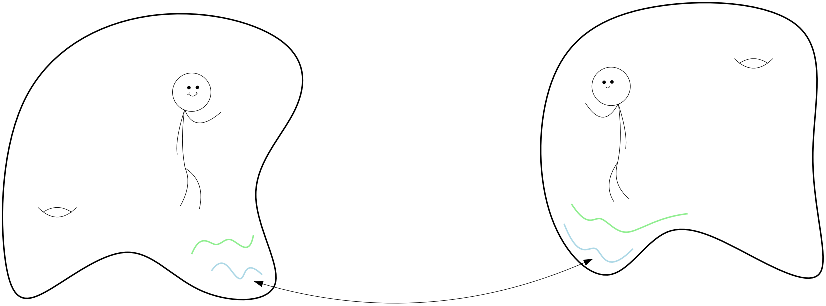
Exotic Manifolds

Smooth manifolds M and N, are *exotic* if M and N are *homeomorphic* but *not diffeomorphic*.

Key Idea: If M and N are an exotic pair, we can *continuously* deform M into N and vice versa, but we can't do it *smoothly*!

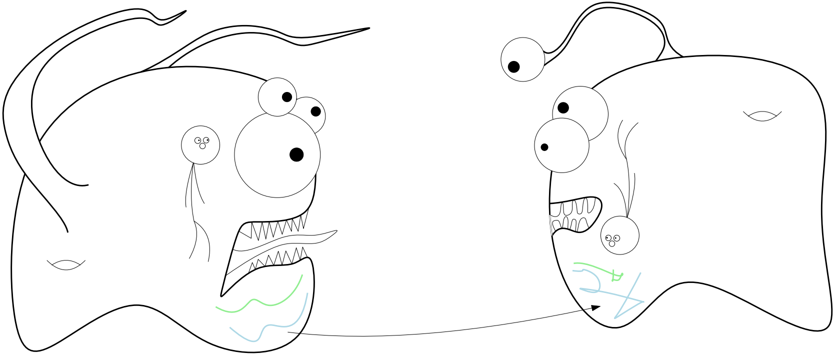
Fact: $\mathbb{R}^n$ admits a unique smooth structure for $n \neq 4$, but $\mathbb{R}^4$ admits *uncountably many distinct* smooth structures.

Exotic Structures



Same when viewed topologically...

Exotic Structures



...different beasts differentiably.

Big Picture Motivation

Theorem 2 (Generalised Poincaré Conjecture)

Fix a category of manifold: TOP *(topological),* PL *(piecewise-linear) or* DIFF *(differentiable). Then every closed n-manifold homotopy equivalent to the n-sphere in the chosen category is isomorphic in the chosen category to the standard n-sphere.*

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- Theorem 2 known to be either true or false, depending on category of manifold, as well as the dimension:
  - TOP: True in all dimensions.
  - PL: True in dimensions other than 4 where it is currently unsettled (as of 2021).
  - DIFF: False in general. True in dimensions 1, 2, 3, 5 and 6. The first known counterexample is in dimension 7. Unknown in dimension 4 where DIFF is equivalent to PL.

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- Finding an exotic $\mathbb{S}^4$ would be a counterexample to the smooth Poincaré conjecture and would be the final brick in this particular wall.

Hope

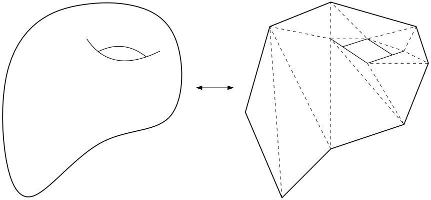
Theorem 3 (Whitehead)

Every smooth manifold is uniquely triangulizable (admits a PL structure).

Theorem 4 (Hirsch-Mazur, Cerf, et. al.)

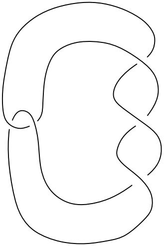
For $n \leq 6$, PL-manifolds can be smoothed (admit a smooth structure).

So in particular, *PL* and *smooth* structures are *equivalent* for 4-manifolds.

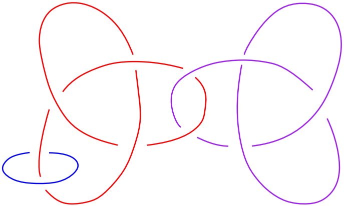


Hope: Gain an understanding of exotic 4-manifolds via combinatorial/computational methods.

