Ch 3: Interpolation -- how to think about it

Wednesday, October 15, 2025 10:51 AM

We've discussed Lagrange interpolation and Newton Divided Difference interpolation.

There are other forms too.

What's the difference? The basis

we can change this

Interpolating polynomial p can be written as $p = \sum_{j=0}^{n} c_j p_j$

Assume degree is n or less.

i.e., $(\forall x)\ p(x) = \sum_{j=0}^{n} c_j\, p(x)$

Fact: $\dim(\underbrace{\text{all polynomials of degree } n \text{ or less}}_{\text{vector space } V_n}) = n+1$

proof: $\{\underbrace{1, x, \dots x^n}_{\text{"monomial basis"}}\}$ is clearly a basis

V_n = vector space of all degree $\le n$ polynomials

Fact: the $\underbrace{\text{interpolating}}_{\text{on } n+1 \text{ points}}$ polynomial p is degree n or less

So we know we can represent it as a combination of basis functions for V_n

So,

LAGRANGE INTERPOLATION

uses the basis for V_n consisting of Lagrange polynomials

$\mathcal{B}_{Lag.} = \{ L_{n,k} : k=0,1,\dots,n\}$ is a basis for V_n

$$L_{n,k}(x) := \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}$$

Scaling doesn't affect whether it's a basis (but can make it more convenient to work with)

How to prove $\mathcal{B}_{Lag}$ is a basis for V_n?

1. Check $\mathcal{B}_{Lag.} \subseteq V_n$ ✓ (ie., make sure we don't have degree too large)
2. Check $|\mathcal{B}_{Lag.}| = \dim(V_n)$ ✓
3. check $\mathcal{B}_{Lag.}$ is linearly independent (yes, we'll discuss later)

MONOMIAL BASIS

$\mathcal{B}_{mon.} = \{1, x, x^2, \dots x^n\}$ a basis for V_n

but doesn't lead to stable algorithms

Note: $\tilde{\mathcal{B}}_{mon} = \{1, (x-x_0), (x-x_0)^2, \dots, (x-x_0)^n\}$ is also a basis.

Wednesday, October 15, 2025 10:53 AM

NEWTON BASIS

$B_{New.} = \{ 1, (x-x_0), (x-x_0)(x-x_1), \ldots, (x-x_0)\cdot(x-x_1)\cdots(x-x_{n-1}) \}$

i.e., elements either 1 or $\prod_{i=1}^{k} (x-x_{i-1})$ for $k=1,2,\ldots,n$

Is this a basis?

① $B_{Newton} \subseteq V_n$? ✓

② $|B_{Newton}| = \dim(V_n) = n+1$? ✓

③ Lin. Independent? Yes

LINEAR INDEPENDENCE

if $B = \{v_0, v_1, \ldots, v_n\}$ is a subset of a vector space V, we say

B is "linearly independent" if $\sum_{j=0}^{n} c_j \cdot v_j = 0 \Rightarrow c_j = 0, j=0,1,\ldots,n$

What does this mean if our "vector" is a function or polynomial?

Same thing: if $\sum_{j=0}^{n} c_j\, v_j(x) = 0 \;\; \forall x \Rightarrow c_j = 0, j=0,1,\ldots,n$

How do we verify in practice?

Method 1

true $\forall x$, so in particular it's true for $\{x_0, x_1, \ldots, x_n\}$

so we have $(n+1)$ equations: $\sum_{j=0}^{n} c_j\, v_j(x_i) = 0$ for $i=0,1,\ldots,n$

In the special case that B is monomials, then A is the Vandermonde matrix.

i.e., solve the linear system

$A \cdot \vec{c} = \vec{0}$ for $A_{i,j} = v_j(x_i)$

If $\underbrace{A \text{ is invertible}}_{\text{iff } \det(A) \neq 0}$, then there's a unique solution $\vec{c} = \underbrace{A^{-1}\cdot\vec{0}}_{=0}$

$\Rightarrow c_j = 0$

$\Rightarrow$ the set is linearly independent.

