APPENDIX C

COORDINATE TRANSFORMATIONS

C.1 NOTATION

We use the notation $\mathbf{C}_{\text{to}}^{\text{from}}$ to denote a coordinate transformation matrix from one coordinate frame (designated by "from") to another coordinated frame (designated by "to"). For example,

$\mathbf{C}_{\text{ENU}}^{\text{ECI}}$ denotes the coordinate transformation matrix from earth-centered inertial (ECI) coordinates to earth-fixed east-north-up (ENU) local coordinates and

$\mathbf{C}_{\text{NED}}^{\text{RPY}}$ denotes the coordinate transformation matrix from vehicle body-fixed roll-pitch-yaw (RPY) coordinates to earth-fixed north-east-down (NED) coordinates.

Coordinate transformation matrices satisfy the composition rule

$$\mathbf{C}_C^B \mathbf{C}_B^A = \mathbf{C}_C^A,$$

where A, B, and C represent different coordinate frames.

What we mean by a coordinate transformation matrix is that if a vector $\mathbf{v}$ has the representation

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{C.1}$$

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in *XYZ* coordinates and the same vector **v** has the alternative representation

$$\mathbf{v} = \begin{bmatrix} v_u \\ v_v \\ v_w \end{bmatrix} \tag{C.2}$$

in *UVW* coordinates, then

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \mathbf{C}_{XYW}^{UVW} \begin{bmatrix} v_u \\ v_v \\ v_w \end{bmatrix}, \tag{C.3}$$

where “*XYZ*” and “*UVW*” stand for any two Cartesian coordinate systems in three-dimensional space.

The components of a vector in either coordinate system can be expressed in terms of the vector components along unit vectors parallel to the respective coordinate axes. For example, if one set of coordinate axes is labeled X, Y and Z, and the other set of coordinate axes are labeled U, V, and W, then the same vector **v** can be expressed in either coordinate frame as

$$\mathbf{v} = v_x \mathbf{1}_x + v_y \mathbf{1}_y + v_z \mathbf{1}_z \tag{C.4}$$

$$= v_u \mathbf{1}_u + v_v \mathbf{1}_v + v_w \mathbf{1}_w, \tag{C.5}$$

where

- the unit vectors $\mathbf{1}_x$, $\mathbf{1}_y$, and $\mathbf{1}_z$ are along the *XYZ* axes;
- the scalars v_x, v_y, and v_z are the respective components of **v** along the *XYZ* axes;
- the unit vectors $\mathbf{1}_u$, $\mathbf{1}_v$, and $\mathbf{1}_w$ are along the *UVW* axes; and
- the scalars v_u, v_v, and v_w are the respective components of **v** along the *UVW* axes.

The respective components can also be represented in terms of dot products of **v** with the various unit vectors,

$$v_x = \mathbf{1}_x^{\mathrm{T}} \mathbf{v} = v_u \mathbf{1}_x^{\mathrm{T}} \mathbf{1}_u + v_v \mathbf{1}_x^{\mathrm{T}} \mathbf{1}_v + v_w \mathbf{1}_x^{\mathrm{T}} \mathbf{1}_w, \tag{C.6}$$

$$v_y = \mathbf{1}_y^{\mathrm{T}} \mathbf{v} = v_u \mathbf{1}_y^{\mathrm{T}} \mathbf{1}_u + v_v \mathbf{1}_y^{\mathrm{T}} \mathbf{1}_v + v_w \mathbf{1}_y^{\mathrm{T}} \mathbf{1}_w, \tag{C.7}$$

$$v_z = \mathbf{1}_z^{\mathrm{T}} \mathbf{v} = v_u \mathbf{1}_z^{\mathrm{T}} \mathbf{1}_u + v_v \mathbf{1}_z^{\mathrm{T}} \mathbf{1}_v + v_w \mathbf{1}_z^{\mathrm{T}} \mathbf{1}_w, \tag{C.8}$$

which can be represented in matrix form as

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \mathbf{1}_x^{\mathrm{T}}\mathbf{1}_u & \mathbf{1}_x^{\mathrm{T}}\mathbf{1}_v & \mathbf{1}_x^{\mathrm{T}}\mathbf{1}_w \\ \mathbf{1}_y^{\mathrm{T}}\mathbf{1}_u & \mathbf{1}_y^{\mathrm{T}}\mathbf{1}_v & \mathbf{1}_y^{\mathrm{T}}\mathbf{1}_w \\ \mathbf{1}_z^{\mathrm{T}}\mathbf{1}_u & \mathbf{1}_z^{\mathrm{T}}\mathbf{1}_v & \mathbf{1}_z^{\mathrm{T}}\mathbf{1}_w \end{bmatrix} \begin{bmatrix} v_u \\ v_v \\ v_w \end{bmatrix} \tag{C.9}$$

$$\stackrel{\text{def}}{=} \mathbf{C}_{XYZ}^{UVW} \begin{bmatrix} v_u \\ v_v \\ v_w \end{bmatrix}, \tag{C.10}$$

which defines the coordinate transformation matrix $\mathbf{C}_{XYZ}^{UVW}$ from UVW to XYZ coordinates in terms of the dot products of unit vectors. However, dot products of unit vectors also satisfy the cosine rule (defined in Section B.1.2.5)

$$\mathbf{1}_a^T \mathbf{1}_b = \cos(\theta_{ab}), \tag{C.11}$$

where θ_{ab} is the angle between the unit vectors $\mathbf{1}_a$ and $\mathbf{1}_b$. As a consequence, the coordinate transformation matrix can also be written in the form

$$C_{XYZ}^{UVW} = \begin{bmatrix} \cos(\theta_{xu}) & \cos(\theta_{xv}) & \cos(\theta_{xw}) \\ \cos(\theta_{yu} & \cos(\theta_{yv}) & \cos(\theta_{yw}) \\ \cos(\theta_{zu}) & \cos(\theta_{zv}) & \cos(\theta_{zw}) \end{bmatrix}, \tag{C.12}$$

which is why coordinate transformation matrices are also called "direction cosines matrices."

Navigation makes use of coordinates that are natural to the problem at hand: inertial coordinates for inertial navigation, orbital coordinates for GPS navigation, and earth-fixed coordinates for representing locations on the earth.

The principal coordinate systems used in navigation, and the transformations between these different coordinate systems, are summarized in this appendix. These are primarily Cartesian (orthogonal) coordinates, and the transformations between them can be represented by orthogonal matrices. However, the coordinate transformations can also be represented by rotation vectors or quaternions, and all representations are used in the derivations and implementation of GPS/INS integration.

C.2 INERTIAL REFERENCE DIRECTIONS

C.2.1 Vernal Equinox

The equinoxes are those times of year when the length of the day equals the length of the night (the meaning of "equinox"), which only happens when the sun is over the equator. This happens twice a year: when the sun is passing from the Southern Hemisphere to the Northern Hemisphere (vernal equinox) and again when it is passing from the Northern Hemisphere to the Southern Hemisphere (autumnal equinox). The time of the vernal equinox defines the beginning of spring (the meaning of "vernal") in the Northern Hemisphere, which usually occurs around March 21–23.

The direction from the earth to the sun at the instant of the vernal equinox is used as a "quasi-inertial" direction in some navigation coordinates. This direction is defined by the intersection of the equatorial plane of the earth with the ecliptic (earth-sun plane). These two planes are inclined at about 23.45°, as illustrated in Fig. C.1. The inertial direction of the vernal equinox is changing ever so slowly, on the order of 5 arc seconds per year, but the departure from truly inertial directions is neglible over the time periods of most navigation problems. The vernal equinox was in the constellation Pisces in the year 2000. It was in the constellation Aries at the time of Hipparchus (190-120 BCE) and is sometimes still called "the first point of Aries."

C.2.2 Polar Axis of Earth

The one inertial reference direction that remains invariant in earth-fixed coordinates as the earth rotates is its polar axis, and that direction is used as a reference direction in inertial coordinates. Because the polar axis is (by definition) orthogonal to the earth's equatorial plane and the vernal equinox is (by definition) in the earth's equatorial plane, the earth's polar axis will always be orthogonal to the vernal equinox.

A third orthogonal axis can then be defined (by their cross-product) such that the three axes define a right-handed (defined in Section B.2.11) orthogonal coordinate system.

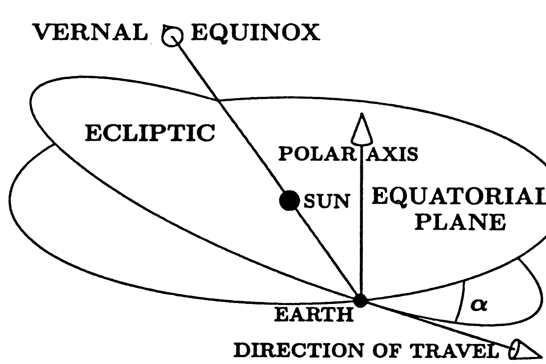

Fig. C.1 Direction of vernal equinox.

C.3 COORDINATE SYSTEMS

Although we are concerned exclusively with coordinate systems in the three dimensions of the observable world, there are many ways of representing a location in that world by a set of coordinates. The coordinates presented here are those used in navigation with GPS and/or INS.

C.3.1 Cartesian and Polar Coordinates

René Descartes (1596–1650) introduced the idea of representing points in three-dimensional space by a triplet of coordinates, called "Cartesian coordinates" in his honor. They are also called "Euclidean coordinates," but not because Euclid discovered them first. The Cartesian coordinates (x, y, z) and polar coordinates (θ, ϕ, r) of a common reference point, as illustrated in Fig. C.2, are related by the equations

$$x = r\cos(\theta)\cos(\phi), \tag{C.13}$$

$$y = r\sin(\theta)\cos(\phi), \tag{C.14}$$

$$z = r\sin(\phi), \tag{C.15}$$

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{C.16}$$

$$\phi = \arcsin\left(\frac{z}{r}\right) \qquad \left(-\frac{1}{2}\pi \le \phi \le +\frac{1}{2}\pi\right), \tag{C.17}$$

$$\theta = \arctan\left(\frac{y}{x}\right) \qquad (-\pi < \theta \le +\pi), \tag{C.18}$$

with the angle θ (in radians) undefined if $\phi = \pm\frac{1}{2}\pi$.

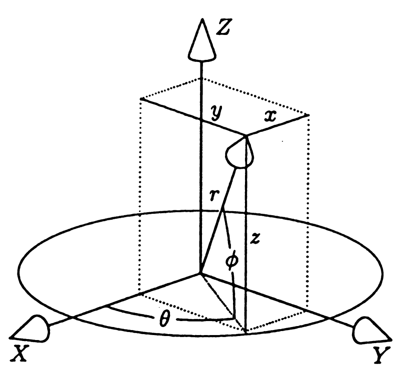

Fig. C.2 Cartesian and polar coordinates.

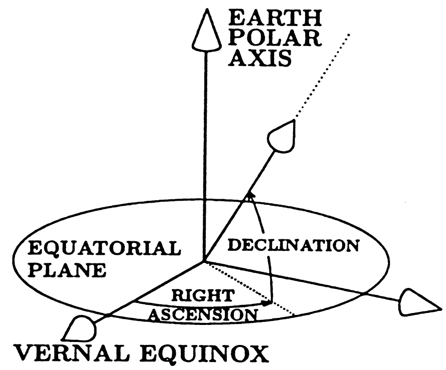

Fig. C.3 Celestial coordinates.

C.3.2 Celestial Coordinates

The "celestial sphere" is a system for inertial directions referenced to the polar axis of the earth and the vernal equinox. The polar axis of these celestial coordinates is parallel to the polar axis of the earth and its prime meridian is fixed to the vernal equinox. Polar celestial coordinates are *right ascension* (the celestial analog of longitude, measured eastward from the vernal equinox) and *declination* (the celestial analog of latitude), as illustrated in Fig. C.3. Because the celestial sphere is used primarily as a reference for direction, no origin need be specified.

Right ascension is zero at the vernal equinox and increases eastward (in the direction the earth turns). The units of right ascension (RA) can be radians, degrees, or hours (with 15 deg/h as the conversion factor).

By convention, declination is zero in the equatorial plane and increases toward the north pole, with the result that celestial objects in the Northern Hemisphere have positive declinations. Its units can be degrees or radians.

C.3.3 Satellite Orbit Coordinates

Johannes Kepler (1571–1630) discovered the geometric shapes of the orbits of planets and the minimum number of parameters necessary to specify an orbit (called "Keplerian" parameters). Keplerian parameters used to specify GPS satellite orbits in terms of their orientations relative to the equatorial plane and the vernal equinox (defined in Section C.2.1 and illustrated in Fig. C.1) include the following:

- Right ascension of the ascending node and orbit inclination, specifying the orientation of the orbital plane with respect to the vernal equinox and equatorial plane, is illustrated in Fig. C.4.
  - (a) Right ascension is defined in the previous section and is shown in Fig. C.3.

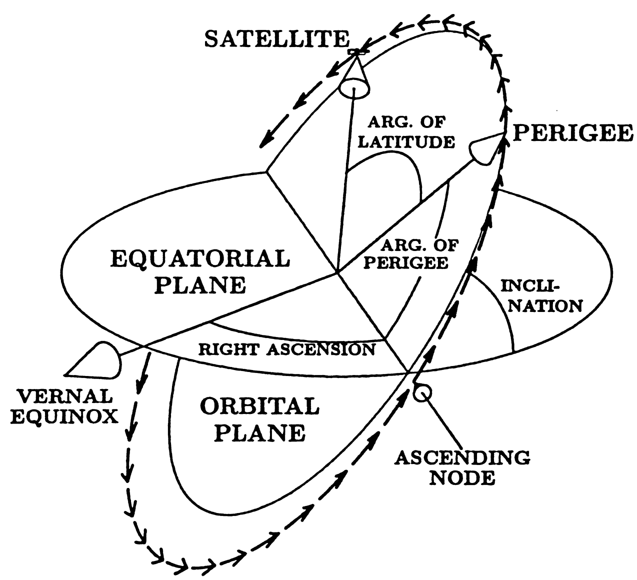

Fig. C.4 Keplerian parameters for satellite orbit.

(b) The intersection of the orbital plane of a satellite with the equatorial plane is called its "line of nodes," where the "nodes" are the two intersections of the satellite orbit with this line. The two nodes are dubbed "ascending"[1] (i.e., ascending from the Southern Hemisphere to the Northern Hemisphere) and "descending". The right ascension of the ascending node (RAAN) is the angle in the equatorial plane from the vernal equinox to the ascending node, measured counterclockwise as seen looking down from the north pole direction.

(c) Orbital inclination is the dihedral angle between the orbital plane and the equatorial plane. It ranges from zero (orbit in equatorial plane) to 90° (polar orbit).

- Semimajor axis a and semiminor axis b (defined in Section C.3.5.2 and illustrated in Fig. C.6) specify the size and shape of the elliptical orbit within the orbital plane.
- Orientation of the ellipse within its orbital plane, specified in terms of the "argument of perigee," the angle between the ascending node and the perigee of the orbit (closest approach to earth), is illustrated in Fig. C.4.
- Position of the satellite relative to perigee of the elliptical orbit, specified in terms of the angle from perigee, called the "argument of latitude" or "true anomaly," is illustrated in Fig. C.4.

[1]The astronomical symbol for the ascending node is ☊, often read as "earphones."

For computer simulation demonstrations, GPS satellite orbits can usually be assumed to be circular with radius $a = b = R = 26{,}560$ km and inclined at 55° to the equatorial plane. This eliminates the need to specify the orientation of the elliptical orbit within the orbital plane. (The argument of perigee becomes overly sensitive to orbit perturbations when eccentricity is close to zero.)

C.3.4 ECI Coordinates

Earth-centered inertial (ECI) coordinates are the favored inertial coordinates in the near-earth environment. The origin of ECI coordinates is at the center of gravity of the earth, with (Fig. C.5)

1. axis in the direction of the vernal equinox,
2. axis direction parallel to the rotation axis (north polar axis) of the earth, and
3. an additional axis to make this a right-handed orthogonal coordinate system, with the polar axis as the third axis (hence the numbering).

The equatorial plane of the earth is also the equatorial plane of ECI coordinates, but the earth itself is rotating relative to the vernal equinox at its sidereal rotation rate of about $7{,}292{,}115{,}167 \times 10^{-14}$ rad/s, or about 15.04109 deg/h, as illustrated in Fig. C.5.

