Radar and Remote Sensing Equation Sheet

Rachel Middleton
Rachel.Middleton@colorado.edu

Radar Fundamentals

1. Transmit Signal, T_x for a monostatic radar block diagram:

$$T_x(t) = A_0 U(t) \cos[2\pi f(t)t + \phi_{TX}(t)]$$

where:

A_0 = amplitude

$U(t)$ = pulse train signal

$f(t)$ = transmit frequency

$\phi_{TX}(t)$ = transmit phase

2. Received Signal, R_x for a monostatic radar block diagram:

$$R_x(t) = k(A_0 U(t - \nabla t) \cos[2\pi f(t - \nabla t) + \psi] + n(t))$$

where:

ψ = the sum of the phase shifts of the target and with the radar

k = is order of 10^{-18}

3. The Radar Power Equation

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma}{(4\pi)^3 R^4}$$

where:

Pt – Transmitted power (at antenna terminals) [W]

Gt – Transmitter antenna gain [unitless]

R – Range to target [m]

RCS – Target's radar cross section (also expressed as σ) [m^2]

Gr – Receiver antenna gain [unitless]

Pr – Received power (at antenna terminals) [W]

$\lambda = \dfrac{c}{f}$ = wavelength [m]

The following quantities can be derived from the RPE, see ASEN5245_01c_Radar_Fundamentals_2024_0118 for more equations

(a) Power density for an Isotropic Antenna

$$Q_i = \frac{P_t}{4\pi R^2} \text{ [Watts}/m^2]$$

where:

Q_i is the incident power density for and isotropic antenna($G_t = 1$)

$4\pi R^2$ is the area of a sphere

(b) Radar Cross Section (RCS)

$$\sigma(\theta, \phi) = \frac{P_{reflected}[W]}{Q_i[W/m^2]}[m^2]$$

(c) Radiation Scattered by a Target

(d) Power Backscattered by a Target

(e) Power Density at Receive Antenna

(f) Power Backscattered by Target

(g) Power Collected by Receive Antenna

(h) Received Power at Antenna Ports

4. Calculations in dB

$$P_{dB} = 10\log(\frac{P_{linear}}{P_0})$$

where:

P_{linear} is a value in "linear" units and P_0 represents the reference value (typically 1)

When the reference has physical meaning, then 'dB' is replaced with a special units:

Reference to milliWatts ($P_0 = 10^{-3}W$) dBm

Reference to Watts dBW

Reference to square meters (RCS, λ,) dBsm

Reference to isotropic radiator dBi

Reference to dipole radiator dBd

5. Pulse Radar Range

$$R_{target} = \frac{cT_{delay}}{2} \text{ [m]}$$

where c is the speed of light, $3x10^8$

6. Pulse Radar Waveform

τ [s] - the pulse width or duration the Tx is transmitting

T_{IPP} or IPP - the inter-pulse period [s], time between pulses

$PRF = 1/T_{IPP}$ [Hz] - the pulse repetition frequency

Duty Cycle $= 100(\frac{\tau}{T_{IPP}})$ [%]

7. Range Resolution

$$\nabla R = R_2 - R_1 = c\frac{T_2 - T_1}{2} = c\frac{\delta t}{2}$$

where δt is the minimum time such that two targets at R1 and at R 2 will appear completely resolved

therefore:

$$\nabla R = \frac{c\tau}{2}$$

8. Unambiguous Range (Pulse Radar)

$$R_{max} = \frac{cT_{IPP}}{2}$$

Range aliasing implies that a target observed at range, $R_{obs} < R_{max}$ could have come from a different inter-pulse period

Noise and Losses

1. Noise Power of a Radar System

$$P_n = kT_{sys}B \text{ [W]}$$

   where:

   P_n is the noise power in Watts

   k - Boltzmann's constant: $1.38x10^{-23}$ [J/K], note: [J] = [Watt second]

   T_{sys} - system noise temperature [K]

   B - receiver bandwidth [Hz] (typically B $1/\tau$, where τ is the pulse width [s]

2. Random Antenna Noise

$$T_{sys} = T_{antenna} + T_e = (T_a + T_A) + T_e$$

   T_a - External noise. Noise picked up by the antenna (e.g., Galactic noise, sun, moon, ground and other thermal emitters. This noise is part of the antenna temperature

   T_A - Thermal Emission. Noise due to physical antenna temperature. Also part of the antenna temperature

   T_e - Receiver noise. Internally generated noise by Rx components (e.g., amplifiers, mixers). This noise is quantified by the Rx effective noise temperature

3. Radar Losses

$$L_{total} = L_{sys}L_{prop} \text{ where } L_{total} \geq 1$$

   Two kinds of losses may occur: Propagation Path Losses – losses associated with the propagation medium between target and radar, such as, atmospheric attenuation and multipath interference

   System Losses – losses within the radar system itself, such as insertion loss, as waves propagate through the radar system. (Losses within the antenna are accounted for in the antenna efficiency)

   effciency, ρ is expressed as $L = \dfrac{1}{\rho}$

   Losses are factored into the RPE as follows:

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma}{(4\pi)^3 L_{FSPL}^2 R^4}$$

   where L_{FSPL} is the one-way propagation loss (notice this quantity squared is the two-way propagation loss)

4. Specific Attenuation Model for Rain

   see reference doc: ITU-R P.838-1

$$\gamma_R = kR^\alpha \text{ [dB/km]}$$

   where $\gamma_R$is the specific attenuation for rain [dB/km]

   k and α are frequency dependent constants tabulated for horizontal (H) and vertical (V) propagation polarizations

