This $\LaTeX$ example is from a mathematics writeup I did in high school, for a Calculus III course. If you have any questions, feel free to contact me at savim2020@gmail.com.

1. Let G be the rectangular box defined by the inequalities $a \le x \le b$, $c \le y \le d$, $k \le z \le l$. Show that

$$\iiint\limits_G f(x)g(y)h(z)\ dV = \left[\int_a^b f(x)\ dx\right]\left[\int_c^d g(y)\ dy\right]\left[\int_k^l h(z)\ dz\right]$$

Answer

We will define some values:

$$X = \int_a^b f(x)\ dx$$

$$Y = \int_c^d g(y)\ dy$$

$$Z = \int_k^l h(z)\ dz$$

Because X, Y, and Z are all definite integrals with constant bounds, they are all constants.

$$\left[\int_a^b f(x)\ dx\right]\left[\int_c^d g(y)\ dy\right]\left[\int_k^l h(z)\ dz\right] = XYZ$$

$$\iiint\limits_G f(x)g(y)h(z)\ dV = \int_k^l \int_c^d \int_a^b f(x)g(y)h(z)\ dx\ dy\ dz$$

$g(y)$ and $h(z)$ are treated as constants with respect to x. We can then rewrite this as

$$\int_k^l \int_c^d \left[g(y)h(z)\int_a^b f(x)\ dx\right]\ dy\ dz$$

$$= \int_k^l \int_c^d (g(y)h(z)X)\ \ dy\ dz$$

$$= X\int_k^l \int_c^d g(y)h(z)\ dy\ dz$$

$h(z)$ is treated as a constant with respect to y. We can then rewrite this as

$$X\int_k^l \left[h(z)\int_c^d g(y)\ dy\right]\ dz$$

$$= X \int_k^l h(z) Y \ dz$$
$$= XY \int_k^l h(z) \ dz$$
$$= XYZ \blacksquare$$

2. Use the result of the previous exercise to evaluate the following integrals.

(a) $\displaystyle\iiint\limits_G xy^2 \sin z \ dV$ where $G = \left\{(x, y, z) | -1 \le x \le 1, 0 \le y \le 1, 0 \le z \le \frac{\pi}{2}\right\}$.

Answer

$$\iiint\limits_G xy^2 \sin z \ dV = \int_{-1}^1 \int_0^1 \int_0^{\frac{\pi}{2}} xy^2 \sin z \ dz \ dy \ dx$$
$$= \left[\int_{-1}^1 x \ dx\right] \left[\int_0^1 y^2 \ dy\right] \left[\int_0^{\frac{\pi}{2}} \sin z \ dz\right]$$

Because x is an odd function, $\displaystyle\int_{-1}^1 x \ dx = 0$. And because y^2 and $\sin z$ are continuous fuctions along all real numbers, their integrals exist and are defined.

$$= 0 \left[\int_0^1 y^2 \ dy\right] \left[\int_0^{\frac{\pi}{2}} \sin z \ dz\right]$$
$$= 0$$

(b) $\displaystyle\iiint\limits_G e^{2x+y-z} \ dV$ where $G = \{(x, y, z) | 0 \le x \le 1, 0 \le y \le \ln 3, 0 \le z \le \ln 2\}$.

Answer

$$\iiint\limits_G e^{2x+y-z} \ dV = \iiint\limits_G \left(e^{2x}\right) \left(e^y\right) \left(e^{-z}\right) \ dV$$
$$= \int_0^1 \int_0^{\ln 3} \int_0^{\ln 2} \left(e^{2x}\right) \left(e^y\right) \left(e^{-z}\right) \ dz \ dy \ dx$$
$$= \int_0^1 e^{2x} \ dx \int_0^{\ln 3} e^y \ dy \int_0^{\ln 2} e^{-z} \ dz$$

$$= \left[\frac{1}{2}e^{2x}\right]_{x=0}^{1} \left[e^{y}\right]_{y=0}^{\ln 3} \left[-e^{-z}\right]_{z=0}^{\ln 2}$$

$$= \frac{1}{2}(e^{2}-1)\left(e^{\ln 3}-1\right)\left(-e^{-\ln 2}+1\right)$$

$$-\frac{1}{2}(e^{2}-1)(3-1)\left(-\frac{1}{2}+1\right)$$

$$= \frac{1}{2}(e^{2}-1)(2)\left(\frac{1}{2}\right)$$

$$= \frac{e^{2}-1}{2}$$

3. Let G be the solid in the first octant bounded by the sphere $x^2+y^2+z^2=4$ and the coordinate planes. Find

$$\iiint\limits_{G} xyz\ dV$$

using

(a) Rectangular coordinates

Answer

Because we are in the first octant, $x \geq 0$, $y \geq 0$, and $z \geq 0$.

