Radar and Remote Sensing Equation Sheet

Rachel Middleton
Rachel.Middleton@colorado.edu

Radar Fundamentals

1. Transmit Signal, T_x for a monostatic radar block diagram:

   $$T_x(t) = A_0 U(t) \cos[2\pi f(t)t + \phi_{TX}(t)]$$

   where:

   A_0 = amplitude

   $U(t)$ = pulse train signal

   $f(t)$ = transmit frequency

   $\phi_{TX}(t)$ = transmit phase

2. Received Signal, R_x for a monostatic radar block diagram:

   $$R_x(t) = k(A_0 U(t - \nabla t) \cos[2\pi f(t - \nabla t) + \psi] + n(t))$$

   where:

   ψ = the sum of the phase shifts of the target and with the radar

   k = is order of 10^{-18}

3. The Radar Power Equation

   $$P_r = \frac{P_t G_t G_r \lambda^2 \sigma}{(4\pi)^3 R^4}$$

   where:

   Pt – Transmitted power (at antenna terminals) [W]

   Gt – Transmitter antenna gain [unitless]

   R – Range to target [m]

   RCS – Target's radar cross section (also expressed as σ) [m^2]

   Gr – Receiver antenna gain [unitless]

   Pr – Received power (at antenna terminals) [W]

   $\lambda = \frac{c}{f}$ = wavelength [m]

The following quantities can be derived from the RPE, see ASEN5245_01c_Radar_Fundamentals_2024_0118 for more equations

(a) Power density for an Isotropic Antenna

$$Q_i = \frac{P_t}{4\pi R^2} \text{ [Watts}/m^2]$$

where:

Q_i is the incident power density for and isotropic antenna($G_t = 1$)

$4\pi R^2$ is the area of a sphere

(b) Radar Cross Section (RCS)

$$\sigma(\theta, \phi) = \frac{P_{reflected}[W]}{Q_i[W/m^2]}[m^2]$$

(c) Radiation Scattered by a Target

(d) Power Backscattered by a Target

(e) Power Density at Receive Antenna

(f) Power Backscattered by Target

(g) Power Collected by Receive Antenna

(h) Received Power at Antenna Ports

4. Calculations in dB

$$P_{dB} = 10\log(\frac{P_{linear}}{P_0})$$

where:

P_{linear} is a value in "linear" units and P_0 represents the reference value (typically 1)

When the reference has physical meaning, then 'dB' is replaced with a special units:

Reference to milliWatts ($P_0 = 10^{-3}W$) dBm

Reference to Watts dBW

Reference to square meters (RCS, λ,) dBsm

Reference to isotropic radiator dBi

Reference to dipole radiator dBd

5. Pulse Radar Range

$$R_{target} = \frac{cT_{delay}}{2} \text{ [m]}$$

where c is the speed of light, $3x10^8$

6. Pulse Radar Waveform

τ [s] - the pulse width or duration the Tx is transmitting

T_{IPP} or IPP - the inter-pulse period [s], time between pulses

$PRF = 1/T_{IPP}$ [Hz] - the pulse repetition frequency

Duty Cycle $= 100(\frac{\tau}{T_{IPP}})$ [%]

7. Range Resolution

$$\nabla R = R_2 - R_1 = c\frac{T_2 - T_1}{2} = c\frac{\delta t}{2}$$

where δt is the minimum time such that two targets at R1 and at R 2 will appear completely resolved

therefore:

$$\nabla R = \frac{c\tau}{2}$$

8. Unambiguous Range (Pulse Radar)

$$R_{max} = \frac{cT_{IPP}}{2}$$

Range aliasing implies that a target observed at range, $R_{obs} < R_{max}$ could have come from a different inter-pulse period

Noise and Losses

1. Noise Power of a Radar System

$$P_n = kT_{sys}B \text{ [W]}$$

where:

P_n is the noise power in Watts

k - Boltzmann's constant: $1.38x10^{-23}$ [J/K], note: [J] = [Watt second]

T_{sys} - system noise temperature [K]

B - receiver bandwidth [Hz] (typically B $1/\tau$, where τ is the pulse width [s]

2. Random Antenna Noise

$$T_{sys} = T_{antenna} + T_e = (T_a + T_A) + T_e$$

T_a - External noise. Noise picked up by the antenna (e.g., Galactic noise, sun, moon, ground and other thermal emitters. This noise is part of the antenna temperature

T_A - Thermal Emission. Noise due to physical antenna temperature. Also part of the antenna temperature

T_e - Receiver noise. Internally generated noise by Rx components (e.g., amplifiers, mixers). This noise is quantified by the Rx effective noise temperature

3. Radar Losses

$$L_{total} = L_{sys}L_{prop} \text{ where } L_{total} \geq 1$$

Two kinds of losses may occur: Propagation Path Losses – losses associated with the propagation medium between target and radar, such as, atmospheric attenuation and multipath interference

System Losses – losses within the radar system itself, such as insertion loss, as waves propagate through the radar system. (Losses within the antenna are accounted for in the antenna efficiency)

effciency, ρ is expressed as $L = \dfrac{1}{\rho}$

Losses are factored into the RPE as follows:

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma}{(4\pi)^3 L_{FSPL}^2 R^4}$$

where L_{FSPL} is the one-way propagation loss (notice this quantity squared is the two-way propagation loss)

4. Specific Attenuation Model for Rain

see reference doc: ITU-R P.838-1

$$\gamma_R = kR^{\alpha} \text{ [dB/km]}$$

where $\gamma_R$is the specific attenuation for rain [dB/km]

k and α are frequency dependent constants tabulated for horizontal (H) and vertical (V) propagation polarizations

R is rain rate [mm/hr]

Then, we factor these losses into the RPE as:

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma}{(4\pi)^3 L_{FSPL}^2 L_{rain}^2 R^4}$$

where L_{rain} is the one-way propagation loss for rain (distance * attenuation rate) (notice this quantity squared is the two-way propagation loss)

Similar attenuation rates can be calculated for other atmospheric mediums, such as cloud and fog (Recommendation ITU-R P.840-3)

5. Coax Cable, Insertion loss (Attenuation) (a system loss)

6. Signal to Noise Ratio (SNR)

$$SNR = \frac{P_r}{P_n} = \frac{P_t G^2 \lambda^2 \sigma}{(4\pi)^3 L_{sys} L_{prop}^2 R^4 k T_{sys} B}$$

7. Coherent integration

$$(SNR)_{coherent}(n_p) = n_p SNR(1)$$

where n_p is the number of consecutive pulses

8. Maximum Detectable Range

$$\text{Detection Threshold} = SNR_{min} = \frac{P_t \tau G^2 \lambda^2 \sigma}{(4\pi)^3 L_{sys} L_{prop}^2 R^4 k T_{sys}}$$
$$\text{Max Detectable range} = R_m ax = [\frac{P_t \tau G^2 \lambda^2 \sigma}{(4\pi)^3 L_{sys} L_{prop}^2 SNR_{min} k T_{sys}}]^{1/4}$$
$$\text{min RCS} = \sigma_{min} = \frac{(4\pi)^3 L_{sys} L_{prop}^2 SNR_{min} k T_{sys} R^4}{P_t \tau G^2 \lambda^2}$$

9. Fourier Transform (FT)

The Fourier transform can be viewed as a projection onto an orthogonal basis function.

Fourier Transform Pair:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad X(f) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi f t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

The basis functions are the complex exponentials.

Note that both $F(\omega)$ and $f(t)$ are continuous functions and can be either real or complex (as with I and Q voltages).

From Eulier's Identity, we can represent cosines and sines with complex exponentials.