   R is rain rate [mm/hr]

   Then, we factor these losses into the RPE as:

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma}{(4\pi)^3 L_{FSPL}^2 L_{rain}^2 R^4}$$

   where L_{rain} is the one-way propagation loss for rain (distance * attenuation rate) (notice this quantity squared is the two-way propagation loss)

   Similar attenuation rates can be calculated for other atmospheric mediums, such as cloud and fog (Recommendation ITU-R P.840-3)

5. Coax Cable, Insertion loss (Attenuation) (a system loss)

6. Signal to Noise Ratio (SNR)

$$SNR = \frac{P_r}{P_n} = \frac{P_t G^2 \lambda^2 \sigma}{(4\pi)^3 L_{sys} L_{prop}^2 R^4 k T_{sys} B}$$

7. Coherent integration

$$(SNR)_{coherent}(n_p) = n_p SNR(1)$$

where n_p is the number of consecutive pulses

8. Maximum Detectable Range

$$\text{Detection Threshold} = SNR_{min} = \frac{P_t \tau G^2 \lambda^2 \sigma}{(4\pi)^3 L_{sys} L_{prop}^2 R^4 k T_{sys}}$$

$$\text{Max Detectable range} = R_max = [\frac{P_t \tau G^2 \lambda^2 \sigma}{(4\pi)^3 L_{sys} L_{prop}^2 SNR_{min} k T_{sys}}]^{1/4}$$

$$\text{min RCS} = \sigma_{min} = \frac{(4\pi)^3 L_{sys} L_{prop}^2 SNR_{min} k T_{sys} R^4}{P_t \tau G^2 \lambda^2}$$

9. Fourier Transform (FT)

The Fourier transform can be viewed as a projection onto an orthogonal basis function.

Fourier Transform Pair:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad X(f) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi f t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

The basis functions are the complex exponentials.

Note that both $F(\omega)$ and $f(t)$ are continuous functions and can be either real or complex (as with I and Q voltages).

From Eulier's Identity, we can represent cosines and sines with complex exponentials.

Negative frequencies represent clockwise (CW) rotation on complex plane

Positive frequencies represent counter-clockwise (CCW) rotation on complex plane 11

Refer to ASEN5245_Intro_Matched_Filter_Compression_013024.pdf for examples and further explanation.

10. Pulse Compression Ratio [TODO]

11. Decoding Phase Modulation [TODO]

EM Basics

1. Electrostatics

   - Ohms Law: $V = RI$
     where:
     V = Voltage [V]
     R = Resistance [Ω]
     I = Current [Ampere]
   - Electric Field
     $\tilde{E} = \rho \tilde{J}$
     where:
     $\tilde{E}$ = electric field [V/m]
     $\tilde{J}$ = current density [A/m^2]
     ρ = resistivity of the medium [Ω/m]
     $\sigma = 1/\rho$ = conductivity

- Coulomb's Law: $\tilde{E} = \frac{q}{4\pi\epsilon_0 r^2}\vec{R}[V/m^2]$

  where:

  ϵ_0 = Permittivity of Free Space = $\frac{1}{36\pi} * 1-^{-9}$ [Farads/m]

  Dielectric Constant = $\epsilon = \epsilon_r \epsilon_0$

  $\tilde{E}$ = electric field [V/m]

  $\tilde{D}$ = Electric Flux Desnsity: $\tilde{D} = \epsilon\tilde{E}$

  r = Distance between the centers of the two charges [m]

2. Magnetostatics

- Biot-Savart Law: $\vec{B} = \mu\tilde{H}$

  where:

  $\tilde{B}$ = Magnetic flux density [Tesla] = $\tilde{B} = \frac{\mu_0 I}{2\pi r}$

  $\tilde{H}$ = Magnetic field intensity [A/m]

  μ_0 = Permeability of free space $\approx 4\pi \times 10^{-7}$ T m/A $\mu = \mu_r\mu_0$ and $\mu_r = 1$ for this class I = Electric current [A] = $\frac{dq}{dt}$

  $\vec{r}$ = Position vector from the length element to the point of observation [m]

3. Electromagnetics

Table 1-6: The three branches of electromagnetics.

Branch	Condition	Field Quantities (Units)
Electrostatics	Stationary charges ($\partial q/\partial t = 0$)	Electric field intensity **E** (V/m) Electric flux density **D** (C/m^2) $\mathbf{D} = \varepsilon\mathbf{E}$
Magnetostatics	Steady currents ($\partial I/\partial t = 0$)	Magnetic flux density **B** (T) Magnetic field intensity **H** (A/m) $\mathbf{B} = \mu\mathbf{H}$
Dynamics (Time-varying fields)	Time-varying currents ($\partial I/\partial t \neq 0$)	**E**, **D**, **B**, and **H** ($\mathbf{E}, \mathbf{D}$) coupled to ($\mathbf{B}, \mathbf{H}$)

Table 1-7: Constitutive parameters of materials.

Parameter	Units	Free-space Value
Electrical permittivity ε	F/m	$\varepsilon_0 = 8.854 \times 10^{-12}$ (F/m) $\simeq \frac{1}{36\pi} \times 10^{-9}$ (F/m)
Magnetic permeability μ	H/m	$\mu_0 = 4\pi \times 10^{-7}$ (H/m)
Conductivity σ	S/m	0

4. Maxwell's Equations

Table 6-1: Maxwell's equations.

