Stat 6021: Interpretation of Regression Coefficients with Log Transformation on Response Variable

1 Interpreting Regression Coefficients: Log Transformation on Response Variable

One of the reasons a log transformation is a popular transformation is that regression coefficients are still fairly easy to interpret. Consider a log transformation applied to the response variable. The least-squares regression equation becomes

$$\begin{aligned} \log \hat{y} &= \hat{\beta}_0 + \hat{\beta}_1 x \\ \Longrightarrow \hat{y} &= \exp(\hat{\beta}_0 + \hat{\beta}_1 x) \end{aligned} \tag{1}$$

When the predictor variable increases by one unit, (1) becomes

$$\begin{aligned} \hat{y}_{new} &= \exp(\hat{\beta}_0 + \hat{\beta}_1 (x+1)) \\ &= \exp(\hat{\beta}_0 + \hat{\beta}_1 x) \times \exp(\hat{\beta}_1) \end{aligned} \tag{2}$$

Consider the ratio of $\hat{y}_{new}$ and $\hat{y}$ using (1) and (2), i.e.,

$$\begin{aligned} \frac{\hat{y}_{new}}{\hat{y}} &= \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1 x) \times \exp(\hat{\beta}_1)}{\exp(\hat{\beta}_0 + \hat{\beta}_1 x)} \\ &= \exp(\hat{\beta}_1) \\ \Longrightarrow \hat{y}_{new} &= \exp(\hat{\beta}_1) \times \hat{y} \end{aligned} \tag{3}$$

From (3), we can see how to interpret the estimated slope when a log transformation is applied to the response variable:

1. The predicted response variable is multiplied by a factor of $\exp(\hat{\beta}_1)$ when the predictor increases by one unit.
2. We can also subtract 1 from $\exp(\hat{\beta}_1)$ to express the change as a percentage.
    - If $\hat{\beta}_1$ is positive, we have a percent increase. The predicted response variable increases by $(\exp(\hat{\beta}_1) - 1) \times 100$ percent for a one-unit increase in the predictor.
    - If $\hat{\beta}_1$ is negative, we have a percent decrease. The predicted response variable decreases by $(1 - \exp(\hat{\beta}_1)) \times 100$ percent for a one-unit increase in the predictor.

Let us use some numerical examples:

- Suppose $\hat{\beta}_1 = 0.1$, then $\exp(\hat{\beta}_1) = 1.105$.
    - When the predictor increases by one unit, the predicted response is multiplied by a factor of 1.105.

- Since $\hat{\beta}_1$ is positive, we have $(\exp(\hat{\beta}_1) - 1) \times 100 = 10.5$. So when the predictor increases by one unit, the predicted response increases by 10.5%.

- Suppose $\hat{\beta}_1 = -0.2$, then $\exp(\hat{\beta}_1) = 0.819$.
  - When the predictor increases by one unit, the predicted response is multiplied by a factor of 0.819.
  - Since $\hat{\beta}_1$ is negative, we have $(1 - \exp(\hat{\beta}_1)) \times 100 = 18.1$. So when the predictor increases by one unit, the predicted response decreases by 18.1%.