

Radar and Remote Sensing Equation Sheet

Radar Fundamentals

1. Transmit Signal, T_x for a monostatic radar block diagram:

$$T_x(t) = A_0 U(t) \cos[2\pi f(t)t + \phi_{TX}(t)]$$

where:

A_0 = amplitude

$U(t)$ = pulse train signal

$f(t)$ = transmit frequency

$\phi_{TX}(t)$ = transmit phase

2. Received Signal, R_x for a monostatic radar block diagram:

$$R_x(t) = k(A_0 U(t - \nabla t) \cos[2\pi f(t - \nabla t) + \psi] + n(t))$$

where:

ψ = the sum of the phase shifts of the target and with the radar

k = is order of 10^{-18}

3. The Radar Power Equation

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma}{(4\pi)^3 R^4}$$

where:

P_t – Transmitted power (at antenna terminals) [W]

G_t – Transmitter antenna gain [unitless]

R – Range to target [m]

RCS – Target's radar cross section (also expressed as σ) [m^2]

G_r – Receiver antenna gain [unitless]

P_r – Received power (at antenna terminals) [W]

$\lambda = \frac{c}{f}$ = wavelength [m]

The following quantities can be derived from the RPE, see ASEN5245_01c_Radar_Fundamentals_2024.0118 for more equations

- (a) Power density for an Isotropic Antenna

$$Q_i = \frac{P_t}{4\pi R^2} \text{ [Watts}/m^2]$$

where:

Q_i is the incident power density for and isotropic antenna ($G_t = 1$)

$4\pi R^2$ is the area of a sphere

- (b) Radar Cross Section (RCS)

$$\sigma(\theta, \phi) = \frac{P_{reflected}[W]}{Q_i[W/m^2]} [m^2]$$

- (c) Radiation Scattered by a Target

- (d) Power Backscattered by a Target

- (e) Power Density at Receive Antenna

- (f) Power Backscattered by Target

- (g) Power Collected by Receive Antenna

- (h) Received Power at Antenna Ports

4. Calculations in dB

$$P_{dB} = 10 \log\left(\frac{P_{linear}}{P_0}\right)$$

where:

P_{linear} is a value in "linear" units and P_0 represents the reference value (typically 1)

When the reference has physical meaning, then 'dB' is replaced with a special units:

Reference to milliWatts ($P_0 = 10^{-3}W$) dBm

Reference to Watts dBW

Reference to square meters (RCS, λ ,) dBsm

Reference to isotropic radiator dBi

Reference to dipole radiator dBd

5. Pulse Radar Range

$$R_{target} = \frac{cT_{delay}}{2} \text{ [m]}$$

where c is the speed of light, 3×10^8

6. Pulse Radar Waveform

τ [s] - the pulse width or duration the Tx is transmitting

T_{IPP} or IPP - the inter-pulse period [s], time between pulses

$PRF = 1/T_{IPP}$ [Hz] - the pulse repetition frequency

Duty Cycle = $100\left(\frac{\tau}{T_{IPP}}\right)$ [%]

7. Range Resolution

$$\nabla R = R_2 - R_1 = c \frac{T_2 - T_1}{2} = c \frac{\delta t}{2}$$

where δt is the minimum time such that two targets at R_1 and at R_2 will appear completely resolved

therefore:

$$\nabla R = \frac{c\tau}{2}$$

8. Unambiguous Range (Pulse Radar)

$$R_{max} = \frac{cT_{IPP}}{2}$$

Range aliasing implies that a target observed at range, $R_{obs} < R_{max}$ could have come from a different inter-pulse period

Noise and Losses

1. Noise Power of a Radar System

$$P_n = kT_{sys}B \text{ [W]}$$

where:

P_n is the noise power in Watts

k - Boltzmann's constant: 1.38×10^{-23} [J/K], note: [J] = [Watt second]

T_{sys} - system noise temperature [K]

B - receiver bandwidth [Hz] (typically $B = 1/\tau$, where τ is the pulse width [s])

2. Random Antenna Noise

$$T_{sys} = T_{antenna} + T_e = (T_a + T_A) + T_e$$

T_a - External noise. Noise picked up by the antenna (e.g., Galactic noise, sun, moon, ground and other thermal emitters. This noise is part of the antenna temperature

T_A - Thermal Emission. Noise due to physical antenna temperature. Also part of the antenna temperature

T_e - Receiver noise. Internally generated noise by Rx components (e.g., amplifiers, mixers). This noise is quantified by the Rx effective noise temperature

3. Radar Losses

$$L_{total} = L_{sys}L_{prop} \text{ where } L_{total} \geq 1$$

Two kinds of losses may occur: Propagation Path Losses – losses associated with the propagation medium between target and radar, such as, atmospheric attenuation and multipath interference

System Losses – losses within the radar system itself, such as insertion loss, as waves propagate through the radar system. (Losses within the antenna are accounted for in the antenna efficiency)

efficiency, ρ is expressed as $L = \frac{1}{\rho}$

Losses are factored into the RPE as follows:

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma}{(4\pi)^3 L_{FSP L}^2 R^4}$$

where $L_{FSP L}$ is the one-way propagation loss (notice this quantity squared is the two-way propagation loss)