Reference	Differential Form	Integral Form	
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	(6.1)
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$	(6.2)†
No magnetic charges (Gauss's law for magnetism)	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	(6.3)
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$	(6.4)

†For a stationary surface S.

5. Phasor notation

$$\bar{F}(x,y,z;t) = F(x,y,z)cos\left(\omega t + \left[k_x \cdot \hat{x} + k_y \cdot \hat{y} + k_z \cdot \hat{z}\right]\right)$$
$$\bar{F}(x,y,z;t) = F(x,y,z)cos\left(\omega t + \bar{k} \cdot \bar{x}\right)$$

Euler's Identity: $Ae^{j\xi} = Acos(\xi) + jAsin(\xi) \qquad j = \sqrt{-1}$

Real part: $Re\left(e^{j\xi}\right) = Acos(\xi)$

$$\bar{F}(x,y,z;t) = F(x,y,z)\cos(\omega t + \bar{k}\cdot\bar{x}) = Re\left[F(x,y,z)\, e^{j\omega t + j\bar{k}\cdot\bar{x}}\right]$$
$$= Re\left[F(x,y,z)\, e^{j\bar{k}\cdot\bar{x}} e^{j\omega t}\right]$$

$$\left.\begin{array}{l} \tilde{F}(x,y,z) = Re\left[F(x,y,z)\, e^{j\omega t}\right] \\ e^{j\phi} = e^{j\bar{k}\cdot\bar{x}} \end{array}\right\} \Rightarrow \begin{array}{l} \bar{F}(x,y,z;t) = \tilde{F}(x,y,z)e^{j} \\ \tilde{\bar{F}} = Ae^{j\phi} \end{array}$$

A = AMPLITUDE & ϕ = PHASE

24

6. Complex permittivity

(1)

$$\nabla \cdot \bar{\tilde{E}} = \frac{\rho_v}{\epsilon}$$

$$\nabla \times \bar{\tilde{E}} = -\mu \frac{\partial \bar{H}}{\partial t}$$

$$\nabla \cdot \bar{\tilde{H}} = 0$$

$$\nabla \times \bar{\tilde{H}} = \bar{\tilde{J}} + \epsilon \frac{\partial E}{\partial t}$$

KNOWING THAT

$$\bar{\tilde{F}} = Ae^{j\phi} e^{j\omega t}$$

(2)

$$\nabla \cdot \bar{\tilde{E}} = \frac{\rho_v}{\epsilon}$$

$$\nabla \times \bar{\tilde{E}} = -j\omega\mu \bar{\tilde{H}}$$

$$\nabla \cdot \bar{\tilde{H}} = 0$$

$$\nabla \times \bar{\tilde{H}} = \bar{\tilde{J}} + j\omega\epsilon \bar{\tilde{E}}$$

(3) SINCE $\bar{J} = \sigma \bar{E}$, THEN $\bar{\tilde{J}} = \sigma \bar{\tilde{E}}$

$$\nabla \times \bar{\tilde{H}} = \sigma \bar{\tilde{E}} + j\omega\epsilon \bar{\tilde{E}}$$

$$= (\sigma + j\omega\epsilon)\bar{\tilde{E}}$$

$$= j\omega\left(\frac{\sigma}{j\omega} + \epsilon\right)\bar{\tilde{E}}$$

$$\nabla \times \bar{\tilde{H}} = j\omega\left(\epsilon - j\frac{\sigma}{\omega}\right)\bar{\tilde{E}}$$

$$\nabla \times \bar{\tilde{H}} = j\omega\epsilon_c \bar{\tilde{E}}$$

$$\epsilon_c \triangleq \epsilon - j\frac{\sigma}{\omega}$$ PERMITTIVITY

$$= \epsilon' - j\epsilon''$$

WITH $\epsilon'' = \dfrac{\sigma}{\omega}$

FOR LOSSLESS MEDIUM, WITH A "PERFECT DIELECTRIC" $\sigma = 0$

$$\epsilon'' = \frac{0}{\omega} = 0 \; ; \quad \epsilon_c = \epsilon' = \epsilon$$

25

7. Propagating Waves

HOW DO WAVES PROPAGATE IN A CHARGE FREE ENVIRONMENT? $\tilde{\rho}_v = 0 \leftarrow$ NO CHARGES

(1)

$$\left.\begin{aligned} \nabla \cdot \bar{\tilde{E}} &= 0 \\ \nabla \times \bar{\tilde{E}} &= -j\omega\mu \bar{\tilde{H}} \\ \nabla \cdot \bar{\tilde{H}} &= 0 \\ \nabla \times \bar{\tilde{H}} &= j\omega\epsilon_c \bar{\tilde{E}} \end{aligned}\right\}$$ COMBINE EQUATIONS

(2)

$$\Rightarrow \bar{\tilde{H}} = \nabla \times \left[\frac{\bar{\tilde{E}}}{-j\omega\mu}\right]$$

$$\nabla \times \left\{\nabla \times \left[\frac{\bar{\tilde{E}}}{-j\omega\mu}\right]\right\} = j\omega\epsilon_c \bar{\tilde{E}}$$

$$\nabla \times (\nabla \times \bar{\tilde{E}}) = -j^2\omega^2\mu\epsilon_c \bar{\tilde{E}}$$

$$\nabla \times (\nabla \times \bar{\tilde{E}}) = \omega^2\mu\epsilon_c \bar{\tilde{E}}$$

(3) VECTOR IDENTITY:

$$\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 F$$

∇^2 IS LAPLACIAN OPERATOR

$$\nabla^2 \bar{F} = \frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$$

(SCALAR)

