24 oct temă

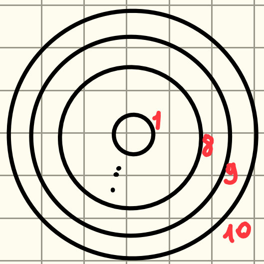
PS-Seminar 1

1, 2, 3 - la seminar

4. $\Omega = \{H, T\}$ o aruncare

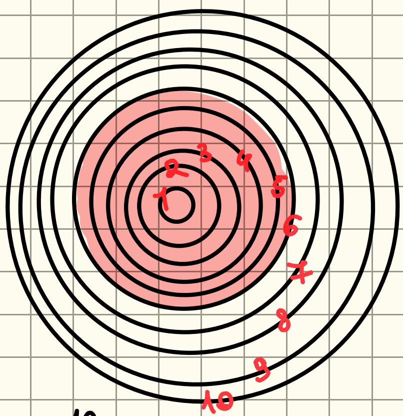
$\Omega = \{HH, HT, TH, TT\}$ 2 aruncări.

5.



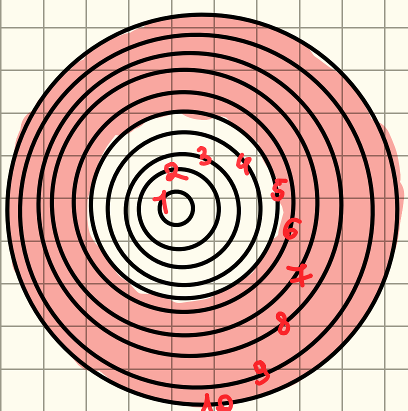
a) $A = \bigcup_{k=1}^{6} A_k = A_6$

ținta să lovească de la cercul 6 spre interior, oricare dintre primele $\overline{1,6}$ cercuri.



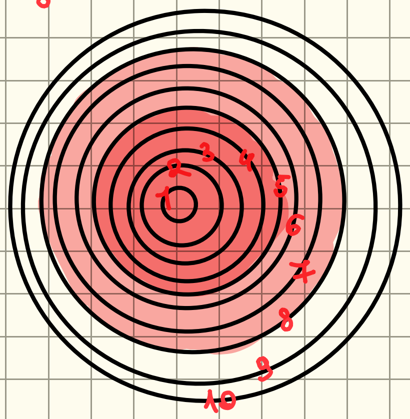
b) $B = \bigcap_{k=5}^{10} A_k = A_{10} \setminus A_5$

Intersecția cercurilor $\overline{5,10}$, cel puțin r_5 aproape de centru



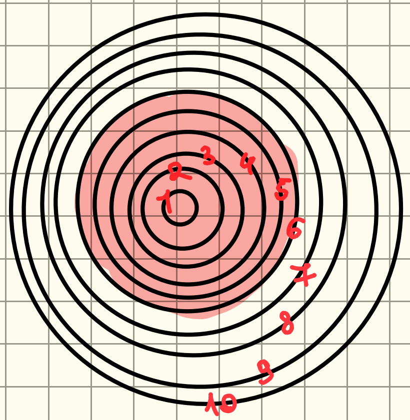
c) $C = \bigcup_{k=5}^{8} A_k = A_8$

Reuniunea cercurilor $\overline{5,8}$, dar $r_{\overline{1,4}} \subset r_5$, Oriunde dar maxim r_8



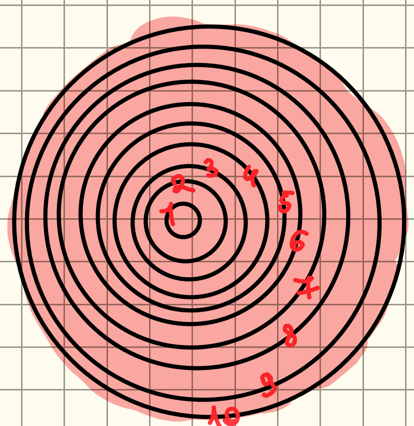
d) $D = \bigcap_{k=5}^{8} A_k = A_5$

Maxim r_5 spre exterior



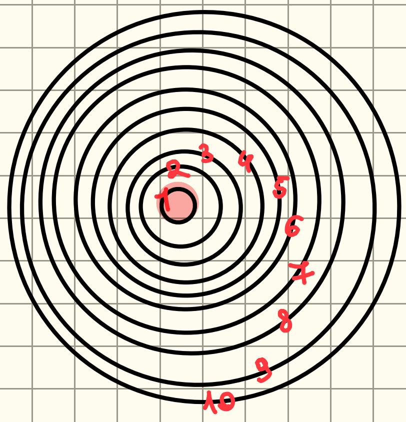
e) $F = \bigcup_{k=1}^{10} A_k = A_{10}$

Reuniunea tuturor cercurilor, adică tot cercul.



f) $F = \bigcap_{k=1}^{10} A_k = A_1$

Partea comună a tuturor cercurilor, A_1



Probleme – prob

I proprietăți axiomatice de bază.

1–5 ⇒ Seminar.

6. $\Omega = A \cup \bar{A}$, disjunctă

$\Rightarrow P(\Omega) = 1$

$\Downarrow$ disj

$P(A) + P(\bar{A}) = 1$

$P(\bar{A}) = 1 - P(A)$

7. $P(A \setminus B) = P(A) - P(A \cap B)$

$A = \underbrace{(A \cap B)}_{\text{disj}} \cup \underbrace{(A \setminus B)}_{\text{disj}}$

$P(A) = P(A \cap B) + P(A \setminus B)$

$P(A \setminus B) = P(A) - P(A \cap B)$

8. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$(A \cap C) \cup (B \cap C)$

$\Uparrow$

$P((A \cup B) \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)$

$= P(A) + P(B) - P(A \cap B) - (P(A \cap C) + P(B \cap C)$

$+ P(C)$

??

9. A, B incompatibile $\Rightarrow A \cap B = \emptyset$

$$P(A \cap B) = 0.$$

$$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{0} = P(A) + P(B)$$

10. $A_1 \subset A_2 \subset A_3 \ldots \Leftrightarrow P(\bigcup_1^{\infty} A_n) = \lim_{n \to \infty} P(A_n)$

$\Leftrightarrow P(A_n) \nearrow \lim P(A_n)$

Fie $B_1 = A_1$

$B_n = A_n \setminus A_{n-1}$

B_i, $i = \overline{1,n}$ disjuncte

$$\Rightarrow \bigcup_1^n A_n = \bigcup_1^n B_n$$

$$P(\bigcup_1^n A_n) = \sum_{n=1}^{\infty} P(B_n) = P(A_n)$$

II Probleme Numerice elementare.

11. $P(A) = 0,2$
$P(B) = 0,5$
$P(A \cap B) = 0,1$

$P(A \cup B) = ?$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0,2 + 0,5 - 0,1$$
$$= 0,6.$$

12. $P(A) = 0,8$

$$P(\bar{A}) = 1 - P(A)$$
$$= 1 - 0,8$$
$$= 0,2.$$

13. $P(A) = 0,3$

$P(A \cup B) = 0,7$

$P(A \cap B) = 0,1$

$P(B)=?$

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$0,7=0,3+P(B)-0,1$

$P(B)=0,5.$

14. $P(A)=0,5$

$P(B)=0,6$

$P(A\cap B)=0,4$

$P(A\setminus B)=?$

$P(A\setminus B)=P(A)-P(A\cap B)$

$=0,5-0,4$

$=0,1.$

15. A,B - compatibile

$P(A\cup B)<P(A)+P(B)$

A,B - comp $\Rightarrow P(A\cap B)>0$

$P(A\cap B)-P(A)-P(B)>-P(A)-P(B)$

$P(A\cup B)<P(A)+P(B)$

16. $\left.\begin{array}{l} P(A)=0,4 \\ P(B)=0,5 \\ P(A\cap B)=0,2 \end{array}\right\} \Rightarrow P(A\cup B)=0,4+0,5-0,2=0,7.$

$P(A\cup B)\leq P(A)+P(B)$

$0,7\leq 0,4+0,5$

$0,7\leq 0,9$ ✓

17. A,B independente

$\Rightarrow P(A\cap B)=P(A)\cdot P(B)$?

$\underbrace{P(A|B)}_{\substack{=P(A) \\ \text{indep}}}=\frac{P(A\cap B)}{P(B)}$

$P(A\cap B)=P(A)\cdot P(B)$.

18. $A\subset B$

$P(B\setminus A)=P(B)-P(A)$?

$B=B\setminus A\cup A$ disjuncte

$P(B)=P(B\setminus A)+P(A)$

$P(B\setminus A)=P(B)-P(A)$

19. $P(\bar{A}\cap\bar{B})=1-P(A\cup B)$

$\overline{A\cup B}=\bar{A}\cap\bar{B}$

$P(\bar{A}\cap\bar{B})=P(\overline{A\cup B})=1-P(A\cup B)$

20. Seminar

III Aditivitate și inegalități

21. $A_1, A_2, \dots, A_n$ disjuncte

$$B_n:\ P\left(\bigcup_{i=1}^{n} A_i\right)=\sum_{i=1}^{n} P(A_i)$$

1. Verificare: $B_1: P(A_1)=P(A_1)$

$P(A_1\cup A_2)=P(A_1)+P(A_2)-P(A_1\cap A_2)$

dar A_1, A_2 disjuncte $\Rightarrow P(A_1\cap A_2)=0$.

$\Rightarrow P(A_1\cup A_2)=P(A_1)+P(A_2)$

21. $B_k\to B_{k+1}$

$P(A_k\cup A_{k+1})=P(A_k)+P(A_{k+1})-P(A_k\cap A_{k+1})$; A_k, A_{k+1} disjuncte.

$=P(A_k)+P(A_{k+1})-0$ $\qquad\Rightarrow P(A_k\cap A_{k+1})=0$

$=P(A_k)+P(A_{k+1})$

$$=P(A_1)+\dots+P(A_k)+P(A_{k+1})=\sum_{i=1}^{k+1} P(A_i)$$

$\Rightarrow$ Rezultă prin metoda inducției matematice că B_n adevărată.

22. $A_1, A_2, \dots, A_n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

1. $B_n: P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

2. Verificare B_1: $P(A_1) \leq P(A_1)$

$B_2: P(A_1 \cap A_2) \geq 0$

$-P(A_1 \cap A_2) \leq 0$

$P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2)$

$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$

22. $P_k \Rightarrow P_{k+1}$.

$P(A_k \cap A_{k+1}) \geq 0$

$-P(A_k \cap A_{k+1}) \leq 0$.

$P(A_k) + P(A_{k+1}) - P(A_k \cap A_{k+1}) \leq P(A_k) + P(A_{k+1})$

$P(A_k \cup A_{k+1}) \leq \sum_{i=1}^{k+1} P(A_{k+1})$ $\quad$ $\overset{\shortparallel}{P(A_1) + \dots P(A_k)}$

$\Rightarrow$ Rezultă prin metoda inducției matematice că B_n adevărată.

23. $\lim_{n \to \infty} P(A_n)$; dacă $A_n \downarrow \emptyset$

0? dar cum???

24. $P(A) = 0,7$
$P(B) = 0,5$
$P(C) = 0,4$

$$P(A \cup B \cup C) = \underbrace{P(A) + P(B) + P(C)}_{1,6 \leq 1} - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

25. $P(A_1 \cup A_2 \cup A_3)$

$P(A_1 \cap A_2) \geq 0$

$-P(A_1 \cap A_2) \leq 0$

$P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2)$

$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$

$P((A_1 \cup A_2) \cap A_3) \geq 0$

$-P((A_1 \cup A_2) \cap A_3) \leq 0$

$P(A_1 \cup A_2) + P(A_3) - P((A_1 \cup A_2) \cap A_3) \leq P(A_1 \cup A_2) + P(A_3)$

$P(A_1 \cup A_2 \cup A_3) \leq P(A_1 \cup A_2) + P(A_3)$

$P(A_1 \cup A_2 \cup A_3) - P(A_3) \leq P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$

$P(A_1 \cup A_2 \cup A_3) \leq P(A_1) + P(A_2) + P(A_3)$

26. ex 22

27. ex: $P(A) = 0.3$
$P(B) = 0.4$
$P(C) = 0.4$

$P(A \cap B) = 0$
$P(B \cap C) = 0.1$
$P(C \cap A) = 0.$

$P(A \cap B \cap C) = 0.$

$P(A \cup B \cup C) = 1.$

$P(A \cup B \cup C) \leq 0.3 + 0.4 + 0.4 - 0.1$
$\leq 1.$

28. $A =$ "nr par"

$2 \leq 2k \leq 100, k \in \mathbb{Z}$
$1 \leq k \leq 50 \Rightarrow 50 - 1 + 1 = 50$ nr.

$P(A) = 0.5$

$B =$ „multiplu de 5"

$5 \le 5k \le 100;\ k \in \mathbb{Z}$
$1 \le k \le 20 \Rightarrow 20-1+1 = 20$ nr

$\Rightarrow P(B) = 0.2.$

$P(A \cup B) \le 0.2 + 0.5 = 0.7.$

Verificare

$A \cup B =$" multiplu de 10

$10 \le k \cdot 10 \le 100,\ k \in \mathbb{Z}$
$1 \le k \le 10 \Rightarrow 10$ nr.
$\Rightarrow$

$P(A \cup B) = 0.1 \le 0.7.$ ADEVĂRAT.