$$G = \left\{(x,y,z)|0 \leq x \leq 2, 0 \leq y \leq \sqrt{4-x^2}, 0 \leq z \leq \sqrt{4-x^2-y^2}\right\}$$

$$\iiint\limits_{G} xyz\ dV = \int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} xyz\ dz\ dy\ dx$$

$$= \int_{0}^{2}\int_{0}^{\sqrt{4-x^2}} \left[\frac{1}{2}xyz^2\right]_{z=0}^{\sqrt{4-x^2-y^2}} dy\ dx$$

$$= \frac{1}{2}\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}} xy(4-x^2-y^2)\ dy\ dx$$

$$= \frac{1}{2}\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}} (4xy-x^3y-xy^3)\ dy\ dx$$

$$= \frac{1}{2}\int_0^2 \left[2xy^2 - \frac{1}{2}x^3y^2 - \frac{1}{4}xy^4\right]_{y=0}^{\sqrt{4-x^2}} \ dx$$

$$= \frac{1}{2}\int_0^2 \left(2x(4-x^2) - \frac{1}{2}x^3(4-x^2) - \frac{1}{4}x(4-x^2)^2\right) \ dx$$

$$= \frac{1}{2}\int_0^2 \left((8x-2x^3) - \frac{1}{2}(4x^3-x^5) - \frac{1}{4}(16x-8x^3+x^5)\right) \ dx$$

$$= \frac{1}{2}\int_0^2 \left(8x - 2x^3 - 2x^3 + \frac{1}{2}x^5 - 4x + 2x^3 - \frac{1}{4}x^5\right) \ dx$$

$$= \frac{1}{2}\int_0^2 \left(\frac{1}{4}x^5 - 2x^3 + 4x\right) \ dx$$

$$= \frac{1}{2}\left[\frac{1}{24}x^6 - \frac{1}{2}x^4 + 2x^2\right]_0^2$$

$$= \frac{1}{2}\left(\frac{64}{24} - \frac{16}{2} + 2(2)^2\right)$$

$$= \frac{1}{2}\left(\frac{8}{3} - 8 + 8\right)$$

$$= \frac{1}{2}\cdot\frac{8}{3} = \frac{4}{3}$$

(b) Cylindrical Cooridnates

Answer

The sphere has a radius of 2. Because we are in the first octant, $0 \le \theta \le \frac{\pi}{2}$.

$$G = \left\{(r,\theta,z)|0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}, 0 \le z \le \sqrt{4-r^2}\right\}$$

$$\iiint\limits_G xyz \ dV = \int_0^{\frac{\pi}{2}}\int_0^2\int_0^{\sqrt{4-r^2}} (r\cos\theta)(r\sin\theta) zr \ dz \ dr \ d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^2\int_0^{\sqrt{4-r^2}} r^3 z\sin(\theta)\cos(\theta) \ dz \ dr \ d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^2\int_0^{\sqrt{4-r^2}} \frac{1}{2}r^3 z\sin(2\theta) \ dz \ dr \ d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^2 \left[\frac{1}{2}r^3z^2\sin(2\theta)\right]_{z=0}^{\sqrt{4-r^2}} \ dr \ d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}}\int_0^2 \left[r^3z^2\sin(2\theta)\right]_{z=0}^{\sqrt{4-r^2}} \ dr \ d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}}\int_0^2 r^3(4-r^2)\sin(2\theta) \ dr \ d\theta$$

$$= \frac{1}{4}\left[\int_0^{\frac{\pi}{2}}\sin(2\theta) \ d\theta\right]\left[\int_0^2 (4r^3-r^5) \ dr\right]$$