(4)

$$\nabla \times (\nabla \times \bar{\tilde{E}}) = \nabla(\nabla \cdot \bar{\tilde{E}}) - \nabla^2 \bar{\tilde{E}} = \omega^2\mu\epsilon_c \bar{\tilde{E}}$$

$$\nabla^2 \bar{\tilde{E}} + \omega^2\mu\epsilon_c \bar{\tilde{E}} = 0$$

HELMHOLTZ EQN (WAVE EQUATION)

26

Wave Propagation and Scattering

1. The Homogenous Wave Equation

   $$\nabla^2 \tilde{\boldsymbol{E}} - \gamma^2 \tilde{\boldsymbol{E}} = 0 \text{ and } \nabla^2 \tilde{\boldsymbol{H}} - \gamma^2 \tilde{\boldsymbol{H}} = 0$$

   where γ^2 is defined as the **propagation constant**: $\gamma^2 = -\omega^2 \mu \varepsilon_c$

   Define the **wavenumber** as $k = \omega\sqrt{\mu\varepsilon_c}$ so that $\gamma^2 = -k^2$.

   The wave equations become:

   $$\nabla^2 \tilde{\boldsymbol{E}} + k^2 \tilde{\boldsymbol{E}} = 0 \text{ and } \nabla^2 \tilde{\boldsymbol{H}} + k^2 \tilde{\boldsymbol{H}} = 0$$

   Recall from Maxwell's equations that E and H are orthogonal:

   $$\bar{H} = \frac{1}{\eta}\vec{k} \times \bar{E}$$
   $$\bar{E} = -\eta\vec{k} \times \bar{H}$$
   $$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

   and $\vec{k}$ is the unit vector in the direction of propagation.

   - One solution to the wave equation with $\tilde{E}_y = 0$ has the form:

     $\boldsymbol{E}(z,t) = \hat{x}A\cos(\omega t - kz + \phi) + \hat{y}(0)$ [V/m]

     $\bar{H} = \frac{1}{\eta}\hat{k} \times \bar{E}$

     $\boldsymbol{H}(z,t) = \hat{x}(0) + \hat{y}\frac{E(z,t)}{\eta} = \hat{y}(2.65)\cos(\omega t - kz + \phi)$ [mA/m]

2. Propagation through multiple elements

   Losses of medium are usually expressed one-way path loses in dB.

   $$L_i = \alpha_i d_i$$

   where α_i is the attenuation coefficient [dB/km] and d_i is the path length [km]

   Loss along the full path length is the Free Space Propagation Loss (FSPL) and is the sum of the Losses $(L_1 + L_2 + L_3 + ... = L_{total})$

3. Plane Wave Propagation in Lossless Media

From Maxwell's equations, $\bar{H} = \frac{1}{\eta}\hat{k}\times\bar{E}$, the **intrinsic impedance** of a lossless medium is

$$\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon_r\varepsilon_0}} = \sqrt{\frac{\mu}{\varepsilon_r\varepsilon_0}} \qquad \text{[Ohms]}$$

The **phase velocity** of the wave traveling through the medium is

$$u_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon_r\varepsilon_0}} = \frac{1}{\sqrt{\mu\varepsilon_r\varepsilon_0}} \qquad \text{[m/s]}$$

The **wavelength** is given as

$$\lambda = \frac{2\pi}{k} = \frac{u_p}{f} \qquad \text{[m]}$$

where f is the frequency in Hz.

In a **vacuum**, $\varepsilon_r = 1$ and $\mu = \mu_0$ (non-magnetic material):

$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377 \approx 120\pi \qquad \text{[Ohms]}$$

$$u_p = c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 299{,}792{,}458 \approx 3x10^8 \qquad \text{[m/s]}$$

In a **vacuum**, $\varepsilon_r = 1$ and $\mu = \mu_0$, intrinsic impedance & phase velocity:

$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377 \approx 120\pi \qquad \text{[Ohms]}$$

$$u_p = c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 299{,}792{,}458 \approx 3x10^8 \qquad \text{[m/s]}$$

In a **dielectric medium**, $\varepsilon_r > 1$ and $\mu = \mu_0$ (non-magnetic):

$$\eta = \sqrt{\frac{\mu}{\varepsilon_r\varepsilon_0}} = \eta_0\sqrt{\frac{1}{\varepsilon_r}} = 377\sqrt{\frac{1}{\varepsilon_r}} \qquad \text{[Ohms]}$$

$$u_p = \frac{1}{\sqrt{\mu\varepsilon_r\varepsilon_0}} = c\frac{1}{\sqrt{\varepsilon_r}} \approx (3x10^8)\frac{1}{\sqrt{\varepsilon_r}} \qquad \text{[m/s]}$$

Example: $\varepsilon_r = 1.38$

$$u_p = c\frac{1}{\sqrt{\varepsilon_r}} \sim c\frac{1}{\sqrt{1.38}} \sim (0.85)c$$

Electrical Specifications

Performance Property	Units	US	(metric)
Cutoff Frequency	GHz		16.2
Velocity of Propagation	%		85
Dielectric Constant	NA		1.38

The wave propagates slower in the cable than in free-space.