4. Specific Attenuation Model for Rain

see reference doc: ITU-R P.838-1

$$\gamma_R = kR^\alpha \text{ [dB/km]}$$

where γ_R is the specific attenuation for rain [dB/km]

k and α are frequency dependent constants tabulated for horizontal (H) and vertical (V) propagation polarizations

R is rain rate [mm/hr]

Then, we factor these losses into the RPE as:

$$P_r = \frac{P_t G_t G_r \lambda^2 \sigma}{(4\pi)^3 L_{FSP L}^2 L_{rain}^2 R^4}$$

where L_{rain} is the one-way propagation loss for rain (distance * attenuation rate) (notice this quantity squared is the two-way propagation loss)

Similar attenuation rates can be calculated for other atmospheric mediums, such as cloud and fog (Recommendation ITU-R P.840-3)

5. Coax Cable, Insertion loss (Attenuation) (a system loss)

6. Signal to Noise Ratio (SNR)

$$SNR = \frac{P_r}{P_n} = \frac{P_t G^2 \lambda^2 \sigma}{(4\pi)^3 L_{sys} L_{prop}^2 R^4 k T_{sys} B}$$

7. Coherent integration

$$(SNR)_{coherent}(n_p) = n_p SNR(1)$$

where n_p is the number of consecutive pulses

8. Maximum Detectable Range

$$\text{Detection Threshold} = SNR_{min} = \frac{P_t \tau G^2 \lambda^2 \sigma}{(4\pi)^3 L_{sys} L_{prop}^2 R^4 k T_{sys}}$$

$$\text{Max Detectable range} = R_{max} = \left[\frac{P_t \tau G^2 \lambda^2 \sigma}{(4\pi)^3 L_{sys} L_{prop}^2 SNR_{min} k T_{sys}} \right]^{1/4}$$

$$\text{min RCS} = \sigma_{min} = \frac{(4\pi)^3 L_{sys} L_{prop}^2 SNR_{min} k T_{sys} R^4}{P_t \tau G^2 \lambda^2}$$

9. Fourier Transform (FT)

The Fourier transform can be viewed as a projection onto an orthogonal basis function.

Fourier Transform Pair:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$X(f) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi f t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

The basis functions are the complex exponentials.

Note that both $F(\omega)$ and $f(t)$ are continuous functions and can be either real or complex (as with I and Q voltages).

From Euler's Identity, we can represent cosines and sines with complex exponentials.

Negative frequencies represent clockwise (CW) rotation on complex plane

Positive frequencies represent counter-clockwise (CCW) rotation on complex plane 11

Refer to ASEN5245.Intro_Matched_Filter_Compression_013024.pdf for examples and further explanation.

10. Pulse Compression Ratio [TODO]

11. Decoding Phase Modulation [TODO]

EM Basics

1. Electrostatics

- Ohms Law: $V = RI$

where:

V = Voltage [V]

R = Resistance [Ω]

I = Current [Ampere]

- Electric Field

$$\vec{E} = \rho \vec{J}$$

where:

\vec{E} = electric field [V/m]

\vec{J} = current density [A/m²]

ρ = resistivity of the medium [Ω /m]

$\sigma = 1/\rho$ = conductivity

- Coulomb's Law: $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \vec{R} [V/m^2]$
 where:
 ϵ_0 = Permittivity of Free Space = $\frac{1}{36\pi} * 10^{-9}$ [Farads/m]
 Dielectric Constant = $\epsilon = \epsilon_r \epsilon_0$
 \vec{E} = electric field [V/m]
 \vec{D} = Electric Flux Density: $\vec{D} = \epsilon \vec{E}$
 r = Distance between the centers of the two charges [m]

2. Magnetostatics

- Biot-Savart Law: $\vec{B} = \mu \vec{H}$
 where:
 \vec{B} = Magnetic flux density [Tesla] = $\vec{B} = \frac{\mu_0 I}{2\pi r}$
 \vec{H} = Magnetic field intensity [A/m]
 μ_0 = Permeability of free space $\approx 4\pi \times 10^{-7}$ T m/A $\mu = \mu_r \mu_0$ and $\mu_r = 1$ for this class $I =$
 Electric current [A] = $\frac{dq}{dt}$
 \vec{r} = Position vector from the length element to the point of observation [m]

3. Electromagnetics

Table 1-6: The three branches of electromagnetics.

Branch	Condition	Field Quantities (Units)
Electrostatics	Stationary charges ($\partial q / \partial t = 0$)	Electric field intensity E (V/m) Electric flux density D (C/m ²) D = $\epsilon \mathbf{E}$
Magnetostatics	Steady currents ($\partial I / \partial t = 0$)	Magnetic flux density B (T) Magnetic field intensity H (A/m) B = $\mu \mathbf{H}$
Dynamics (Time-varying fields)	Time-varying currents ($\partial I / \partial t \neq 0$)	E, D, B, and H (E, D) coupled to (B, H)

Table 1-7: Constitutive parameters of materials.