29. A, B, C incompatibile

$\Rightarrow A \cap B = \emptyset \Rightarrow P(A \cap B) = 0$
$A \cap C = \emptyset \Rightarrow P(A \cap C) = 0$
$B \cap C = \emptyset \Rightarrow P(B \cap C) = 0$
$A \cap B \cap C = \emptyset \Rightarrow P(A \cap B \cap C) = 0$

$\Rightarrow P(\cup A \cup B \cup C) = \sum P(A_i).$

30. A, B, C complementare

$\Rightarrow A \cup B \cup C = \Omega$

$\Rightarrow P(A \cup B \cup C) = 1.$

31. $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

$$\left.\begin{array}{l} P(A) = 0,4 \\ P(A \cap B) = 0,2 \end{array}\right\} \Rightarrow P(B|A) = \frac{0,2}{0,4} = \frac{\frac{1}{5}}{\frac{2}{5}} = \frac{1}{2}.$$

32. $P(A) = 0,6$
$P(B) = 0,3$
$P(A \cap B) = 0,18.$

$$P(B|A) = \frac{0,18}{0,6} = \frac{\frac{18}{\cancel{100}}}{\frac{6}{\cancel{100}}} = \frac{18}{60} = 0.3$$

$$P(A|B) = \frac{0{,}18}{0.3} = \frac{18}{30} = \frac{3}{5} = 0{,}6$$

33. $P(A) = 0.3$

$P(A\cup B) = 0.6$

$P(A\cap B) = 0.15$

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$0.6 = 0.3 + P(B) - 0.15$

$P(B) = 0.45$.

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0{,}15}{0{,}45} = \frac{1}{3} = 0.(3)$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{0{,}15}{0{,}3} = \frac{1}{2} = 0.5$$

34. $P(B|A) = \dfrac{P(A\cap B)}{P(A)} \;\Big|\cdot P(A)$

$P(A\cap B) = P(B|A)\cdot P(A)$?

35. $P(A) = 0.8$

$P(B|A) = 0.75$.

$P(A\cap B) = ?$

$$P(A\cap B) = 0.8\cdot 0.75 = \frac{8}{10}\cdot\frac{75}{100} = \frac{4}{5}\cdot\frac{3}{4} = \frac{3}{5}.$$

36. A, B, C

$P(\underbrace{A\cap B}_{D}\cap C) = P(A)\cdot P(B|A)P(C|A\cap B)$

$P(D\cap C) = P(D)\cdot P(C|D)$

$P(D\cap C) = \underbrace{P(A\cap B)}_{D}\cdot P(C|A\cap B)$

$P(D\cap C) = P(A)\cdot P(B|A)\cdot P(C|A\cap B)$

37. 60% au trecut la examen $= A$

70% au trecut la cel practic $= B$.

$P(A) = 60\% = 0.6$
$P(B|A) = 70\% = 0.7$

$P(A \cap B) = P(A) \cdot P(B|A) = 0.6 \cdot 0.7 = 0.42$.

$\downarrow$

cei care au trecut la ambele

38. $\left.\begin{array}{l} P(U_1) = 0.4 \\ P(U_2) = 0.6 \end{array}\right\} \Rightarrow P(U_1) + P(U_2) = 1 \xrightarrow{\text{formula prob totale}}$

$P(R|U_1) = 0.3$
$P(R|U_2) = 0.1$ $\Rightarrow P(R) = P(U_1) \cdot P(R|U_1) + P(U_2) \cdot P(R|U_2)$

$P(R) = ?$

$= 0.4 \cdot 0.3 + 0.6 \cdot 0.1$

$= 0.12 + 0.06$

$= 0.18$.

39. $P(U_1|R) = ?$

$$P(U_1|R) = \frac{P(U_1 \cap R)}{P(R)} = \frac{P(U_1) \cdot P(R|U_1)}{P(R} = \frac{0.4 \cdot 0.3}{0.18}$$

$$= \frac{0.12}{0.18} = \frac{12}{18} = \frac{2}{3} = 0,(6)$$

40. Inducție

1: $B_n: P(\bigcap_{i=1}^{n} A_i) = P(A_1) P(A_2|A_1) \dots P(A_n|A_1 \cap \dots A_{n-1})$

2: B_1: $P(A_1 \cap A_2) = P(A_1) \cdot P(A_2|A_1)$ Adevărat.

3: P_k - adev

$P_k \Rightarrow P_{k+1}$: $P((A_1 \cap A_2 \dots A_k) \cap A_{k+1}) = P(A_k) \cdot P(A_{k+1}|A_k)$

$= P(A_1) \cdot P(A_2|A_1) \cdot \ldots \cdot P(A_{k+1}|A_1 \cap \ldots \cap A_k)$ ADEVĂRAT.

V. PROBABILITATEA TOTALĂ ȘI TEOREMA LUI BAYES

41. $(A_1, A_2, \ldots, A_n)$ Sistem complet evenimente.

$$P(B) = \sum_{i=1}^{n} P(A_i) P(B|A_i)$$

$A_i \cap A_j = \emptyset$, $i \neq j$

$\bigcup_{i=1}^{n} A_i = \Omega$ $\quad$ A_i - partiții.

$$P(B) = P\left(\bigcup_{i=1}^{n} (B \cap A_i)\right) = \sum_{i=1}^{n} P(B \cap A_i)$$

dar $P(B \cap A_i) = P(A_i) \cdot P(B|A_i)$

$$\Rightarrow P(B) = \sum_{i=1}^{n} P(A_i) \cdot P(B|A_i)$$

42. $P(U_1) = 0.2$
$P(U_2) = 0.5$
$P(U_3) = 0.3$

$P(B|U_i) = 0,9; 0,7; 0,2$

$$P(B) = 0.2 \cdot 0.9 + 0.5 \cdot 0.7 + 0.3 \cdot 0.2$$
$$= 0.18 + 0.35 + 0.06$$
$$= 0.59.$$

43. $P(U_2|B) = ?$

$$P(U_2|B) = \frac{P(U_2 \cap B)}{P(B)} = \frac{P(U_2) \cdot P(B|U_2)}{P(B)}$$

$$= \frac{0.5 \cdot 0.7}{0.59} = \frac{35}{59}.$$

44. ?

45. B: evenim în care pers e bolnavă

T^+: evenimentul să fie pozitiv

PS-lab 10.pdf

1. $X : \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 3p & 4p & 2p & p & p \end{pmatrix}$ $p \in \mathbb{R}$;

a) $p = ?$

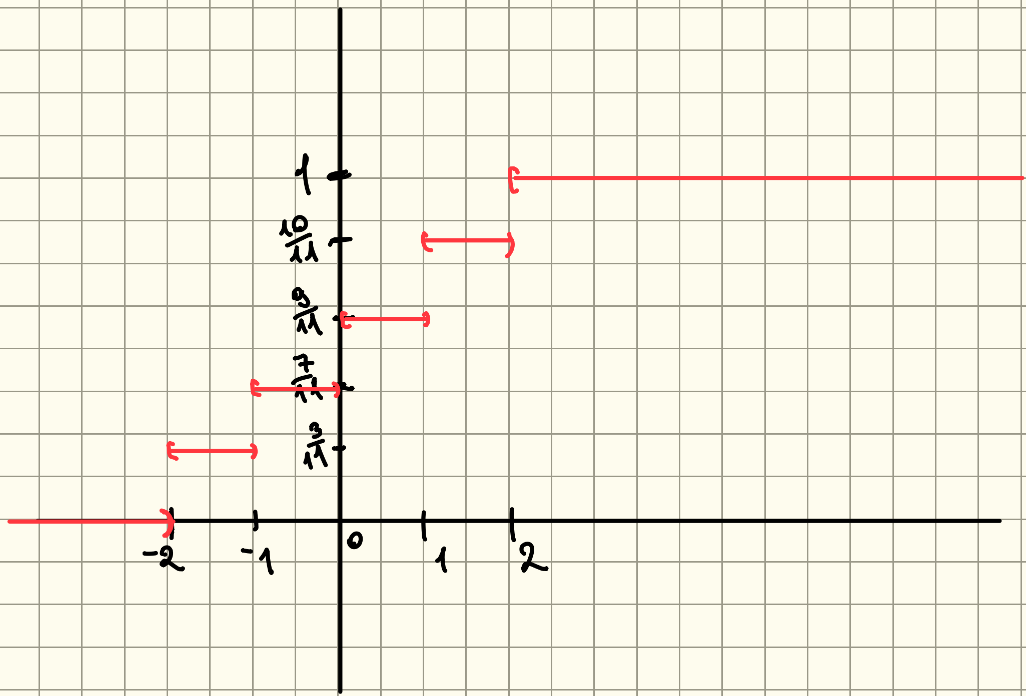
Cond:

$\sum_{i=1}^{5} p_i = 1 \iff 3p + 4p + 2p + p + p = 1$

$11p = 1 \Rightarrow p = \frac{1}{11}$

b) $X : \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{3}{11} & \frac{4}{11} & \frac{2}{11} & \frac{1}{11} & \frac{1}{11} \end{pmatrix}$

$$F(x) = P(X \le x) = \begin{cases} 0 ; & x < -2 \\ \frac{3}{11} ; & -2 \le x < -1 \\ \frac{7}{11} ; & -1 \le x < 0 \\ \frac{9}{11} ; & 0 \le x < 1 \\ \frac{10}{11} ; & 1 \le x < 2 \\ 1 ; & x \ge 2 \end{cases}$$



c) $\underline{16X-23}$

$$\mathbb{E}(16X-23) = \mathbb{E}(16X) = 16\,\mathbb{E}(X) - 23$$

$$\mathbb{E}(X) = -2 \cdot \frac{3}{11} - 1 \cdot \frac{4}{11} + \frac{1}{11} + 2 \cdot \frac{1}{11}$$

$$= \frac{-6 - 4 + 1 + 2}{11} = \frac{-7}{11}$$

$$\mathbb{E}(16X-23) = 16 \cdot \frac{-7}{11} - 23 = \frac{-112 - 253}{11} = -\frac{365}{11}$$

$$\mathrm{Var}(16X-23) = 16^2\,\mathrm{Var}(X)$$

$$= 16^2\left(\mathbb{E}(X^2) - (\mathbb{E}(X))^2\right)$$

$$= 256\left(\frac{21}{11} - \frac{49}{121}\right) = 256\left(\frac{231-49}{121}\right) = 256 \cdot \frac{182}{121}$$

$$\mathbb{E}(X^2) = 4\left(\frac{3}{11} + \frac{1}{11}\right) + 1\left(\frac{4}{11} + \frac{1}{11}\right)$$

$$= \frac{16}{11} + \frac{5}{11} = \frac{21}{11}$$

$\underline{3X-2}$

$$\mathbb{E}(3X-2) = 3\,\mathbb{E}(X) - 2 = 3 \cdot \left(-\frac{7}{11}\right) - 2$$

$$= -\frac{21}{11} - \frac{11}{2}$$

$$= -\frac{21}{11} - \frac{22}{11}$$

$$= -\frac{43}{11}.$$

$$Var(3X-2) = 9Var(X)$$
$$= 9 \cdot \frac{182}{121}.$$

2. $X: \begin{pmatrix} 0 & 1 \\ 0.4 & 0.6 \end{pmatrix}$ $Y: \begin{pmatrix} -1 & 1 \\ 0.5 & 0.5 \end{pmatrix}.$ $Y^2: \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$k = P(X=1, Y=-1)$

a) Rep comună

x \ y	-1	1	
0	0.5-k	k-0.1	0.4
1	k	0.6-k	0.6
	0.5	0.5	

$\mathbb{E}(X) = 0.6$

$\mathbb{E}(Y) = 0$

b) Coef de corelație

$$\rho(X,y) = \frac{cov(X,y)}{\sqrt{var(x)\, var(y)}}$$

$$cov(x,y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$X \cdot Y: \begin{pmatrix} -1 & 1 & 0 \\ k & 0.6-k & 0.4 \end{pmatrix}$$

$$cov(x,y) = -k + 0.6 - k - 0$$
$$= 0.6 - 2k.$$

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$
$$= 0.6 - 0.36$$
$$= 0.24$$

$$Var(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2$$
$$= 1 - 0$$
$$= 1.$$

$$\rho(x,y) = \frac{0.6-2k}{\sqrt{0.24}}$$

c) X Y necorelate $\Leftrightarrow \rho(x,y)=0 \Leftrightarrow 0.6-2k=0$

$2k = 0.6$

$k = 0.3.$

x \ y	-1	1	
0	0.2	0.2	0.4
1	0.3	0.3	0.6
	0.5	0.5	

$\pi_{11} = 0.2 = 0.5 \cdot 0.4 = p_1 \cdot q_1$ V

$\pi_{12} = 0.2 = 0.5 \cdot 0.4 = p_1 \cdot q_2$ V

$\pi_{21} = 0.3 = 0.5 \cdot 0.6 = p_2 \cdot q_1$ V $\Rightarrow$ X, Y independente.

$\pi_{22} = 0.3 = 0.5 \cdot 0.6 = p_2 \cdot q_2$ V

3. $X: \begin{pmatrix} a & 1 & 2 \\ \frac{1}{3} & p & q \end{pmatrix}$ $Y: \begin{pmatrix} a+1 & 1 & 2 \\ \frac{1}{3} & \frac{2}{3}-q & p \end{pmatrix}$ $p, q, a \in \mathbb{R}$

X, Y independente.

