PH C240B: HW 2

Yue You

Problem 2

Let (W, A, Y) be the observed data with distribution, $P_0 \in M$ nonparametric. Define the following parameter mapping for $P \in M : \Psi(P) = E_P[(1, 1, W)\beta]$ where $\beta = \Psi^1(P) = \text{argmin}_\gamma E_P(Y - (1, A, W)\gamma)^2$.

(a) The empirical distribution, $\mathbf{P}_n$, is the NPMLE for the true distribution. We use $\mathbf{P}_n$ as a plug-in estimator, $\Psi(\mathbf{P}_n)$, for the true parameter, $\Psi(P_0)$. This is called the NPMLE for $\Psi(P_0)$. Derive this estimator's influence curve. You may derive the efficient influence curve, $D^*_\Psi(P)$ first and then your answer is $D^*_\Psi(\mathbf{P}_n)$ This is a valid approach but not the only approach.

Because $\Psi(P)$ is a function of $\Psi^1(P)$ we first derive the efficient influence curve for $\Psi^1(P) = \beta = \text{argmin}_\gamma E_P(Y - (1, A, W)\gamma)^2$. First we need to compute the pathwise derivative (also let's assume that W is d-dimensional:

We can use the delta method to derive the efficient influence curve for this parameter mapping:

$D^*_\Psi(P) = \frac{d\Psi}{d\beta} D^*_\beta(P)(O)$
$= E_P(1, 1, W)D^*_\beta(P)(O)$

Next we compute $D^*_\beta(P)(O)$: $\because \Psi'(P) = \beta = \text{argmin}_\gamma E_P(Y - (1, A, W)\gamma)^2$
$\therefore \beta\ satisfies\ \frac{d}{d\beta}\text{argmin}_\gamma E_P(Y - (1, A, W)\beta)^2 = 0_{d\times 1}$
$\Rightarrow 2[E_P(Y - (1, A, W)\beta)(1, A, W)^T] = 0_{d\times 1}$
$\Rightarrow E_P(Y - (1, A, W)\beta)(1, A, W)^T = 0_{d\times 1}$..................(1)

And $\because \Psi'(P_\epsilon) = \beta = \text{argmin}_\gamma E_{P_\epsilon}(Y - (1, A, W)\gamma)^2$
$\therefore \beta_\epsilon\ satisfies\ \frac{d}{d\beta_\epsilon}\text{argmin}_\gamma E_{P_\epsilon}(Y - (1, A, W)\beta_\epsilon)^2 = 0_{d\times 1}$
$\Rightarrow 2[E_{P_\epsilon}(Y - (1, A, W)\beta_\epsilon)(1, A, W)^T] = 0_{d\times 1}$
$\Rightarrow E_{P_\epsilon}(Y - (1, A, W)\beta_\epsilon)(1, A, W)^T = 0_{d\times 1}$....................(2)
from (1)(2)$\therefore E_P(Y - (1, A, W)\beta)(1, A, W)^T - E_{P_\epsilon}(Y - (1, A, W)\beta_\epsilon)(1, A, W)^T = 0_{d\times 1}$
$\Rightarrow E_P(Y - (1, A, W)\beta)(1, A, W)^T - E_{P_\epsilon}(Y - (1, A, W)\beta)(1, A, W)^T$
$+E_{P_\epsilon}(Y - (1, A, W)\beta)(1, A, W)^T - E_{P_\epsilon}(Y - (1, A, W)\beta_\epsilon)(1, A, W)^T = 0_{d\times 1}$
$\therefore E_{P_\epsilon}(Y - (1, A, W)\beta_\epsilon)(1, A, W)^T - E_{P_\epsilon}(Y - (1, A, W)\beta)(1, A, W)^T$.......(3)
$= E_P(Y - (1, A, W)\beta)(1, A, W)^T - E_{P_\epsilon}(Y - (1, A, W)\beta)(1, A, W)^T$..........(4)
$\because lim_{\epsilon\to 0}\frac{(4)}{\epsilon}$
$= E_{P_\epsilon}(Y - (1, A, W)\frac{\beta_\epsilon - \beta}{\epsilon})(1, A, W)^T$
$= lim_{\epsilon\to 0}E_{P_\epsilon}(1, A, W)^T(1, A, W)\frac{\beta_\epsilon - \beta}{\epsilon})$
$= E_{P_\epsilon}(1, A, W)^T(1, A, W)\frac{d\Psi'(P_\epsilon)}{d\epsilon}|_{\epsilon=0}$.........(5)
And $\because lim_{\epsilon\to 0}\frac{(3)}{\epsilon}$
$= lim_{\epsilon\to 0}\int(Y - (1, A, W))\beta(1, A, W)^T\frac{P_\epsilon(O) - P(O)}{\epsilon}dv(O)(\because P_\epsilon(O) = (1 + \epsilon h)P(O))$
$= \int(Y - (1, A, W))\beta(1, A, W)^T S(O)P(O)dv(O)$
$= E_p[(Y - (1, A, W))\beta(1, A, W)^T S(O)]$.........(6)
$\therefore (5) = (6)$
$\Rightarrow E_{P_\epsilon}(1, A, W)^T(1, A, W)\frac{d\Psi'(P_\epsilon)}{d\epsilon}|_{\epsilon=0}$
$= E_p[(Y - (1, A, W))\beta(1, A, W)^T S(O)]$
$\Rightarrow \frac{d\Psi'(P_\epsilon)}{d\epsilon}|_{\epsilon=0}$
$= [E_{P_\epsilon}(1, A, W)^T(1, A, W)]^{-1}E_p[(Y - (1, A, W))\beta(1, A, W)^T S(O)]$

$\Rightarrow D^*_\beta(P)(O) = [E_{P_\epsilon}(1,A,W)^T(1,A,W)]^{-1}E_p[(Y-(1,A,W))\beta(1,A,W)^T]$
$\Rightarrow IC = E_{P_n}(1,1,W)[E_{P_{n,\epsilon}}(1,A,W)^T(1,A,W)]^{-1}E_{p_n}[(Y-(1,A,W))\beta(1,A,W)^T] - \Psi(P)$

Collaborators & Resources

Rachael Philips, Chris Kennedy, Tommy Carpenito