- A *knot* K is a copy of $\mathbb{S}^1$ embedded in $\mathbb{S}^3$, considered up to isotopy.
- A *link* is a disjoint union of (possibly interlinking) knots.



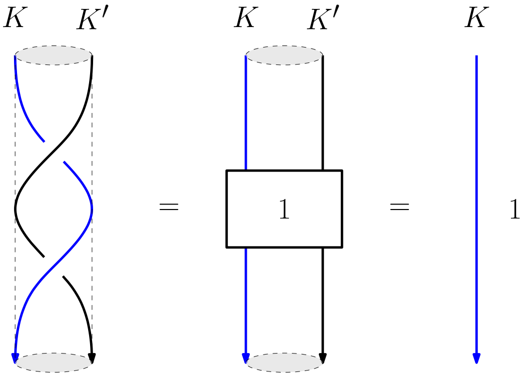
(a) A knot



(b) A link

4-Manifolds from Links

- A *framing* on a knot is a choice of trivialisation of the normal (disk) bundle νK.
- Framings are parameterised by the number of times, $c \in \mathbb{Z}$, a tubular neighbourhood of the knot "twists" around the knot.



4-Manifolds from Links

Let (L, c) be a framed link, with l components, and $c = (c_1, \ldots, c_l)$.

We build a 4-manifold, called the *2-handlebody* (or *link trace*) associated to (L, c) via

$$M^4(L, c) = D^4 \cup_\varphi (H_1 \cup \cdots \cup H_l),$$

where $H_i = D^2 \times D^2$, and

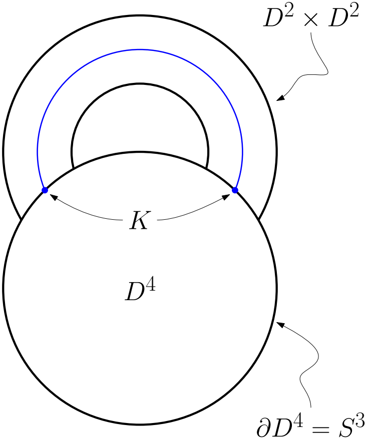
$$\varphi_i : \partial D^2 \times D^2 \to \partial D^4$$

is such that:

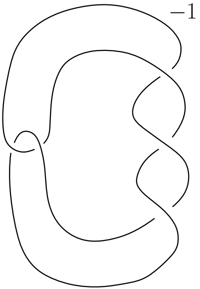
- $\phi_i(\mathbb{S}^1 \times \{0\}) = L_i$, and
- L_i has linking number c_i with $\phi_i(\mathbb{S}^1 \times \{x\})$, $\forall x \in D^2 \backslash \{0\}$ (i.e. has framing c_i).

4-Manifolds from Links

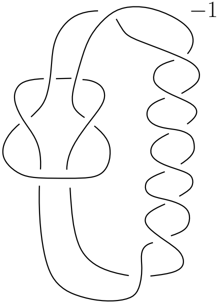
Key Idea: These are 4-manifolds we can visualise as link diagrams decorated with integers on each component.



Some Examples



A_1



A_2

Exotic Homotopy $\overline{\mathbb{CP}^2}_0$ (Akbulut, 1991).

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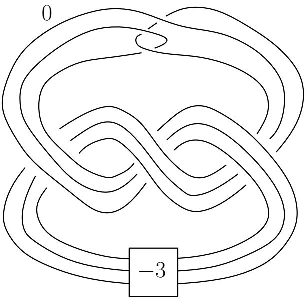
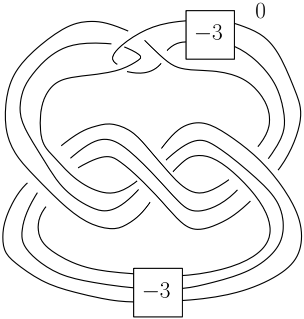


Y_1



Y_2

Exotic Homotopy $\mathbb{S}^2 \times D^2 - D^4$ (Yasui, 2015).

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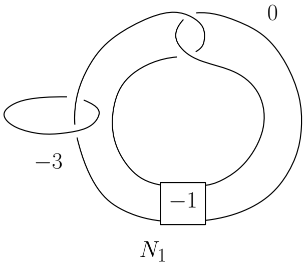
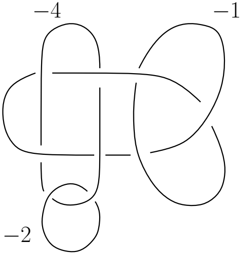
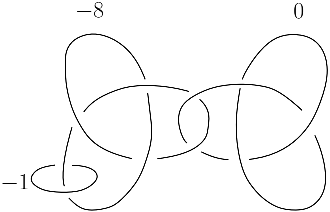


Exotic "Nuclei" of the Elliptic Surface $E(n)$, ($n \geq 3$ odd) (Gompf, 1991).