$a = ?$ $Var(X-Y) = \frac{4}{9}$

a influențează $\rho(x,y)$

Cond:

$\frac{1}{3} + p + q = 1$

$\frac{1}{3} + \frac{2}{3} - q + p = 1$

+

$\frac{2}{3} + \frac{2}{3} + 2p = 2$

$\frac{4}{3} + 2p = 2$

$\frac{2}{3} + p = 1$

$p = \frac{1}{3}. \Rightarrow \frac{2}{3} + q = 1$

$q = \frac{1}{3}.$

$$X:\begin{pmatrix} a & 1 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \qquad Y:\begin{pmatrix} a+1 & 1 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$X-Y:\begin{pmatrix} a-a-1 & a-1 & a-2 & 1-a-1 & 1-1 & 1-2 & 2-a-1 & 2-1 & 2-2 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}$$

$$X-Y:\begin{pmatrix} -1 & a-1 & a-2 & -a & 0 & -1 & 1-a & 1 & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}$$

$$X-Y:\begin{pmatrix} -a & a-2 & a-1 & 1-a & -1 & 0 & 1 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

$$Var(x-y) \overset{x,y \text{ independente}}{=} Var(X) + Var(y)$$

$$\mathbb{E}(X^2) = a^2 \cdot \frac{1}{3}$$

$$= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 + \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2$$

$$= a^2 \cdot \frac{1}{3} + \frac{1}{3} + \frac{4}{3} - \left(a \cdot \frac{1}{3} + \frac{1}{3} + \frac{2}{3}\right)^2 + (a+1)^2 \cdot \frac{1}{3} + \frac{1}{3} + \frac{4}{3} - \left(\frac{1}{3}(a+1) + \frac{1}{3} + \frac{2}{3}\right)^2$$

$$= \frac{a^2}{3} + \frac{5}{3} - \left(\frac{a+3}{3}\right)^2 + (a^2+2a+1)\frac{1}{3} + \frac{5}{3} - \left(\frac{a+4}{3}\right)^2$$

$$= \frac{a^2}{3} + \frac{5}{3} - \frac{a^2+6a+9}{9} + \frac{a^2+2a+1}{3} + \frac{5}{3} - \frac{a^2+8a+16}{9}$$

$$= \frac{3a^2 + 15 - a^2 - \cancel{6a} - 9 + 3a^2 + \cancel{6a} + 3 + 15 - a^2 - 8a - 16}{9}$$

$$= \frac{4a^2 - 8a + 8}{9}$$

$21 - 16 = 5$

$14 - 9$

$$Var(X-Y) = \frac{4}{9}$$

$$4a^2 - 8a + 8 = 4 \quad |:4$$

$$a^2 - 2a + 2 = 1$$

$$a^2 - 2a + 1 = 0$$

$$(a-1)^2 = 0 \Rightarrow \underline{\underline{a = 1}}$$

X, Y independente $\Leftrightarrow \rho(X,Y) = 0 \ \forall a \in \mathbb{R}$

Deci a nu influențează $\rho(X,Y)$

4. $X:\begin{pmatrix} -2 & 3 & 4 & 6 \\ 6p & 2p & 9p & p \end{pmatrix}$ $p \in \mathbb{R}$

$6p+2p+9p+p=1.$

$18p=1$

$p=\frac{1}{18}.$

$a, b = ?$ $Y=aX+b$ a.î. $\mathbb{E}(Y)=57$

$Var(Y)=75.$

funcție rep a lui X, grafic.

$X:\begin{pmatrix} -2 & 3 & 4 & 6 \\ \frac{6}{18} & \frac{2}{18} & \frac{3}{18} & \frac{1}{18} \end{pmatrix}$

$\mathbb{E}(Y)=\mathbb{E}(aX+b)=a\mathbb{E}(X)+b$

$=2a+b$

$\Rightarrow 2a+b=57.$

$\mathbb{E}(X)=-\cancel{\frac{12}{18}}+\cancel{\frac{6}{18}}+\frac{36}{18}+\cancel{\frac{6}{18}}$

$=\frac{36}{18}=2.$

$\mathbb{E}(X^2)=\frac{24}{18}+\frac{18}{18}+\frac{144}{18}+\frac{36}{18}$

$=\frac{222}{18}=111$

$Var(Y)=Var(aX+b)=a^2\,Var(X)$

$=a^2\left(\mathbb{E}(X^2)-\mathbb{E}(X)^2\right)$

$=a^2\left(\frac{222}{18}-\frac{72}{18}\right)$

$=a^2\cdot\frac{150}{18}$

$\Rightarrow a^2\cdot\frac{150}{18}=75$

$a^2=\frac{75\cdot 18}{150}=9$

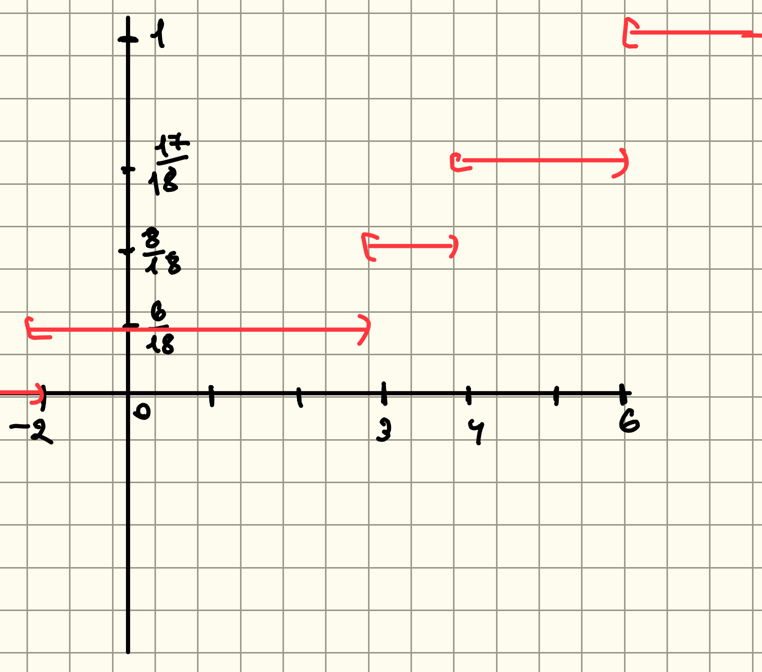
$a=\pm 3.$

Pt $a=3 \Rightarrow 6+b=51 \Rightarrow b=51$

Pt $a=-3 \Rightarrow -6+b=57 \Rightarrow b=63.$

$X:\begin{pmatrix} -2 & 3 & 4 & 6 \\ \frac{6}{18} & \frac{2}{18} & \frac{3}{18} & \frac{1}{18} \end{pmatrix}$

$$F(x)=\mathbb{P}(X<x)=\begin{cases} 0 ; & x<-2 \\ \frac{6}{18} ; & -2\le x<3 \\ \frac{8}{18} ; & 3\le x<4 \\ \frac{17}{18} ; & 4\le x<6 \\ 1 ; & x\ge 6 \end{cases}.$$



5. făcut la seminar

6. $\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 3p & 4p & 2p & p & p \end{pmatrix}$ $\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{3}{11} & \frac{4}{11} & \frac{2}{11} & \frac{1}{11} & \frac{1}{11} \end{pmatrix}$

a) $p = ?$

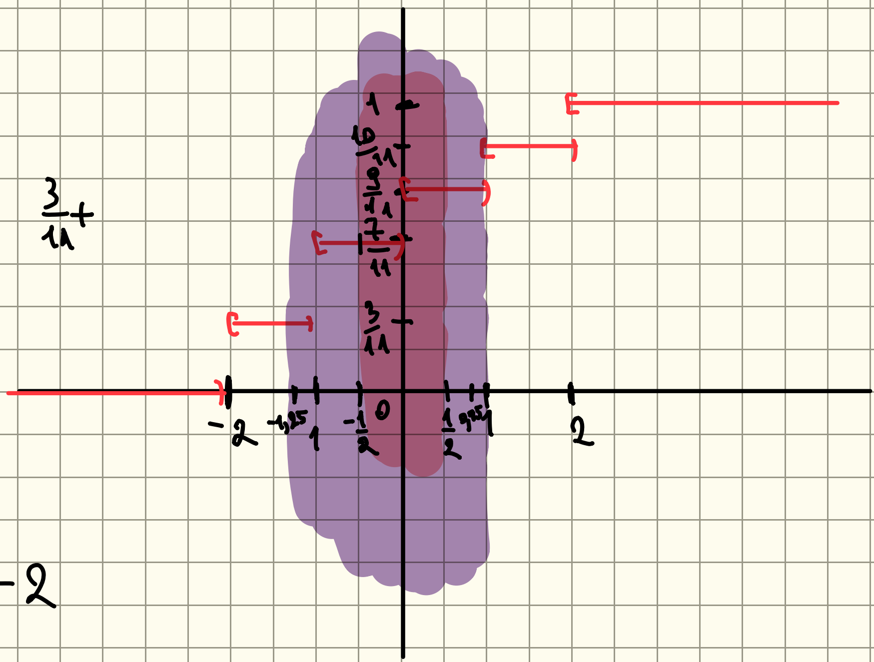
Cond: $11p = 1$

$p = \frac{1}{11}$

b) funcţie rep + Grafic?

$$F(x) = P(X \le x) = \begin{cases} 0, & x < -2 \\ \frac{3}{11}, & -2 \le x < -1 \\ \frac{7}{11}, & -1 \le x < 0 \\ \frac{9}{11}, & 0 \le x < 1 \\ \frac{10}{11}, & 1 \le x < 2 \\ 1; & x \ge 2 \end{cases}$$

$\frac{3}{11}+$



c) $\mathbb{E}(3X-2) = 3\mathbb{E}(X) - 2$

$$= 3\left(-\frac{6}{11} - \frac{4}{11} + \frac{1}{11} + \frac{2}{11}\right) - 2$$

$$= -\frac{21}{11} - \overset{11)}{2} = \frac{-21-22}{11} = \frac{-43}{11}.$$

$Var(6X-3) = 36\, Var(X)$

$$= 36\left(\mathbb{E}(X^2) - \mathbb{E}^2(X)\right)$$

$$= 36\left(4 \cdot \frac{3}{11} + 1 \cdot \frac{4}{11} + 1 \cdot \frac{1}{11} + 4 \cdot \frac{1}{11}\right)$$

$$= 36\left(\frac{12+4+1+4}{11} - \frac{49}{121}\right)$$

$$= 36\, \frac{231-49}{121}$$

$$= \frac{36 \cdot 182}{121}$$

$21 \cdot 11$, 21, 21, $231 -$ 64, 167

$231 - 49 = 182$

$182 + 49 = 231$

$\mathbb{E}(X+X^2)=\mathbb{E}(X)+\mathbb{E}(X^2)$

$= -\frac{7}{11}+\frac{21}{11}=\frac{14}{11}.$

d) $\mathbb{P}(|X|<\frac{1}{2} \mid -1.25<X<0.75)$

$\mathbb{P}(-\frac{1}{2}<X<\frac{1}{2} \mid -1.25<X<0.75) = \dfrac{\mathbb{P}(|X|<\frac{1}{2} \cap -1.25<x<0.25)}{\mathbb{P}(-1.25<x<0.75)}$

$= \dfrac{\frac{2}{11}}{\frac{2}{11}+\frac{4}{11}} = \frac{2}{6}=\frac{1}{3}$

7.

X \ Y	−2	0	9	$\mathbb{P}(X=x_i)$
−1	b	2b	0	
0	3b	4b	5b	
$\mathbb{P}(Y=y_j)$				

$b \in \mathbb{R}$

a) Cond: $3b+12b=1$

$15b=1$

$b=\frac{1}{15}$

X \ Y	−2	0	9	$\mathbb{P}(X=x_i)$
−1	$\frac{1}{15}$	$\frac{2}{15}$	0	$\frac{3}{15}$.
0	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$	$\frac{12}{15}$
$\mathbb{P}(Y=y_j)$	$\frac{4}{15}$	$\frac{6}{15}$	$\frac{5}{15}$	1

b) indep X, Y

$\mathbb{P}(X\cdot Y\neq 0)=?$

$\pi_{11}=\frac{1}{15}\neq\frac{3}{15}\cdot\frac{4}{15}\neq p_1\cdot q_1 \Rightarrow$ X, Y nu sunt independente.

$\mathbb{P}(X\cdot Y\neq 0)=\mathbb{P}(X=-1, Y=-2)+\mathbb{P}(X=-1, Y=9)$

$=\frac{1}{15}+0=\frac{1}{15}$

c) $3X-2Y$

$Var(3X-2Y)=9\,Var(X)+4\,Var(Y)-12\,Cov(X,Y)$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X)$$
$$= \left(1 \cdot \frac{3}{15}\right) - \frac{9}{225}$$
$$= \frac{45-9}{225} = \frac{36}{225}$$

$$\left(\frac{-8+45}{15}\right)^2 = \frac{37}{15}$$

$$\mathrm{Van}(Y) = \mathbb{E}(Y^2) - \mathbb{E}^2(Y)$$
$$= \left(\frac{16}{15} + 81 \cdot \frac{5}{15}\right) - \left(\frac{37}{15}\right)^2 = \frac{4946}{225}.$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$X \cdot Y = \begin{pmatrix} 2 & 0 & -9 \\ \frac{1}{15} & \frac{2}{15}+\frac{3}{15}+\frac{4}{15}+\frac{5}{15} & 0 \end{pmatrix}$$

$$X \cdot Y = \begin{pmatrix} 2 & 0 \\ \frac{1}{15} & \frac{14}{15} \end{pmatrix}$$

$$\mathbb{E}(X \cdot Y) = \frac{2}{15} + 0 = \frac{2}{15}.$$

$$\mathbb{E}(X) = -\frac{3}{15}$$
$$\mathbb{E}(Y) = -\frac{8}{15} + \frac{45}{15} = \frac{37}{15}.$$
$$\mathrm{COV}(X,Y) = \frac{2}{15} + \frac{3}{15} \cdot \frac{37}{15} = \frac{30+111}{225} = \frac{141}{225}$$

$$\mathrm{Var}(3X-2Y) = 9 \cdot \frac{36}{225} + 4 \cdot \frac{4946}{225} - 12 \cdot \frac{141}{225} = \ldots$$

10.