$$= \frac{1}{4}\left[-\frac{1}{2}\cos(2\theta)\right]_0^{\frac{\pi}{2}}\left[r^4-\frac{1}{6}r^6\right]_0^2$$

$$= -\frac{1}{8}\left[\cos(2\theta)\right]_0^{\frac{\pi}{2}}\left(16-\frac{64}{6}\right)$$

$$-\frac{1}{8}(-1-1)\left(16-\frac{64}{6}\right)$$

$$= \frac{1}{4}\left(16-\frac{64}{6}\right)$$

$$= 4-\frac{16}{6}$$

$$= \frac{12}{3}-\frac{8}{3}=\frac{4}{3}$$

(c) Spherical coordinates

Answer

Because we are in the first octant, $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \varphi \le \frac{\pi}{2}$.

$$G = \left\{(\rho,\theta,\varphi) | 0 \le \rho \le 2, 0 \le \theta \le \frac{\pi}{2}, 0 \le \varphi \le \frac{\pi}{2}\right\}$$

$$\iiint_G xyz \ dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^2 (\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta)(\rho\cos\varphi)(\rho^2\sin\varphi) \ d\rho \ d\varphi \ d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^2 \rho^5\sin^3\varphi\cos\varphi\sin\theta\cos\theta \ d\rho \ d\varphi \ d\theta$$

$$= \left[\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \ d\theta\right]\left[\int_0^{\frac{\pi}{2}} \sin^3\varphi\cos\varphi \ d\varphi\right]\left[\int_0^{2} \rho^5 \ d\rho\right]$$

$$= \left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}}\left[\frac{1}{4}\sin^4\varphi\right]_0^{\frac{\pi}{2}}\left[\frac{1}{6}\rho^6\right]_0^{2}$$

$$= \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{64}{6}$$

$$= \frac{8}{6} = \frac{4}{3}$$

4. Evaluate $\iint_R \sin(9x^2+4y^2) \ dA$, where R is the region in the first quadrant bounded by the ellipse $9x^2+4y^2=1$, by making an appropriate change in variables.

Answer

Change x and y into terms of u and v:

$$x = \frac{u}{3}, \quad y = \frac{v}{2}$$

$$9x^2+4y^2 = u^2+v^2 = 1$$

$$S = \left\{(u,v)|0 \le u \le 1, 0 \le v \le 1, u^2+v^2 \le 1\right\}$$

Find the Jacobian.

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{vmatrix}$$

$$\iint\limits_R \sin(9x^2+4y^2) \ dA = \iint\limits_S \sin(u^2+v^2)\begin{vmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{vmatrix} \ dA$$

$$= \frac{1}{6}\iint\limits_S \sin(u^2+v^2) \ dA$$

Because S is a circular region, we can convert the double integral into a double polar integral.

$$r^2 = u^2+v^2$$

$$S = \left\{(r,\theta)|0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}\right\}$$

$$\frac{1}{6}\iint\limits_S \sin(u^2+v^2) \ dA = \frac{1}{6}\iint\limits_S r\sin(r^2) \ dA = \frac{1}{6}\int_0^{\frac{\pi}{2}}\int_0^{1} r\sin(r^2) \ dr \ d\theta$$

$$= \frac{1}{6}\left[\int_0^{\frac{\pi}{2}}\ d\theta\right]\left[\int_0^1 r\sin(r^2)\ dr\right]$$

Use u substitution here.

$$u = r^2$$

$$du = 2r\ dr$$

$$dr = \frac{du}{2r}$$

$$\int_0^1 r\sin(r^2)\ dr = \int_0^1 \frac{1}{2}\sin(u)\ du$$

$$= \left[-\frac{1}{2}\cos(r^2)\right]_{r=0}^1$$

$$\frac{1}{6}\left[\int_0^{\frac{\pi}{2}}\ d\theta\right]\left[\int_0^1 r\sin(r^2)\ dr\right] = \frac{1}{12}\pi\left[-\frac{1}{2}\cos(r^2)\right]_{r=0}^1$$

$$= -\frac{1}{24}\pi\left(\cos 1 - \cos 0\right)$$

$$= \frac{1}{24}\pi\left(\cos 0 - \cos 1\right)$$

$$= \frac{1}{24}\pi\left(1 - \cos 1\right)$$