Some Examples



W_1



W_2

Exotic Homotopy $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2} - D^4$ (Naoe, 2017).

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- *One* readily available example, due to **Kronheimer** and **Mrowka** (1994): $(\mathbb{R}P^4 \# K3,\ \mathbb{R}P^4 \# 11(S^2 \times S^2))$.
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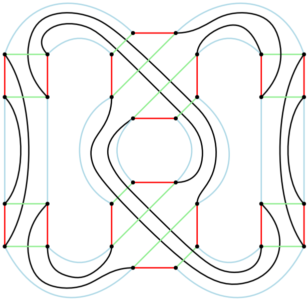
- Have triangulations of $\mathbb{R}P^4$, $K3$ and $S^2 \times S^2$, so easy to construct the above example.
- **Main Problem:** No systematic way of generating triangulations of (exotic) 4-manifolds.

A Method for Triangulating 4-Manifolds

- *Crystallisation theory* allows us to represent PL-manifolds by *edge-coloured* graphs.
- An $(n+1)$-*coloured graph* (Γ, γ) consists of:
  - A finite multigraph $\Gamma = (V, E)$,
  - A surjective map $\gamma : E \to \Delta_n$, where $\Delta_n = \{0, \dots, n\}$ is a set of colours, and we have $\gamma(e) \neq \gamma(f)$ whenever e and f are adjacent.



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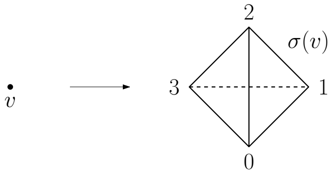

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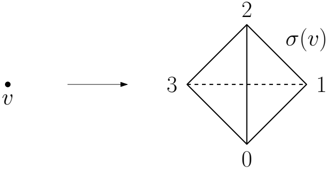
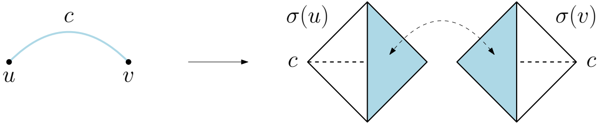
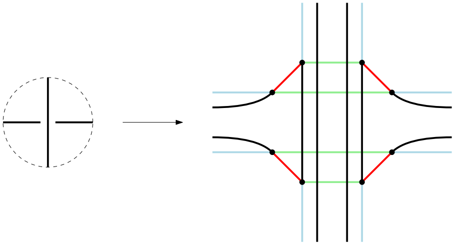
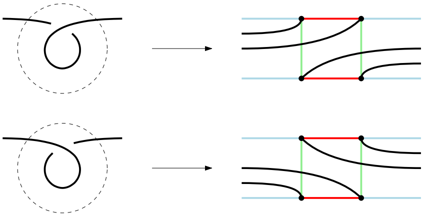


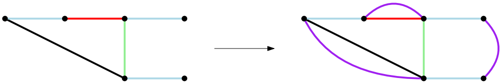
  - If $u, v \in V(\Gamma)$ are joined by a c-coloured edge, identify the $(n-1)$-facets of $\sigma(u)$ and $\sigma(v)$ opposite to the c-labelled vertices such that equally labelled vertices coincide.



Casali (2000) described an algorithm for producing a coloured graph representing $M^4(L, c)$ directly from its link diagram, however no implementation was available.







Results

- We have now implemented this algorithm (in C++).
- This provides a systematic way of quickly generating triangulations of 2-handlebodies.
- In particular this has allowed us to generate triangulations of the exotic 4-manifold pairs shown before.

Manifold	Vertices	Edges	Triangles	Tetrahedra	Pentachora
A_1	5	72	206	207	70
A_2	5	156	458	459	154
Y_1	5	440	1310	1311	438
Y_2	5	508	1514	1515	506
N_1	7	88	254	258	88
N_2	7	144	422	426	144
W_1	7	62	228	280	112
W_2	7	64	236	290	116

Ongoing Research

- Find more exotic 2-handlebodies.
- Construct minimal examples.
- Extend to a wider class of 4-manifold (e.g. 4-manifolds with 1-handles).
- Investigation of corks and related objects (currently understood as what can give rise to "exoticness") in a purely PL/computational setting.
- Investigation into computationally feasible PL invariants.

Thank You