Negative frequencies represent clockwise (CW) rotation on complex plane

Positive frequencies represent counter-clockwise (CCW) rotation on complex plane 11

Refer to ASEN5245_Intro_Matched_Filter_Compression_013024.pdf for examples and further explanation.

10. Pulse Compression Ratio [TODO]

11. Decoding Phase Modulation [TODO]

EM Basics

1. Electrostatics

   - Ohms Law: $V = RI$
     where:
     V = Voltage [V]
     R = Resistance [Ω]
     I = Current [Ampere]
   - Electric Field
     $\tilde{E} = \rho \tilde{J}$
     where:
     $\tilde{E}$ = electric field [V/m]
     $\tilde{J}$ = current density [A/m^2]
     ρ = resistivity of the medium [Ω/m]
     $\sigma = 1/\rho$ = conductivity

- Coulomb's Law: $\tilde{E} = \frac{q}{4\pi\epsilon_0 r^2}\vec{R}[V/m^2]$
  where:
  ϵ_0 = Permittivity of Free Space = $\frac{1}{36\pi} * 1-^{-9}$ [Farads/m]
  Dielectric Constant = $\epsilon = \epsilon_r\epsilon_0$
  $\tilde{E}$ = electric field [V/m]
  $\tilde{D}$ = Electric Flux Desnsity: $\tilde{D} = \epsilon\tilde{E}$
  r = Distance between the centers of the two charges [m]

2. Magnetostatics

- Biot-Savart Law: $\vec{B} = \mu\tilde{H}$
  where:
  $\tilde{B}$ = Magnetic flux density [Tesla] = $\tilde{B} = \frac{\mu_0 I}{2\pi r}$
  $\tilde{H}$ = Magnetic field intensity [A/m]
  μ_0 = Permeability of free space $\approx 4\pi \times 10^{-7}$ T m/A $\mu = \mu_r\mu_0$ and $\mu_r = 1$ for this class I = Electric current [A] = $\frac{dq}{dt}$
  $\vec{r}$ = Position vector from the length element to the point of observation [m]

3. Electromagnetics

Table 1-6: The three branches of electromagnetics.

Branch	Condition	Field Quantities (Units)
Electrostatics	Stationary charges ($\partial q/\partial t = 0$)	Electric field intensity **E** (V/m) Electric flux density **D** (C/m^2) $\mathbf{D} = \varepsilon\mathbf{E}$
Magnetostatics	Steady currents ($\partial I/\partial t = 0$)	Magnetic flux density **B** (T) Magnetic field intensity **H** (A/m) $\mathbf{B} = \mu\mathbf{H}$
Dynamics (Time-varying fields)	Time-varying currents ($\partial I/\partial t \neq 0$)	**E**, **D**, **B**, and **H** $(\mathbf{E},\mathbf{D})$ coupled to $(\mathbf{B},\mathbf{H})$

Table 1-7: Constitutive parameters of materials.

Parameter	Units	Free-space Value
Electrical permittivity ε	F/m	$\varepsilon_0 = 8.854 \times 10^{-12}$ (F/m) $\simeq \frac{1}{36\pi} \times 10^{-9}$ (F/m)
Magnetic permeability μ	H/m	$\mu_0 = 4\pi \times 10^{-7}$ (H/m)
Conductivity σ	S/m	0

4. Maxwell's Equations

Table 6-1: Maxwell's equations.

Reference	Differential Form	Integral Form	
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	(6.1)
Faraday's law	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \dfrac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$	(6.2)†
No magnetic charges (Gauss's law for magnetism)	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	(6.3)
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$	(6.4)

†For a stationary surface S.