$X \backslash Y$	1	2	3	4	p_i
0	$\frac{4}{40}$	$\frac{3}{40}$	$\frac{2}{40}$	$\frac{1}{40}$	$\frac{1}{4}$
1	$\frac{1}{40}$	$\frac{4}{40}$	$\frac{3}{40}$	$\frac{2}{40}$	$\frac{1}{4}$.
2	$\frac{2}{40}$	$\frac{1}{40}$	$\frac{4}{40}$	$\frac{3}{40}$	$\frac{1}{4}$
3	$\frac{3}{40}$	$\frac{2}{40}$	$\frac{1}{40}$	$\frac{4}{40}$	$\frac{1}{4}$
q_i	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1

a)

b) $\rho(X,Y) = \dfrac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$

$$\mathrm{Cov}(X,Y) = \mathbb{E}(X\cdot Y) - \mathbb{E}(X)\cdot\mathbb{E}(Y)$$

$$\mathbb{E}(X,Y) = 1\cdot\frac{1}{40} + 2\cdot\frac{4}{40} + 3\cdot\frac{3}{40} + 4\cdot\frac{2}{40} + 2\cdot\frac{2}{40} + 4\cdot\frac{1}{40} + 6\cdot\frac{4}{40} + 8\cdot\frac{3}{40}$$
$$+ 3\cdot\frac{3}{40} + 6\cdot\frac{2}{40} + 9\cdot\frac{1}{40} + 12\cdot\frac{4}{40}$$

$$= \frac{1+8+9+8+4+4+24+24+9+12+9+48}{40} = \frac{160}{40} = 4$$

$$\mathbb{E}(X) = \frac{1}{4} + \frac{2}{4} + \frac{3}{4} = \frac{6}{4}$$

$$\mathbb{E}(Y) = \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} = \frac{10}{4}.$$

$$\mathrm{Cov}(X,Y) = 4 - \frac{\cancel{10}^{\,5}}{\cancel{4}_{\,2}}\cdot\frac{\cancel{6}^{\,3}}{\cancel{4}_{\,2}} = 4 - \frac{15}{4} = \frac{16-15}{4} = \frac{1}{4}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X)$$
$$= \frac{1}{4} + 4\cdot\frac{1}{4} + 9\cdot\frac{1}{4} - \frac{36}{16}$$
$$= \frac{14\cdot4 - 36}{16} = \frac{20}{16}$$

$14\cdot4 = 56$

$$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}^2(Y)$$
$$= \frac{1}{4} + 4\cdot\frac{1}{4} + 9\cdot\frac{1}{4} + 16\cdot\frac{1}{4} - \frac{100}{16}$$

$$= \frac{4 \cdot 30 - 100}{16}$$

$$= \frac{120 - 100}{16} = \frac{20}{16}$$

$$\rho(X, Y) = \frac{\frac{1}{4}}{\sqrt{\frac{20^2}{16^2}}} = \frac{\frac{1}{4}}{\frac{20}{16}} = \frac{1}{\cancel{4}} \cdot \frac{\cancel{16}^{4}}{20} = \frac{4}{20} = \frac{1}{5}$$

c) v. a cond $X \mid Y=3$ şi $Y \mid X=1$

$\mathbb{E}(X \mid Y=3) = ?$

$\mathbb{E}(Y \mid X=1) = ?$

$\frac{2}{40} \cdot 4$

$$X \mid Y=3 : \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{8}{40} & \frac{12}{40} & \frac{16}{40} & \frac{4}{40} \end{pmatrix} \quad \mathbb{E}(X) = \frac{12+32+12}{40} = \frac{56}{40}$$

$$Y \mid X=1 : \begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{4}{40} & \frac{16}{40} & \frac{12}{40} & \frac{8}{40} \end{pmatrix} \quad \mathbb{E}(Y) = \frac{4+32+36+32}{40} = \frac{104}{40}$$

d) $Var(-X+5) = (-1)^2 Var(X) = \frac{20}{16}$

e) $\mathbb{P}(X<1, Y>3) =$

$\mathbb{P}(X=0, Y=4) = \frac{1}{40}.$

V.a. discrete / TEMA

1. a) $X : \begin{pmatrix} 2 & 3 \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix} \quad Y : \begin{pmatrix} -3 & -2 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$

$$3X = \begin{pmatrix} 6 & 9 \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

$$X^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

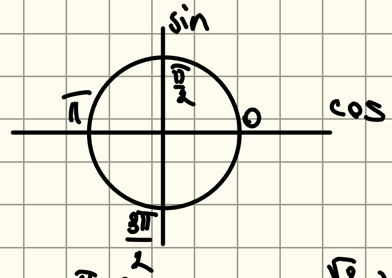
$$\cos\left(\frac{\pi}{2} \cdot X\right) = \cos\left(\begin{pmatrix} \frac{2\pi}{2} & \frac{3\pi}{2} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}\right) = \begin{pmatrix} -1 & -\frac{1}{2} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

$$Y^2=\begin{pmatrix} 9 & 4 \\ \frac{4}{5} & \frac{1}{5}\end{pmatrix}$$

$$Y+3=\begin{pmatrix} 0 & 1 \\ \frac{4}{5} & \frac{1}{5}\end{pmatrix}$$

b) $X:\begin{pmatrix} 0 & 9 \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$ $\qquad Y:\begin{pmatrix} -3 & 1 \\ \frac{1}{7} & \frac{6}{7}\end{pmatrix}$

$$X-1=\begin{pmatrix} -1 & 8 \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$$



$$X^{-2}=\begin{pmatrix} 0 & \frac{1}{81} \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

$$\sin\left(\frac{\pi}{2}\cdot X\right)=\sin\left(\begin{pmatrix} 0 & \frac{9\pi}{2} \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\right)=\sin\left(\begin{pmatrix} 0 & 2\pi+\frac{\pi}{4} \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\right)=\begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

$$Y\cdot 5=\begin{pmatrix} -15 & 5 \\ \frac{1}{7} & \frac{6}{7}\end{pmatrix}$$

$$e^{Y}\underset{g(Y)=e^{Y}}{=}\begin{pmatrix} e^{-3} & e \\ \frac{1}{7} & \frac{6}{7}\end{pmatrix}$$

c) $X:\begin{pmatrix} 5 & 8 \\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$ $\qquad Y:\begin{pmatrix} -1 & 1 \\ \frac{1}{6} & \frac{5}{6}\end{pmatrix}$

$2X=\begin{pmatrix} 10 & 16 \\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$ $\qquad X^{-3}=\begin{pmatrix} \frac{1}{5^3} & \frac{1}{8^3} \\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$

$$tg(\pi\cdot X)=tg\left(\begin{pmatrix} 5\pi & 8\pi \\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}\right)=\begin{pmatrix} 0 \\ 1\end{pmatrix}$$

$Y-2=\begin{pmatrix} -3 & -2 \\ \frac{1}{6} & \frac{5}{6}\end{pmatrix}$ $\qquad |Y|=\begin{pmatrix} 1 \\ 0\end{pmatrix}$

d) $X : \begin{pmatrix} -3 & 6 \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix} \quad Y : \begin{pmatrix} e & e^3 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$

$$2 - X = 2 + \begin{pmatrix} -6 & 3 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix} \qquad X^3 = \begin{pmatrix} -3^3 & 6^3 \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix}$$

$$\cos\left(\frac{\pi}{6} \cdot X\right) = \cos\left(\begin{pmatrix} -\frac{3\pi}{6} & \frac{6\pi}{6} \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix}\right) = \cos\left(\begin{pmatrix} -\frac{\pi}{2} & \pi \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix}\right)$$

$$= \begin{pmatrix} -\frac{1}{2} & -1 \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix}$$

a) $2X + 3Y = \begin{pmatrix} 4 & 6 \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix} + \begin{pmatrix} -9 & -6 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$

$$= \begin{pmatrix} -5 & -3 & -2 & 0 \\ \frac{4}{25} & \frac{16}{25} & \frac{1}{25} & \frac{4}{25} \end{pmatrix}$$

$$3X - Y = \begin{pmatrix} 6 & 9 \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 9 & 11 & 12 \\ \frac{1}{25} & \frac{4}{25} & \frac{4}{25} & \frac{16}{25} \end{pmatrix}$$

$$X^2 \cdot Y^3 = \begin{pmatrix} 4 & 9 \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix} \cdot \begin{pmatrix} -81 & -8 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -4 \cdot 81 & -4 \cdot 8 & -9 \cdot 81 & -9 \cdot 8 \\ \frac{4}{5} & \frac{1}{5} & \frac{16}{5} & \frac{4}{5} \end{pmatrix} = \begin{pmatrix} -243 & -108 & -72 & -32 \\ \frac{16}{25} & \frac{4}{25} & \frac{4}{25} & \frac{1}{25} \end{pmatrix}$$

b) $X - Y = \begin{pmatrix} 0 & 9 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} -1 & 3 \\ \frac{6}{7} & \frac{1}{7} \end{pmatrix}$

$$= \begin{pmatrix} -1 & 3 & 8 & 12 \\ \frac{6}{14} & \frac{1}{14} & \frac{6}{14} & \frac{1}{14} \end{pmatrix}$$

$$\cos(\pi \cdot X \cdot Y) = \cos\left(\pi \begin{pmatrix} 0 & 0 & -27 & 9 \\ \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{6}{14} \end{pmatrix}\right)$$

$$\cos\left(\begin{pmatrix} -27\pi & 0 & 9\pi \\ \frac{1}{14} & \frac{7}{14} & \frac{6}{14} \end{pmatrix}\right)$$

$$= \begin{pmatrix} -1 & 1 \\ \frac{1}{14}+\frac{6}{14} & \frac{7}{14} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ \frac{7}{14} & \frac{7}{14} \end{pmatrix}$$

$$X^2 + 3Y = \begin{pmatrix} 0 & 81 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} -9 & 3 \\ \frac{1}{7} & \frac{6}{7} \end{pmatrix}$$

$$= \begin{pmatrix} -9 & 3 & 72 & 84 \\ \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{6}{14} \end{pmatrix}$$

c) $$X + Y = \begin{pmatrix} 5 & 8 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ \frac{1}{6} & \frac{5}{6} \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 6 & 7 & 9 \\ \frac{1}{18} & \frac{5}{18} & \frac{2}{18} & \frac{10}{18} \end{pmatrix}$$

$$\sin\left(\frac{\pi}{2} \cdot X \cdot Y\right) = \sin\left(\frac{\pi}{2}\begin{pmatrix} -5 & 5 & -8 & 8 \\ \frac{1}{18} & \frac{5}{18} & \frac{2}{18} & \frac{10}{18} \end{pmatrix}\right)$$

$$= \sin\begin{pmatrix} -\frac{8\pi}{2} & -\frac{5\pi}{2} & \frac{5\pi}{2} & \frac{8\pi}{2} \\ \frac{1}{18} & \frac{5}{18} & \frac{2}{18} & \frac{10}{18} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{18} & \frac{12}{18} & \frac{5}{18} \end{pmatrix}$$

$$\frac{1}{X} + \frac{1}{Y} = X^{-1} + Y^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{8} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ \frac{1}{6} & \frac{5}{6} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{5} & \frac{6}{5} & -\frac{7}{8} & \frac{9}{8} \\ \frac{1}{18} & \frac{5}{18} & \frac{2}{18} & \frac{10}{18} \end{pmatrix}$$

d) $$X \cdot Y = \begin{pmatrix} -3e & -3e^3 & 6e & 6e^3 \\ \frac{1}{32} & \frac{3}{32} & \frac{7}{32} & \frac{21}{32} \end{pmatrix}$$

$$\frac{X}{Y} = X \cdot Y^{-1} = \begin{pmatrix} -3 & 6 \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{e} & \frac{1}{e^3} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$= \begin{pmatrix} -3\frac{1}{e} & -3\frac{1}{e^3} & 6\frac{1}{e} & 6\frac{1}{e^3} \\ \frac{1}{32} & \frac{3}{32} & \frac{7}{32} & \frac{21}{32} \end{pmatrix}$$

$$|X - Y^2| = \left| \begin{pmatrix} -3 & 6 \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix} - \begin{pmatrix} e^2 & e^6 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right|$$

$$=\left|\begin{pmatrix} -3 & 6 \\ \frac{1}{8} & \frac{7}{8}\end{pmatrix} + \begin{pmatrix} -e^2 & -e^6 \\ \frac{1}{4} & \frac{3}{4}\end{pmatrix}\right|$$

$$=\left|\begin{pmatrix} -3-e^2 & -3-e^6 & 6-e^2 & 6-e^6 \\ \frac{1}{32} & \frac{3}{32} & \frac{7}{32} & \frac{21}{4}\end{pmatrix}\right|$$

$$=\begin{pmatrix} 3+e^2 & 3+e^6 & e^2-6 & e^6-6 \\ \frac{1}{32} & \frac{3}{32} & \frac{7}{32} & \frac{21}{4}\end{pmatrix}$$

3. $p=?;\ q=?$

a) $X:\begin{pmatrix} 1 & 2 \\ p & q\end{pmatrix}$ $\quad Y:\begin{pmatrix} 3 & 9 \\ 0{,}1 & \frac{p^2+0{,}02}{q}\end{pmatrix}$

cond: $p+q=1 \Rightarrow p=1-q$

$0{,}1+\frac{p^2+0{,}02}{q}=1 \Leftrightarrow \frac{p^2+0{,}02}{q}=0{,}9$

$\Leftrightarrow p^2+0{,}02=0{,}9\,q$

$\Rightarrow (1-q)^2+0{,}02=0{,}9\,q$.

