CS 61 Discrete Math
Exam 2 Review

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1 Combinatorial Proofs

$1 + 3 + 5 + ... + 2n - 1 = n^2$

Question: Area of a Square

class: $2^0 + 2^1 + 2^2 + ... + 2^{n-1} = 2^n - 1$

Question: All non-zero base-2 numbers of length < n

13.3: $23^0 + 23^1 + 23^2 + ... + 23^{n-1} = 3^n - 1$

Question: All non-zero base-3 numbers of length < n

$(x-1)x^0 + (x-1)x^1 + (x-1)x^2 + ... + (x-1)x^{n-1} = x^n - 1$

Question: All non-zero base-x numbers of length < n

13.5 $n^2 = n(n-1) + n$

Question: Number of length-2 lists from n elements w repetition

recitation wksht 5: $n^3 = n(n-1)(n-2) + 3n(n-1) + n$

Question: Number of length-3 lists from n elements w repetition

from mock exam 2:
Problem: $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n}$

Question: How many elements are in a powerset with size n OR how many subsets does a set with n elements have?

from exam 2 worksheet: $\binom{n}{k}\binom{k}{4} = \binom{n}{4}\binom{n-4}{k-4}$

Question: number of ways to choose a committee of k members from n people, and choosing a subcommittee of 4 members from committee of k members

from lecture slides: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ where $n >= 1$

Question: pascal's triangle and binomial coefficient or "how many k-element subsets does the set $1, 2, 3...n$ have?"

From worksheet 5: Give a combinatorial proof of Pythagorean theorem given picture
LHS: c^2
RHS: area triangle $*4+$ area little square $= 4 * ab/2 + (a-b)(a-b)$

Question: What is area of the square

2 Relations, Equivalence Relations, and Partitions

Equivalence Relation

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A relation R defined on a set A is called an EQUIVALENCE RELATION if R satisfies the following:

- Reflexive : $(a, a) \in R$, for all $a \in A$
- Symmetric : $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$
- Transitive : $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$, for all $a, b, c \in A$

A partition is the same as a set of **equivalence classes**. = parts of partition.

$\{\{1,2\},\{3\},\{4,5,6\},\{7\}\}$

3 Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

$$\text{e.g.}\ \binom{5}{3} = \frac{5!}{3! \times 2!}$$

4 Proof by Contradiction or Contrapositive

Proof by Contradiction, Template #1

PROOF BY CONTRADICTION

Template #1: Generic

Claim
p

Proof.
Suppose* $\neg p$
...
Contradiction. $\rightarrow\leftarrow$
Thus we conclude p. $\square$

*or more clearly, "Suppose for sake of contradiction"

Proof by Contradiction, Template #2

PROOF BY CONTRADICTION

Template #2: If-then statements, negating original claim

Claim
$q \rightarrow r$

Proof.
Suppose* $\neg(q \rightarrow r)$,
in other words $q \wedge \neg r$.
...
Contradiction. $\rightarrow\leftarrow$
Thus $q \rightarrow r$. $\square$

Proof by Contradiction, Template #3

PROOF BY CONTRADICTION

Template #3: If-then statements, contradiction within proof

Claim
$q \rightarrow r$

Proof.
Assume q.
Suppose* $\neg r$,
...
Contradiction. $\rightarrow\leftarrow$
Thus r. $\square$

Proof by Contrapositive - Template

PROOF BY CONTRAPOSITIVE TEMPLATE

Claim

$p \to q$

Proof.

Assume $\neg q$.

...

Then $\neg p$.

By the contrapositive, $p \to q$. $\square$

5 Proof by Induction or Strong Induction

Proof by Induction - Template

PROOF BY INDUCTION: TEMPLATE

Claim

$\forall n \in \mathbb{N}, p(n)$

Proof.

Base case: Show $p(0)$ is true.

Inductive step:

Assume $p(k)$ is true for some $k \geq 0$

...