5. Phasor notation

$$\bar{F}(x,y,z;t) = F(x,y,z)cos(\omega t + [k_x \cdot \hat{x} + k_y \cdot \hat{y} + k_z \cdot \hat{z}])$$
$$\bar{F}(x,y,z;t) = F(x,y,z)cos(\omega t + \bar{k} \cdot \bar{x})$$

Euler's Identity: $Ae^{j\xi} = Acos(\xi) + jAsin(\xi) \qquad j = \sqrt{-1}$

Real part: $Re(e^{j\xi}) = Acos(\xi)$

$$\bar{F}(x,y,z;t) = F(x,y,z)\cos(\omega t + \bar{k}\cdot\bar{x}) = Re\left[F(x,y,z)\, e^{j\omega t + j\bar{k}\cdot\bar{x}}\right]$$
$$= Re\left[F(x,y,z)\, e^{j\bar{k}\cdot\bar{x}} e^{j\omega t}\right]$$

$$\left.\begin{array}{l} \tilde{F}(x,y,z) = Re\left[F(x,y,z)\, e^{j\omega t}\right] \\ e^{j\phi} = e^{j\bar{k}\cdot\bar{x}} \end{array}\right\} \Rightarrow \begin{array}{l} \bar{F}(x,y,z;t) = \tilde{F}(x,y,z)e^{j} \\ \bar{\tilde{F}} = Ae^{j\phi} \end{array}$$

A = AMPLITUDE & ϕ = PHASE

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6. Complex permittivity

(1)

$\nabla \cdot \tilde{\bar{E}} = \frac{\rho_v}{\epsilon}$

$\nabla \times \tilde{\bar{E}} = -\mu \frac{\partial \tilde{\bar{H}}}{\partial t}$

$\nabla \cdot \tilde{\bar{H}} = 0$

$\nabla \times \tilde{\bar{H}} = \tilde{\bar{J}} + \epsilon \frac{\partial E}{\partial t}$

KNOWING THAT

$\tilde{\bar{F}} = Ae^{j\phi} e^{j\omega t}$

(2)

$\nabla \cdot \tilde{\bar{E}} = \frac{\rho_v}{\epsilon}$

$\nabla \times \tilde{\bar{E}} = -j\omega\mu \tilde{\bar{H}}$

$\nabla \cdot \tilde{\bar{H}} = 0$

$\nabla \times \tilde{\bar{H}} = \tilde{\bar{J}} + j\omega\epsilon \tilde{\bar{E}}$

(3) SINCE $\bar{J} = \sigma \bar{E}$, THEN $\tilde{\bar{J}} = \sigma \tilde{\bar{E}}$

$\nabla \times \tilde{\bar{H}} = \sigma \tilde{\bar{E}} + j\omega\epsilon \tilde{\bar{E}}$

$= (\sigma + j\omega\epsilon) \tilde{\bar{E}}$

$= j\omega \left(\frac{\sigma}{j\omega} + \epsilon\right) \tilde{\bar{E}}$

$\nabla \times \tilde{\bar{H}} = j\omega \left(\epsilon - j\frac{\sigma}{\omega}\right) \tilde{\bar{E}}$

$\nabla \times \tilde{\bar{H}} = j\omega \epsilon_c \tilde{\bar{E}}$

$\epsilon_c \triangleq \epsilon - j\frac{\sigma}{\omega}$ PERMITTIVITY

$= \epsilon' - j\epsilon''$

WITH $\epsilon'' = \frac{\sigma}{\omega}$

FOR LOSSLESS MEDIUM, WITH A "PERFECT DIELECTRIC" $\sigma = 0$

$\epsilon'' = \frac{0}{\omega} = 0$; $\epsilon_c = \epsilon' = \epsilon$

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7. Propagating Waves

HOW DO WAVES PROPAGATE IN A CHARGE FREE ENVIRONMENT? $\tilde{\rho}_v = 0$ ← NO CHARGES

(1)

$\nabla \cdot \tilde{\bar{E}} = 0$

$\nabla \times \tilde{\bar{E}} = -j\omega\mu \tilde{\bar{H}}$

$\nabla \cdot \tilde{\bar{H}} = 0$

$\nabla \times \tilde{\bar{H}} = j\omega\epsilon_c \tilde{\bar{E}}$

COMBINE EQUATIONS

(2)