??

b) $X:\begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{3} & p & q^2\end{pmatrix}$ $\quad X^2:\begin{pmatrix} 1 & 4 & 9 \\ p & p & p^2\end{pmatrix}$

Cond: $p,q\in(0,1)$

$\frac{1}{3}+p+q^2=1$

$p+p+p^2=1$

$p^2+2p-1=0$ $\qquad \Delta=4+4\cdot 1=8$

$p_{12}=\frac{-2\pm 2\sqrt{2}}{2}=-1\pm\sqrt{2}$

$p\in(0,1)$ $\qquad \Rightarrow p=-1+\sqrt{2}$

$p\approx 0.41.$

$\Rightarrow \frac{1}{3}+(\sqrt{2}-1)+q^2=1$

$q^2=1-\frac{1}{3}-\sqrt{2}+1$

$q^2=\frac{5}{3}-\sqrt{2}>0$

$$q = \pm\sqrt{\frac{5}{3} - \sqrt{2}}$$

c) $X : \begin{pmatrix} -1 & 0 & 1 \\ p & p^2 & q \end{pmatrix}$ $\quad X^4 : \begin{pmatrix} 0 & 1 \\ \frac{9}{25} & \frac{16}{25} \end{pmatrix}$

$X^2 : \begin{pmatrix} 0 & 1 \\ p^2 & p+q \end{pmatrix}$

$X^4 : \begin{pmatrix} 0 & 1 \\ p^2 & p+q \end{pmatrix} \Rightarrow$ $p+q = \frac{16}{25}$

$p^2 = \frac{9}{25} \Rightarrow p = \pm\frac{3}{5}$

$p > 0$

$\Rightarrow p = \frac{3}{5}.$

$\Rightarrow q = \frac{16}{25} - \frac{3}{5} = \frac{16-15}{25} = \frac{1}{25}$

d) $X = \begin{pmatrix} -1 & 1 \\ 2p & q \end{pmatrix}$ $\quad Y = \begin{pmatrix} 0 & 1 \\ q & 7q \end{pmatrix}$

cond: $2p, 7q, q \in (0,1)$

$2p + q = 1$

$q + 7q = 1 \Rightarrow 8q = 1$

$q = \frac{1}{8}$

$2p + \frac{1}{8} = 1$

$2p = \frac{7}{8} \quad |:2$

$p = \frac{7}{16}$

4) a) $\mathbb{P}(2X+3Y>1) = 0$ $\quad \begin{pmatrix} -5 & -3 & -2 & 0 \\ \frac{4}{25} & \frac{16}{25} & \frac{1}{25} & \frac{4}{25} \end{pmatrix}$

$$\mathbb{P}(2X+3Y>1 \mid X>0) = \frac{\mathbb{P}(2X+3Y \cap X>0)}{\mathbb{P}(X>0)}$$

$$= \frac{0}{1} = 0.$$

$$\mathbb{P}(2X+3Y<3 \mid Y<-2) = \frac{\mathbb{P}(2X+3Y<3 \cap Y<-2)}{\mathbb{P}(Y<-2)}$$

$2X + 3(-3) < 3$

$2X - 9 < 3$

$2X < 12$

$X < 6$

$\mathbb{P}(2X+3Y<3) = 1$

$$= \frac{\frac{4}{5}}{\frac{4}{5}} = 1.$$

$\mathbb{P}(X^2 \cdot Y^3 > 3) = 0$

$\mathbb{P}(X^2 \cdot Y^3 \leq 3) = 1.$

$$\begin{pmatrix} -4\cdot 81 & -4\cdot 8 & -9\cdot 81 & -9\cdot 8 \\ \frac{4}{5} & \frac{1}{5} & \frac{16}{5} & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -243 & -108 & -72 & -32 \\ \frac{16}{25} & \frac{4}{25} & \frac{4}{25} & \frac{1}{25} \end{pmatrix}$$

$\mathbb{P}(2X + 3Y < 3X - Y) = ?$

$\mathbb{P}(2X + 3Y < 3X - Y) = 1$

$$\begin{pmatrix} 8 & 9 & 11 & 12 \\ \frac{1}{25} & \frac{4}{25} & \frac{4}{25} & \frac{16}{25} \end{pmatrix}$$

$$\begin{pmatrix} -5 & -3 & -2 & 0 \\ \frac{4}{25} & \frac{16}{25} & \frac{1}{25} & \frac{4}{25} \end{pmatrix}$$

b) $\mathbb{P}(X - Y > 0)$

$= \frac{1}{14} + \frac{6}{14} + \frac{1}{14} = \frac{8}{14}$

$$\left(\begin{array}{c|ccc} -1 & 3 & 8 & 12 \\ \frac{6}{14} & \frac{1}{14} & \frac{6}{14} & \frac{1}{14} \end{array}\right)$$

$$\mathbb{P}(X - Y < 0 \mid X > 0) = \frac{\mathbb{P}(X - Y < 0 \cap X > 0)}{\mathbb{P}(X > 0)}$$

$$= \frac{0}{\frac{1}{2}} = 0.$$

$$\mathbb{P}(X - Y > 0 \mid Y \leq 0) = \frac{\mathbb{P}((X - Y > 0) \cap (Y \leq 0))}{\mathbb{P}(Y \leq 0)}$$

$$= \frac{0}{\frac{1}{2}} = 0$$

$\mathbb{P}(\cos(\pi X Y) < \frac{1}{2}) = ?$

$$\left(\begin{array}{c|c} -1 & 1 \\ \frac{7}{14} & \frac{7}{14} \end{array}\right)$$

$\mathbb{P}(\cos(\pi X Y) < \frac{1}{2}) = \frac{7}{14} = \frac{1}{2}.$

$\mathbb{P}(X^2+3Y \geq 3)=?$

$\mathbb{P}(X^2+3Y \geq 3) = \frac{13}{14}$

$$\begin{pmatrix} -9 & 3 & 72 & 84 \\ \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{6}{14} \end{pmatrix}$$

$\mathbb{P}(X-Y < X^2+3Y)=?$

$\mathbb{P}(X-X^2 < 4Y) = \frac{6}{14} + \frac{1}{14} + \frac{6}{14} = \frac{13}{14}$

$X: \begin{pmatrix} 0 & 9 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$Y: \begin{pmatrix} -3 & 1 \\ \frac{1}{7} & \frac{6}{7} \end{pmatrix}$

$(0,-3)$: $0 < -12$ Fals

$(0,1)$: $0 < 4$ Adev

$(9,-3)$: $-72 < -12$ adev

$(9,1)$: $-72 < 4$

c) $\mathbb{P}(X+Y<2)=?$

$$\begin{pmatrix} 4 & 6 & 7 & 9 \\ \frac{1}{18} & \frac{5}{18} & \frac{2}{18} & \frac{10}{18} \end{pmatrix}$$

$\mathbb{P}(X+Y<2)=0$

$$\mathbb{P}(X+Y<12 \mid Y<0) = \frac{\mathbb{P}((X+Y<12)\cap(Y<0))}{\mathbb{P}(Y<0)}$$

$$= \frac{\mathbb{P}((X+Y<12)\cap(Y=-1))}{\mathbb{P}(Y=-1)}$$

$$= \frac{\frac{1}{6}}{\frac{1}{6}} = 1.$$

$\mathbb{P}\left(\sin\left(\frac{\pi}{2}XY\right) \leq \frac{1}{2}\right) = \frac{1}{18} + \frac{12}{18} = \frac{13}{18}$

$$\begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{18} & \frac{12}{18} & \frac{5}{18} \end{pmatrix}$$

$\mathbb{P}\left(\frac{1}{X}+\frac{1}{Y}<1 \mid Y<0\right)=?$

$$= \frac{\mathbb{P}\left(\left(\frac{1}{X}+\frac{1}{Y}<1\right)\cap(Y<0)\right)}{\mathbb{P}(Y<0)}$$

$$\begin{pmatrix} -\frac{4}{5} & \frac{6}{5}, & -\frac{7}{8} & \frac{9}{8} \\ \frac{1}{18} & \frac{5}{8} & \frac{2}{18} & \frac{10}{18} \end{pmatrix}$$

$$= \frac{\frac{1}{18} + \frac{2}{18}}{\frac{1}{6}} = 1.$$

$$\begin{pmatrix} -\frac{7}{8} & -\frac{4}{5} & \frac{9}{8} & \frac{6}{5} \\ \frac{2}{18} & \frac{1}{18} & \frac{10}{18} & \frac{5}{8} \end{pmatrix}$$

$$\mathbb{P}\left(\frac{1}{x} + \frac{1}{y} < X + Y\right) = \frac{1}{18} + \frac{5}{18} + \frac{2}{18} + \frac{10}{18} = 1.$$

$$\begin{pmatrix} 4 & 6 & 7 & 9 \\ \frac{1}{18} & \frac{5}{18} & \frac{2}{18} & \frac{10}{18} \end{pmatrix}$$

2,5

d) $\mathbb{P}(X \cdot Y \le e^4) = \frac{1}{32} + \frac{3}{32} + \frac{7}{32} = \frac{11}{32}$

$$\mathbb{P}(X \cdot Y \geq 7 \mid X < 0) = \frac{\mathbb{P}((X \cdot Y \geq 7) \cap (X < 0))}{\mathbb{P}(X < 0)}$$

$Y \le -\frac{7}{3}$

$Y \le -3{,}5$

$$= \frac{0}{\frac{1}{8}} = 0$$

$$\mathbb{P}(X \cdot Y < 9 \mid Y > 3) = \frac{\mathbb{P}((X \cdot Y < 9) \cap (Y > 3))}{\mathbb{P}(Y > 3)}$$

$$= \frac{\mathbb{P}((X \cdot Y < 9) \cap (Y = e^3))}{\mathbb{P}(Y = 3)}$$

$$= \frac{1}{8}.$$

$\mathbb{P}$

2. Fie $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} k\, x^3 e^{-\frac{x}{2}}, & x \geq 0 \\ 0 & \text{în rest} \end{cases}$$

b)

$$f(x) = \begin{cases} \frac{1}{96} \cdot x^3 \cdot e^{-\frac{x}{2}}, & x \geq 0 \\ 0 & \text{în rest} \end{cases}$$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_{-\infty}^{0} \underbrace{x \cdot 0}_{0}\, dx + \int_{0}^{\infty} x \cdot \frac{1}{96} \cdot x^3 \cdot e^{-\frac{x}{2}} dx$$

$$= \frac{1}{96} \int_0^{\infty} x^4 \cdot e^{-\frac{x}{2}} dx$$

Notăm $t = \frac{x}{2}$

$$\Rightarrow x = 2t, \quad dx = 2dt.$$

$$\Rightarrow \mathbb{E}(X) = \frac{1}{96} \int_0^{\infty} (2t)^4 \cdot e^{-t}\, 2dt.$$

$$\mathbb{E}(X) = \frac{1}{96} \cdot 2^5 \int_0^{\infty} t^4 \cdot e^{-t} dt$$

$$\mathbb{E}(X) = \frac{2^5}{96} \cdot \Gamma(5)$$

$$\mathbb{E}(X) = \frac{2^5}{96} \cdot 4!$$

$$\mathbb{E}(X) = 8.$$

$$Var(X) = \mathbb{E}(X^2) - E(X)^2$$

$$= \int_0^{\infty} \frac{1}{96} x^5 \cdot e^{-\frac{x}{2}} dx - 8^2$$

Notăm $t = \frac{x}{2}$

$$x = 2t$$

$$dx = 2dt.$$

$$= \frac{1}{96}\int_0^\infty (2t)^5 \cdot e^{-t}\, 2\,dt - 8^2$$

$$= \frac{1}{96}\cdot 2^6 \int_0^\infty t^5 \cdot e^{-t}\,dt - 8^2$$

$$= \frac{1}{96}\cdot 2^6 \cdot \Gamma'(6) - 8^2$$

$120 \mid 3$ 40

$$= \frac{2^6}{96}\cdot 5! - 8^2$$

$96 \mid 2$ $48 \mid 2$ 24; 16

$$= \frac{2}{3}\cdot 5! - 8^2$$

$24 \mid 2$ 12

$$= 2\cdot 40 - 8^2$$

$12 \mid 2$ 6

$$= 16$$

$6 \mid 2$ 3

Tema v.a continue. pdf

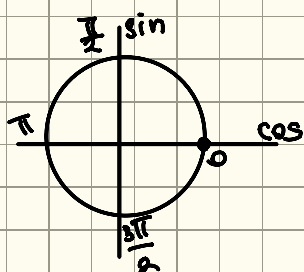
1. Fie X o v.a

$$f(x) = \begin{cases} A\cdot \sin x, & x\in[0,\pi] \\ 0, & \text{în rest.} \end{cases}$$

a) $A = ?$

f densitate de prob $\iff$ $f(x) \geq 0 \ \forall x \in \mathbb{R}$

$\int_{-\infty}^{\infty} f(x)\,dx = 1.$



1) $f(x) \geq 0 \ \forall x\in\mathbb{R} \iff A\cdot \sin x \geq 0 \ \forall x\in[0,\pi]$

$x\in[0,\pi]$

$\Rightarrow \sin x \geq 0$

Cond: $A \geq 0$.