Parameter	Units	Free-space Value
Electrical permittivity ϵ	F/m	$\epsilon_0 = 8.854 \times 10^{-12}$ (F/m) $\simeq \frac{1}{36\pi} \times 10^{-9}$ (F/m)
Magnetic permeability μ	H/m	$\mu_0 = 4\pi \times 10^{-7}$ (H/m)
Conductivity σ	S/m	0

4. Maxwell's Equations

Table 6-1: Maxwell's equations.

Reference	Differential Form	Integral Form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ (6.1)
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$ (6.2) [†]
No magnetic charges (Gauss's law for magnetism)	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ (6.3)
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$ (6.4)

[†]For a stationary surface S .

5. Phasor notation

$$\bar{F}(x, y, z; t) = F(x, y, z) \cos(\omega t + [k_x \cdot \hat{x} + k_y \cdot \hat{y} + k_z \cdot \hat{z}])$$

$$\bar{F}(x, y, z; t) = F(x, y, z) \cos(\omega t + \bar{k} \cdot \bar{x})$$

Euler's Identity: $Ae^{j\xi} = A\cos(\xi) + jA\sin(\xi)$ $j = \sqrt{-1}$

Real part: $Re(e^{j\xi}) = A\cos(\xi)$

$$\bar{F}(x, y, z; t) = F(x, y, z) \cos(\omega t + \bar{k} \cdot \bar{x}) = Re[F(x, y, z) e^{j\omega t + j\bar{k} \cdot \bar{x}}]$$

$$= Re[F(x, y, z) e^{j\bar{k} \cdot \bar{x}} e^{j\omega t}]$$

$$\left. \begin{aligned} \tilde{F}(x, y, z) &= Re[F(x, y, z) e^{j\omega t}] \\ e^{j\phi} &= e^{j\bar{k} \cdot \bar{x}} \end{aligned} \right\} \Rightarrow \bar{F}(x, y, z; t) = \tilde{F}(x, y, z) e^{j\omega t}$$

$$\tilde{F} = A e^{j\phi}$$

$A = \text{AMPLITUDE}$ & $\phi = \text{PHASE}$

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6. Complex permittivity

① $\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon}$
 $\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$
 $\nabla \cdot \vec{H} = 0$
 $\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}$

KNOWING THAT $\vec{F} = A e^{j\phi} e^{j\omega t}$

② $\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon}$
 $\nabla \times \vec{E} = -j\omega \mu \vec{H}$
 $\nabla \cdot \vec{H} = 0$
 $\nabla \times \vec{H} = \vec{J} + j\omega \epsilon \vec{E}$

③ SINCE $\vec{J} = \sigma \vec{E}$, THEN $\vec{J} = \sigma \vec{E}$
 $\nabla \times \vec{H} = \sigma \vec{E} + j\omega \epsilon \vec{E}$
 $= (\sigma + j\omega \epsilon) \vec{E}$
 $= j\omega (\frac{\sigma}{\omega} + \epsilon) \vec{E}$
 $\nabla \times \vec{H} = j\omega (\epsilon - j\frac{\sigma}{\omega}) \vec{E}$

$\epsilon_c \triangleq \epsilon - j\frac{\sigma}{\omega}$ PERMITTIVITY
 $= \epsilon' - j\epsilon''$
 WITH $\epsilon'' = \frac{\sigma}{\omega}$

FOR LOSSLESS MEDIUM, WITH A
 "PERFECT DIELECTRIC" $\sigma = 0$
 $\epsilon'' = \frac{\sigma}{\omega} = 0$; $\epsilon_c = \epsilon' = \epsilon$

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7. Propagating Waves

HOW DO WAVES PROPAGATE IN A CHARGE FREE ENVIRONMENT? $\rho_v = 0 \leftarrow$ NO CHARGES

① $\nabla \cdot \vec{E} = 0$
 $\nabla \times \vec{E} = -j\omega \mu \vec{H}$
 $\nabla \cdot \vec{H} = 0$
 $\nabla \times \vec{H} = j\omega \epsilon_c \vec{E}$

COMBINE EQUATIONS

② $\Rightarrow \vec{H} = \nabla \times \left[\frac{\vec{E}}{-j\omega \mu} \right]$
 $\nabla \times \left[\nabla \times \left[\frac{\vec{E}}{-j\omega \mu} \right] \right] = j\omega \epsilon_c \vec{E}$
 $\nabla \times (\nabla \times \vec{E}) = -j^2 \omega^2 \mu \epsilon_c \vec{E}$
 $\nabla \times (\nabla \times \vec{E}) = \omega^2 \mu \epsilon_c \vec{E}$

③ VECTOR IDENTITY:
 $\nabla \times (\nabla \times \vec{F}) = \nabla (\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$
 ∇^2 IS LAPLACIAN OPERATOR
 $\nabla^2 \vec{F} = \frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2}$
 (SCALAR)

④ $\nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \omega^2 \mu \epsilon_c \vec{E}$

$\nabla^2 \vec{E} + \omega^2 \mu \epsilon_c \vec{E} = 0$

HENKOLTZ EQN
 (WAVE EQUATION)

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Wave Propagation and Scattering

1.

Antenna Properties

1.

Scattering