C.3.5 ECEF Coordinates

Earth-centered, earth-fixed (ECEF) coordinates have the same origin (earth center) and third (polar) axis as ECI coordinates but rotate with the earth, as shown

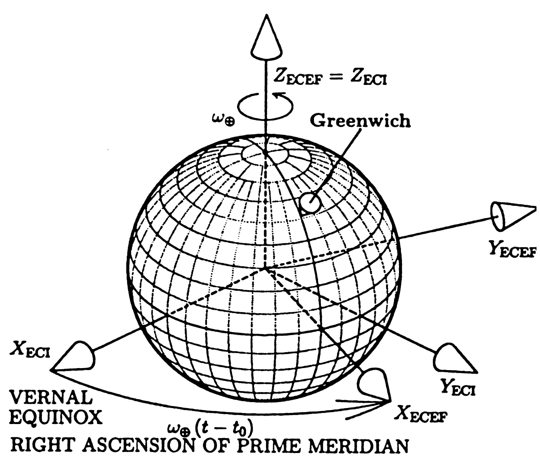

Fig. C.5 ECI and ECEF Coordinates.

in Fig. C.5. As a consequence, ECI and ECEF longitudes differ only by a linear function of time.

Longitude in ECEF coordinates is measured east (+) and west (−) from the prime meridian passing through the principal transit instrument at the observatory at Greenwich, UK, a convention adopted by 41 representatives of 25 nations at the International Meridian Conference, held in Washington, DC, in October of 1884.

Latitudes are measured with respect to the equatorial plane, but there is more than one kind of "latitude." *Geocentric latitude* would be measured as the angle between the equatorial plane and a line from the reference point to the center of the earth, but this angle could not be determined accurately (before GPS) without running a transit survey over vast distances. The angle between the pole star and the local vertical direction could be measured more readily, and that angle is more closely approximated as *geodetic latitude*. There is yet a third latitude (parametric latitude) that is useful in analysis. The latter two latitudes are defined in the following subsections.

C.3.5.1 Ellipsoidal Earth Models *Geodesy* is the study of the size and shape of the earth and the establishment of physical control points defining the origin and orientation of coordinate systems for mapping the earth. Earth shape models are very important for navigation using either GPS or INS, or both. INS alignment is with respect to the local vertical, which does not generally pass through the center of the earth. That is because the earth is not spherical.

At different times in history, the earth has been regarded as being flat (first-order approximation), spherical (second-order), and ellipsoidal (third-order). The third-order model is an ellipsoid of revolution, with its shorter radius at the poles and its longer radius at the equator.

C.3.5.2 Parametric Latitude For geoids based on ellipsoids of revolution, every meridian is an ellipse with equatorial radius a (also called "semimajor axis") and polar radius b (also called "semiminor axis"). If we let z be the Cartesian coordinate in the polar direction and $x_{\text{meridional}}$ be the equatorial coordinate in the meridional plane, as illustrated in Fig. C.6, then the equation for this ellipse will be

$$\frac{x_{\text{meridional}}^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{C.19}$$

$$= \cos^2(\phi_{\text{parametric}}) + \sin^2(\phi_{\text{parametric}}) \tag{C.20}$$

$$= \frac{a^2\cos^2(\phi_{\text{parametric}})}{a^2} + \frac{b^2\sin^2(\phi_{\text{parametric}})}{b^2} \tag{C.21}$$

$$= \frac{[a\cos(\phi_{\text{parametric}})]^2}{a^2} + \frac{[b\sin(\phi_{\text{parametric}})]^2}{b^2}. \tag{C.22}$$

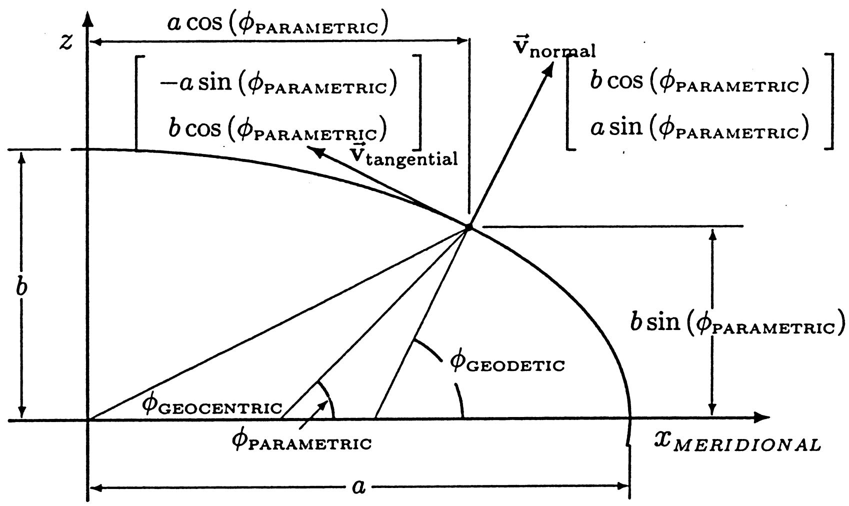

Fig. C.6 Geocentric, parametric, and geodetic latitudes in meridional plane.

That is, a parametric solution for the ellipse is

$$x_{\text{meridional}} = a\cos(\phi_{\text{parametric}}), \tag{C.23}$$

$$z = b\sin(\phi_{\text{parametric}}), \tag{C.24}$$

as illustrated in Fig. C.6. Although the parametric latitude $\phi_{\text{parametric}}$ has no physical significance, it is quite useful for relating geocentric and geodetic latitude, which do have physical significance.

C.3.5.3 Geodetic Latitude Geodetic latitude is defined as the elevation angle above (+) or below (−) the equatorial plane of the normal to the ellipsoidal surface. This direction can be defined in terms of the parametric latitude, because it is orthogonal to the meridional tangential direction.

The vector tangential to the meridian will be in the direction of the derivative to the elliptical equation solution with respect to parametric latitude:

$$\mathbf{v}_{\text{tangential}} \propto \frac{\partial}{\partial \phi_{\text{parametric}}} \begin{bmatrix} a\cos(\phi_{\text{parametric}}) \\ b\sin(\phi_{\text{parametric}}) \end{bmatrix} \tag{C.25}$$

$$= \begin{bmatrix} -a\sin(\phi_{\text{parametric}}) \\ b\cos(\phi_{\text{parametric}}) \end{bmatrix}, \tag{C.26}$$

and the meridional normal direction will be orthogonal to it, or

$$\mathbf{v}_{\text{normal}} \propto \begin{bmatrix} b\cos(\phi_{\text{parametric}}) \\ a\sin(\phi_{\text{parametric}}) \end{bmatrix}, \tag{C.27}$$

as illustrated in Fig. C.6.

The tangent of geodetic latitude is then the ratio of the z- and x-components of the surface normal vector, or

$$\tan(\phi_{\mathrm{geodetic}}) = \frac{a\sin(\phi_{\mathrm{parametric}})}{b\cos(\phi_{\mathrm{parametric}})} \tag{C.28}$$

$$= \frac{a}{b}\tan(\phi_{\mathrm{parametric}}), \tag{C.29}$$

from which, using some standard trigonometric identities,

$$\sin(\phi_{\mathrm{geodetic}}) = \frac{\tan(\phi_{\mathrm{geodetic}})}{\sqrt{1+\tan^2(\phi_{\mathrm{geodetic}})}} \tag{C.30}$$

$$= \frac{a\sin(\phi_{\mathrm{parametric}})}{\sqrt{a^2\sin^2(\phi_{\mathrm{parametric}}) + b^2\cos^2(\phi_{\mathrm{parametric}})}}, \tag{C.31}$$

$$\cos(\phi_{\mathrm{geodetic}}) = \frac{1}{\sqrt{1+\tan^2(\phi_{\mathrm{geodetic}})}} \tag{C.32}$$

$$= \frac{b\cos(\phi_{\mathrm{parametric}})}{\sqrt{a^2\sin^2(\phi_{\mathrm{parametric}}) + b^2\cos^2(\phi_{\mathrm{parametric}})}}. \tag{C.33}$$

The inverse relationship is

$$\tan(\phi_{\mathrm{parametric}}) = \frac{b}{a}\tan(\phi_{\mathrm{geodetic}}), \tag{C.34}$$

from which, using the same trigonometric identities as before,

$$\sin(\phi_{\mathrm{parametric}}) = \frac{\tan(\phi_{\mathrm{parametric}})}{\sqrt{1+\tan^2(\phi_{\mathrm{parametric}})}} \tag{C.35}$$

$$= \frac{b\sin(\phi_{\mathrm{geodetic}})}{\sqrt{a^2\cos^2(\phi_{\mathrm{geodetic}}) + b^2\sin^2(\phi_{\mathrm{geodetic}})}}, \tag{C.36}$$

$$\cos(\phi_{\mathrm{parametric}}) = \frac{1}{\sqrt{1+\tan^2(\phi_{\mathrm{parametric}})}} \tag{C.37}$$

$$= \frac{a\cos(\phi_{\mathrm{geodetic}})}{\sqrt{a^2\cos^2(\phi_{\mathrm{geodetic}}) + b^2\sin^2(\phi_{\mathrm{geodetic}})}}, \tag{C.38}$$

and the two-dimensional X-Z Cartesian coordinates in the meridional plane of a point on the geoid surface will

$$x_{\text{meridional}} = a\cos(\phi_{\text{parametric}}) \tag{C.39}$$

$$= \frac{a^2\cos(\phi_{\text{geodetic}})}{\sqrt{a^2\cos^2(\phi_{\text{geodetic}}) + b^2\sin^2(\phi_{\text{geodetic}})}}, \tag{C.40}$$

$$z = b\sin(\phi_{\text{parametric}}) \tag{C.41}$$

$$= \frac{b^2\sin(\phi_{\text{geodetic}})}{\sqrt{a^2\cos^2(\phi_{\text{geodetic}}) + b^2\sin^2(\phi_{\text{geodetic}})}} \tag{C.42}$$

in terms of geodetic latitude.

Equations C.40 and C.42 apply only to points on the geoid surface. Orthometric height h above (+) or below (−) the geoid surface is measured along the surface normal, so that the X-Z coordinates for a point with altitude h will be

$$x_{\text{meridional}} = \cos(\phi_{\text{geodetic}}) \times \left(h + \frac{a^2}{\sqrt{a^2\cos^2(\phi_{\text{geodetic}}) + b^2\sin^2(\phi_{\text{geodetic}})}}\right), \tag{C.43}$$

$$z = \sin(\phi_{\text{geodetic}}) \times \left(h + \frac{b^2}{\sqrt{a^2\cos^2(\phi_{\text{geodetic}}) + b^2\sin^2(\phi_{\text{geodetic}})}}\right). \tag{C.44}$$

In three-dimensional ECEF coordinates, with the X-axis passing through the equator at the prime meridian (at which longitude $\theta = 0$),

$$x_{\text{ECEF}} = \cos(\theta)x_{\text{meridional}} \tag{C.45}$$

$$= \cos(\theta)\cos(\phi_{\text{geodetic}}) \times \left(h + \frac{a^2}{\sqrt{a^2\cos^2(\phi_{\text{geodetic}}) + b^2\sin^2(\phi_{\text{geodetic}})}}\right), \tag{C.46}$$

$$y_{\text{ECEF}} = \sin(\theta)x_{\text{meridional}} \tag{C.47}$$

$$= \sin(\theta)\cos(\phi_{\text{geodetic}}) \times \left(h + \frac{a^2}{\sqrt{a^2\cos^2(\phi_{\text{geodetic}}) + b^2\sin^2(\phi_{\text{geodetic}})}} \right), \quad \text{(C.48)}$$

$$z_{\text{ECEF}} = \sin(\phi_{\text{geodetic}}) \times \left(h + \frac{b^2}{\sqrt{a^2\cos^2(\phi_{\text{geodetic}}) + b^2\sin^2(\phi_{\text{geodetic}})}} \right), \quad \text{(C.49)}$$

in terms of geodetic latitude ϕ_{geodetic}, longitude θ, and orthometric altitude h with respect to the reference geoid.

The inverse transformation, from ECEF *XYZ* to geodetic longitude–latitude–altitude coordinates, is

$$\theta = \text{atan2}(y_{\text{ECEF}}, x_{\text{ECEF}}), \quad \text{(C.50)}$$

$$\phi_{\text{geodetic}} = \text{atan2}\left(z_{\text{ECEF}} + \frac{e^2 a^2 \sin^3(\zeta)}{b}, \xi - e^2 a \cos^3(\zeta) \right), \quad \text{(C.51)}$$

$$h = \frac{\xi}{\cos(\phi)} - r_T, \quad \text{(C.52)}$$

where atan2 is the four-quadrant arctangent function in MATLAB and

$$\zeta = \text{atan2}(a z_{\text{ECEF}}, b\xi), \quad \text{(C.53)}$$

$$\xi = \sqrt{x_{\text{ECEF}}^2 + y_{\text{ECEF}}^2}, \quad \text{(C.54)}$$

$$r_T = \frac{a}{\sqrt{1 - e^2 \sin(\phi)}}, \quad \text{(C.55)}$$

where r_T is the transverse radius of curvature on the ellipsoid, a is the equatorial radius, b is the polar radius, and e is elliptical eccentricity.

C.3.5.4 Geocentric Latitude For points on the geoid surface, the tangent of geocentric latitude is the ratio of distance above (+) or below (−) the equator $[z = b\sin(\phi_{\text{parametric}})]$ to the distance from the polar axis $[(x_{\text{meridional}} = a\cos(\phi_{\text{parametric}})]$, or

$$\tan(\phi_{\text{GEOCENTRIC}}) = \frac{b\sin(\phi_{\text{parametric}})}{a\cos(\phi_{\text{parametric}})} \quad \text{(C.56)}$$

$$= \frac{b}{a}\tan(\phi_{\text{parametric}}) \quad \text{(C.57)}$$

$$= \frac{b^2}{a^2}\tan(\phi_{\text{geodetic}}), \quad \text{(C.58)}$$

from which, using the same trigonometric identities as were used for geodetic latitude,

$$\sin(\phi_{\text{geocentric}}) = \frac{\tan(\phi_{\text{geocentric}})}{\sqrt{1+\tan^2(\phi_{\text{geocentric}})}} \tag{C.59}$$

$$= \frac{b\sin(\phi_{\text{parametric}})}{\sqrt{a^2\cos^2(\phi_{\text{parametric}}) + b^2\sin^2(\phi_{\text{parametric}})}} \tag{C.60}$$

$$= \frac{b^2\sin(\phi_{\text{geodetic}})}{\sqrt{a^4\cos^2(\phi_{\text{geodetric}}) + b^4\sin^2(\phi_{\text{geodetic}})}}, \tag{C.61}$$

$$\cos(\phi_{\text{geocentric}}) = \frac{1}{\sqrt{1+\tan^2(\phi_{\text{geocentric}})}} \tag{C.62}$$

$$= \frac{a\cos(\phi_{\text{parametric}})}{\sqrt{a^2\cos^2(\phi_{\text{parametric}}) + b^2\sin^2(\phi_{\text{parametric}})}} \tag{C.63}$$

$$= \frac{a^2\cos(\phi_{\text{geodetic}})}{\sqrt{a^4\cos^2(\phi_{\text{geodetic}}) + b^4\sin^2(\phi_{\text{geodetic}})}}. \tag{C.64}$$

The inverse relationships are

$$\tan(\phi_{\text{parametric}}) = \frac{a}{b}\tan(\phi_{\text{geocentric}}), \tag{C.65}$$

$$\tan(\phi_{\text{geodetic}}) = \frac{a^2}{b^2}\tan(\phi_{\text{geocentric}}), \tag{C.66}$$

from which, using the same trigonometric identities again,

$$\sin(\phi_{\text{parametric}}) = \frac{\tan(\phi_{\text{parametric}})}{\sqrt{1+\tan^2(\phi_{\text{parametric}})}} \tag{C.67}$$

$$= \frac{a\sin(\phi_{\text{geocentric}})}{\sqrt{a^2\sin^2(\phi_{\text{geocentric}}) + b^2\cos^2(\phi_{\text{geocentric}})}}, \tag{C.68}$$

$$\sin(\phi_{\text{geodetic}}) = \frac{a^2\sin(\phi_{\text{geocentric}})}{\sqrt{a^4\sin^2(\phi_{\text{geocentric}}) + b^4\cos^2(\phi_{\text{geocentric}})}}, \tag{C.69}$$

$$\cos(\phi_{\text{parametric}}) = \frac{1}{\sqrt{1+\tan^2(\phi_{\text{parametric}})}} \tag{C.70}$$

$$= \frac{b\cos(\phi_{\text{geocentric}})}{\sqrt{a^2\sin^2(\phi_{\text{geocentric}}) + b^2\cos^2(\phi_{\text{geocentric}})}}, \tag{C.71}$$

$$\cos(\phi_{\text{geodetic}}) = \frac{b^2\cos(\phi_{\text{geocentric}})}{\sqrt{a^4\sin^2(\phi_{\text{geocentric}}) + b^4\cos^2(\phi_{\text{geocentric}})}}. \tag{C.72}$$

C.3.6 LTP Coordinates

Local tangent plane (LTP) coordinates, also called "locally level coordinates," are a return to the first-order model of the earth as being flat, where they serve as local reference directions for representing vehicle attitude and velocity for operation on or near the surface of the earth. A common orientation for LTP coordinates has one horizontal axis (the north axis) in the direction of increasing latitude and the other horizontal axis (the east axis) in the direction of increasing longitude, as illustrated in Fig. C.7. Horizontal location components in this local coordinate frame are called "relative northing" and "relative easting."