2. $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$\iff \int_{-\infty}^{0} f(x)\,dx + \int_0^\pi f(x)\,dx + \int_\pi^\infty f(x)\,dx = 1$

$\iff \int_{-\infty}^{0} 0\,dx + \int_0^\pi A\cdot \sin x\,dx + \int_\pi^\infty 0\,dx = 1.$

$\Leftrightarrow \int_0^{\pi} A\cdot \sin x\, dx = 1$

$\Leftrightarrow -A\cdot \cos x\Big|_0^{\pi} = 1$

$-A\cdot(-1-1) = 1$

$A\cdot 2 = 1$

$A = \frac{1}{2} > 0.$

$\Rightarrow f(x) = \begin{cases} \frac{1}{2}\sin x, & x\in[0,\pi] \\ 0, & \text{în rest.} \end{cases}$

b) $\mathbb{P}\left(X<\frac{\pi}{3}\right) = \int_{-\infty}^{\frac{\pi}{3}} f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_0^{\frac{\pi}{3}} \frac{1}{2}\sin x\,dx$

$= 0 + \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin x\,dx$

$\cos 60 \; \frac{1}{2}$

$= -\frac{1}{2}\cdot \cos\Big|_0^{\frac{\pi}{3}}$

$= -\frac{1}{2}\left(\frac{1}{2}-1\right)$

$= \frac{1}{2}$

$\mathbb{P}\left(X<\frac{\pi}{4}\,\middle|\, X>\frac{\pi}{6}\right) = \dfrac{\mathbb{P}\left(X<\frac{\pi}{4} \cap X>\frac{\pi}{6}\right)}{\mathbb{P}\left(X<\frac{\pi}{6}\right)}$

$= \dfrac{\mathbb{P}\left(\frac{\pi}{6}<X<\frac{\pi}{4}\right)}{\mathbb{P}\left(X<\frac{\pi}{6}\right)}$

$= \dfrac{\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} f(x)\,dx}{\int_{-\infty}^{\frac{\pi}{6}} f(x)\,dx}$

$= \dfrac{\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} f(x)\,dx}{\int_{-\infty}^{0} f(x)\,dx + \int_0^{\frac{\pi}{6}} f(x)\,dx}$

$= \dfrac{\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin x\,dx}{\int_{-\infty}^{0} 0\,dx + \frac{1}{2}\int_0^{\frac{\pi}{6}} \sin x\,dx}$

$$= \frac{\cancel{\frac{1}{2}} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin x \, dx}{\cancel{\frac{1}{2}} \int_{0}^{\frac{\pi}{6}} \sin x \, dx}$$

$$= \frac{-\cos x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}}{-\cos x \Big|_{0}^{\frac{\pi}{6}}}$$

$$= \frac{-\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\right)}{-\left(\frac{\sqrt{3}}{2} - 1\right)}$$

$$= \frac{\sqrt{2} - \sqrt{3}}{\sqrt{3} - 2}$$

c) $\mathbb{E}(X) = ?$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx$$

$$\mathbb{E}(X) = \int_{-\infty}^{0} x \cdot 0 \, dx + \int_{0}^{\pi} \frac{1}{2} x \sin x \, dx + \int_{\pi}^{\infty} x \cdot 0 \, dx$$

$$\mathbb{E}(X) = \frac{1}{2} \int_{0}^{\pi} x \cdot \sin x \, dx$$

$$u(x) = x \Rightarrow u'(x) = 1.$$

$$v'(x) = \sin x \Rightarrow v(x) = -\cos x$$

$$\mathbb{E}(X) = \frac{1}{2}\left[-x \cdot \cos x \Big|_{0}^{\pi} - \int_{0}^{\pi} -\cos x \, dx\right]$$

$$= \frac{1}{2}\left[-(\pi \cdot \cos \pi - 0) + \sin x \Big|_{0}^{\pi}\right]$$

$$= \frac{1}{2}\left(\pi + (\sin \pi - \sin 0)\right)$$

$$= \frac{\pi}{2}.$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx.$$

$$= \int_{-\infty}^{0} x^2 \cdot 0 \, dx + \int_{0}^{\pi} \frac{1}{2} \cdot x^2 \cdot \sin x \, dx + \int_{\pi}^{\infty} x^2 \cdot 0 \, dx.$$

$$= \frac{1}{2} \int_{0}^{\pi} x^2 \cdot \sin x \, dx$$

$u(x) = x^2 \Rightarrow u'(x) = 2x$

$v'(x) = \sin x \Rightarrow v(x) = -\cos x$.

$E(x) = -x^2 \cos x \Big|_0^{\pi} - \int_0^{\pi} 2x(-\cos x)dx$

$= -(\pi^2 \cos\pi - 0) + 2\underbrace{\int_0^{\pi} x\cos x\, dx}_{I}.$

$I = \int_0^{\pi} x\cos x\, dx$

$u(x) = x \Rightarrow u'(x) = 1$
$v'(x) = \cos x \Rightarrow v(x) = \sin x$

$\Rightarrow I = x\sin x \Big|_0^{\pi} - \int_0^{\pi} \sin x dx$

$I = 0 - \pi\sin\pi - \left(-\cos x \Big|_0^{\pi}\right)$

$= 0 - 0 + (\cos\pi - \cos 0)$

$= -1 - 1 = -2.$

$\Rightarrow \mathbb{E}(X^2) = (\pi^2 - 2)\cdot\frac{1}{2}$

$\Rightarrow Var(X) = \frac{\pi^2 - 4}{2} - \left(\frac{\pi}{2}\right)^2 = \frac{2\pi^2 - 8}{4}$

$\sigma(X) = \sqrt{Var(X)} = \frac{\sqrt{2\pi^2 - 8}}{2}$

d) Funcția de repartiție a v.a X.

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \begin{cases} \frac{1}{2}\sin x, & x \in [0, \pi] \\ 0, & \text{în rest} \end{cases}$.

$F(x) = \int_{-\infty}^{x} f(t)\, dt.$

I $x < 0 \Rightarrow F(x) = \int_{-\infty}^{x} 0\, dt = 0$.

II $0 \leq x < \pi \Rightarrow F(x) = \int_{-\infty}^{0} 0 \, dt + \int_{0}^{x} \frac{1}{2} \sin t \, dt = \frac{1}{2} \int_{0}^{x} \sin t \, dt$

$$= -\frac{1}{2}(\cos x - \cos 0)$$

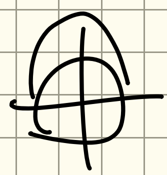
$$= -\frac{1}{2}(\cos x - 1)$$

III $x \geq 2$ $\quad F(x) = \int_{-\infty}^{0} 0 \, dt + \int_{0}^{\pi} \frac{1}{2} \sin t \, dt + \int_{\pi}^{x} 0 \, dt = 1.$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ -\frac{1}{2}(\cos x - 1) & 0 \leq x < \pi \\ 1, & x \geq \pi. \end{cases}$$

e) Mediana și modulul va X

Modul:



$f'(x) = \frac{1}{2} \cos x \quad \forall x \in [0, \pi]$

$f'(x) = 0 \Leftrightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$

Modul va X este $\frac{\pi}{2}$.

Mediana:

$F(Me) = \frac{1}{2}$

$-\frac{1}{2}(\cos x - 1) = \frac{1}{2} \quad | \cdot (-2)$

$\cos x - 1 = -1$

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}.$

$$f(x) = \begin{cases} A\cos x, & x \in [0, \frac{\pi}{2}] \\ 0, & \text{în rest.} \end{cases}$$

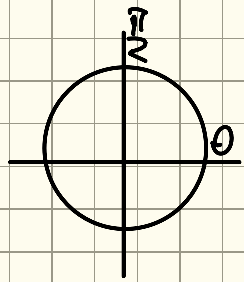
a) $A = ?$

f densitatea de prob $\Leftrightarrow$ 1. $f(x) \geq 0 \ \forall x \in \mathbb{R}$

2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

1. pt $x \notin [0, \frac{\pi}{2}]$ $f(x) = 0 \geq 0$

pt $x \in [0, \frac{\pi}{2}]$ $f(x) = A\cos x$



pt $x \in [0, \frac{\pi}{2}]$

$\cos x \geq 0 \Rightarrow$

$A \geq 0$

2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\frac{\pi}{2}} f(x)\,dx + \int_{\frac{\pi}{2}}^{\infty} f(x)\,dx = 1$

$\int_{-\infty}^{0} 0\,dx + \int_{0}^{\frac{\pi}{2}} A\cos x\,dx + \int_{\frac{\pi}{2}}^{\infty} 0\,dx = 1$

$0 + A\int_{0}^{\frac{\pi}{2}} \cos x\,dx + 0 = 1$

$A \cdot \sin x \Big|_{0}^{\frac{\pi}{2}} = 1$

$A(\sin \frac{\pi}{2} - \sin 0) = 1$

$A(1 - 0) = 1$

$A = 1 > 0$

$$f(x) = \begin{cases} \cos x, & x \in [0, \frac{\pi}{2}] \\ 0, & \text{în rest.} \end{cases}$$

b) $\mathbb{P}(X < \frac{\pi}{3}) = \int_{-\infty}^{\frac{\pi}{3}} f(x)\,dx$

$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\frac{\pi}{3}} f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\frac{\pi}{3}} \cos x\,dx$

$= 0 + \sin x \Big|_{0}^{\frac{\pi}{3}}$

$= \sin \frac{\pi}{3} - \sin 0$

$= \frac{\sqrt{3}}{2} - 0 = \frac{\sqrt{3}}{2}$.

$$\mathbb{P}(X < \frac{\pi}{4} \mid X > \frac{\pi}{6}) = \frac{\mathbb{P}(\frac{\pi}{6} < X < \frac{\pi}{4})}{\mathbb{P}(X > \frac{\pi}{6})}$$

$$= \frac{\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} f(x)\,dx}{\int_{\frac{\pi}{6}}^{\infty} f(x)\,dx}$$

$$= \frac{\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x\,dx}{\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\,dx + \int_{\frac{\pi}{2}}^{\infty} 0\,dx}$$

$$= \frac{\sin x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}}{\sin x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} + 0}$$

$$= \frac{\sin\frac{\pi}{4} - \sin\frac{\pi}{6}}{\sin\frac{\pi}{2} - \sin\frac{\pi}{6}}$$

$$= \frac{\frac{\sqrt{2}}{2} - \frac{1}{2}}{1 - \frac{1}{2}}$$

$$= \frac{\sqrt{2} - 1}{2 - 1} = \sqrt{2} - 1$$

c) $\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

$$\mathbb{E}(X) = \int_{-\infty}^{0} x \cdot f(x)\,dx + \int_{0}^{\frac{\pi}{2}} x \cdot f(x)\,dx + \int_{\frac{\pi}{2}}^{\infty} x \cdot f(x)\,dx$$

$$= \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{\frac{\pi}{2}} x \cdot \cos x\,dx + \int_{\frac{\pi}{2}}^{\infty} x \cdot 0\,dx$$

$$= \int_{0}^{\frac{\pi}{2}} x \cdot \cos x\,dx.$$

$u(x) = x \Rightarrow u'(x) = 1$

$v'(x) = \cos x \Rightarrow v(x) = \sin x$

$$\mathbb{E}(X) = x \sin x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= \frac{\pi}{2} \sin\frac{\pi}{2} - 0 - \left(-\cos x \Big|_0^{\frac{\pi}{2}}\right)$$

$$= \frac{\pi}{2} + (0 - 1)$$

$$= \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$$

$$= \int_{-\infty}^{0} x^2 \cdot \underbrace{f(x)}_{0}\,dx + \int_{0}^{\frac{\pi}{2}} x^2 \cdot f(x)\,dx + \int_{\frac{\pi}{2}}^{\infty} x^2 \cdot \underbrace{f(x)}_{0}\,dx$$

$$= \int_0^{\frac{\pi}{2}} x^2 \cdot \cos x\,dx$$

$u(x) = x^2 \Rightarrow u'(x) = 2x$

$v'(x) = \cos x \Rightarrow v(x) = \sin x.$

$$\mathbb{E}(X) = x^2 \sin x \Big|_0^{\frac{\pi}{2}} - 2\underbrace{\int_0^{\frac{\pi}{2}} x \sin x\,dx}_{I}$$

$$I = \int_0^{\frac{\pi}{2}} x \sin x \, dx.$$

$u(x) = x \Rightarrow u'(x) = 1$

$v'(x) = \sin x \Rightarrow v(x) = -\cos x$

$$I = -x\cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$I = -\frac{\pi}{2}\cos\frac{\pi}{2} - 0 + \sin\frac{\pi}{2} - \sin 0$$

$$I = 1$$

$$E(X^2) = \frac{\pi^2}{4}\sin\frac{\pi}{2} - 0 - 2$$

$$E(X^2) = \frac{\pi^2}{4} - 2 = \frac{\pi^2 - 4}{4}.$$

$$Var(X) = \frac{\cancel{\pi^2} - 4}{4} - \frac{\cancel{\pi^2} - 4\pi + 4}{4}$$

$$= \frac{4\pi - 8}{4}$$

$$= \pi - 2.$$

$$\sigma = \sqrt{Var(X)} = \sqrt{\pi - 2}.$$

d) funcția rep X.

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

pt $x \in [-\infty, 0]$ $\quad F(x) = \int_{-\infty}^{x} 0 \, dt = 0.$

pt $x \in [0, \frac{\pi}{2}]$ $\quad F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^x \cos t \, dt = 0 + \sin x - \sin 0$

$$= \sin x.$$

pt $x \in [\frac{\pi}{2}, \infty]$ $\quad F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^{\frac{\pi}{2}} \cos t\,dt + \int_{\frac{\pi}{2}}^{x} 0\,dt = 1$

$$F(x) = \begin{cases} 0, & x \in [-\infty, 0) \\ \sin x, & x \in [0, \frac{\pi}{2}) \\ 1, & x \in [\frac{\pi}{2}, \infty) \end{cases}$$

e) $x \in [0, \frac{\pi}{2}]$; $f'(x) = -\sin x$;

$f'(x) = 0 \iff \sin x = 0 \Rightarrow x = 0 \in [0, \frac{\pi}{2}]$

$M_0 = 0$.

Mediana:

$F(M_e) = \frac{1}{2}$

$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6} \in [0, \frac{\pi}{2}]$

2. $f(x) = \begin{cases} k(e^{-\frac{1}{2}x} + e^{\frac{1}{2}x}); & x \in [0, 2] \\ 0, \text{ în rest} \end{cases}$; $k \in \mathbb{R}$

a) $k = ?$

f densitate $\iff$ 1. $f(x) \geq 0 \ \forall x \in \mathbb{R}$

2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\iff \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{0} + \int_0^2 f(x)\,dx + \underbrace{\int_2^{\infty} f(x)\,dx}_{0} = 1$

$\iff k\int_0^2 e^{-\frac{1}{2}x} + e^{\frac{1}{2}x}\,dx = 1$

$\iff k\left(\int_0^2 e^{-\frac{1}{2}x}\,dx + \int_0^2 e^{\frac{1}{2}x}\,dx\right) = 1$.