4. Atmospheric Refraction

Refraction - radio waves travel in a different direction because of the **index of refraction**

$$\text{The index of refraction, } n = \frac{c}{v_p}$$

where c is the speed of light and v_p is the phase speed of the wave in the medium

In a vacuum, $v_{phase} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{(4\pi x10^{-7})(36\pi x10^{-9})}} \sim 3x10^8 = c$

In a medium with dielectric $\epsilon = \epsilon_r \epsilon_0$, the propagation speed is:

$$v_{phase} = \frac{1}{\sqrt{\mu_0 \epsilon_r \epsilon_0}} = \frac{1}{\sqrt{(4\pi x10^{-7})(36\pi x10^{-9})\epsilon_r}} = \frac{c}{\sqrt{\epsilon_r}}$$

Thus, the index of refraction is:

$$n = \frac{c}{v_p} = \left(\frac{1}{\sqrt{\mu_0 \epsilon_0}}\right)\left(\frac{\sqrt{\mu_0 \epsilon_r \epsilon_0}}{1}\right) = \sqrt{\epsilon_r}$$

The index of refraction is a complex quantity with real and imaginary parts ($n = \sqrt{\epsilon_r} = n' - jn''$) and is also defined as:

$$n = 1 + 10^{-6} N$$

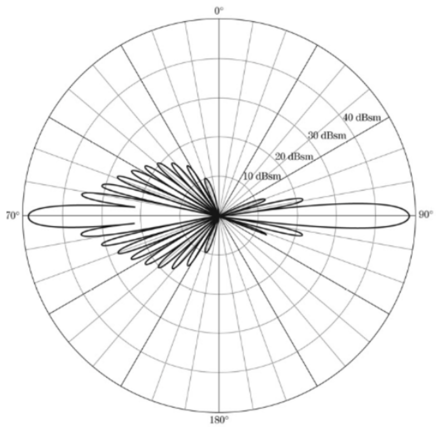
where N is the refractivity

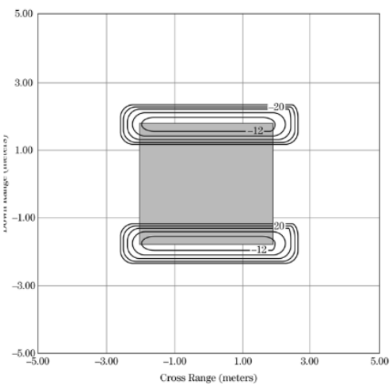
5. *Scattering

- Refraction - waves change direction as they pass from one medium to another
- Diffraction - waves change direction as they pass through an opening or navigate around a barrierrr
- Reflection - waves bounce off barrier
  – Surface, multi-path, and plate reflection
- Radar cross-section
  For example, a plate has the cross section:

  Radar Cross section of a plate: $\sigma_{plate} = \frac{4\pi a^2 b^2}{\lambda^2}$ [m^2]

  But the edges produce reflections, too.





Antenna Properties

1. Radiation Properties

- Reciprocity
- Antenna Pattern
- Gain
- Polarization

2. Impedance Properties

- Radiation resistance
- Loss resistance
- Voltage Standing Wave Ratio (VSWR)

3. Antenna Radiated Power

Power Density [W/m²]

Time averaged Poynting vector of an Electromagnetic (EM) wave is:

$$\vec{Q}_{ave}(r,\theta,\phi) = \frac{1}{2}Re[\vec{E}(r,\theta,\phi)\times\vec{H}(r,\theta,\phi)^*]$$

Total Power Radiated [W]

Total power radiated by the antenna at a distance R is:

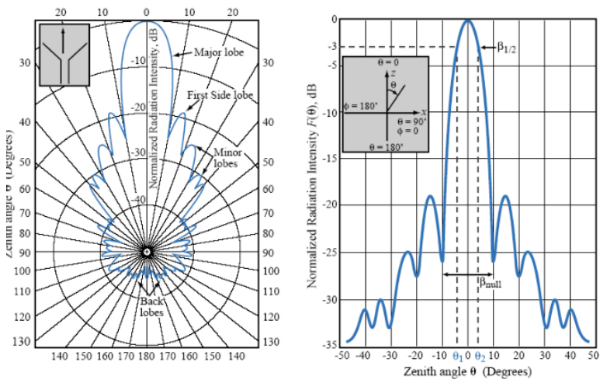
$$P_{rad} = R^2 \iint_{4\pi} Q_{ave}(r,\theta,\phi)d\Omega$$

After normalizing the radiation pattern, with R separable from θ and ϕ:

$$P_{rad} = R^2 Q_{max} \iint_{4\pi} F_n(\theta,\phi)d\Omega$$

4. Radiation Pattern

The normalized radiation pattern is as follows:



(a) Polar Diagram (b) Rectangular Plot

Normalized Radiation Pattern:

$$F_n(\theta,\phi) = \frac{Q(\theta,\phi)}{Q_{max}}$$

(Dimensionless)

Maximum value = 1.
Normally expressed as 10log(F).

5. Beam width

HPBW (half power beam width) is the angle between the two angles where the poer is half of the max value. In dB, the power drops about 3 dB from the max value. In Linear Voltage, the power drops by a factor of 0.707.