C.3.6.1 Alpha Wander Coordinates Maintaining east–north orientation was a problem for some INSs at the poles, where north and east directions change by 180°. Early gimbaled inertial systems could not slew the platform axes fast enough for near-polar operation. This problem was solved by letting the platform axes "wander" from north but keeping track of the angle α between north and a reference platform axis, as shown in Fig. C.8. This LTP orientation came to be called "alpha wander."'

C.3.6.2 ENU/NED Coordinates East–north–up (ENU) and north–east–down (NED) are two common right-handed LTP coordinate systems. ENU coordinates may be preferred to NED coordinates because altitude increases in the upward direction. But NED coordinates may also be preferred over ENU coordinates because the direction of a right (clockwise) turn is in the positive direction

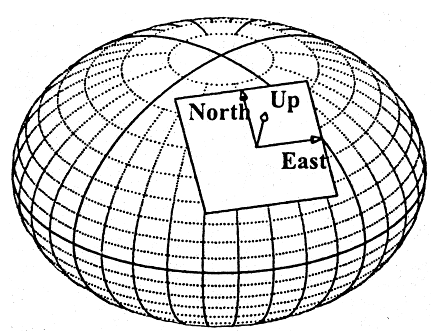

Fig. C.7 ENU coordinates.

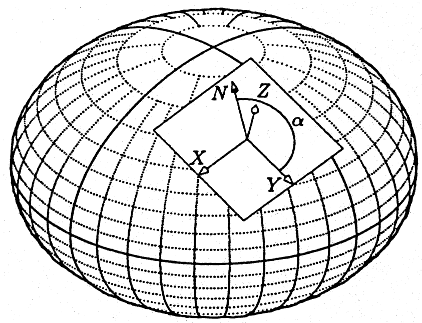

Fig. C.8 Alpha wander.

with respect to a downward axis, and NED coordinate axes coincide with vehicle-fixed roll–pitch–yaw (RPY) coordinates (Section C.3.7) when the vehicle is level and headed north.

The coordinate transformation matrix $\mathbf{C}_{\text{NED}}^{\text{ENU}}$ from ENU to NED coordinates and the transformation matrix $\mathbf{C}_{\text{ENU}}^{\text{NED}}$ from NED to ENU coordinates are one and the same:

$$\mathbf{C}_{\text{NED}}^{\text{ENU}} = \mathbf{C}_{\text{ENU}}^{\text{NED}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{C.73}$$

C.3.6.3 ENU/ECEF Coordinates The unit vectors in local *east*, *north*, and *up* directions, as expressed in ECEF Cartesian coordinates, will be

$$\mathbf{1}_E = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}, \tag{C.74}$$

$$\mathbf{1}_N = \begin{bmatrix} -\cos(\theta)\sin(\phi_{\text{geodetic}}) \\ -\sin(\theta)\sin(\phi_{\text{geodetic}}) \\ \cos(\phi_{\text{geodetic}}) \end{bmatrix}, \tag{C.75}$$

$$\mathbf{1}_U = \begin{bmatrix} \cos(\theta)\cos(\phi_{\text{geodetic}}) \\ \sin(\theta)\cos(\phi_{\text{geodetic}}) \\ \sin(\phi_{\text{geodetic}}) \end{bmatrix}, \tag{C.76}$$

and the unit vectors in the ECEF X, Y, and Z directions, as expressed in ENU coordinates, will be

$$\mathbf{1}_X = \begin{bmatrix} -\sin(\theta) \\ -\cos(\theta)\sin(\phi_{\text{geodetic}}) \\ \cos(\theta)\cos(\phi_{\text{geodetic}}) \end{bmatrix}, \tag{C.77}$$

$$\mathbf{1}_Y = \begin{bmatrix} \cos(\theta) \\ -\sin(\theta)\sin(\phi_{\text{geodetic}}) \\ \sin(\theta)\cos(\phi_{\text{geodetic}}) \end{bmatrix}, \tag{C.78}$$

$$\mathbf{1}_Z = \begin{bmatrix} 0 \\ \cos(\phi_{\text{geodetic}}) \\ \sin(\phi_{\text{geodetic}}) \end{bmatrix}. \tag{C.79}$$

C.3.6.4 NED/ECEF Coordinates It is more natural in some applications to use NED directions for locally level coordinates. This coordinate system coincides with vehicle-body-fixed RPY coordinates (shown in Fig. C.9) when the vehicle is level headed north. The unit vectors in local *north*, *east*, and *down* directions, as expressed in ECEF Cartesian coordinates, will be

$$\mathbf{1}_N = \begin{bmatrix} -\cos(\theta)\sin(\phi_{\text{geodetic}}) \\ -\sin(\theta)\sin(\phi_{\text{geodetic}}) \\ \cos(\phi_{\text{geodetic}}) \end{bmatrix}, \tag{C.80}$$

$$\mathbf{1}_E = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}, \tag{C.81}$$

$$\mathbf{1}_D = \begin{bmatrix} -\cos(\theta)\cos(\phi_{\text{geodetic}}) \\ -\sin(\theta)\cos(\phi_{\text{geodetic}}) \\ -\sin(\phi_{\text{geodetic}}) \end{bmatrix}, \tag{C.82}$$

and the unit vectors in the ECEF X, Y, and Z directions, as expressed in NED coordinates, will be

$$\mathbf{1}_X = \begin{bmatrix} -\cos(\theta)\sin(\phi_{\text{geodetic}}) \\ -\sin(\theta) \\ -\cos(\theta)\cos(\phi_{\text{geodetic}}) \end{bmatrix}, \tag{C.83}$$

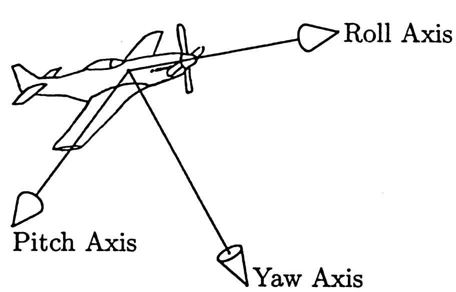

Fig. C.9 Roll-pitch-yaw axes.

$$\mathbf{1}_Y = \begin{bmatrix} -\sin(\theta)\sin(\phi_{\text{geodetic}}) \\ \cos(\theta) \\ -\sin(\theta)\cos(\phi_{\text{geodetic}}) \end{bmatrix}, \tag{C.84}$$

$$\mathbf{1}_Z = \begin{bmatrix} \cos(\phi_{\text{geodetic}}) \\ 0 \\ -\sin(\phi_{\text{geodetic}}) \end{bmatrix}, \tag{C.85}$$

C.3.7 RPY Coordinates

The RPY coordinates are vehicle fixed, with the roll axis in the nominal direction of motion of the vehicle, the pitch axis out the right-hand side, and the yaw axis such that turning to the right is positive, as illustrated in Fig.C.9. The same orientations of vehicle-fixed coordinates are used for surface ships and ground vehicles. They are also called "SAE coordinates," because they are the standard body-fixed coordinates used by the Society of Automotive Engineers.

For rocket boosters with their roll axes vertical at lift-off, the pitch axis is typically defined to be orthogonal to the plane of the boost trajectory (also called the "pitch plane" or "ascent plane").

C.3.8 Vehicle Attitude Euler Angles

The attitude of the vehicle body with respect to local coordinates can be specified in terms of rotations about the vehicle roll, pitch, and yaw axes, starting with these axes aligned with NED coordinates. The angles of rotation about each of these axes are called *Euler angles*, named for the Swiss mathematician Leonard Euler (1707–1783). It is always necessary to specify the order of rotations when specifying Euler (pronounced "oiler") angles.

A fairly common convention for vehicle attitude Euler angles is illustrated in Fig. C.10, where, starting with the vehicle level with roll axis pointed north:

1. *Yaw/Heading*. Rotate through the yaw angle (Y) about the vehicle yaw axis to the intended azimuth (heading) of the vehicle roll axis. Azimuth is measured clockwise (east) from north.

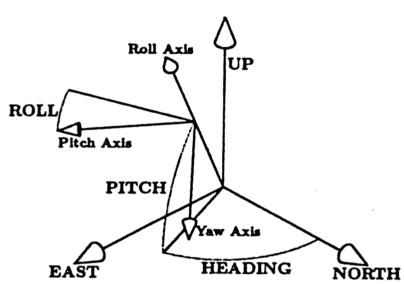

Fig. C.10 Vehicle Euler angles.

2. *Pitch*. Rotate through the pitch angle (P) about the vehicle pitch axis to bring the vehicle roll axis to its intended elevation. Elevation is measured positive upward from the local horizontal plane.
3. *Roll*. Rotate through the roll angle (R) about the vehicle roll axis to bring the vehicle attitude to the specified orientation.

Euler angles are redundant for vehicle attitudes with 90° pitch, in which case the roll axis is vertical. In that attitude, heading changes also rotate the vehicle about the roll axis. This is the attitude of most rocket boosters at lift-off. Some boosters can be seen making a roll maneuver immediately after lift-off to align their yaw axes with the launch azimuth in the ascent plane. This maneuver may be required to correct for launch delays on missions for which launch azimuth is a function of launch time.

C.3.8.1 RPY/ENU Coordinates With vehicle attitude specified by yaw angle (Y), pitch angle (P), and roll angle (R) as specified above, the resulting unit vectors of the roll, pitch, and yaw axes in ENU coordinates will be

$$\mathbf{1}_R = \begin{bmatrix} \sin(Y)\cos(P) \\ \cos(Y)\cos(P) \\ \sin(P) \end{bmatrix}, \tag{C.86}$$

$$\mathbf{1}_P = \begin{bmatrix} \cos(R)\cos(Y) + \sin(R)\sin(Y)\sin(P) \\ -\cos(R)\sin(Y) + \sin(R)\cos(Y)\sin(P) \\ -\sin(R)\cos(P) \end{bmatrix}, \tag{C.87}$$

$$\mathbf{1}_Y = \begin{bmatrix} -\sin(R)\cos(Y) + \cos(R)\sin(Y)\sin(P) \\ \sin(R)\sin(Y) + \cos(R)\cos(Y)\sin(P) \\ -\cos(R)\cos(P) \end{bmatrix}; \tag{C.88}$$

the unit vectors of the east, north, and up axes in RPY coordinates will be

$$\mathbf{1}_E = \begin{bmatrix} \sin(Y)\cos(P) \\ \cos(R)\cos(Y) + \sin(R)\sin(Y)\sin(P) \\ -\sin(R)\cos(Y) + \cos(R)\sin(Y)\sin(P) \end{bmatrix}, \tag{C.89}$$

$$\mathbf{1}_N = \begin{bmatrix} \cos(Y)\cos(P) \\ -\cos(R)\sin(Y) + \sin(R)\cos(Y)\sin(P) \\ \sin(R)\sin(Y) + \cos(R)\cos(Y)\sin(P) \end{bmatrix}, \tag{C.90}$$

$$\mathbf{1}_U = \begin{bmatrix} \sin(P) \\ -\sin(R)\cos(P) \\ -\cos(R)\cos(P) \end{bmatrix}; \tag{C.91}$$

and the coordinate transformation matrix from RPY coordinates to ENU coordinates will be

$$C_{\text{ENU}}^{\text{RPY}} = [\mathbf{1}_R \quad \mathbf{1}_P \quad \mathbf{1}_Y] = \begin{bmatrix} \mathbf{1}_E^{\text{T}} \\ \mathbf{1}_N^{\text{T}} \\ \mathbf{1}_U^{\text{T}} \end{bmatrix} \tag{C.92}$$

$$= \begin{bmatrix} S_Y C_P & C_R C_Y + S_R S_Y S_P & -S_R C_Y + C_R S_Y S_P \\ C_Y C_P & -C_R S_Y + S_R C_Y S_P & S_R S_Y + C_R C_Y S_P \\ S_P & -S_R C_P & -C_R C_P \end{bmatrix}, \tag{C.93}$$

where

$$S_R = \sin(R), \tag{C.94}$$

$$C_R = \cos(R), \tag{C.95}$$

$$S_P = \sin(P), \tag{C.96}$$

$$C_P = \cos(P), \tag{C.97}$$

$$S_Y = \sin(Y), \tag{C.98}$$

$$C_Y = \cos(Y). \tag{C.99}$$

C.3.9 GPS Coordinates

The parameter Ω in Fig. C.12 is the RAAN, which is the ECI longitude where the orbital plane intersects the equatorial plane as the satellite crosses from the Southern Hemisphere to the Northern Hemisphere. The orbital plane is specified by Ω and α, the inclination of the orbit plane with respect to the equatorial plane ($\alpha \approx 55^\circ$ for GPS satellite orbits). The θ parameter represents the location of the satellite within the orbit plane, as the angular phase in the circular orbit with respect to ascending node.

For GPS satellite orbits, the angle θ changes at a nearly constant rate of about 1.4584×10^{-4} rad/s and a period of about 43,082s (half a day).

The nominal satellite position in ECEF coordinates is then given as

$$x = R[\cos\theta\cos\Omega - \sin\theta\sin\Omega\cos\alpha], \tag{C.100}$$

$$y = R[\cos\theta\cos\Omega + \sin\theta\sin\Omega\cos\alpha], \tag{C.101}$$

$$z = R\sin\theta\sin\alpha, \tag{C.102}$$

$$\theta = \theta_0 + (t - t_0)\frac{360}{43,082}\text{deg}, \tag{C.103}$$

$$\Omega = \Omega_0 - (t - t_0)\frac{360}{86,164}\text{deg}, \tag{C.104}$$

$$R = 26{,}560{,}000 \text{ m}. \tag{C.105}$$

GPS satellite positions in the transmitted navigation message are specified in the ECEF coordinate system of WGS 84. A locally level x^1, y^1, z^1 reference coordinate system (described in Section C.3.6) is used by an observer location on the earth, where the $x^1 - y^1$ plane is tangential to the surface of the earth, x^1 pointing east, y^1 pointing north, and z^1 normal to the plane. See Fig. C.11. Here,

$$X_{\text{ENU}} = \mathbf{C}_{\text{ENU}}^{\text{ECEF}} X_{\text{ECEF}} + \mathbf{S},$$

$$\mathbf{C}_{\text{ENU}}^{\text{ECEF}} = \text{coordinate transformation matrix from ECEF to ENU},$$

$$\mathbf{S} = \text{coordinate origin shift vector from ECEF to local reference},$$

$$\mathbf{C}_{\text{ENU}}^{\text{ECEF}} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \end{bmatrix},$$

$$S = \begin{bmatrix} X_U \sin\theta - Y_U \cos\theta \\ X_U \sin\phi\cos\theta - Y_U \sin\phi\sin\theta - Z_U \cos\phi \\ -X_U \cos\phi\cos\theta - Y_U \cos\phi\sin\theta - Z_U \sin\phi \end{bmatrix},$$

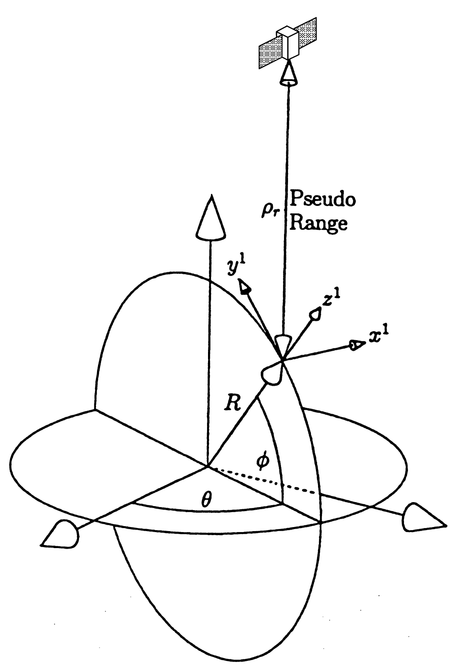

Fig. C.11 Pseudorange.