$\Rightarrow \tilde{\bar{H}} = \nabla \times \left[\frac{\tilde{\bar{E}}}{-j\omega\mu}\right]$

$\nabla \times \left\{\nabla \times \left[\frac{\tilde{\bar{E}}}{-j\omega\mu}\right]\right\} = j\omega\epsilon_c \tilde{\bar{E}}$

$\nabla \times (\nabla \times \tilde{\bar{E}}) = -j^2 \omega^2 \mu \epsilon_c \tilde{\bar{E}}$

$\nabla \times (\nabla \times \tilde{\bar{E}}) = \omega^2 \mu \epsilon_c \tilde{\bar{E}}$

(3) VECTOR IDENTITY:

$\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 F$

∇^2 IS LAPLACIAN OPERATOR

$\nabla^2 \bar{F} = \frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$

(SCALAR)

(4) $\nabla \times (\nabla \times \tilde{\bar{E}}) = \nabla(\nabla \cdot \tilde{\bar{E}}) - \nabla^2 \tilde{\bar{E}} = \omega^2 \mu \epsilon_c \tilde{\bar{E}}$

$\nabla^2 \tilde{\bar{E}} + \omega^2 \mu \epsilon_c \tilde{\bar{E}} = 0$

HELMHOLTZ EQN

(WAVE EQUATION)

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Wave Propagation and Scattering

1. The Homogenous Wave Equation

   $\nabla^2 \tilde{\boldsymbol{E}} - \gamma^2 \tilde{\boldsymbol{E}} = 0$ and $\nabla^2 \tilde{\boldsymbol{H}} - \gamma^2 \tilde{\boldsymbol{H}} = 0$

   where γ^2 is defined as the **propagation constant**: $\gamma^2 = -\omega^2 \mu \varepsilon_c$

   Define the **wavenumber** as $k = \omega\sqrt{\mu\varepsilon_c}$ so that $\gamma^2 = -k^2$.

   The wave equations become:

   $\nabla^2 \tilde{\boldsymbol{E}} + k^2 \tilde{\boldsymbol{E}} = 0$ and $\nabla^2 \tilde{\boldsymbol{H}} + k^2 \tilde{\boldsymbol{H}} = 0$

   Recall from Maxwell's equations that E and H are orthogonal:

   $$\bar{H} = \frac{1}{\eta}\vec{k} \times \bar{E}$$
   $$\bar{E} = -\eta\vec{k} \times \bar{H}$$
   $$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

   and $\vec{k}$ is the unit vector in the direction of propagation.

   - One solution to the wave equation with $\tilde{E}_y = 0$ has the form:

     $\boldsymbol{E}(z,t) = \hat{x}A\cos(\omega t - kz + \phi) + \hat{y}(0)$ [V/m]

     $\bar{H} = \frac{1}{\eta}\hat{k}\times\bar{E}$

     $\boldsymbol{H}(z,t) = \hat{x}(0) + \hat{y}\frac{E(z,t)}{\eta} = \hat{y}(2.65)\cos(\omega t - kz + \phi)$ [mA/m]

2. Propagation through multiple elements

   Losses of medium are usually expressed one-way path loses in dB.

   $$L_i = \alpha_i d_i$$

   where α_i is the attenuation coefficient [dB/km] and d_i is the path length [km]

   Loss along the full path length is the Free Space Propagation Loss (FSPL) and is the sum of the Losses ($L_1 + L_2 + L_3 + ... = L_{total}$)

3. Plane Wave Propagation in Lossless Media

From Maxwell's equations, $\overline{H} = \frac{1}{\eta}\hat{k}\times\overline{E}$, the **intrinsic impedance** of a lossless medium is

$$\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon_r\varepsilon_0}} = \sqrt{\frac{\mu}{\varepsilon_r\varepsilon_0}} \qquad \text{[Ohms]}$$

The **phase velocity** of the wave traveling through the medium is

$$u_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon_r\varepsilon_0}} = \frac{1}{\sqrt{\mu\varepsilon_r\varepsilon_0}} \qquad \text{[m/s]}$$

The **wavelength** is given as

$$\lambda = \frac{2\pi}{k} = \frac{u_p}{f} \qquad \text{[m]}$$

where f is the frequency in Hz.