6. Directivity (Isotropic radiator and Pattern solid angle)

How well an antenna directs energy relative to an isotropic antenna

Radiation intensity, the power radiated per unit solid angle:

$$I_{isotropic} = R^2 Q_{isotropic}(R) = \frac{R^2 P_{radiated}}{4\pi R^2} = \frac{P_{radiated}}{4\pi} \quad \left[\frac{W}{steradian}\right]$$

And the power density at range R:

$$Q_{density}(R) = \frac{P_{radiated}}{\Omega_p R^2}[\frac{W}{m^2}]$$

where Ω_p is the pattern solid angle and can be approximated as:

$$\Omega_p = \theta_{HPBW}\phi_{HPBW}$$

Directivity, D:

$$D = \frac{I_{antenna}}{I_{isotropic}} = \frac{\frac{P_{radiated}}{\Omega_p}}{\frac{P_{radiated}}{4\pi}} = \frac{4\pi}{\Omega_p}$$

Note that we make the following assumptions:

- Ignore all sidelobes
- All energy is within HPBW
- Power density is uniform across beam solid angle

7. Gain

   Antenna efficiency, ρ:

   $\rho = \dfrac{P_{radiated}}{P_t}$

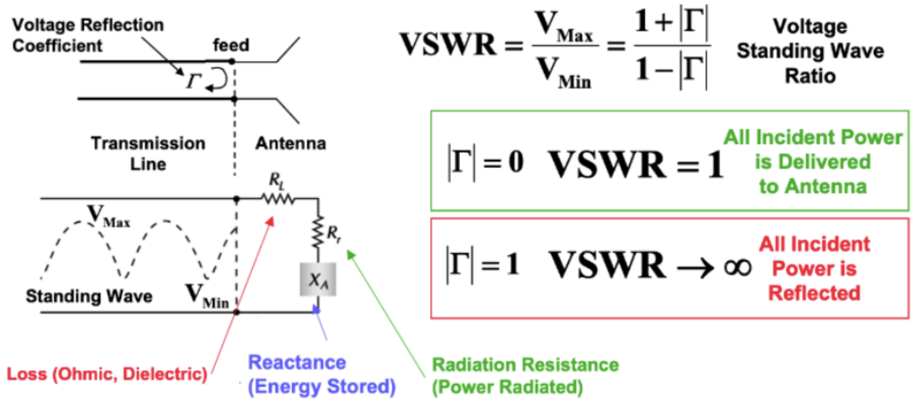
   where P_t is the power supplied by the transmitter to the antenna terminals.

   Gain, G:

   $G = \rho D$

8. VSWR

   Considered the antenna as an impedance, or the ratio of voltage to current at the feed port. We want to maximize the power transfer, but a VSWR of 2:1 is generally good enough:



   where Γ is the reflection coefficient

9. Effective Aperture, A_e

   The solid angle Ω_p is related to the area an antenna would use to capture propagating power density, Q.

   The effective aperture is scaled by wavelength.

   $\Omega_p = \dfrac{\lambda^2}{A_e}$

   Thus: $A_e = \dfrac{\lambda^2}{\Omega_p} = \dfrac{\lambda^2}{\theta_{HPBW}\phi_{HPBW}}$

   So Directivity, D, can also be calculated as:

   $D = \dfrac{4\pi}{\Omega_p} = \dfrac{4\pi}{\lambda^2}A_e$

10. Beam Width

    Let beam width be a function of antenna coordinates (x,y) instead of θ and ϕ.

    From $\Omega_p = \theta_{HPBW}\phi_{HPBW} = \dfrac{\lambda^2}{A_e}$, we can write:

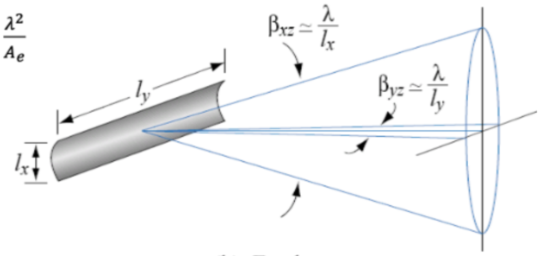
    $\beta_x\beta_y \approx \dfrac{\lambda}{l_x}\dfrac{\lambda}{l_y}$

    So, beam width is approximately: $\beta \approx \dfrac{\lambda}{length of antenna}$ [radians]

Beam widths: $\beta_x \simeq \frac{\lambda}{l_x}$, $\beta_y \simeq \frac{\lambda}{l_y}$

2-D beam width: $\Omega_p \simeq \beta_x \beta_y \simeq \frac{\lambda^2}{l_x l_y} \simeq \frac{\lambda^2}{A_e}$

Directivity: $D = \frac{4\pi}{\Omega_p} = \frac{4\pi A_e}{\lambda^2}$



gure 9-24

(b) Fan beam

11. Physical Aperture, A_p

    For dish and horn antennas, it can be useful to relate effective and physical apertures through the aperture efficiency $\eta_{aperature}$

    $A_e = \eta_{aperature} A_p$

12. Aperture illumination

    More amplitude taper increases the antenna beam width by factor k_x. and reduces the side-lobe amplitude

    $$\beta_x = k_x \frac{\lambda}{l_x}$$

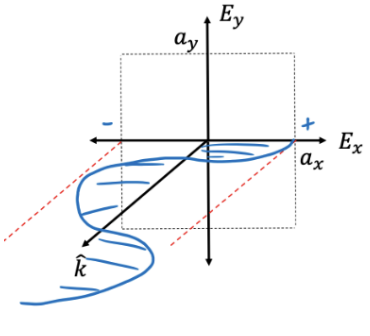
    Antenna efficiency is also related to these constants:

    $$\eta_{aperature} = \frac{1}{k_x k_y}$$

13. Polarization

    Types of polarizations:

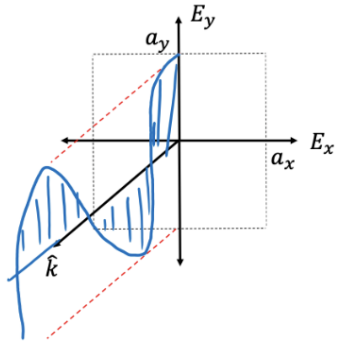
    - Linear polarizations (horizontal and vertical)