Ex Newton Basis

(this completes the claim we made in those lecture notes, since it proves it is a basis)

$B_{New} = \{ 1, x-x_0, (x-x_0)(x-x_1), \ldots, (x-x_0)(x-x_1)\cdots(x-x_{n-1}) \}$

plug in the points $\underbrace{\{x_0, x_1, \ldots, x_n\}}_{\text{other points work but then it's messy and complicated.}}$ and our matrix A is

$$\begin{array}{c} \\ \text{row } i \\ \text{is plugging} \\ \text{in } x = x_i \\ \\ \end{array} \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & & 0 \\ 1 & x_1-x_0 & 0 & 0 & \cdots & 0 \\ 1 & x_2-x_0 & (x_2-x_0)(x_2-x_1) & 0 & & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ 1 & x_n-x_0 & (x_n-x_0)(x_n-x_1) & & & (x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1}) \end{array}\right]$$

Column j = function $p_j \in B_{Newton}$

then

A is lower triangular, so $\det(A)$ = product of diagonal entries

and all diagonal terms are nonzero since $\{x_0, x_1, \ldots, x_n\}$ are distinct.

$\Rightarrow \det(A) \neq 0 \Rightarrow A$ is invertible $\Rightarrow B_{Newton}$ is lin. independent. ✓

p. 3

Wednesday, October 15, 2025 10:54 AM

Aside: Mairhuber–Curtis Thm, 1950's

If we do $\vec{x} \in \mathbb{R}^d$ for $d>1$, then there is no (fixed) basis such that for distinct points $\{x_0, x_1, \dots, x_n\}$ the "A" matrix is always invertible.

(of course it must be sometimes invertible)

Implication: basis should adapt to nodes so it's not fixed.

Only non-adaptive one is (not shifted) monomials.

Related to ill-conditioning

Method 2

If $\underbrace{\sum_{j=0}^{n} c_j v_j}_{f} = 0$, then $f' = 0$, i.e., $\sum c_j v_j' = 0$

and similarly $\sum c_j v_j'' = 0$, etc.

So,

pick any point x, and make

$$A = \begin{bmatrix} v_0(x) & v_1(x) & \dots & v_n(x) \\ v_0'(x) & v_1'(x) & \dots & v_n'(x) \\ \vdots & & & \\ v_0^{(n)}(x) & v_1^{(n)}(x) & \dots & v_n^{(n)}(x) \end{bmatrix} \Bigg\} \text{ the Wronskian at point } x.$$

So if $A(x)$ is invertible at any point x then you can deduce $\vec{c} = 0$ and hence $\{v_0, v_1, \dots, v_n\}$ is lin. independent.

Back to ~~basis~~ bases,

1) Lagrange polynomials ← (all have degree n)

2) Monomials or Shifted Monomials — each polynomial in the basis has a different degree

3) Newton polynomials

NEW (4) Legendre polynomials (Gram-Schmidt orthogonalize monomials, $\int_{-1}^{1} v_i(x) v_j(x)\,dx = \delta_{ij}$)

5) Chebyshev polynomials (orthogonal with a new meaning of "inner product",
$$\int_{-1}^{1} v_i(x) v_j(x)\, w(x)\, dx = \delta_{ij}$$
for a weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ Chebyshev of the 1st kind

or $w(x) = \sqrt{1-x^2}$ Chebyshev of the 2nd kind)

we'll discuss more later in this class

Alternatives to polynomial bases:

RBF = Radial Basis Function. Choose a "kernel" $\varphi: \mathbb{R} \to \mathbb{R}$

ex: $\varphi(r) = e^{-r^2/2}$

Define $v_j(\vec{x}) = \varphi(\|\vec{x} - \vec{x}_j\|_2)$ and interpolate by solving a linear system

Nice in high dimensions where polynomials struggle