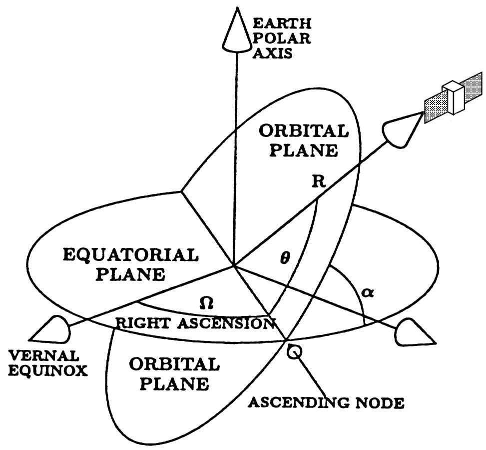

Fig. C.12 Satellite coordinates.

$$X_U, Y_U, Z_U = \text{user's position},$$
$$\theta = \text{local reference longitude},$$
$$\phi = \text{local geometric latitude}.$$

C.4 COORDINATE TRANSFORMATION MODELS

Coordinate transformations are methods for transforming a vector represented in one coordinate system into the appropriate representation in another coordinate system. These coordinate transformations can be represented in a number of different ways, each with its advantages and disadvantages.

These transformations generally involve translations (for coordinate systems with different origins) and rotations (for Cartesian coordinate systems with different axis directions) or transcendental transformations (between Cartesian and polar or geodetic coordinates). The transformations between Cartesian and polar coordinates have already been discussed in Section C.3.1 and translations are rather obvious, so we will concentrate on the rotations.

C.4.1 Euler Angles

Euler angles were used for defining vehicle attitude in Section C.3.8, and vehicle attitude representation is a common use of Euler angles in navigation.

Euler angles are used to define a coordinate transformation in terms of a set of three angular rotations, performed in a specified sequence about three specified orthogonal axes, to bring one coordinate frame to coincide with another.

The coordinate transformation from RPY coordinates to NED coordinates, for example, can be composed from three Euler rotation matrices:

$$C_{\text{NED}}^{\text{RPY}} = \overbrace{\begin{bmatrix} C_Y & -S_Y & 0 \\ S_Y & C_Y & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{Yaw}} \overbrace{\begin{bmatrix} C_P & 0 & S_P \\ 0 & 1 & 0 \\ -S_P & 0 & C_P \end{bmatrix}}^{\text{Pitch}} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & C_R & -S_R \\ 0 & S_R & C_R \end{bmatrix}}^{\text{Roll}} \tag{C.106}$$

$$= \underbrace{\begin{bmatrix} C_Y P_P & -S_Y C_R + C_Y S_P S_R & S_Y S_R + C_Y S_P C_R \\ S_Y C_P & C_Y C_R + S_Y S_P S_R & -C_Y S_R + S_Y S_P C_R \\ -S_P & C_P S_R & C_P C_R \\ (\text{rollaxis}) & (\text{pitchaxis}) & (\text{yawaxis}) \end{bmatrix}}_{\text{in NED coordinates}}, \tag{C.107}$$

where the matrix elements are defined in Eqs. C.94–C.99. This matrix also rotates the NED coordinate axes to coincide with RPY coordinate axes. (Compare this with the transformation from RPY to ENU coordinates in Eq. C.93.)

For example, the coordinate transformation for nominal booster rocket launch attitude (roll axis straight up) would be given by Eq. with pitch angle $P = \frac{1}{2}\pi$ ($C_P = 0$, $S_P = 1$), which becomes

$$\mathbf{C}_{\text{NED}}^{\text{RPY}} = \begin{bmatrix} 0 & \sin(R - Y) & \cos(R - Y) \\ 0 & \cos(R - Y) & -\sin(R - Y) \\ 1 & 10 & 0 \end{bmatrix}.$$

That is, the coordinate transformation in this attitude depends only on the difference between roll angle (R) and yaw angle (Y). Euler angles are a concise representation for vehicle attitude. They are handy for driving cockpit displays such as compass cards (using Y) and artificial horizon indicators (using R and P), but they are not particularly handy for representing vehicle attitude dynamics. The reasons for the latter include the following:

- Euler angles have discontinuities analogous to "gimbal lock" (Section 6.4.1.2) when the vehicle roll axis is pointed upward, as it is for launch of many rockets. In that orientation, tiny changes in vehicle pitch or yaw cause $\pm 180^\circ$ changes in heading angle. For aircraft, this creates a slewing rate problem for electromechanical compass card displays.
- The relationships between sensed body rates and Euler angle rates are mathematically complicated.

C.4.2 Rotation Vectors

All right-handed orthogonal coordinate systems with the same origins in three dimensions can be transformed one onto another by single rotations about fixed axes. The corresponding *rotation vectors* relating two coordinate systems are defined by the direction (rotation axis) and magnitude (rotation angle) of that transformation.

For example, the rotation vector for rotating ENU coordinates to NED coordinates (and vice versa) is

$$\boldsymbol{\rho}_{\text{NED}}^{\text{ENU}} = \begin{bmatrix} \pi/\sqrt{2} \\ \pi/\sqrt{2} \\ 0 \end{bmatrix}, \tag{C.108}$$

which has magnitude $|\boldsymbol{\rho}_{\text{NED}}^{\text{ENU}}| = \pi(180^\circ)$ and direction north—east, as illustrated in Fig. C.13. (For illustrative purposes only, NED coordinates are shown as being translated from ENU coordinates in Fig. C.13. In practice, rotation vectors represent pure rotations, without any translation.)

The rotation vector is another minimal representation of a coordinate transformation, along with Euler angles. Like Euler angles, rotation vectors are concise but also have some drawbacks:

1. It is not a unique representation, in that adding multiples of $\pm 2\pi$ to the magnitude of a rotation vector has no effect on the transformation it represents.
2. It is a nonlinear and rather complicated representation, in that the result of one rotation followed by another is a third rotation, the rotation vector for which is a fairly complicated function of the first two rotation vectors.

But, unlike Euler angles, rotation vector models do not exhibit "gimbal lock."

C.4.2.1 Rotation Vector to Matrix The rotation represented by a rotation vector

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} \tag{C.109}$$

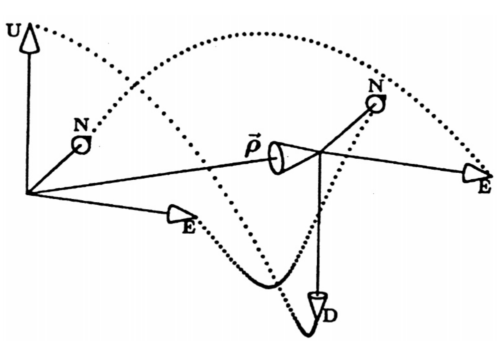

Fig. C.13 Rotation from ENU to NED coordinates.

can be implemented as multiplication by the matrix

$$\mathbf{C}(\boldsymbol{\rho}) \stackrel{\text{def}}{=} \exp(\boldsymbol{\rho}\otimes) \tag{C.110}$$

$$\stackrel{\text{def}}{=} \exp\left(\begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix}\right) \tag{C.111}$$

$$= \cos(|\boldsymbol{\rho}|)\mathbf{I}_3 + \frac{1-\cos(|\boldsymbol{\rho}|)}{|\boldsymbol{\rho}|^2}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}} + \frac{\sin(|\boldsymbol{\rho}|)}{|\boldsymbol{\rho}|}\begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix} \tag{C.112}$$

$$= \cos(\theta)\mathbf{I}_3 + (1-\cos(\theta))\mathbf{1}_\rho\mathbf{1}_\rho^{\mathrm{T}} + \sin(\theta)\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}, \tag{C.113}$$

$$\theta \stackrel{\text{def}}{=} |\boldsymbol{\rho}|, \tag{C.114}$$

$$\mathbf{1}_\rho \stackrel{\text{def}}{=} \frac{\boldsymbol{\rho}}{\boldsymbol{\rho}|}, \tag{C.115}$$

which was derived in Eq. B.17. That is, for any three-rowed column vector $\mathbf{v}$, $\mathbf{C}(\boldsymbol{\rho})\mathbf{v}$ rotates it through an angle of $|\boldsymbol{\rho}|$ radians about the vector $\boldsymbol{\rho}$.

The form of the matrix in Eq. C.113[2] is better suited for computation when $\theta \approx 0$, but the form of the matrix in Eq. C.112 is useful for computing sensitivities using partial derivatives (used in Chapter 8).

For example, the rotation vector $\rho_{\mathrm{NED}}^{\mathrm{ENU}}$ in Eq. C.108 transforming between ENU and NED has magnitude and direction

$$\theta = \pi \qquad [\sin(\theta) = 0, \cos(\theta) = -1],$$

$$\mathbf{1}_\rho = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix},$$

respectively, and the corresponding rotation matrix

$$\mathbf{C}_{\mathrm{NED}}^{\mathrm{ENU}} = \cos(\pi)\mathbf{I}_3 + [1-\cos(\pi)]\mathbf{1}_\rho\mathbf{1}_\rho + \sin(\pi)\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

$$= -\mathbf{I}_3 + 2\mathbf{1}_\rho\mathbf{1}_\rho^{\mathrm{T}} + 0$$

[2]Linear combinations of the sort $a_1\mathbf{I}_{3\times 3} + a_2[\mathbf{1}_\rho\otimes] + a_3\mathbf{1}_\rho\mathbf{1}_\rho^{\mathrm{T}}$, where $\mathbf{1}$ is a unit vector, form a subalgebra of 3×3 matrices with relatively simple rules for multiplication, inversion, etc.

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

transforms from ENU to NED coordinates. (Compare this result to Eq. C.73.) Because coordinate transformation matrices are orthogonal matrices and the matrix $\mathbf{C}_{\mathrm{NED}}^{\mathrm{ENU}}$ is also symmetric, $\mathbf{C}_{\mathrm{NED}}^{\mathrm{ENU}}$ is its own inverse. That is,

$$\mathbf{C}_{\mathrm{NED}}^{\mathrm{ENU}} = \mathbf{C}_{\mathrm{ENU}}^{\mathrm{NED}}. \tag{C.116}$$

C.4.2.2 Matrix to Rotation Vector Although there is a unique coordinate transformation matrix for each rotation vector, the converse is not true. Adding multiples of 2π to the magnitude of a rotation vector has no effect on the resulting coordinate transformation matrix. The following approach yields a unique rotation vector with magnitude $|\boldsymbol{\rho}| \leq \pi$.

The trace tr($\mathbf{C}$) of a square matrix $\mathbf{M}$ is the sum of its diagonal values. For the coordinate transformation matrix of Eq. C.112,

$$\mathrm{tr}[\mathbf{C}(\boldsymbol{\rho})] = 1 + 2\cos(\theta), \tag{C.117}$$

from which the rotation angle

$$|\boldsymbol{\rho}| = \theta \tag{C.118}$$

$$= \arccos\left(\frac{\mathrm{tr}[\mathbf{C}(\boldsymbol{\rho})] - 1}{2}\right), \tag{C.119}$$

a formula that will yield a result in the range $0 < \theta < \pi$, but with poor fidelity near where the derivative of the cosine equals zero at $\theta = 0$ and $\theta = \pi$.

The values of θ near $\theta = 0$ and $\theta = \pi$ can be better estimated using the sine of θ, which can be recovered using the antisymmetric part of $\mathbf{C}(\boldsymbol{\rho})$,

$$\mathbf{A} = \begin{bmatrix} 0 & -a_{21} & a_{13} \\ a_{21} & 0 & -a_{32} \\ -a_{13} & a_{32} & 0 \end{bmatrix} \tag{C.120}$$

$$\stackrel{\mathrm{def}}{=} \tfrac{1}{2}[\mathbf{C}(\boldsymbol{\rho}) - \mathbf{C}^{\mathrm{T}}(\boldsymbol{\rho})] \tag{C.121}$$

$$= \frac{\sin(\theta)}{\theta} \begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix}, \tag{C.122}$$

from which the vector

$$\begin{bmatrix} a_{32} \\ a_{13} \\ a_{21} \end{bmatrix} = \sin(\theta)\frac{1}{|\boldsymbol{\rho}|}\boldsymbol{\rho} \tag{C.123}$$

will have magnitude

$$\sqrt{a_{32}^2 + a_{13}^2 + a_{21}^2} = \sin(\theta) \tag{C.124}$$

and the same direction as $\boldsymbol{\rho}$. As a consequence, one can recover the magnitude θ of $\boldsymbol{\rho}$ from

$$\theta = \mathrm{atan2}\left(\sqrt{a_{32}^2 + a_{13}^2 + a_{21}^2}, \frac{\mathrm{tr}[\mathbf{C}(\boldsymbol{\rho})] - 1}{2}\right) \tag{C.125}$$

using the MATLAB function atan2, and then the rotation vector $\boldsymbol{\rho}$ as

$$\boldsymbol{\rho} = \frac{\theta}{\sin(\theta)}\begin{bmatrix} a_{32} \\ a_{13} \\ a_{21} \end{bmatrix} \tag{C.126}$$

when $0 < \theta < \pi$.

***C.4.2.3 Special Cases for* $\sin(\boldsymbol{\theta}) \approx \mathbf{0}$** For $\theta \approx 0$, $\boldsymbol{\rho} \approx 0$, although Eq. C.126 may still work adequately for $\theta > 10^{-6}$, say.

For $\theta \approx \pi$, the symmetric part of $\mathbf{C}(\boldsymbol{\rho})$,

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} \tag{C.127}$$

$$\stackrel{\mathrm{def}}{=} \tfrac{1}{2}[\mathbf{C}(\boldsymbol{\rho}) + \mathbf{C}^{\mathrm{T}}(\boldsymbol{\rho})] \tag{C.128}$$

$$= \cos(\theta)\mathbf{I}_3 + \frac{1 - \cos(\theta)}{\theta_2}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}} \tag{C.129}$$

$$\approx -\mathbf{I}_3 + \frac{2}{\theta^2}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}} \tag{C.130}$$

and the unit vector

$$\mathbf{1}_\rho \stackrel{\mathrm{def}}{=} \frac{1}{\theta}\boldsymbol{\rho} \tag{C.131}$$

satisfies

$$\mathbf{S} \approx \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 & 2u_1u_3 \\ 2u_1u_2 & 2u_2^2 - 1 & 2u_2u_3 \\ 2u_1u_3 & 2u_2u_3 & 2u_3^2 - 1 \end{bmatrix}, \tag{C.132}$$

which can be solved for a unique $\mathbf{u}$ by assigning $u_k > 0$ for

$$k = \operatorname{argmax}\left(\begin{bmatrix} s_{11} \\ s_{22} \\ s_{33} \end{bmatrix}\right), \tag{C.133}$$

$$u_k = \sqrt{\tfrac{1}{2}(s_{kk} + 1)} \tag{C.134}$$

then, depending on whether $k = 1$, $k = 2$, or $k = 3$,

$$\left.\begin{array}{cccc} & k=1 & k=2 & k=3 \\ u_1 \approx & \sqrt{\dfrac{s_{11}+1}{2}} & \dfrac{s_{12}}{2u_2} & \dfrac{s_{13}}{2u_3} \\ u_2 \approx & \dfrac{s_{12}}{2u_1} & \sqrt{\dfrac{s_{22}+1}{2}} & \dfrac{s_{23}}{2u_2} \\ u_3 \approx & \dfrac{s_{13}}{2u_1} & \dfrac{s_{23}}{2u_2} & \sqrt{\dfrac{s_{11}+1}{2}} \end{array}\right\} \tag{C.135}$$

and

$$\boldsymbol{\rho} = \theta \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \tag{C.136}$$

C.4.2.4 Time Derivatives of Rotation Vectors The mathematical relationships between rotation rates ω_k and the time derivatives of the corresponding rotation vector $\boldsymbol{\rho}$ are fairly complicated, but they can be derived from Eq. C.221 for the dynamics of coordinate transformation matrices.