$k\left(-2 \cdot e^{-\frac{1}{2}x}\Big|_0^2 + 2\, e^{\frac{1}{2}x}\Big|_0^2\right) = 1$

$2k(-e^{-1} + e^0 + e^1 - e^0) = 1$

$2k\left(e - \frac{1}{e}\right) = 1$

$$k = \frac{1}{2\left(e - \frac{1}{e}\right)}$$

$$k = \frac{1}{\frac{2e^2 - 2}{e}} = \frac{e}{2e^2 - 2} \text{ ; } f(x) > 0 .$$

b) $F(x) = ?$

pt $x \in (-\infty, 0)$ $\quad F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0.$

pt $x \in [0, 2)$ $\quad F(x) = \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} \frac{e}{2e^2-2} \cdot \left(e^{-\frac{1}{2}t} + e^{\frac{1}{2}t}\right) dt$

$$= 0 + \frac{e}{2e^2-2} \cdot 2\left(-e^{-\frac{1}{2}t}\Big|_0^x + e^{\frac{1}{2}t}\Big|_0^x\right)$$

$$= \frac{e}{2e^2-2} \cdot 2\left(-e^{-\frac{1}{2}x} + \cancel{e^0} + e^{\frac{1}{2}x} - \cancel{e^0}\right)$$

$$= \frac{e}{e^2-1}\left(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}\right)$$

pt $x \in [2, \infty)$ $\quad F(x) = 1.$

$$F(x) = \begin{cases} 0 , & x < 0 \\ \frac{e}{e^2-1}\left(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}\right) , & 0 \le x < 2 \\ 1 & x \geq 2 \end{cases}$$

$$\mathbb{P}\left(X < \frac{1}{2} \,\middle|\, X > \frac{1}{4}\right) = \frac{\mathbb{P}\left(\frac{1}{4} < X < \frac{1}{2}\right)}{\mathbb{P}\left(X > \frac{1}{4}\right)}$$

$$= \frac{\int_{\frac{1}{4}}^{\frac{1}{2}} f(x)\,dx}{\int_{\frac{1}{4}}^{\infty} f(x)\,dx}$$

$$= \frac{\frac{e}{\cancel{2e^2-2}} \int_{\frac{1}{4}}^{\frac{1}{2}} e^{-\frac{1}{2}x} + e^{\frac{1}{2}x}\, dx}{\frac{e}{\cancel{2e^2-2}} \int_{\frac{1}{4}}^{2} e^{-\frac{1}{2}x} + e^{\frac{1}{2}x}\, dx + 0}$$

$$= \frac{\cancel{2}\left(-e^{-\frac{1}{2}x}\Big|_{\frac{1}{4}}^{\frac{1}{2}} + e^{\frac{1}{2}x}\Big|_{\frac{1}{4}}^{\frac{1}{2}}\right)}{\cancel{2}\left(-e^{-\frac{1}{2}x}\Big|_{\frac{1}{4}}^{2} + e^{\frac{1}{2}x}\Big|_{\frac{1}{4}}^{2}\right)}$$

$$= \dots\dots$$

c) $\mathbb{E}(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

$$\mathbb{E}(x) = \underbrace{\int_{-\infty}^{0} x \cdot f(x)\, dx}_{} + \int_{0}^{2} x \cdot f(x)\, dx + \underbrace{\int_{2}^{\infty} x \cdot f(x)\, dx}_{0}$$

$$\mathbb{E}(x) = \frac{e}{e^2-1} \cdot \int_0^2 x \cdot e^{-\frac{1}{2}x} + x \cdot e^{\frac{1}{2}x}\, dx.$$

$$\mathbb{E}(x) = \frac{e}{e^2-1} \cdot \left(\int_0^2 x \cdot e^{-\frac{1}{2}x}\, dx + \int_0^2 x \cdot e^{\frac{1}{2}x}\, dx\right)$$

$$\int_0^2 x \cdot e^{-\frac{1}{2}x}\, dx$$

$u_1(x) = x \Rightarrow u_1'(x) = 1$ $\qquad$ $u_2(x) = x \Rightarrow u_2'(x) = 1$

$v_1'(x) = e^{-\frac{1}{2}x} \Rightarrow v_1(x) = -2e^{-\frac{1}{2}x}$ $\qquad$ $v_2'(x) = e^{\frac{1}{2}x} \Rightarrow v_2 = 2e^{\frac{1}{2}x}$

$$\Rightarrow \mathbb{E}(x) = \frac{e}{e^2-1}\left(2xe^{-\frac{1}{2}x}\Big|_0^2 - \int_0^2 -2e^{-\frac{1}{2}x}\, dx + 2xe^{\frac{1}{2}x}\Big|_0^2 - \int_0^2 2e^{\frac{1}{2}x}\, dx\right)$$

$$\mathbb{E}(x) = \frac{e}{e^2-1}\left(2 \cdot 2 \cdot e^{-1} - 0 + 2 \cdot (-2)e^{-\frac{1}{2}x}\Big|_0^2 + 2 \cdot 2 \cdot e^1 - 0 - 2 \cdot 2\, e^{\frac{1}{2}x}\Big|_0^2\right)$$

$$\mathbb{E}(x) = \frac{4}{e+1}$$

$$\mathbb{E}(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$\mathbb{E}(x^2) = \int_{-\infty}^{0} x^2 \cdot \underbrace{f(x)}_{0}\, dx + \int_0^2 x^2 f(x)\, dx + \int_2^{\infty} x^2 \underbrace{f(x)}_{0}\, dx.$$

$$\mathbb{E}(x^2) = \frac{e}{e^2-1} \int_0^2 x^2\left(e^{-\frac{x}{2}} + e^{\frac{x}{2}}\right) dx$$

$$\mathbb{E}(X^2) = \frac{e}{e^2-1}\left(\int_0^2 x^2 e^{-\frac{x}{2}}dx + \int_0^2 x^2 e^{\frac{x}{2}}dx\right)$$

$u_1(x) = x^2 \Rightarrow u_1'(x) = 2x$ $\qquad$ $u_2(x) = x^2 \Rightarrow u_2'(x) = 2x$

$v_1'(x) = e^{-\frac{x}{2}} \Rightarrow v_1(x) = -2e^{-\frac{x}{2}}$ $\qquad$ $v_2'(x) = e^{\frac{x}{2}} \Rightarrow v_2(x) = 2e^{\frac{x}{2}}$

$$\mathbb{E}(X^2) = \frac{e}{e^2-1}\left(-2x^2e^{-\frac{x}{2}}\Big|_0^2 - \int_0^2 2x\cdot(-2e^{-\frac{x}{2}})dx + 2x^2e^{\frac{x}{2}}\Big|_0^2 - \int_0^2 2x\cdot 2e^{\frac{x}{2}}dx\right)$$

$$\mathbb{E}(X^2) = \frac{e}{e^2-1}\left(-8e^{-1} + 4\underbrace{\int_0^2 x\cdot e^{-\frac{x}{2}}dx}_{\text{mai sus}} + 8e^1 - 4\underbrace{\int_0^2 xe^{\frac{x}{2}}dx}_{\text{mai sus}}\right)$$

$$\mathbb{E}(X^2) = \frac{4(e^2-5)}{e^2-1}$$

$$Var(X) = \frac{4(e^2-5)}{e^2-1} - \left(\frac{4}{e+1}\right)^2.$$

3) $f:\mathbb{R}\to\mathbb{R}$

$$f(x) = \begin{cases} k\cdot x^{a-1}\cdot(1-x)^{b-1}, & x\in(0,1) \\ 0 & \text{în rest} \end{cases}$$

$a,b>0$
$k\in\mathbb{R}.$

a) f densitate de prob $\Leftrightarrow$ 1. $f(x)\geq 0\ \forall x\in\mathbb{R}$

2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Leftrightarrow \underbrace{\int_{-\infty}^0 f(x)dx}_{0} + \int_0^1 f(x)dx + \underbrace{\int_1^\infty f(x)dx}_{0}$$

$$\Leftrightarrow k\cdot\int_0^1 x^{a-1}\cdot(1-x)^{b-1}dx$$

$$\Leftrightarrow k\cdot B(a,b) = 1$$

$$k = \frac{1}{B(a,b)}$$

$$k = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)}$$

b) $\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

$$\mathbb{E}(X) = \int_{-\infty}^{0} x \cdot \underbrace{f(x)}_{0} dx + \int_{0}^{1} x \cdot f(x) dx + \int_{1}^{\infty} x \cdot \underbrace{f(x)}_{0} dx$$

$$\mathbb{E}(X) = \frac{1}{\beta(a,b)} \int_0^1 x \cdot x^{a-1} (1-x)^{b-1} dx$$

$$\mathbb{E}(X) = \frac{1}{\beta(a,b)} \underbrace{\int_0^1 x^a (1-x)^{b-1} dx}_{\beta(a+1,b)}$$

$$\mathbb{E}(X) = \frac{\beta(a+1,b)}{\beta(a,b)}$$

$$\mathbb{E}(X) = \frac{\frac{\Gamma(a+1)\,\Gamma(b)}{\Gamma(a+1+b)}}{\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}}$$

$$\mathbb{E}(X) = \frac{\Gamma(a+1)}{\Gamma(a+b+1)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)}$$

$$\mathbb{E}(x) = \frac{a}{a+b}$$

$$\mathbb{E}(X^2) = \frac{1}{\beta(a,b)} \int_0^1 x^2 \cdot x^{a-1} (1-x)^{b-1} dx$$

$$= \frac{1}{\beta(a,b)} \int_0^1 x^{a+1} (1-x)^{b-1} dx$$

$$= \frac{B(a+2,b)}{B(a,b)}$$

$$\mathbb{E}(x^2) = \frac{a(a+1)}{(a+b)(a+b+1}$$

$$\mathrm{Var}(x) = \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2$$

$$E(X^r) = \frac{1}{\beta(a,b)} \int_0^1 x^r \cdot x^{a-1} \cdot (1-x)^{b-1} dx$$

$$= \frac{1}{\beta(a,b)} \int_0^1 x^{a+r-1} \cdot (1-x)^{b-1} dx$$

$$= \frac{\beta(a+r,b)}{\beta(a,b)}$$

$$= \frac{\frac{\Gamma(a+r)\Gamma(b)}{\Gamma(a+r+b)}}{\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}}$$

$$= \frac{\Gamma(a+r)\Gamma(a+b)}{\Gamma(a)+\Gamma(a+b+r)}$$

c) Funcție repartiție a v.a. X

$$f(x) = \begin{cases} \frac{1}{\beta(2,3)} \cdot x(1-x)^2, & x \in (0,1) \\ 0 & \text{în rest.} \end{cases}$$

$$f(x) = \begin{cases} 12 \cdot x(1-x)^2, & x \in (0,1) \\ 0, & \text{în rest} \end{cases}$$

$$f(x) = \begin{cases} 12x - 24x^2 + 12x^3, & x \in (0,1) \\ 0, & \text{în rest} \end{cases}$$

pt $x \in (-\infty, 0)$ $\quad F(x) = \int_{-\infty}^{x} f(t)\,dt = 0$

pt $x \in [0,1)$ $\quad F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_0^x f(t)\,dt$

$$= 0 + 12\frac{x^2}{2} - 24\frac{x^3}{3} + 12\frac{x^4}{4}$$

$$= 6x^2 - 8x^3 + 3x^4$$

pt $x \in [1, \infty)$ $F(x) = 1$

$$F(x) = \begin{cases} 0 & x \le 0 \\ 6x^2 - 8x^3 + 3x^4 & x \in (0,1) \\ 1 & x \geq 1. \end{cases}$$

$$\mathbb{P}\left(x < \frac{1}{2}\right) = F\left(\frac{1}{2}\right) = 6 \cdot \left(\frac{1}{2}\right)^2 - 8 \cdot \left(\frac{1}{2}\right)^3 + 3 \cdot \left(\frac{1}{2}\right)^4$$

$$= 6 \cdot \frac{1}{4} - 8 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} = \frac{11}{16}$$

$$\mathbb{P}\left(X > \frac{1}{3}\right) = 1 - \mathbb{P}\left(x \le \frac{1}{3}\right)$$

$$= 1 - F\left(\frac{1}{3}\right)$$

$$= 1 - \frac{6}{9} + \frac{8}{27} - \frac{3}{81} = \frac{16}{27}.$$

$$\mathbb{P}\left(X \le \frac{1}{2} \mid X > \frac{1}{4}\right) = \frac{\mathbb{P}\left(\frac{1}{4} < X \le \frac{1}{2}\right)}{\mathbb{P}\left(X > \frac{1}{4}\right)} = \frac{F\left(\frac{1}{2}\right) - F\left(\frac{1}{4}\right)}{1 - F\left(\frac{1}{4}\right)} = \frac{109}{189}$$

$$F\left(\frac{1}{4}\right) = \frac{6}{16} - \frac{8}{64} + \frac{3}{256} = \frac{67}{256}$$

4. $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} k \cdot x \cdot e^{-\frac{x^2}{2a^2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a) f densitate prob $\Leftrightarrow$ 1. $f(x) \geq 0 \quad \forall (x)$

2. $\int_{-\infty}^{\infty} f(x) = 1.$

$$\Leftrightarrow \int_{-\infty}^{0} \underbrace{f(x)}_{0} dx + \int_{0}^{+\infty} f(x)\, dx = 1$$

$$\Leftrightarrow k \int_0^{\infty} x \cdot e^{-\frac{x^2}{2a^2}} dx = 1$$

$$t=\frac{x^2}{2a^2} \qquad x=0 \Rightarrow t=0$$

$$dt=\frac{2x}{2a^2}dx=\frac{x}{a^2}dx \qquad x=\infty \Rightarrow t=\infty.$$

$$a^2\,dt=x\,dx$$

$$\Leftrightarrow k\cdot a^2\int_0^\infty e^{-t}\,dt=1$$

$$ka^2(-e^{-t})\Big|_0^\infty=1$$

$$ka^2(0-(-1))=1$$

$$ka^2=1$$

$$k=\frac{1}{a^2}$$

b) Funcția de repartiție.

