Mathematical Framework

In this project, we explore three quantum sampling schemes implemented using Qiskit: (2) the n-layer Quantum Galton Box, which produces a binomial distribution that approximates a Gaussian in the large-n limit; (3a) a direct state-preparation method that generates an exponential target distribution via a noiseless all-to-all sampler; and (3b) a coined discrete-time Hadamard quantum walk on a one-dimensional lattice, demonstrating the ballistic spread characteristic of quantum walks. The following sections formalize the mathematical underpinnings of these schemes, including state definitions, unitaries, target distributions, and measurement procedures.

Common notation

Let $\mathcal{H}_2 = \mathrm{span}\{|0\rangle, |1\rangle\}$ and $\mathcal{H}_n = \mathbb{C}^{n+1}$ for n layers/steps. We label the $n+1$ output "bins" by $k \in \{0, 1, \ldots, n\}$. For a bit string $b \in \{0,1\}^n$, denote its Hamming weight by $w(b)$. Projective measurement in the computational basis is $M = \{|z\rangle\langle z|\}_z$.

(2) n-layer Quantum Galton Box $\Rightarrow$ Binomial/ Gaussian

The circuit uses n independent "coins" (qubits) each prepared by a Hadamard. The state before measurement is

$$|\psi_n\rangle = \bigotimes_{i=1}^{n} H|0\rangle = \frac{1}{2^{n/2}} \sum_{b \in \{0,1\}^n} |b\rangle .$$

After measuring all n qubits, outcomes are uniformly distributed over $\{0,1\}^n$. Grouping by Hamming weight induces a distribution on bins $k = 0, \ldots, n$:

$$\Pr\{K = k\} = \sum_{b:\, w(b)=k} \frac{1}{2^n} = \frac{\binom{n}{k}}{2^n} =: \mathrm{Bin}\big(k\,;\, n, \tfrac{1}{2}\big) .$$

Hence $\mathbb{E}[K] = n/2$ and $\mathrm{Var}(K) = n/4$. By the Central Limit Theorem,

$$\frac{K - \frac{n}{2}}{\sqrt{n/4}} \xrightarrow{d} \mathcal{N}(0,1) \qquad (n \to \infty),$$

so the histogram over k approaches a Gaussian with mean $n/2$ and variance $n/4$.

Complexity. Depth $=1$ (all H in parallel), width $=n$, with classical post-processing by Hamming weight.

(3a) Exponential target via a noiseless all-to-all sampler

We target the probability law on $\{0, \ldots, n\}$ given by

$$p_k \propto e^{-\lambda k}, \qquad p_k = \frac{e^{-\lambda k}}{\sum_{j=0}^{n} e^{-\lambda j}} = \frac{(e^{-\lambda})^k (1 - e^{-\lambda})}{1 - e^{-(n+1)\lambda}}.$$

Prepare a state on $\mathcal{H}_n$ whose computational amplitudes realize $\sqrt{p_k}$:

$$|\phi_\lambda\rangle = \sum_{k=0}^{n} \sqrt{p_k}\; |k\rangle\,, \qquad \sum_{k=0}^{n} p_k = 1.$$

Any exact state-preparation unitary U_λ with $U_\lambda |0\rangle = |\phi_\lambda\rangle$ implements a *noiseless all-to-all* sampler: measuring M yields $\Pr\{K = k\} = p_k$. (Our code uses Initialize to realize U_λ directly.)

Remarks. This approach changes the target *distribution* without imposing a specific walk/coin dynamics; it is optimal in depth for the noiseless model and provides a clean baseline for later noise studies.

(3b) Hadamard Quantum Walk on the line

Consider a coined discrete-time quantum walk on the 1D lattice with position space $\mathcal{H}_X = \mathrm{span}\{|x\rangle : x \in \mathbb{Z}\}$ (truncated to $x \in \{-t, \ldots, t\}$ for t steps) and coin space $\mathcal{H}_C = \mathcal{H}_2$. Let H be the Hadamard on $\mathcal{H}_C$ and define the conditional shift

$$S = \sum_x \Big(|0\rangle\langle 0| \otimes |x-1\rangle\langle x| \;+\; |1\rangle\langle 1| \otimes |x+1\rangle\langle x| \Big).$$

One time step is the unitary

$$U := S\,(H \otimes I_X).$$

Starting from $|\Psi_0\rangle = |\chi\rangle \otimes |0\rangle$ with coin seed $|\chi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, the state after t steps is

$$|\Psi_t\rangle = U^t\,|\Psi_0\rangle\,.$$

The position distribution is

$$P_t(x) \;=\; \sum_{c \in \{0,1\}} |\langle c|\langle x|\;\Psi_t\rangle|^2\,.$$

For the symmetric choice $|\chi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ one obtains a symmetric, *ballistic* spread with standard deviation $\Theta(t)$ and characteristic twin peaks (contrast with the $\Theta(\sqrt{t})$ diffusive spread of classical walks). On a finite window $x \in \{-t, \ldots, t\}$ the above definition is implemented by reflecting or by encoding x in a finite register; our circuit uses a unary encoding with CSWAP chains to realize S exactly.

Measurement. Measuring $I_C \otimes M_X$ gives the empirical histogram $\{P_t(x)\}_{x=-t}^{t}$, which matches the plotted distribution.

Distance-to-target (optional, if reported)

Given an empirical histogram $\hat{p}$ and a target q, report, e.g.,

$$\mathrm{TV}(\hat{p}, q) = \frac{1}{2} \sum_k |\hat{p}_k - q_k|, \qquad \mathrm{H}^2(\hat{p}, q) = 1 - \sum_k \sqrt{\hat{p}_k q_k},$$

and account for finite-sample fluctuations via multinomial error bars $\mathrm{SE}[\hat{p}_k] = \sqrt{q_k(1 - q_k)/N}$ for N shots.