Let $\boldsymbol{\rho}_{\mathrm{ENU}}$ be the rotation vector represented in earth-fixed ENU coordinates that rotates earth-fixed ENU coordinate axes into vehicle body-fixed RPY axes, and let $\mathbf{C}(\boldsymbol{\rho})$ be the corresponding rotation matrix, so that, in ENU coordinates,

$$\mathbf{1}_E = [\ 1 \quad 0 \quad 0\ ]^{\mathrm{T}}, \quad \mathbf{1}_N = [\ 0 \quad 1 \quad 0\ ]^{\mathrm{T}}, \quad \mathbf{1}_U = [\ 0 \quad 0 \quad 1\ ]^{\mathrm{T}},$$

$$\mathbf{C}(\boldsymbol{\rho}_{\mathrm{ENU}})\mathbf{1}_E = \mathbf{1}_R, \qquad \mathbf{C}(\boldsymbol{\rho}_{\mathrm{ENU}})\mathbf{I}_N = \mathbf{1}_P, \qquad \mathbf{C}(\boldsymbol{\rho}_{\mathrm{ENU}})\mathbf{1}_U = \mathbf{1}_Y,$$

$$\begin{aligned}
\mathbf{C}_{\text{ENU}}^{\text{RPY}} &= [\ \mathbf{1}_R \quad \mathbf{1}_P \quad \mathbf{1}_Y\ ], \\
&= [\mathbf{C}(\boldsymbol{\rho}_{\text{ENU}})\mathbf{1}_E \quad \mathbf{C}(\boldsymbol{\rho}_{\text{ENU}})\mathbf{1}_N \quad \mathbf{C}(\boldsymbol{\rho}_{\text{ENU}})\mathbf{1}_U] \\
&= \mathbf{C}(\boldsymbol{\rho}_{\text{ENU}})[\ \mathbf{1}_E \quad \mathbf{1}_N \quad \mathbf{1}_U\ ] \qquad \text{(C.137)} \\
&= \mathbf{C}(\boldsymbol{\rho}_{\text{ENU}}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathbf{C}_{\text{ENU}}^{\text{RPY}} &= \mathbf{C}(\boldsymbol{\rho}_{\text{ENU}}). \qquad \text{(C.138)}
\end{aligned}$$

That is, $\mathbf{C}(\boldsymbol{\rho}_{\text{ENU}})$ is the coordinate transformation matrix from RPY coordinates to ENU coordinates. As a consequence, from Eq. C.221,

$$\begin{aligned}
\frac{d}{dt}\mathbf{C}(\boldsymbol{\rho}_{\text{ENU}}) &= \frac{d}{dt}\mathbf{C}_{\text{ENU}}^{\text{RPY}} \qquad \text{(C.139)} \\
&= \begin{bmatrix} 0 & \omega_U & -\omega_N \\ -\omega_U & 0 & \omega_E \\ \omega_N & -\omega_E & 0 \end{bmatrix} \mathbf{C}_{\text{ENU}}^{\text{RPY}} \\
&\quad + \mathbf{C}_{\text{ENU}}^{\text{RPY}} \begin{bmatrix} 0 & -\omega_Y & \omega_P \\ \omega_Y & 0 & -\omega_R \\ -\omega_P & \omega_R & 0 \end{bmatrix}, \qquad \text{(C.140)} \\
\frac{d}{dt}\mathbf{C}(\boldsymbol{\rho}_{\text{ENU}}) &= \begin{bmatrix} 0 & \omega_U & -\omega_N \\ -\omega_U & 0 & \omega_E \\ \omega_N & -\omega_E & 0 \end{bmatrix} \mathbf{C}(\boldsymbol{\rho}_{\text{ENU}}) \\
&\quad + \mathbf{C}(\boldsymbol{\rho}_{\text{ENU}}) \begin{bmatrix} 0 & -\omega_Y & \omega_P \\ \omega_Y & 0 & -\omega_R \\ -\omega_P & \omega_R & 0 \end{bmatrix}, \qquad \text{(C.141)}
\end{aligned}$$

where

$$\boldsymbol{\omega}_{\text{RPY}} = \begin{bmatrix} \omega_R \\ \omega_P \\ \omega_Y \end{bmatrix} \qquad \text{(C.142)}$$

is the vector of inertial rotation rates of the vehicle body, expressed in RPY coordinates, and

$$\boldsymbol{\omega}_{\text{ENU}} = \begin{bmatrix} \omega_E \\ \omega_N \\ \omega_U \end{bmatrix} \qquad \text{(C.143)}$$

is the vector of inertial rotation rates of the ENU coordinate frame, expressed in ENU coordinates.

The 3×3 matrix equation C.141 is equivalent to nine scalar equations:

$$
\begin{aligned}
\frac{\partial c_{11}}{\partial \rho_E}\dot{\rho}_E + \frac{\partial c_{11}}{\partial \rho_N}\dot{\rho}_N + \frac{\partial c_{11}}{\partial \rho_U}\dot{\rho}_U &= -c_{1,3}\omega_P + c_{1,2}\omega_Y - c_{3,1}\omega_N + c_{2,1}\omega_U,\\
\frac{\partial c_{12}}{\partial \rho_E}\dot{\rho}_E + \frac{\partial c_{12}}{\partial \rho_N}\dot{\rho}_N + \frac{\partial c_{12}}{\partial \rho_U}\dot{\rho}_U &= c_{1,3}\omega_R - c_{1,1}\omega_Y - c_{3,2}\omega_N + c_{2,2}\omega_U,\\
\frac{\partial c_{13}}{\partial \rho_E}\dot{\rho}_E + \frac{\partial c_{13}}{\partial \rho_N}\dot{\rho}_N + \frac{\partial c_{13}}{\partial \rho_U}\dot{\rho}_U &= -c_{1,2}\omega_R + c_{1,1}\omega_P - c_{3,3}\omega_N + c_{2,3}\omega_U,\\
\frac{\partial c_{21}}{\partial \rho_E}\dot{\rho}_E + \frac{\partial c_{21}}{\partial \rho_N}\dot{\rho}_N + \frac{\partial c_{21}}{\partial \rho_U}\dot{\rho}_U &= -c_{2,3}\omega_P + c_{2,2}\omega_Y + c_{3,1}\omega_E - c_{1,1}\omega_U,\\
\frac{\partial c_{22}}{\partial \rho_E}\dot{\rho}_E + \frac{\partial c_{22}}{\partial \rho_N}\dot{\rho}_N + \frac{\partial c_{22}}{\partial \rho_U}\dot{\rho}_U &= c_{2,3}\omega_R - c_{2,1}\omega_Y + c_{3,2}\omega_E - c_{1,2}\omega_U,\\
\frac{\partial c_{23}}{\partial \rho_E}\dot{\rho}_E + \frac{\partial c_{23}}{\partial \rho_N}\dot{\rho}_N + \frac{\partial c_{23}}{\partial \rho_U}\dot{\rho}_U &= -c_{2,2}\omega_R + c_{2,1}\omega_P + c_{3,3}\omega_E - c_{1,3}\omega_U,\\
\frac{\partial c_{31}}{\partial \rho_E}\dot{\rho}_E + \frac{\partial c_{31}}{\partial \rho_N}\dot{\rho}_N + \frac{\partial c_{31}}{\partial \rho_U}\dot{\rho}_U &= -c_{3,3}\omega_P + c_{3,2}\omega_Y - c_{2,1}\omega_E + c_{1,1}\omega_N,\\
\frac{\partial c_{32}}{\partial \rho_E}\dot{\rho}_E + \frac{\partial c_{32}}{\partial \rho_N}\dot{\rho}_N + \frac{\partial c_{32}}{\partial \rho_U}\dot{\rho}_U &= c_{3,3}\omega_R - c_{3,1}\omega_Y - c_{2,2}\omega_E + c_{1,2}\omega_N,\\
\frac{\partial c_{33}}{\partial \rho_E}\dot{\rho}_E + \frac{\partial c_{33}}{\partial \rho_N}\dot{\rho}_N + \frac{\partial c_{33}}{\partial \rho_U}\dot{\rho}_U &= -c_{3,2}\omega_R + c_{3,1}\omega_P - c_{2,3}\omega_E + c_{1,3}\omega_N,
\end{aligned}
$$

where

$$
\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \stackrel{\text{def}}{=} \mathbf{C}(\boldsymbol{\rho}_{\text{ENU}})
$$

and the partial derivatives

$$
\begin{aligned}
\frac{\partial c_{11}}{\partial \rho_E} &= \frac{u_E(1-u_E^2)\{2[1-\cos(\theta)] - \theta\sin(\theta)\}}{\theta},\\
\frac{\partial c_{11}}{\partial \rho_N} &= \frac{u_N\{-2u_E^2[1-\cos(\theta)] - \theta\sin(\theta)(1-u_E^2)\}}{\theta},\\
\frac{\partial c_{11}}{\partial \rho_U} &= \frac{u_U\{-2u_E^2[1-\cos(\theta)] - \theta\sin(\theta)(1-u_E^2)\}}{\theta},\\
\frac{\partial c_{12}}{\partial \rho_E} &= \frac{u_N(1-2u_E^2)[1-\cos(\theta)] + u_E u_U\sin(\theta) - \theta u_E u_U\cos(\theta) + \theta u_N u_E^2\sin(\theta)}{\theta},\\
\frac{\partial c_{12}}{\partial \rho_N} &= \frac{u_E(1-2u_N^2)[1-\cos(\theta)] + u_U u_N\sin(\theta) - \theta u_N u_U\cos(\theta) + \theta u_E u_N^2\sin(\theta)}{\theta},
\end{aligned}
$$

$$\frac{\partial c_{12}}{\partial \rho_U} = \frac{-2u_E u_N u_U[1-\cos(\theta)] - (1-u_U^2)\sin(\theta) - \theta u_U^2\cos(\theta) + \theta u_U u_N u_E \sin(\theta)}{\theta},$$

$$\frac{\partial c_{13}}{\partial \rho_E} = \frac{u_U(1-2u_E^2)[1-\cos(\theta)] - u_E u_N \sin(\theta) + \theta u_E u_N \cos(\theta) + \theta u_U u_E^2 \sin(\theta)}{\theta},$$

$$\frac{\partial c_{13}}{\partial \rho_N} = \frac{-2u_E u_N u_U[1-\cos(\theta)] + (1-u_N^2)\sin(\theta) + \theta u_N^2\cos(\theta) + \theta u_U u_N u_E \sin(\theta)}{\theta},$$

$$\frac{\partial c_{13}}{\partial \rho_U} = \frac{u_E(1-2u_U^2)[1-\cos(\theta)] - u_U u_N \sin(\theta) + \theta u_N u_U \cos(\theta) + \theta u_E u_U^2 \sin(\theta)}{\theta},$$

$$\frac{\partial c_{21}}{\partial \rho_E} = \frac{u_N(1-2u_E^2)[1-\cos(\theta)] - u_E u_U \sin(\theta) + \theta u_E u_U \cos(\theta) + \theta u_N u_E^2 \sin(\theta)}{\theta},$$

$$\frac{\partial c_{21}}{\partial \rho_N} = \frac{u_E(1-2u_N^2)[1-\cos(\theta)] - u_U u_N \sin(\theta) + \theta u_N u_U \cos(\theta) + \theta u_E u_N^2 \sin(\theta)}{\theta},$$

$$\frac{\partial c_{21}}{\partial \rho_U} = \frac{-2u_E u_N u_U[1-\cos(\theta)] + \sin(\theta)(1-u_U^2) + \theta u_U^2\cos(\theta) + \theta u_U u_N u_E \sin(\theta)}{\theta},$$

$$\frac{\partial c_{22}}{\partial \rho_E} = \frac{u_E\{-2u_N^2[1-\cos(\theta)] - \theta(1-u_N^2)\sin(\theta)\}}{\theta},$$

$$\frac{\partial c_{22}}{\partial \rho_N} = \frac{u_N(1-u_N^2)\{2[1-\cos(\theta)] - \theta\sin(\theta)\}}{\theta},$$

$$\frac{\partial c_{22}}{\partial \rho_U} = \frac{u_U\{-2u_N^2[1-\cos(\theta)] - \theta(1-u_N^2)\sin(\theta)\}}{\theta},$$

$$\frac{\partial c_{23}}{\partial \rho_E} = \frac{-2u_E u_N u_U[1-\cos(\theta)] - (1-u_E^2)\sin(\theta) - \theta u_E^2\cos(\theta) + \theta u_E u_N u_U \sin(\theta)}{\theta},$$

$$\frac{\partial c_{23}}{\partial \rho_N} = \frac{u_U(1-2u_N^2)[1-\cos(\theta)] + u_E u_N \sin(\theta) - \theta u_E u_N \cos(\theta) + \theta u_N^2 u_U \sin(\theta)}{\theta},$$

$$\frac{\partial c_{23}}{\partial \rho_U} = \frac{u_N(1-2u_U^2)[1-\cos(\theta)] + u_E u_U \sin(\theta) - \theta u_E u_U \cos(\theta) + \theta u_U^2 u_N \sin(\theta)}{\theta},$$

$$\frac{\partial c_{31}}{\partial \rho_E} = \frac{u_U(1-2u_E^2)[1-\cos(\theta)] + u_E u_N \sin(\theta) - \theta u_E u_N \cos(\theta) + \theta u_U u_E^2 \sin(\theta)}{\theta},$$

$$\frac{\partial c_{31}}{\partial \rho_N} = \frac{-2u_E u_N u_U[1-\cos(\theta)] - (1-u_N^2)\sin(\theta) - \theta u_N^2\cos(\theta) + \theta u_U u_N u_E \sin(\theta)}{\theta},$$

$$\frac{\partial c_{31}}{\partial \rho_U} = \frac{u_E(1-2u_U^2)[1-\cos(\theta)] + u_U u_N \sin(\theta) - \theta u_N u_U \cos(\theta) + \theta u_E u_U^2 \sin(\theta)}{\theta},$$

$$\frac{\partial c_{32}}{\partial \rho_E} = \frac{-2u_E u_N u_U[1-\cos(\theta)] + (1-u_E^2)\sin(\theta) + \theta u_E^2\cos(\theta) + \theta u_U u_N u_E \sin(\theta)}{\theta},$$

$$\frac{\partial c_{32}}{\partial \rho_N} = \frac{u_U(1-2u_N^2)[1-\cos(\theta)] - u_E u_N \sin(\theta) + \theta u_E u_N \cos(\theta) + \theta u_N^2 u_U \sin(\theta)}{\theta},$$

$$
\begin{aligned}
\frac{\partial c_{32}}{\partial \rho_U} &= \frac{u_N(1-2u_U^2)[1-\cos(\theta)] - u_E u_U \sin(\theta) + \theta u_E u_U \cos(\theta) + \theta u_U^2 u_N \sin(\theta)}{\theta}, \\
\frac{\partial c_{33}}{\partial \rho_E} &= \frac{u_E\{-2u_U^2[1-\cos(\theta)] - \theta \sin(\theta)(1+u_U^2)\}}{\theta}, \\
\frac{\partial c_{33}}{\partial \rho_N} &= \frac{u_N\{-2u_U^2[1-\cos(\theta)] - \theta \sin(\theta)(1+u_U^2)\}}{\theta}, \\
\frac{\partial c_{33}}{\partial \rho_U} &= \frac{u_U(1-u_U^2)\{2[1-\cos(\theta)] - \theta \sin(\theta)\}}{\theta}
\end{aligned}
$$

for

$$
\begin{aligned}
&\theta \stackrel{\text{def}}{=} |\boldsymbol{\rho}_{\text{ENU}}|, \\
&u_E \stackrel{\text{def}}{=} \frac{\rho_E}{\theta}, \qquad u_N \stackrel{\text{def}}{=} \frac{\rho_N}{\theta}, \qquad u_u \stackrel{\text{def}}{=} \frac{\rho_U}{\theta}.
\end{aligned}
$$

These nine scalar linear equations can be put into matrix form and solved in least squares fashion as

$$
\mathbf{L}\begin{bmatrix} \dot{\rho}_E \\ \dot{\rho}_N \\ \dot{\rho}_U \end{bmatrix} = \mathbf{R}\begin{bmatrix} \omega_R \\ \omega_P \\ \omega_Y \\ \omega_E \\ \omega_N \\ \omega_U \end{bmatrix}, \tag{C.144}
$$

$$
\begin{bmatrix} \dot{\rho}_E \\ \dot{\rho}_N \\ \dot{\rho}_U \end{bmatrix} = \underbrace{[\mathbf{L}^{\mathrm{T}}\mathbf{L}]\backslash[\mathbf{L}^{\mathrm{T}}\mathbf{R}]}_{\partial\dot{\boldsymbol{\rho}}/\partial\boldsymbol{\omega}}\begin{bmatrix} \boldsymbol{\omega}_{\text{RPY}} \\ \boldsymbol{\omega}_{\text{ENU}} \end{bmatrix}. \tag{C.145}
$$

The matrix product $\mathbf{L}^T\mathbf{L}$ will always be invertible because its determinant

$$
\det[\mathbf{L}^{\mathrm{T}}\mathbf{L}] = 32\frac{[1-\cos(\theta)]^2}{\theta^4}, \tag{C.146}
$$

$$
\lim_{\theta\to 0}\det[\mathbf{L}^{\mathrm{T}}\mathbf{L}] = 8, \tag{C.147}
$$

and the resulting equation for $\boldsymbol{\rho}_{\text{ENU}}$ can be put into the form

$$
\dot{\boldsymbol{\rho}}_{\text{ENU}} = \left[\frac{\partial\dot{\boldsymbol{\rho}}}{\partial\boldsymbol{\omega}}\right]\begin{bmatrix} \boldsymbol{\omega}_{\text{RPY}} \\ \boldsymbol{\omega}_{\text{ENU}} \end{bmatrix}. \tag{C.148}
$$