Horizontal Linear

$E_x(z,t) = a_x \cos(\omega t - kz)$

$E_y = 0$

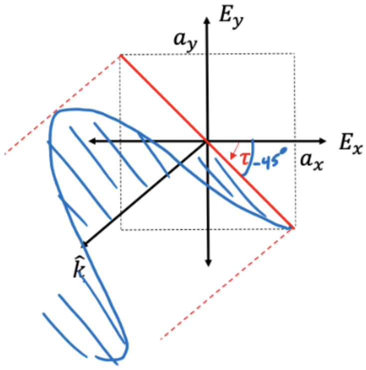


Vertical Linear

$E_x = 0$

$E_y(z,t) = a_y \cos(\omega t - kz)$

    - Slant polarization



Slant-Linear Polarization:
Angle from Horizontal

$$\tau(z,t) = tan^{-1}\left[\frac{E_y(z,t)}{E_x(z,t)}\right]$$
$$\tau(z,t) = tan^{-1}\left[\frac{-a_y\cos(\omega t-kz)}{a_x\cos(\omega t-kz)}\right]$$
$$\tau(z,t) = tan^{-1}\left[\frac{-a_y}{a_x}\right] = -45°$$

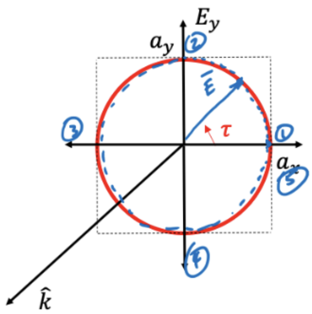
$$E_x(z,t) = a_x\cos(\omega t - kz)$$
$$E_y(z,t) = -a_y\cos(\omega t - kz)$$

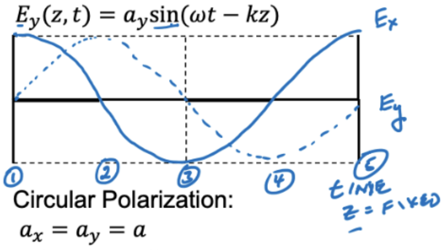
- Circular polarization

  Right hand circular (counter clock-wise rotation)

[redacted]

$$E_x(z,t) = a_x\cos(\omega t - kz)$$
$$E_y(z,t) = a_y\sin(\omega t - kz)$$





Circular Polarization:
$a_x = a_y = a$
90° phase difference between E_x and E_y

$$\tau(z,t) = tan^{-1}\left[\frac{E_y(z,t)}{E_x(z,t)}\right]$$
$$\tau(z,t) = tan^{-1}\left[\frac{\text{asin}(\omega t-kz)}{\text{acos}(\omega t-kz)}\right]$$
$$\tau(z,t) = (\omega t - kz) = \text{constant rotation}$$

$\tau \uparrow 0$

At $z = 0$, $\tau(0,t) = (\omega t - k0) = \omega t$
$E(t)$ rotates counter-clockwise as a function of time.

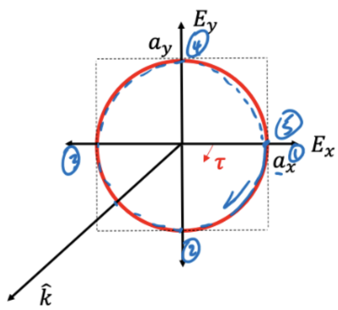
Counter clockwise rotation.
Right Hand Circular (RHC)

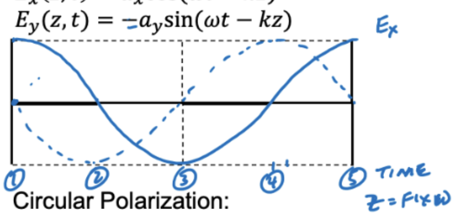
10

Left hand circular (clock-wise rotation)

[redacted]

$$E_x(z,t) = a_x\cos(\omega t - kz)$$
$$E_y(z,t) = -a_y\sin(\omega t - kz)$$





Circular Polarization:
$a_x = a_y = a$
-90° phase difference between E_x and E_y

$$\tau(z,t) = tan^{-1}\left[\frac{E_y(z,t)}{E_x(z,t)}\right]$$
$$\tau(z,t) = tan^{-1}\left[\frac{-\text{asin}(\omega t-kz)}{\text{acos}(\omega t-kz)}\right]$$
$$\tau(z,t) = -(\omega t - kz) = \text{constant rotation}$$

At $z = 0$, $\tau(0,t) = -\omega t$
$E(t)$ rotates clockwise as a function of time.

Clockwise rotation.
Left Hand Circular (LHC)

12

- Elliptical polarization

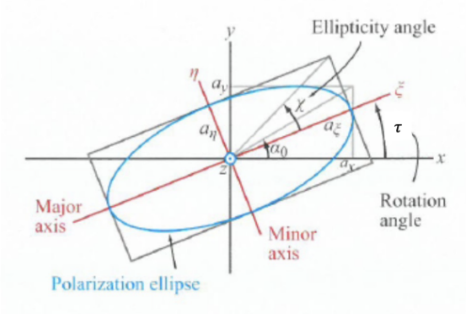


Figure 2-8: Polarization ellipse in the x–y plane, with the wave traveling in the z direction (out of the page).