In a **vacuum**, $\varepsilon_r = 1$ and $\mu = \mu_0$ (non-magnetic material):

$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377 \approx 120\pi \qquad \text{[Ohms]}$$

$$u_p = c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 299{,}792{,}458 \approx 3x10^8 \qquad \text{[m/s]}$$

In a **vacuum**, $\varepsilon_r = 1$ and $\mu = \mu_0$, intrinsic impedance & phase velocity:

$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377 \approx 120\pi \qquad \text{[Ohms]}$$

$$u_p = c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 299{,}792{,}458 \approx 3x10^8 \qquad \text{[m/s]}$$

In a **dielectric medium**, $\varepsilon_r > 1$ and $\mu = \mu_0$ (non-magnetic):

$$\eta = \sqrt{\frac{\mu}{\varepsilon_r\varepsilon_0}} = \eta_0\sqrt{\frac{1}{\varepsilon_r}} = 377\sqrt{\frac{1}{\varepsilon_r}} \qquad \text{[Ohms]}$$

$$u_p = \frac{1}{\sqrt{\mu\varepsilon_r\varepsilon_0}} = c\frac{1}{\sqrt{\varepsilon_r}} \approx (3x10^8)\frac{1}{\sqrt{\varepsilon_r}} \qquad \text{[m/s]}$$

Example: $\varepsilon_r = 1.38$

$$u_p = c\frac{1}{\sqrt{\varepsilon_r}} \sim c\frac{1}{\sqrt{1.38}} \sim (0.85)c$$

Electrical Specifications

Performance Property	Units	US	(metric)
Cutoff Frequency	GHz	16.2	
Velocity of Propagation	%	85	
Dielectric Constant	NA	1.38	

The wave propagates slower in the cable than in free-space.

4. Atmospheric Refraction

Refraction - radio waves travel in a different direction because of the **index of refraction**

The index of refraction, $n = \frac{c}{v_p}$

where c is the speed of light and v_p is the phase speed of the wave in the medium

In a vacuum, $v_{phase} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{(4\pi x 10^{-7})(36\pi x 10^{-9})}} \sim 3x10^8 = c$

In a medium with dielectric $\epsilon = \epsilon_r \epsilon_0$, the propagation speed is:

$$v_{phase} = \frac{1}{\sqrt{\mu_0 \epsilon_r \epsilon_0}} = \frac{1}{\sqrt{(4\pi x 10^{-7})(36\pi x 10^{-9})\epsilon_r}} = \frac{c}{\sqrt{\epsilon_r}}$$

Thus, the index of refraction is:

$$n = \frac{c}{v_p} = \left(\frac{1}{\sqrt{\mu_0 \epsilon_0}}\right)\left(\frac{\sqrt{\mu_0 \epsilon_r \epsilon_0}}{1}\right) = \sqrt{\epsilon_r}$$

The index of refraction is a complex quantity with real and imaginary parts ($n = \sqrt{\epsilon_r} = n' - jn''$) and is also defined as:

$$n = 1 + 10^{-6} N$$

where N is the refractivity

Antenna Properties

1. Radiation Properties
2. Radiation Pattern
3. Beam width
4. Directivity (Isotropic radiator and Pattern solid angle)
5. Gain

Scattering

- Refraction
- Diffraction
- Reflection – Surface, multi-path, and plate reflection
- Aircraft radar cross-section