The 3×6 matrix $\partial\boldsymbol{\rho}/\partial\boldsymbol{\omega}$ can be partitioned as

$$
\left[\frac{\partial\dot{\boldsymbol{\rho}}}{\partial\boldsymbol{\omega}}\right] = \left[\left.\frac{\partial\dot{\boldsymbol{\rho}}}{\partial\boldsymbol{\omega}_{\text{RPY}}}\right|\frac{\partial\dot{\boldsymbol{\rho}}}{\partial\boldsymbol{\omega}_{\text{ENU}}}\right] \tag{C.149}
$$

with 3×3 submatrices

$$\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{\omega}_{\mathrm{RPY}}} = \left[\frac{1}{|\boldsymbol{\rho}|^2} - \frac{\sin(|\boldsymbol{\rho}|)}{2|\boldsymbol{\rho}|[1-\cos(|\boldsymbol{\rho}|)]}\right]\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}} + \frac{|\boldsymbol{\rho}|\sin(|\boldsymbol{\rho}|)}{2[1-\cos(|\boldsymbol{\rho}|)]}\mathbf{I} + \frac{1}{2}[\boldsymbol{\rho}\otimes] \tag{C.150}$$

$$= \mathbf{1}_\rho \mathbf{1}_\rho^{\mathrm{T}} + \frac{\theta \sin(\theta)}{2[1-\cos(\theta)]}[\mathbf{I} - \mathbf{1}_\rho \mathbf{1}\rho^{\mathrm{T}}] + \frac{\theta}{2}[\mathbf{1}_\rho \otimes], \tag{C.151}$$

$$\lim_{P\to 0} |\boldsymbol{\rho}\frac{P\partial \dot{\boldsymbol{\rho}}}{\partial \boldsymbol{\omega}_{\mathrm{RPY}}} = \mathbf{I}, \tag{C.152}$$

$$\frac{\partial \dot{\boldsymbol{\rho}}}{\partial \boldsymbol{\omega}_{\mathrm{ENU}}} = -\left[\frac{1}{|\boldsymbol{\rho}|^2} - \frac{\sin(|\boldsymbol{\rho}|)}{2|\boldsymbol{\rho}|[1-\cos(|\boldsymbol{\rho}|)]}\right]\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}} - \frac{|\boldsymbol{\rho}|\sin(|\boldsymbol{\rho}|)}{2[1-\cos(|\boldsymbol{\rho}|)]}\mathbf{I} + \frac{1}{2}[\boldsymbol{\rho}\otimes]4 \tag{C.153}$$

$$= -\mathbf{1}_\rho \mathbf{1}_\rho^{\mathrm{T}} - \frac{\theta \sin(\theta)}{2[1-\cos(\theta)]}[\mathbf{I} - \mathbf{1}_\rho \mathbf{1}_\rho^{\mathrm{T}}] + \frac{\theta}{\mathbf{2}}[\mathbf{1}_\rho \otimes]. \tag{C.154}$$

$$\lim_{|\boldsymbol{\rho}|\to 0} \frac{\partial \dot{\boldsymbol{\rho}}}{\partial \boldsymbol{\omega}_{\mathrm{ENU}}} = -\mathbf{I}. \tag{C.155}$$

For locally leveled gimbaled systems, $\boldsymbol{\omega}_{\mathrm{RPH}} = \mathbf{0}$. That is, the gimbals normally keep the accelerometer axes aligned to the ENU or NED coordinate axes, a process modeled by $\boldsymbol{\omega}_{\mathrm{ENU}}$ alone.

C.4.2.5 Time Derivatives of Matrix Expressions The Kalman filter implementation for integrating GPS with a strapdown INS in Chapter 8 will require derivatives with respect to time of the matrices

$$\frac{\partial \dot{\boldsymbol{\rho}}_{\mathrm{ENU}}}{\partial \boldsymbol{\omega}_{\mathrm{RPH}}} \quad \text{(Eq.C.150)} \quad \text{and} \quad \frac{\partial \dot{\boldsymbol{\rho}}_{\mathrm{ENU}}}{\partial \boldsymbol{\omega}_{\mathrm{ENU}}} \quad \text{(Eq. C.153)}.$$

We derive here a general-purpose formula for taking such derivatives and then apply it to these two cases.

General Formulas There is a general-purpose formula for taking the time derivatives $(d/dt)\mathbf{M}(\boldsymbol{\rho})$ of matrix expressions of the sort

$$\mathbf{M}(\boldsymbol{\rho}) = \mathbf{M}(s_1(\boldsymbol{\rho}), s_2(\boldsymbol{\rho}), s_3(\boldsymbol{\rho})) \tag{C.156}$$

$$= s_1(\boldsymbol{\rho})\mathbf{I}_3 + s_2(\boldsymbol{\rho})[\boldsymbol{\rho}\otimes] + s_3(\boldsymbol{\rho})\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}, \tag{C.157}$$

that is, as linear combinations of $\mathbf{I}_3$, $\boldsymbol{\rho}\otimes$, and $\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}$ with scalar functions of $\boldsymbol{\rho}$ as the coefficients.

The derivation uses the time derivatives of the basis matrices,

$$\frac{d}{dt}\mathbf{I}_3 = \mathbf{0}_3, \tag{C.158}$$

$$\frac{d}{dt}[\boldsymbol{\rho}\otimes] = [\dot{\boldsymbol{\rho}}\otimes], \tag{C.159}$$

$$\frac{d}{dt}\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}} = \dot{\boldsymbol{\rho}}\boldsymbol{\rho}^{\mathrm{T}} + \boldsymbol{\rho}\dot{\boldsymbol{\rho}}^{\mathrm{T}}, \tag{C.160}$$

where the vector

$$\dot{\boldsymbol{\rho}} = \frac{d}{dt}\boldsymbol{\rho}, \tag{C.161}$$

and then uses the chain rule for differentiation to obtain the general formula

$$\begin{aligned}\frac{d}{dt}\mathbf{M}(\boldsymbol{\rho}) = {} & \frac{\partial s_1(\boldsymbol{\rho})}{\partial\boldsymbol{\rho}}\dot{\boldsymbol{\rho}}\mathbf{I}_3 + \frac{\partial s_2(\boldsymbol{\rho})}{\partial\boldsymbol{\rho}}\dot{\boldsymbol{\rho}}[\boldsymbol{\rho}\otimes] + s_2(\boldsymbol{\rho})[\dot{\boldsymbol{\rho}}\otimes],\\ & + \frac{\partial s_3(\boldsymbol{\rho})}{\partial\boldsymbol{\rho}}\dot{\boldsymbol{\rho}}[\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}] + s_3(\boldsymbol{\rho})[\dot{\boldsymbol{\rho}}\boldsymbol{\rho}^{\mathrm{T}} + \boldsymbol{\rho}\dot{\boldsymbol{\rho}}^{\mathrm{T}}],\end{aligned} \tag{C.162}$$

where the gradients $\partial s_i(\boldsymbol{\rho})/\partial\boldsymbol{\rho}$ are to be computed as row vectors and the inner products $[\partial s_i(\boldsymbol{\rho})/\partial\boldsymbol{\rho}]\dot{\boldsymbol{\rho}}$ will be scalars.

Equation C.162 is the general-purpose formula for the matrix forms of interest, which differ only in their scalar functions $s_i(\boldsymbol{\rho})$. These scalar functions $s_i(\boldsymbol{\rho})$ are generally rational functions of the following scalar functions (shown in terms of their gradients):

$$\frac{\partial}{\partial\boldsymbol{\rho}}|\boldsymbol{\rho}|^p = p|\boldsymbol{\rho}|^{p-2}\boldsymbol{\rho}^{\mathrm{T}}, \tag{C.163}$$

$$\frac{\partial}{\partial\boldsymbol{\rho}}\sin(|\boldsymbol{\rho}|) = \cos(|\boldsymbol{\rho}|)|\boldsymbol{\rho}|^{-1}\boldsymbol{\rho}^{\mathrm{T}}, \tag{C.164}$$

$$\frac{\partial}{\partial\boldsymbol{\rho}}\cos(|\boldsymbol{\rho}|) = -\sin(|\boldsymbol{\rho}|)|\boldsymbol{\rho}|^{-1}\boldsymbol{\rho}^{\mathrm{T}} \tag{C.165}$$

Time Derivative of $\partial\dot{\boldsymbol{\rho}}_{\mathrm{ENU}}/\partial\boldsymbol{\omega}_{\mathrm{RPY}}$ In this case (Eq. C.150).

$$s_1(\boldsymbol{\rho}) = \frac{|\boldsymbol{\rho}|\sin(|\boldsymbol{\rho}|)}{2[1-\cos(|\boldsymbol{\rho}|)]}, \tag{C.166}$$

$$\frac{\partial s_1(\boldsymbol{\rho})}{\partial\boldsymbol{\rho}} = -\frac{1-|\boldsymbol{\rho}|^{-1}\sin(|\boldsymbol{\rho}|)}{2[1-\cos(|\boldsymbol{\rho}|)]}\boldsymbol{\rho}^{\mathrm{T}}, \tag{C.167}$$

$$s_2(\boldsymbol{\rho}) = \tfrac{1}{2}, \tag{C.168}$$

$$\frac{\partial s_2}{\partial \boldsymbol{\rho}} = \mathbf{0}_{1\times 3}, \tag{C.169}$$

$$s_3(\boldsymbol{\rho}) = \left[\frac{1}{|\boldsymbol{\rho}|^2} - \frac{\sin(|\boldsymbol{\rho}|)}{2|\boldsymbol{\rho}|[1-\cos(|\boldsymbol{\rho}|)]}\right], \tag{C.170}$$

$$\frac{\partial s_3(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}} = \frac{1 + |\boldsymbol{\rho}|^{-1}\sin(|\boldsymbol{\rho}|) - 4|\boldsymbol{\rho}|^{-2}[1-\cos(|\boldsymbol{\rho}|)]}{2|\boldsymbol{\rho}|^2[1-\cos(|\boldsymbol{\rho}|)]}\boldsymbol{\rho}^{\mathrm{T}}, \tag{C.171}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial \dot{\boldsymbol{\rho}}_{\mathrm{ENU}}}{\partial \boldsymbol{\omega}_{\mathrm{RPY}}} &= \frac{\partial s_1(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}\dot{\boldsymbol{\rho}}\mathbf{I}_3 + \frac{\partial s_2(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}\dot{\boldsymbol{\rho}}[\boldsymbol{\rho}\otimes] + s_2(\boldsymbol{\rho})[\dot{\boldsymbol{\rho}}\otimes] \\ &\quad + \frac{\partial s_3(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}\dot{\boldsymbol{\rho}}[\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}] + s_3(\boldsymbol{\rho})[\dot{\boldsymbol{\rho}}\boldsymbol{\rho}^{\mathrm{T}} + \boldsymbol{\rho}\dot{\boldsymbol{\rho}}^{\mathrm{T}}], \end{aligned}\tag{C.172}$$

$$\begin{aligned} &= -\left(\frac{1-|\boldsymbol{\rho}|^{-1}\sin(|\boldsymbol{\rho}|)}{2[1-\cos(|\boldsymbol{\rho}|)]}\right)(\boldsymbol{\rho}^{\mathrm{T}}\dot{\boldsymbol{\rho}})\mathbf{I}_3 + \frac{1}{2}[\dot{\boldsymbol{\rho}}\otimes], \\ &\quad + \left(\frac{1 + |\boldsymbol{\rho}|^{-1}\sin(|\boldsymbol{\rho}|) - 4|\boldsymbol{\rho}|^{-2}[1-\cos(|\boldsymbol{\rho}|)]}{2|\boldsymbol{\rho}|^2[1-\cos(|\boldsymbol{\rho}|)]}\right) \times (\boldsymbol{\rho}^{\mathrm{T}}\dot{\boldsymbol{\rho}})[\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}], \\ &\quad + \left(\frac{1}{|\boldsymbol{\rho}|^2} - \frac{\sin(|\boldsymbol{\rho}|)}{2|\boldsymbol{\rho}|[1-\cos(|\boldsymbol{\rho}|)]}\right)[\dot{\boldsymbol{\rho}}\boldsymbol{\rho}^{\mathrm{T}} + \boldsymbol{\rho}\dot{\boldsymbol{\rho}}^{\mathrm{T}}]. \end{aligned}\tag{C.173}$$

Time Derivative of $\partial\dot{\boldsymbol{\rho}}_{\mathrm{ENU}}/\partial\boldsymbol{\omega}_{\mathrm{ENU}}$ *In this case (Eq. C.153)*,

$$s_1(\boldsymbol{\rho}) = -\frac{|\boldsymbol{\rho}|\sin(|\boldsymbol{\rho}|)}{2[1-\cos(|\boldsymbol{\rho}|)]}, \tag{C.174}$$

$$\frac{\partial s_1(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}} = \frac{1-|\boldsymbol{\rho}|^{-1}\sin(|\boldsymbol{\rho}|)}{2[1-\cos(|\boldsymbol{\rho}|)]}\boldsymbol{\rho}^{\mathrm{T}}, \tag{C.175}$$

$$s_2(\boldsymbol{\rho}) = \tfrac{1}{2}, \tag{C.176}$$

$$\frac{\partial s_2}{\partial \boldsymbol{\rho}} = \mathbf{0}_{1\times 3}, \tag{C.177}$$

$$s_3(\boldsymbol{\rho}) = -\left[\frac{1}{|\boldsymbol{\rho}|^2} - \frac{\sin(|\boldsymbol{\rho}|)}{2|\boldsymbol{\rho}|[1-\cos(|\boldsymbol{\rho}|)]}\right], \tag{C.178}$$

$$\frac{\partial s_3(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}} = -\frac{1 + |\boldsymbol{\rho}|^{-1}\sin(|\boldsymbol{\rho}|) - 4|\boldsymbol{\rho}|^{-2}[1-\cos(|\boldsymbol{\rho}|)]}{2|\boldsymbol{\rho}|^2[1-\cos(|\boldsymbol{\rho}|)]}\boldsymbol{\rho}^{\mathrm{T}}, \tag{C.179}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial \dot{\boldsymbol{\rho}}_{\mathrm{ENU}}}{\partial \boldsymbol{\omega}_{\mathrm{ENU}}} &= \frac{\partial s_1(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}\dot{\boldsymbol{\rho}}\mathbf{I}_3 + \frac{\partial s_2(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}\dot{\boldsymbol{\rho}}[\boldsymbol{\rho}\otimes] + s_2(\boldsymbol{\rho})[\dot{\boldsymbol{\rho}}\otimes], \\ &\quad + \frac{\partial s_3(\boldsymbol{\rho})}{\partial \boldsymbol{\rho}}\dot{\boldsymbol{\rho}}[\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}}] + s_3(\boldsymbol{\rho})[\dot{\boldsymbol{\rho}}\boldsymbol{\rho}^{\mathrm{T}} + \boldsymbol{\rho}\dot{\boldsymbol{\rho}}^{\mathrm{T}}] \end{aligned}\tag{C.180}$$

$$= \left(\frac{1 - |\boldsymbol{\rho}|^{-1} \sin(|\boldsymbol{\rho}|)}{2[1 - \cos(|\boldsymbol{\rho}|)]} \right) (\boldsymbol{\rho}^{\mathrm{T}} \dot{\boldsymbol{\rho}}) \mathbf{I}_3 + \frac{1}{2} (\boldsymbol{\rho}) [\dot{\boldsymbol{\rho}} \otimes],$$
$$- \left(\frac{1 + |\boldsymbol{\rho}|^{-1} \sin(|\boldsymbol{\rho}|) - 4|\boldsymbol{\rho}|^{-2} [1 - \cos(|\boldsymbol{\rho}|)]}{2|\boldsymbol{\rho}|^2 [1 - \cos(|\boldsymbol{\rho}|)]} \right) \times (\boldsymbol{\rho}^{\mathrm{T}} \dot{\boldsymbol{\rho}}) [\boldsymbol{\rho} \boldsymbol{\rho}^{\mathrm{T}}],$$
$$- \left(\frac{1}{|\boldsymbol{\rho}|^2} - \frac{\sin(|\boldsymbol{\rho}|)}{2|\boldsymbol{\rho}|[1 - \cos(|\boldsymbol{\rho}|)]} \right) [\dot{\boldsymbol{\rho}} \boldsymbol{\rho}^{\mathrm{T}} + \boldsymbol{\rho} \dot{\boldsymbol{\rho}}^{\mathrm{T}}]. \tag{C.181}$$

C.4.2.6 Partial Derivatives with Respect to Rotation Vectors Calculation of the dynamic coefficient matrices **F** and measurement sensitivity matrices **H** in linearized or extended Kalman filtering with rotation vectors $\boldsymbol{\rho}_{\mathrm{ENU}}$ as part of the system model state vector requires taking derivatives with respect to $\boldsymbol{\rho}_{\mathrm{ENU}}$ of associated vector-valued **f**- or **h**-functions, as

$$\mathbf{F} = \frac{\partial \mathbf{f}(\boldsymbol{\rho}_{\mathrm{ENU}}, \mathbf{v})}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}}, \tag{C.182}$$

$$\mathbf{H} = \frac{\partial \mathbf{h}(\boldsymbol{\rho}_{\mathrm{ENU}}, \mathbf{v})}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}}, \tag{C.183}$$

where the vector-valued functions will have the general form

$$\begin{aligned} &\mathbf{f}(\boldsymbol{\rho}_{\mathrm{ENU}}, \mathbf{v}) \text{ or } \mathbf{h}(\boldsymbol{\rho}_{\mathrm{ENU}}, \mathbf{v}) \\ &\quad = \{ s_0(\boldsymbol{\rho}_{\mathrm{ENU}}) \mathbf{I}_3 + s_1(\boldsymbol{\rho}_{\mathrm{ENU}}) [\boldsymbol{\rho}_{\mathrm{ENU}} \otimes] + s_2(\boldsymbol{\rho}_{\mathrm{ENU}}) \boldsymbol{\rho}_{\mathrm{ENU}} \boldsymbol{\rho}_{\mathrm{ENU}}^{\mathrm{T}} \} \mathbf{v}, \end{aligned} \tag{C.184}$$

and s_0, s_1, s_2 are scalar-valued functions of $\boldsymbol{\rho}_{\mathrm{ENU}}$ and $\mathbf{v}$ is a vector that does not depend on $\boldsymbol{\rho}_{\mathrm{ENU}}$. We will derive here the general formulas that can be used for taking the partial derivatives $\partial \mathbf{f}(\boldsymbol{\rho}_{\mathrm{ENU}}, \mathbf{v}) / \partial \boldsymbol{\rho}_{\mathrm{ENU}}$ or $\partial \mathbf{h}(\boldsymbol{\rho}_{\mathrm{ENU}}, \mathbf{v}) / \partial \boldsymbol{\rho}_{\mathrm{ENU}}$. These formulas can all be derived by calculating the derivatives of the different factors in the functional forms and then using the chain rule for differentiation to obtain the final result.