Elliptical Polarization:
E_x and E_y have different amplitudes
$a_x \neq a_y$

Two phase terms determine rotation direction and shape
τ – rotation angle
(sign determines RHC & LHC)
χ – Ellipticity angle

Transform between linear and circular polarizations:

Linear polarized waves can be transformed into circular polarizations by shifting the phase of one of the two linear components by +/- 90 degrees:

If transmitting E_H^t and E_V^t, then RHC and LHC can be estimated using:

$$\begin{bmatrix} E_{RHC}^t \\ E_{LHC}^t \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & +j \\ 1 & -j \end{bmatrix} \begin{bmatrix} E_H^t \\ E_V^t \end{bmatrix}$$

This transformation yields:

$$E_{RHC}^t = \frac{1}{\sqrt{2}}[E_H^t + jE_V^t]$$

$$E_{LHC}^t = \frac{1}{\sqrt{2}}[E_H^t - jE_V^t]$$

If transmitting RHC and LHC, then E_H^t and E_V^t can be estimated using:

$$\begin{bmatrix} E_H^t \\ E_V^t \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -j & +j \end{bmatrix} \begin{bmatrix} E_{RHC}^t \\ E_{LHC}^t \end{bmatrix}$$

This transformation yields:

$$E_H^t = \frac{1}{\sqrt{2}}[E_{RHC}^t + E_{LHC}^t]$$

$$E_V^t = \frac{1}{\sqrt{2}}[-jE_{RHC}^t + jE_{LHC}^t]$$

Target Scattering Matrix

A scattered wave's polarization is determined by its scattering matrix:

$E^{scat} = S_{target} E^{inc}$

where the scattering angle and incident angle both change the scattering matrix. In the monostatic case, the scattering angle is 180 degrees from the incident angle is called backscattering.

Scattering matrix notation S_{XY}: X is backscatter and Y is incident.

$$\boldsymbol{E}^{scat} = \begin{bmatrix} E_V^{scat} \\ E_H^{scat} \end{bmatrix} = \begin{bmatrix} S_{VV} & S_{VH} \\ S_{HV} & S_{HH} \end{bmatrix} \begin{bmatrix} E_V^{inc} \\ E_H^{inc} \end{bmatrix}$$

For monostatic radars, cross-polarizations are the same, thus $S_{VH} = S_{HV}$.

Volume Scattering

1. Radar Pulse Volume

   The volume of the radar resolution volume is defined by range, and the azimuth and elevation angles of the antenna pattern.

2. Radar Cross Section of volume of distributed targets

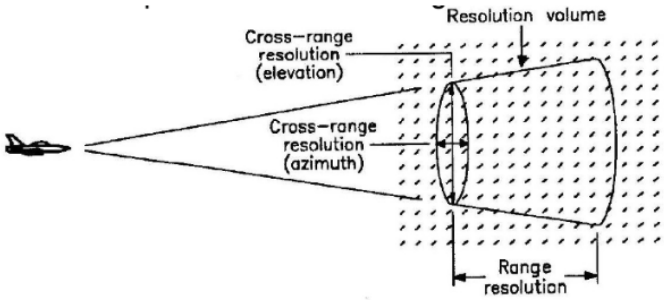
   The RCS of volume targets is the summation of RCS of the individual targets:

   $\sigma_{volume} = \sigma^0 \Delta V = \sum \sigma_{targets}(i) \Delta V = \eta \Delta V$

   where σ^0 is the RCS of the volume of targets per unit volume

3. Radar Power Equation for volume target scattering

   The radar resolution volume (aka resolution cell) is the volume illuminated by the radar.



Volume:

$$\Delta V = \Delta R[\pi ab] = \Delta R\left[\pi\left(\frac{R\theta_{HPBW}}{2}\right)\left(\frac{R\phi_{HPBW}}{2}\right)\right]$$

$$\Delta V = \pi\left(\frac{c\tau}{2}\right)\left(\frac{R\theta_{HPBW}}{2}\right)\left(\frac{R\phi_{HPBW}}{2}\right)$$

$$\Delta V = \left(\frac{c\tau}{2}\right)\left(\frac{\pi R^2\theta_{HPBW}\phi_{HPBW}}{4}\right) \qquad [\mathrm{m}^3]$$

The power returned at the antenna terminals by a volume target is given as:

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma_{volume}}{(4\pi)^3 R^4} = \frac{P_t G_t G_r \lambda^2}{(4\pi)^3 R^4}\left[\left(\frac{c\tau}{2}\right)\left(\frac{\pi R^2\theta_{HPBW}\phi_{HPBW}}{4}\right)\right]\eta \quad [\mathrm{W}]$$

$$P_r = \frac{P_t G_t G_r \lambda^2 c\tau\pi(\theta_{HPBW}\phi_{HPBW})}{8(4\pi)^3 R^2}\eta \qquad [\mathrm{W}]$$

The SNR for a volume target is given as:

$$SNR = \frac{P_r}{Noise_r} = \frac{P_t G_t G_r \lambda^2 c\tau\pi(\theta_{HPBW}\phi_{HPBW})\eta}{8(4\pi)^3 L_{sys} L_{FSPL}^2 (kT_{sys}B) R^2}$$

With a match filter, $B = 1/\tau$

$$SNR = \frac{P_r}{Noise_r} = \frac{P_t \tau^2 G_t G_r \lambda^2 c(\theta_{HPBW}\phi_{HPBW})\eta}{(8)^3(\pi)^2 L_{sys} L_{FSPL}^2 (kT_{sys}) R^2}$$

where τ is the pulse length.

4. For more on estimating RCS depending on the radar grazing angle, see ASEN5245_Scattering_01_Volume_Surface_Particle_022724.pdf