Then $p(k+1)$ is true.

Conclusion: Thus $p(n)$ is true for all $\mathbb{N}$. □

Different start point: If the statement is instead true for all $n \geq n_0$ for some n_0, the base case is $p(n_0)$ and the inductive hypothesis is for some $k \geq n_0$.

Proof by Strong Induction: Template

Claim

$\forall n \in \mathbb{N}, p(n)$

Proof.

Base case: Show $p(0)$ is true.

Inductive step:

Assume $p(0), p(1), \ldots, p(k)$ are all true for some $k \geq 0$.

...

Then $p(k+1)$ is true.

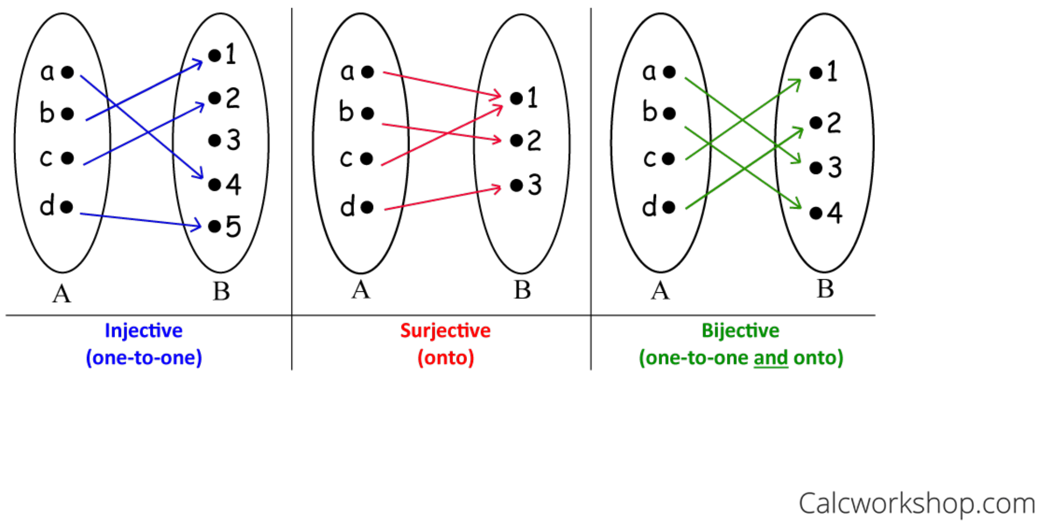
Conclusion: Thus $p(n)$ is true for all $\mathbb{N}$. □

Note #1: **Different base case**. If the statement is instead true for all $n \geq n_0$ for some n_0, the base case is $p(n_0)$ and the inductive hypothesis is $p(n_0) \ldots p(k)$ for some $k \geq n_0$.

Note #2: **Multiple base cases**. For example, some inductive steps need the previous *two* cases to be true, i.e., in order to prove $p(k+1)$ we must know both $p(k)$ and $p(k-1)$. Then you must prove *two* base cases, otherwise the inductive step cannot "get started."

6 Functions

Function: if every input maps to one output



7 Pigeonhole Principle

The Pigeonhole Principle

The Pigeonhole Principle: If $f : A \to B$ where A and B are finite and $|A| > |B|$, then f is not one-to-one. (Thus, two distinct inputs must map to the same output.)

Generalized Pigeonhole Principle: If $f : A \to B$ where $|A| = n$ and $|B| = m$, then $\exists b \in B$ with at least $\lceil \frac{n}{m} \rceil$ elements of A that map to it. In other words, if we are putting n objects into m buckets, there is at least one bucket with $\lceil \frac{n}{m} \rceil$ objects in it.

Proof Template

PROOF TEMPLATE for Pigeonhole Principle

- Identify buckets (how many?)
- Identify objects (how many?)
- Conclude at least one bucket has $\lceil \frac{\#\text{objects}}{\#\text{buckets}} \rceil$
- Thus...?