Derivatives of Scalars The derivatives of the scalar factors s_0, s_1, s_2 are

$$\frac{\partial}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}} s_i(\boldsymbol{\rho}_{\mathrm{ENU}}) = \left[\frac{\partial s_i(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_E} \frac{\partial s_i(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_N} \frac{\partial s_i(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_U} \right], \tag{C.185}$$

a row vector. Consequently, for any vector-valued function $\mathbf{g}(\boldsymbol{\rho}_{\mathrm{ENU}})$ by the chain rule, the derivatives of the vector-valued product $s_i(\boldsymbol{\rho}_{\mathrm{ENU}}) \mathbf{g}(\boldsymbol{\rho}_{\mathrm{ENU}})$ are

$$\frac{\partial \{ s_i(\boldsymbol{\rho}_{\mathrm{ENU}}) \mathbf{g}(\boldsymbol{\rho}_{\mathrm{ENU}}) \}}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}} = \underbrace{\mathbf{g}(\boldsymbol{\rho}_{\mathrm{ENU}}) \frac{\partial s_i(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}}}_{3 \times 3 \text{matrix}} + s_i(\boldsymbol{\rho}_{\mathrm{ENU}}) \underbrace{\frac{\partial \mathbf{g}(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}}}_{3 \times 3 \text{matrix}}, \tag{C.186}$$

the result of which will be the 3×3 Jacobian matrix of that subexpression in **f** or **h**.

Derivatives of Vectors The three potential forms of the vector-valued function $\mathbf{g}$ in Eq. C.186 are

$$\mathbf{g}(\boldsymbol{\rho}_{\mathrm{ENU}}) = \begin{cases} \mathbf{I}\mathbf{v} = \mathbf{v}, \\ \boldsymbol{\rho}_{\mathrm{ENU}} \otimes \mathbf{v}, \\ \boldsymbol{\rho}_{\mathrm{ENU}} \boldsymbol{\rho}_{\mathrm{ENU}}^{T} \mathbf{v}, \end{cases} \tag{C.187}$$

each of which is considered independently:

$$\frac{\partial \mathbf{v}}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}} = 0_{3\times 3}, \tag{C.188}$$

$$\frac{\partial \boldsymbol{\rho}_{\mathrm{ENU}} \otimes \mathbf{v}}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}} = \frac{\partial [-\mathbf{v} \otimes \boldsymbol{\rho}_{\mathrm{ENU}}]}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}}, \tag{C.189}$$

$$= -[\mathbf{v}\otimes], \tag{C.190}$$

$$= -\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}, \tag{C.191}$$

$$\frac{\partial \boldsymbol{\rho}_{\mathrm{ENU}} \boldsymbol{\rho}_{\mathrm{ENU}}^{\mathrm{T}} \mathbf{v}}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}} = (\boldsymbol{\rho}_{\mathrm{ENU}}^{\mathrm{T}} \mathbf{v}) \frac{\partial \boldsymbol{\rho}_{\mathrm{ENU}}}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}} + \boldsymbol{\rho}_{\mathrm{ENU}} \frac{\partial \boldsymbol{\rho}_{\mathrm{ENU}}^{\mathrm{T}} \mathbf{v}}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}}, \tag{C.192}$$

$$= (\boldsymbol{\rho}_{\mathrm{ENU}}^{\mathrm{T}} \mathbf{v}) \mathbf{I}_{3\times 3} + \boldsymbol{\rho}_{\mathrm{ENU}} \mathbf{v}^{\mathrm{T}}. \tag{C.193}$$

General Formula Combining the above formulas for the different parts, one can obtain the following general-purpose formula:

$$\begin{aligned} &\frac{\partial}{\partial \boldsymbol{\rho}_{\mathrm{ENU}}} \{ s_0(\boldsymbol{\rho}_{\mathrm{ENU}}) \mathbf{I}_3 + s_1(\boldsymbol{\rho}_{\mathrm{ENU}}) [\boldsymbol{\rho}_{\mathrm{ENU}} \otimes] + s_2(\boldsymbol{\rho}_{\mathrm{ENU}}) \boldsymbol{\rho}_{\mathrm{ENU}} \boldsymbol{\rho}_{\mathrm{ENU}}^{T} \} \mathbf{v} \\ &\quad = \mathbf{v} \left[\frac{\partial s_0(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_E} \frac{\partial s_0(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_N} \frac{\partial s_0(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_U} \right] \\ &\qquad + [\boldsymbol{\rho}_{\mathrm{ENU}} \otimes \mathbf{v}] \left[ccc \frac{\partial s_1(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_E} \frac{\partial s_1(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_N} \frac{\partial s_1(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_U} \right] \\ &\qquad - s_1(\boldsymbol{\rho}_{\mathrm{ENU}}) [\mathbf{v}\otimes] \\ &\qquad + (\boldsymbol{\rho}_{\mathrm{ENU}}^{\mathrm{T}} \mathbf{v}) \boldsymbol{\rho}_{\mathrm{ENU}} \left[\frac{\partial s_2(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_E} \frac{\partial s_2(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_N} \frac{\partial s_2(\boldsymbol{\rho}_{\mathrm{ENU}})}{\partial \rho_U} \right] \\ &\qquad + s_2(\boldsymbol{\rho}_{\mathrm{ENU}}) [(\boldsymbol{\rho}_{\mathrm{ENU}}^{\mathrm{T}} \mathbf{v}) \mathbf{I}_{3\times 3} + \boldsymbol{\rho}_{\mathrm{ENU}} \mathbf{v}^{\mathrm{T}}], \end{aligned} \tag{C.194}$$

applicable for any differentiable scalar functions s_0, s_1, s_2.

C.4.3 Direction Cosines Matrix

We have demonstrated in Eq.C.12 that the coordinate transformation matrix between one orthogonal coordinate system and another is a matrix of direction cosines between the unit axis vectors of the two coordinate systems,

$$\mathbf{C}_{XYZ}^{UVW} = \begin{bmatrix} \cos(\theta_{XU}) & \cos(\theta_{XV}) & \cos(\theta_{XW}) \\ \cos(\theta_{YU}) & \cos(\theta_{YV}) & \cos(\theta_{YW}) \\ \cos(\theta_{ZU}) & \cos(\theta_{ZV}) & \cos(\theta_{ZW}) \end{bmatrix}. \tag{C.195}$$

Because the angles do not depend on the order of the direction vectors (i.e., $\theta_{ab} = \theta_{ba}$), the inverse transformation matrix

$$\mathbf{C}_{UVW}^{XYZ} = \begin{bmatrix} \cos(\theta_{UX}) & \cos(\theta_{UY}) & \cos(\theta_{UZ}) \\ \cos(\theta_{VX}) & \cos(\theta_{VY}) & \cos(\theta_{VZ}) \\ \cos(\theta_{WX}) & \cos(\theta_{WY}) & \cos(\theta_{WX}) \end{bmatrix}, \tag{C.196}$$

$$= \begin{bmatrix} \cos(\theta_{XU}) & \cos(\theta_{XV}) & \cos(\theta_{XW}) \\ \cos(\theta_{YU}) & \cos(\theta_{YV}) & \cos(\theta_{YW}) \\ \cos(\theta_{ZU}) & \cos(\theta_{ZV}) & \cos(\theta_{ZW}) \end{bmatrix}^{\mathrm{T}}, \tag{C.197}$$

$$= (\mathbf{C}_{XYZ}^{UVW})^{\mathrm{T}}. \tag{C.198}$$

That is, the inverse coordinate transformation matrix is the transpose of the forward coordinate transformation matrix. This implies that the coordinate transformation matrices are orthogonal matrices.

C.4.3.1 Rotating Coordinates Let "rot" denote a set of rotating coordinates, with axes $\mathrm{X_{rot}}$, $\mathrm{Y_{rot}}$, $\mathrm{Z_{rot}}$, and let "non" represent a set of non-rotating (i.e., inertial) coordinates, with axes $\mathrm{X_{non}}$, $\mathrm{Y_{non}}$, $\mathrm{Z_{non}}$, as illustrated in Fig. C.14.

Any vector $\mathbf{v}_{\mathrm{rot}}$ in rotating coordinates can be represented in terms of its nonrotating components and unit vectors parallel to the nonrotating axes, as

$$\mathbf{v}_{\mathrm{rot}} = v_{x,\mathrm{non}}\mathbf{1}_{x,\mathrm{non}} + v_{y,\mathrm{non}}\mathbf{1}_{y,\mathrm{non}} + v_{z,\mathrm{non}}\mathbf{1}_{z,\mathrm{non}} \tag{C.199}$$

$$= [\ \mathbf{1}_{x,\mathrm{non}} \quad \mathbf{1}_{y,\mathrm{non}} \quad \mathbf{1}_{z,\mathrm{non}}\ ] \begin{bmatrix} v_{x,\mathrm{non}} \\ v_{y,\mathrm{non}} \\ v_{z,\mathrm{non}} \end{bmatrix} \tag{C.200}$$

$$= \mathbf{C}_{\mathrm{rot}}^{\mathrm{non}}\mathbf{v}_{\mathrm{non}}, \tag{C.201}$$

where $v_{x,\mathrm{non}}$, $v_{y,\mathrm{non}}$, $v_{z,\mathrm{non}}$ are nonrotating components of the vector, $\mathbf{1}_{x,\mathrm{non}}$, $\mathbf{1}_{y,\mathrm{non}}$, $\mathbf{1}_{z,\mathrm{non}} =$ unit vectors along X_{non}, Y_{non}, Z_{non} axes, as expressed in rotating coordinates

$\mathbf{v}_{\mathrm{rot}} =$ vector $\mathbf{v}$ expressed in RPY coordinates

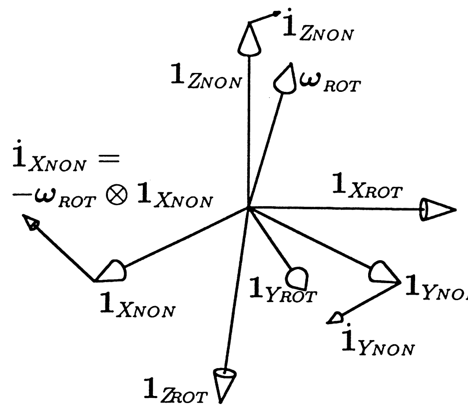

Fig. C.14 Rotating coordinates.

$\mathbf{v}_{\text{non}}$ = vector $\mathbf{v}$ expressed in ECI coordinates,

$\mathbf{C}_{\text{rot}}^{\text{non}}$ = coordinate transformation matrix from nonrotating coordinates to rotating coordinates

and

$$\mathbf{C}_{\text{rot}}^{\text{non}} = [\ \mathbf{1}_{x,\text{non}} \quad \mathbf{1}_{y,\text{non}} \quad \mathbf{1}_{z,\text{non}}\ ]. \tag{C.202}$$

The time derivative of $\mathbf{C}_{\text{rot}}^{\text{non}}$, as viewed from the non-rotating coordinate frame, can be derived in terms of the dynamics of the unit vectors $\mathbf{1}_{x,\text{non}}$, $\mathbf{1}_{y,\text{non}}$ and $\mathbf{1}_{z,\text{non}}$ in rotating coordinates.

As seen by an observer fixed with respect to the nonrotating coordinates, the nonrotating coordinate directions will appear to remain fixed, but the external inertial reference directions will appear to be changing, as illustrated in Fig. C.14. Gyroscopes fixed in the rotating coordinates would measure three components of the inertial rotation rate vector

$$\boldsymbol{\omega}_{\text{rot}} = \begin{bmatrix} \omega_{x,\text{rot}} \\ \omega_{y,\text{rot}} \\ \omega_{z,\text{rot}} \end{bmatrix} \tag{C.203}$$

in rotating coordinates, but the non-rotating unit vectors, as viewed in rotating coordinates, appear to be changing in the opposite sense, as

$$\frac{d}{dt}\mathbf{1}_{x,\text{non}} = -\boldsymbol{\omega}_{\text{rot}} \otimes \mathbf{1}_{x,\text{non}}, \tag{C.204}$$

$$\frac{d}{dt}\mathbf{1}_{y,\text{non}} = -\boldsymbol{\omega}_{\text{rot}} \otimes \mathbf{1}_{y,\text{non}}, \tag{C.205}$$

$$\frac{d}{dt}\mathbf{1}_{z,\text{non}} = -\boldsymbol{\omega}_{\text{rot}} \otimes \mathbf{1}_{z,\text{non}}, \tag{C.206}$$

as illustrated in Fig. C.14. The time-derivative of the coordinate transformation represented in Eq. C.202 will then be

$$\frac{d}{dt}\mathbf{C}^{\text{non}}_{\text{rot}} = \left[\frac{d}{dt}\mathbf{1}_{x,\text{non}} \quad \frac{d}{dt}\mathbf{1}_{y,\text{non}} \quad \frac{d}{dt}\mathbf{1}_{z,\text{non}}\right] \tag{C.207}$$

$$= [-\boldsymbol{\omega}_{\text{rot}} \otimes \mathbf{1}_{x,\text{non}} \quad -\boldsymbol{\omega}_{\text{rot}} \otimes \mathbf{1}_{y,\text{non}} \quad -\boldsymbol{\omega}_{\text{rot}} \otimes \mathbf{1}_{z,\text{non}}]$$

$$= -[\boldsymbol{\omega}_{\text{rot}}\otimes][\mathbf{1}_{x,\text{non}}] \quad \mathbf{1}_{y,\text{non}} \quad \mathbf{1}_{z,\text{non}}]$$

$$= -[\boldsymbol{\omega}_{\text{rot}}\otimes]\mathbf{C}^{\text{non}}_{\text{rot}}, \tag{C.208}$$

$$[\boldsymbol{\omega}_{\text{rot}}\otimes] \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -\omega_{z,\text{rot}} & \omega_{y,\text{rot}} \\ \omega_{z,\text{rot}} & 0 & -\omega_{x,\text{rot}} \\ -\omega_{y,\text{rot}} & \omega_{x,\text{rot}} & 0 \end{bmatrix}. \tag{C.209}$$

The inverse coordinate transformation

$$\mathbf{C}^{\text{rot}}_{\text{non}} = (\mathbf{C}^{\text{non}}_{\text{rot}})^{-1} \tag{C.210}$$

$$= (\mathbf{C}^{\text{non}}_{\text{rot}})^{\text{T}}, \tag{C.211}$$

the transpose of $\mathbf{C}^{\text{non}}_{\text{rot}}$, and its derivative

$$\frac{d}{dt}\mathbf{C}^{\text{rot}}_{\text{non}} = \frac{d}{dt}(\mathbf{C}^{\text{non}}_{\text{rot}})^{\text{T}} \tag{C.212}$$

$$= \left(\frac{d}{dt}\mathbf{C}^{\text{non}}_{\text{rot}}\right)^{\text{T}} \tag{C.213}$$

$$= (-[\boldsymbol{\omega}_{\text{rot}}\otimes]\mathbf{C}^{\text{non}}_{\text{rot}})^{\text{T}} \tag{C.214}$$

$$= -(\mathbf{C}^{\text{non}}_{\text{rot}})^{\text{T}}[\boldsymbol{\omega}_{\text{rot}}\otimes]^{\text{T}}, \tag{C.215}$$

$$= \mathbf{C}^{\text{rot}}_{\text{non}}[\boldsymbol{\omega}_{\text{rot}}\otimes]. \tag{C.216}$$