pt $x\in(-\infty,0)$: $F(x)=\int_{-\infty}^{x}f(t)\,dt=0$

pt $x\in[0,+\infty)$: $F(x)=\int_{-\infty}^{0}f(t)\,dt+\int_0^x f(t)\,dt$

$$=\frac{1}{a^2}\int_0^x t\,e^{-\frac{t^2}{2a^2}}dt$$

$$u=\frac{t^2}{2a^2}$$

$$\vdots$$

$$F(x)=\int_0^{\frac{x^2}{2a^2}}e^{-u}\,du$$

$$F(x)=-e^{-u}\Big|_0^{\frac{x^2}{2a^2}}$$

$$F(x)=-e^{-\frac{x^2}{2a^2}}-(-e^0)$$

$$F(x)=1-e^{-\frac{x^2}{2a^2}}$$

$$F(x)=\begin{cases}0 & x<0\\ 1-e^{-\frac{x^2}{2a^2}} & x\geq 0\end{cases}$$

c) $\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = 0 + \int_0^{\infty} x \cdot \frac{x}{a^2} e^{-\frac{x^2}{2a^2}}\,dx$

$t = \frac{x^2}{2a^2} \qquad dt = \frac{x}{a^2}dx$

$x^2 = 2a^2 t$

$x = a\sqrt{2} \cdot t^{\frac{1}{2}}$

$$\mathbb{E}(X) = \int_0^{\infty} (a\sqrt{2} \cdot t^{\frac{1}{2}}) \cdot e^{-t}\,dt$$

$$= a\sqrt{2} \int_0^{\infty} t^{\frac{1}{2}} e^{-t}\,dt$$

$$= a\sqrt{2}\, \Gamma\left(\frac{3}{2}\right)$$

$$= a\sqrt{2}\, \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right)$$

$$= a\sqrt{2}\, \frac{1}{2}\sqrt{\pi}$$

$$= a\sqrt{\frac{\pi}{2}}$$

$$\mathbb{E}(X^2) = \int_0^{\infty} x^2 \cdot f(x)\,dx$$

$$= \int_0^{\infty} x^2 \cdot \frac{x}{a^2} e^{-\frac{x^2}{2a^2}}\,dx$$

$t = \frac{x^2}{2a^2} \qquad dt = \frac{\cancel{2}x}{\cancel{2}a^2}dx$

$x^2 = 2a^2 \cdot t$

$$\mathbb{E}(X^2) = 2a^2 \int_0^{\infty} t^1 \cdot e^{-t}\,dt$$

$$= 2a^2 \cdot \Gamma(2)$$

$$= 2a^2 \cdot 1$$

$$= 2a^2$$

$\Rightarrow \mathrm{Var}(X) = 2a^2 - a^2\frac{\pi}{2}$

Mom inițial ordin r.

$$\mathbb{E}(X^r) = \int_0^{\infty} x^r \cdot f(x)\,dx$$

$$= \int_0^\infty x^n \left(\frac{x}{a^2} e^{-\frac{x^2}{2a^2}}\right) dx$$

$$= \frac{1}{a^2}\int_0^\infty x^{n+1} \cdot e^{-\frac{x^2}{2a^2}} dx$$

$$t = \frac{x^2}{2a^2}$$

$$x^2 = 2a^2 t$$

$$x = a\sqrt{2} \cdot t^{\frac{1}{2}}$$

$$dx = a\sqrt{2}\,\frac{1}{2}\, t^{-\frac{1}{2}}$$

$$\mathbb{E}(x^n) = \frac{1}{a^2}\int_0^\infty (a\sqrt{2})^{n+1} \cdot t^{\frac{n+1}{2}} \cdot e^{-t}\, \frac{a}{\sqrt{2}}\, t^{-\frac{1}{2}} dt$$

$$\mathbb{E}(x^n) = (a\sqrt{2})^n \int_0^\infty t^{\frac{n}{2}} \cdot e^{-t} dt$$

$$= (a\sqrt{2})^n \cdot \Gamma\left(\frac{n}{2}+1\right)$$

d) $\mathbb{P}(x < 2a) = F(2a) = 1 - e^{-\frac{(2a)^2}{2a^2}} = 1 - e^{-\frac{\cancel{4}^{2}\cancel{a^2}}{\cancel{2a^2}}}$
$= 1 - \frac{1}{e^2}$

$$\mathbb{P}(x > a) = 1 - \mathbb{P}(x \leq a) =$$

$$= 1 - F(a)$$

$$= \cancel{1} - \cancel{1} + e^{-\frac{\cancel{a^2}}{2\cancel{a^2}}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

$$\mathbb{P}(x \leq 4a \mid x > 2a) = \frac{\mathbb{P}(2a < x \leq 4a)}{\mathbb{P}(x > 2a)}$$

$$= \frac{F(4a) - F(2a)}{1 - F(2a)}$$

$$= \frac{(1-e^{-8}) - (1-e^{-2})}{e^{-2}}$$

$$= \frac{e^{-2} - e^{-8}}{e^{-2}}$$

e) Fie M mediana

$$F(M) = \frac{1}{2}$$

$$1 - e^{\frac{-M^2}{2a^2}} = \frac{1}{2}$$

$$e^{-\frac{M^2}{2a}} = \frac{1}{2}$$

$$-\frac{M^2}{2a^2} = -\ln 2$$

$$M^2 = 2a^2 \ln 2$$

$$M = a\sqrt{2\ln 2}$$

8) ?

5.

Probleme cu repartiții v.a. pdf.

1.
$$f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a) $$\mathbb{P}(x > 3) = \int_3^{\infty} f(x)\,dx = \int_3^{\infty} \frac{1}{3} e^{-\frac{x}{3}} dx$$

$$= \frac{1}{\cancel{3}} \cdot -\cancel{3} \cdot e^{-\frac{x}{3}} \Big|_3^{\infty}$$

$$= -(-e^{-\infty}) - (-e^{-\frac{3}{3}})$$

$$= 0 + e^{-1} = \frac{1}{e}.$$

b) $$\mathbb{P}(x > 12 \mid x > 9) = \frac{\mathbb{P}(x > 12 \cap x > 9)}{\mathbb{P}(x > 9)}$$

$$= \frac{\mathbb{P}(x > 12)}{\mathbb{P}(x > 9)} \quad ; \quad \mathbb{P}(X > x) = e^{-\lambda x}$$

$$= \frac{e^{-\frac{12}{3}}}{e^{-\frac{9}{3}}} = \frac{e^{-4}}{e^{-3}} = e^{-1} = \frac{1}{e}$$

2 Fie evenimentele:

E: "vaccinul este eficient"

$\bar{E}$: "vaccinul nu este eficient"

$\mathbb{P}(E) = 0.75$

$\mathbb{P}(\bar{E}) = 0.25.$

X: "nr de viroze ale unei pers într-un an"

$$\mathbb{P}(X = k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

??

Test la prob și statistică

1\. Fie A = "evenimentul ca acțiunea firmei X crește "

A) B = "evenimentul în care acțiunea firmei Y crește"

a) $\mathbb{P}(A) = 0.65$

$\mathbb{P}(B) = 0.55$

$\mathbb{P}(A \cap B) = 0.25$.

$$\mathbb{P}(A \cup B) \underset{\substack{A,B \\ \text{indep}}}{=} \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.65 + 0.55 - 0.25 = 0.95$$

b) $\mathbb{P}((A \cap \overline{B}) \cup (\overline{A} \cap B)) = \mathbb{P}(A \cup B) - \mathbb{P}(A \cap B)$

$= 0.95 - 0.25 = 0.70$

c) R

B) Repartiția

$$X = \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ 0.1 & 0.09 & 0.081 & & 0.9^{k-1} \cdot 0.1 & \end{pmatrix}$$

Fie A = "evenimentul în care ghicește)

$\mathbb{P}(A) = 0.1$

$\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A) = 0.9$.

$\mathbb{P}(X = 1) = \mathbb{P}(A) = 0.1$

$\mathbb{P}(X = 2) = \mathbb{P}(\overline{A}) \cdot \mathbb{P}(A) = 0.09$

$\vdots$

$\mathbb{P}(X = k) = \mathbb{P}(\overline{A})^{k-1} \cdot \mathbb{P}(A) = 0.9^{k-1} \cdot 0.1$

b) $\mathbb{P}(X=3)=0.081$

$$\mathbb{P}\left(X\geq\frac{5}{3}\right)=1-\mathbb{P}\left(X<\frac{5}{3}\right)$$
$$=1-\mathbb{P}(X=1)$$
$$=0.9$$

$$\mathbb{P}\left(X<\frac{10}{3}\right)=\mathbb{P}(X=1)+\mathbb{P}(X=2)+\mathbb{P}(X=3)$$
$$=0.271.$$

$$\mathbb{P}(X\leq 2\mid X>0.3)=\frac{\mathbb{P}((X\leq 2)\cap(X>0.3))}{\mathbb{P}(X>0.3)}=\frac{\mathbb{P}(0.3<X\leq 2)}{1-\mathbb{P}(X\leq 0.3)}$$
$$=\frac{0.1+0.09}{1}$$
$$=0.19.$$

c) $F\left(\frac{7}{3}\right)$; F – funcția de repartiție.

$$F(x)=\begin{cases}0 & x<1\\ 0.1 & 1\leq x<2\\ 0.19 & 2\leq x<3\\ \vdots & \\ & k\leq x<k+1\\ \vdots & \\ 1 & x>\ldots\end{cases}$$

$$F\left(\frac{7}{3}\right)=0.19$$
$$2\leq\frac{7}{3}<3$$

d) $\mathbb{E}(X)=\sum x_i\cdot p_i=1\cdot(0.1)+2\cdot(0.1\cdot 0.9)+3(0.1\cdot 0.9^2)$
$+\ldots k\cdot(0.1\cdot 0.9^{k-1})$

$$\mathrm{Var}(X)=\mathbb{E}(X^2)-\mathbb{E}(X)^2$$

2.

$x \backslash y$	-1	1	2	f_i
-1	0.3	0.2	0.2	0.7
5	0.1	0.1	0.1	0.3
g_j	0.4	0.3	0.3	1

$\mathbb{E}(5X)=4 \iff 5\mathbb{E}(X)=4$

$\mathbb{E}(X)=0.8$

$0.7\cdot(-1)+0.3\cdot a=0.8$

$-0.7+0.3\cdot a=0.8$

$0.3a=1.5$

$a=5$

$\mathbb{E}(10Y)=5$

$10\mathbb{E}(Y)=5$

$\mathbb{E}(Y)=0.5$

$0.4\cdot B+0.3+0.6=0.5$

$0.4B+0.9=0.5$

$0.4B=-0.4$

$B=-1.$

b)

c) v.a XY

$$XY=\begin{pmatrix} 1 & -1 & -2 & -5 & 5 & 10 \\ 0.3 & 0.2 & 0.2 & 0.1 & 0.1 & 0.1 \end{pmatrix}$$

$$XY = \begin{pmatrix} -5 & -2 & -1 & 1 & 5 & 10 \\ 0.1 & 0.2 & 0.2 & 0.3 & 0.1 & 0.1 \end{pmatrix}$$

d) $\text{cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

$= 0.7 - 0.8 \cdot 0.5 = 0.3$

$\text{cov}(3X+5, 2Y-3X)$

$= \text{cov}(3X, 2Y-3X)$

$= \text{cov}(3X, 2Y) + \text{cov}(3X, -3X)$

$= 3 \cdot 2 \, \text{cov}(X,Y) - 3 \cdot 3 \, \text{cov}(X,X)$

$= 6 \cdot \text{cov}(X,Y) - 9 \, \text{Var}(X)$

e) $\pi_{11} = 0.3 \neq 0.7 \cdot 0.4 \Rightarrow X, Y$ nu sunt independente.

B) $f : \mathbb{R} \to \mathbb{R}$

$f(x) = 1 + \alpha x, \ x \in [-2, 0]$

f densitate de prob $\Leftrightarrow f(x) \geq 0 \ \forall x \in \mathbb{R}$.

$$f(x) = \begin{cases} 0 & x \in (-\infty, -2) \\ 1 + 0.5x & x \in [-2, 0] \\ 0 & x \in (0, \infty) \end{cases} \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1 \Leftrightarrow$$

$$\Leftrightarrow \int_{-\infty}^{-2} 0\,dx + \int_{-2}^{0} 1 + \alpha x\,dx + \int_{0}^{\infty} 0\,dx = 1$$

$$\Leftrightarrow \int_{-2}^{0} 1 + \alpha x\,dx = 1$$

$$\Leftrightarrow x \Big|_{-2}^{0} + \alpha \frac{x^2}{2} \Big|_{-2}^{0} = 1$$

$$\Leftrightarrow 2 + \alpha(0 - 2) = 1$$

$$-2\alpha = -1$$

$$\alpha = \frac{1}{2} = 0.5$$

b) $\mathbb{P}(-1 \leq X \leq 1 \mid X < -0.5)$

$$= \frac{\mathbb{P}((-1 \leq X \leq 1) \cap (X < -0.5))}{\mathbb{P}(X < -0.5)}$$

$$= \frac{\mathbb{P}(-1 \leq X < -0.5)}{\mathbb{P}(X < -0.5)}$$

$$= \frac{\int_{-1}^{-0.5} f(x)\,dx}{\int_{-\infty}^{0.5} f(x)\,dx}$$

$$= \frac{\int_{-1}^{-0.5} 1 + 0.5x\,dx}{\int_{-\infty}^{-2} 0\,dx + \int_{-2}^{-0.5} 1 + 0.5x\,dx}$$

$$= \frac{x\Big|_{-1}^{-0.5} + \frac{1}{2}\cdot\frac{x^2}{2}\Big|_{-1}^{-0.5}}{x\Big|_{-2}^{-0.5} + \frac{1}{2}\frac{x^2}{2}\Big|_{-2}^{-0.5}}$$

$$= \frac{5}{9}$$

d) $Y = 3X - 2.$

fie g densitatea lui Y.

dacă $x = -2 \Rightarrow y = -8$

$x = 0 \Rightarrow y = -2$

$\Rightarrow Y \in [-8, -2)$