In the case that "rot" is "RPY" (roll-pitch-yaw coordinates) and "non" is "ECI" (earth centered inertial coordinates), Eq. C.216 becomes

$$\frac{d}{dt}\mathbf{C}^{\text{RPY}}_{\text{ECI}} = \mathbf{C}^{\text{RPY}}_{\text{ECI}}[\boldsymbol{\omega}_{\text{RPY}}\otimes], \tag{C.217}$$

and in the case that "rot" is "ENU" (east-north-up coordinates) and "non" is "ECI" (earth centered inertial coordinates), Eq. C.208 becomes

$$\frac{d}{dt}\mathbf{C}^{\text{ECI}}_{\text{ENU}} = -[\boldsymbol{\omega}_{\text{ENU}}\otimes]\mathbf{C}^{\text{ECI}}_{\text{ENU}}, \tag{C.218}$$

and the derivative of their product

$$\mathbf{C}_{\text{ENU}}^{\text{RPY}} = \mathbf{C}_{\text{ENU}}^{\text{ECI}}\mathbf{C}_{\text{ECI}}^{\text{RPY}}, \tag{C.219}$$

$$\begin{aligned}
\frac{d}{dt}\mathbf{C}_{\text{ENU}}^{\text{RPY}} &= \left[\frac{d}{dt}\mathbf{C}_{\text{ENU}}^{\text{ECI}}\right]\mathbf{C}_{\text{ECI}}^{\text{RPY}} + \mathbf{C}_{\text{ENU}}^{\text{ECI}}\left[\frac{d}{dt}\mathbf{C}_{\text{ECI}}^{\text{RPY}}\right] \qquad \text{(C.220)}\\
&= [[-\boldsymbol{\omega}_{\text{ENU}}\otimes]\mathbf{C}_{\text{ENU}}^{\text{ECI}}]\mathbf{C}_{\text{ECI}}^{\text{RPY}} + \mathbf{C}_{\text{ENU}}^{\text{ECI}}[\mathbf{C}_{\text{ECI}}^{\text{RPY}}[\boldsymbol{\omega}_{\text{RPY}}\otimes]]\\
&= [-\boldsymbol{\omega}_{\text{ENU}}\otimes]\underbrace{\mathbf{C}_{\text{ENU}}^{\text{ECI}}\mathbf{C}_{\text{ECI}}^{\text{RPY}}}_{\mathbf{C}_{\text{ENU}}^{\text{RPY}}} + \underbrace{\mathbf{C}_{\text{ENU}}^{\text{ECI}}\mathbf{C}_{\text{ECI}}^{\text{RPY}}}_{\mathbf{C}_{\text{ENU}}^{\text{RPY}}}[\boldsymbol{\omega}_{\text{RPY}}\otimes],
\end{aligned}$$

$$\frac{d}{dt}\mathbf{C}_{\text{ENU}}^{\text{RPY}} = -[\boldsymbol{\omega}_{\text{ENU}}\otimes]\mathbf{C}_{\text{ENU}}^{\text{RPY}} + \mathbf{C}_{\text{ENU}}^{\text{RPY}}[\boldsymbol{\omega}_{\text{RPY}}\otimes]. \tag{C.221}$$

Equation C.221 was originally used for maintaining vehicle attitude information in strapdown INS implementations, where the variables

$$\boldsymbol{\omega}_{\text{RPY}} = \text{vector of inertial rates measured by the gyroscopes}, \tag{C.222}$$

$$\boldsymbol{\omega}_{\text{ENU}} = \boldsymbol{\omega}_{\text{earthrate}} + \boldsymbol{\omega}_{v_E} + \boldsymbol{\omega}_{v_N},$$

$$\boldsymbol{\omega}_{\oplus} = \omega_{\oplus}\begin{bmatrix} 0 \\ \cos(\phi_{\text{geodetic}}) \\ \sin(\phi_{\text{geodetic}}) \end{bmatrix}, \tag{C.223}$$

$$\boldsymbol{\omega}_{v_E} = \frac{v_E}{r_T + h}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{C.224}$$

$$\boldsymbol{\omega}_{v_N} = \frac{v_N}{r_M + h}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \tag{C.225}$$

and
where

$\omega_{\oplus}$ = earth rotation rate
ϕ_{geodetic} = geodetic latitude
v_E = the east component of velocity with respect to the surface of the earth
r_T = transverse radius of curvature of the ellipsoid (Eq. 6.41)
v_N = north component of velocity with respect to the surface of the earth
r_M = meridional radius of curvature of the ellipsoid (Eq. 6.38)
h = altitude above (+) or below (−) the reference ellipsoid surface (≈mean sea level)

Unfortunately, Eq. C.221 was found to be not particularly well suited for accurate integration in finite-precision arithmetic. This integration problem was eventually solved using quaternions.

C.4.4 Quaternions

The term *quaternions* is used in several contexts to refer to sets of four. In mathematics, it refers to an algebra in four dimensions discovered by the Irish physicist and mathematician Sir William Rowan Hamilton (1805–1865). The utility of quaternions for representing rotations (as points on a sphere in four dimensions) was known before strapdown systems, they soon became the standard representation of coordinate transforms in strapdown systems, and they have since been applied to computer animation.

C.4.4.1 Quaternion Matrices For people already familiar with matrix algebra, the algebra of quaternions can be defined by using an isomorphism between 4×1 *quaternion vectors* $\mathbf{q}$ and real 4×4 *quaternion matrices* $\mathbf{Q}$:

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \leftrightarrow \mathbf{Q} = \begin{bmatrix} q_1 & -q_2 & -q_3 & -q_4, \\ q_2 & q_1 & -q_4 & q_3, \\ q_3 & q_4 & q_1 & -q_2, \\ q_4 & -q_3 & q_2 & q_1 \end{bmatrix} \tag{C.226}$$

$$= q_1\mathcal{Q}_1 + q_2\mathcal{Q}_2 + q_3\mathcal{Q}_3 + q_4\mathcal{Q}_4, \tag{C.227}$$

$$\mathcal{Q}_1 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 & 0, \\ 0 & 1 & 0 & 0, \\ 0 & 0 & 1 & 0, \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{C.228}$$

$$\mathcal{Q}_2 \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{C.229}$$

$$\mathcal{Q}_3 \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tag{C.230}$$

$$\mathcal{Q}_4 \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{C.231}$$

in terms of four 4×4 *quaternion basis matrices*, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, $\mathcal{Q}_4$, the first of which is an identity matrix and the rest of which are antisymmetric.

C.4.4.2 Addition and Multiplication Addition of quaternion vectors is the same as that for ordinary vectors. Multiplication is defined by the usual rules for matrix multiplication applied to the four quaternion basis matrices, the multiplication table for which is given in Table C.1. Note that, like matrix multiplication,

TABLE C.1. Multiplication of Quaternion Basis Matrices

First Factor	Second Factor			
	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$
$\mathcal{Q}_1$	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$
$\mathcal{Q}_2$	$\mathcal{Q}_2$	$-\mathcal{Q}_1$	$\mathcal{Q}_4$	$-\mathcal{Q}_3$
$\mathcal{Q}_3$	$\mathcal{Q}_3$	$-\mathcal{Q}_4$	$-\mathcal{Q}_1$	$\mathcal{Q}_2$
$\mathcal{Q}_4$	$\mathcal{Q}_4$	$\mathcal{Q}_3$	$-\mathcal{Q}_2$	$-\mathcal{Q}_1$

quaternion multiplication is noncommutative. That is, the result depends on the order of multiplication.

Using the quaternion basis matrix multiplication Table (C.1), the ordered product **AB** of two quaternion matrices

$$\mathbf{A} = a_1\mathcal{Q}_1 + a_2\mathcal{Q}_2 + a_3\mathcal{Q}_3 + a_4\mathcal{Q}_4, \tag{C.232}$$

$$\mathbf{B} = b_1\mathcal{Q}_1 + b_2\mathcal{Q}_2 + b_3\mathcal{Q}_3 + b_4\mathcal{Q}_4 \tag{C.233}$$

can be shown to be

$$\begin{aligned}\mathbf{AB} = {} & (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4)\mathcal{Q}_1 \\ & +(a_2b_1 + a_1b_2 - a_4b_3 + a_3b_4)\mathcal{Q}_2 \\ & +(a_3b_1 + a_4b_2 + a_1b_3 - a_2b_4)\mathcal{Q}_3 \\ & +(a_4b_1 - a_3b_2 + a_2b_3 + a_1b_4)\mathcal{Q}_4\end{aligned} \tag{C.234}$$

in terms of the coefficients a_k, b_k and the quaternion basis matrices.

C.4.4.3 Conjugation Conjugation of quaternions is a unary operation analogous to conjugation of complex numbers, in that the real part (the first component of a quaternion) is unchanged and the other parts change sign. For quaternions, this is equivalent to transposition of the associated quaternion matrix

$$\mathbf{Q} = q_1\mathcal{Q}_1 + q_2\mathcal{Q}_2 + q_3\mathcal{Q}_3 + q_4\mathcal{Q}_4, \tag{C.235}$$

so that

$$\mathbf{Q}^{\mathrm{T}} = q_1\mathcal{Q}_1 - q_2\mathcal{Q}_2 - q_3\mathcal{Q}_3 - q_4\mathcal{Q}_4 \tag{C.236}$$

$$\leftrightarrow \mathbf{q}^*, \tag{C.237}$$

$$\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = (q_1^2 + q_2^2 + q_3^2 + q_4^2)\mathcal{Q}_1 \tag{C.238}$$

$$\leftrightarrow \mathbf{q}^*\mathbf{q} = |\mathbf{q}|^2. \tag{C.239}$$

C.4.4.4 Representing Rotations The problem with rotation vectors as representations for rotations is that the rotation vector representing successive rotations $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \ldots, \boldsymbol{\rho}_n$ is not a simple function of the respective rotation vectors.

This representation problem is solved rather elegantly using quaternions, such that the quaternion representation of the successive rotations is represented by the quaternion product $\mathbf{q}_n \times \mathbf{q}_{n-1} \times \cdots \times \mathbf{q}_3 \times \mathbf{q}_2 \times \mathbf{q}_1$. That is, each successive rotation can be implemented by a single quaternion product.

The quaternion equivalent of the rotation vector $\boldsymbol{\rho}$ with $|\boldsymbol{\rho}| = \theta$,

$$\boldsymbol{\rho} \stackrel{\text{def}}{=} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} \stackrel{\text{def}}{=} \theta \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{C.240}$$

(i.e., where $\mathbf{u}$ is a unit vector), is

$$\mathbf{q}(\boldsymbol{\rho}) \stackrel{\text{def}}{=} \begin{bmatrix} \cos\left(\dfrac{\theta}{2}\right) \\ \dfrac{\rho_1 \sin(\theta/2)}{\theta} \\ \dfrac{\rho_2 \sin(\theta/2)}{\theta} \\ \dfrac{\rho_3 \sin(\theta/2)}{\theta} \end{bmatrix} = \begin{bmatrix} \cos\left(\dfrac{\theta}{2}\right) \\ u_1 \sin\left(\dfrac{\theta}{2}\right) \\ u_2 \sin\left(\dfrac{\theta}{2}\right) \\ u_3 \sin\left(\dfrac{\theta}{2}\right) \end{bmatrix}, \tag{C.241}$$

and the vector $\mathbf{w}$ resulting from the rotation of any three-dimensional vector

$$\mathbf{v} \stackrel{\text{def}}{=} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

through the angle θ about the unit vector $\mathbf{u}$ is implemented by the quaternion product

$$\mathbf{q}(\mathbf{w}) \stackrel{\text{def}}{=} \mathbf{q}(\boldsymbol{\rho})\mathbf{q}(\mathbf{v})\mathbf{q}^*(\boldsymbol{\rho}) \tag{C.242}$$

$$\stackrel{\text{def}}{=} \begin{bmatrix} \cos\left(\dfrac{\theta}{2}\right) \\ u_1 \sin\left(\dfrac{\theta}{2}\right) \\ u_2 \sin\left(\dfrac{\theta}{2}\right) \\ u_3 \sin\left(\dfrac{\theta}{2}\right) \end{bmatrix} \times \begin{bmatrix} 0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} \cos\left(\dfrac{\theta}{2}\right) \\ -u_1 \sin\left(\dfrac{\theta}{2}\right) \\ -u_2 \sin\left(\dfrac{\theta}{2}\right) \\ -u_3 \sin\left(\dfrac{\theta}{2}\right) \end{bmatrix} \tag{C.243}$$

$$= \begin{bmatrix} 0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}, \tag{C.244}$$

$$\begin{aligned} w_1 &= \cos(\theta)v_1 + [1-\cos(\theta)][u_1(u_1v_1+u_2v_2+u_3v_3)] \\ &\quad + \sin(\theta)(u_2v_3 - u_3v_2), & \text{(C.245)} \\ w_2 &= \cos(\theta)v_2 + [1-\cos(\theta)][u_2(u_1v_1+u_2v_2+u_3v_3)] \\ &\quad + \sin(\theta)(u_3v_1 - u_1v_3), & \text{(C.246)} \\ w_3 &= \cos(\theta)v_3 + [1-\cos(\theta)][u_3(u_1v_1+u_2v_2+u_3v_3)] \\ &\quad + \sin(\theta)(u_1v_2 - u_2v_1), & \text{(C.247)} \end{aligned}$$

or

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \mathbf{C}(\boldsymbol{\rho}) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \tag{C.248}$$

where the rotation matrix $\mathbf{C}(\boldsymbol{\rho})$ is defined in Eq. C.113 and Eq. C.242 implements the same rotation of $\mathbf{v}$ as the matrix product $\mathbf{C}(\boldsymbol{\rho})\mathbf{v}$. Moreover, if

$$\mathbf{q}(\mathbf{w}_k) \stackrel{\text{def}}{=} \mathbf{v} \tag{C.249}$$

and

$$\mathbf{q}(\mathbf{w}_k) \stackrel{\text{def}}{=} \mathbf{q}(\boldsymbol{\rho}_k)\mathbf{q}(\mathbf{w}_{k-1})\mathbf{q}^*(\boldsymbol{\rho}_k) \tag{C.250}$$

for $k = 1, 2, 3, \ldots, n$, then the nested quaternion product

$$\mathbf{q}(\mathbf{w}_n) = \mathbf{q}(\boldsymbol{\rho}_n)\cdots\mathbf{q}(\boldsymbol{\rho}_2)\mathbf{q}(\boldsymbol{\rho}_1)\mathbf{q}(\mathbf{v})\mathbf{q}^*(\boldsymbol{\rho}_1)\mathbf{q}^*(\boldsymbol{\rho}_2)\cdots\mathbf{q}^*(\boldsymbol{\rho}_n) \tag{C.251}$$

implements the succession of rotations represented by the rotation vectors $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$, $\boldsymbol{\rho}_3, \ldots, \boldsymbol{\rho}_n$, and the single quaternion

$$\begin{aligned} \mathbf{q}_{[n]} &\stackrel{\text{def}}{=} \mathbf{q}(\boldsymbol{\rho}_n)\mathbf{q}(\boldsymbol{\rho}_{n-1})\cdots\mathbf{q}(\boldsymbol{\rho}_3)\mathbf{q}(\boldsymbol{\rho}_2)\mathbf{q}(\boldsymbol{\rho}_1) & \text{(C.252)} \\ &= \mathbf{q}(\boldsymbol{\rho}_n)\mathbf{q}_{[n-1]} & \text{(C.253)} \end{aligned}$$

then represents the net effect of the successive rotations as

$$\mathbf{q}(\mathbf{w}_n) = \mathbf{q}_{[n]}\mathbf{q}(\mathbf{w}_0)\mathbf{q}^*_{[n]}. \tag{C.254}$$

The initial value $\mathbf{q}_{[0]}$ for the rotation quaternion will depend upon the inital orientation of the two coordinate systems. The initial value

$$\mathbf{q}_{[0]} \stackrel{\text{def}}{=} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{C.255}$$

applies to the case that the two coordinate systems are aligned. In strapdown system applications, the initial value $\mathbf{q}_{[0]}$ is determined during the INS alignment procedure.

Equation C.252 is the much-used quaternion representation for successive rotations, and Eq. C.254 is how it is used to perform coordinate transformations of any vector $\mathbf{w}_0$.

This representation uses the four components of a unit quaternion to maintain the transformation from one coordinate frame to another through a succession of rotations. In practice, computer roundoff may tend to alter the magnitude of the alegedly unit quaternion, but it can easily be rescaled to a unit quaternion by dividing by its magnitude.