1. (15 points) **Scaling and shifting**

(a) (10 points) Sketch the following signals for $x(t)$ as shown in the figure below.

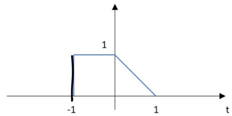


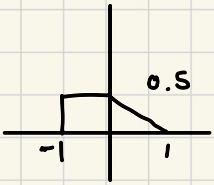
Figure 1: $x(t)$

i. $\frac{1}{2}x(2t-6)$

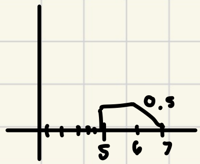
ii. $x(\frac{1}{10}-\frac{1}{5}t)$

1a i) reverse PEMDAS

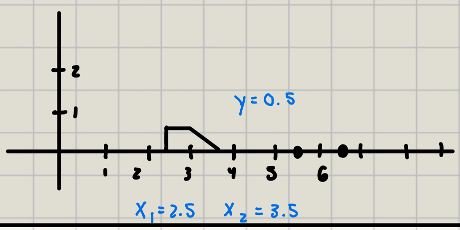
① y compress $\frac{1}{2}$



② shift by 6

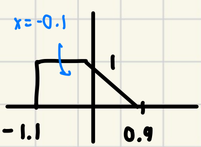


③ x compress $\frac{1}{2}$

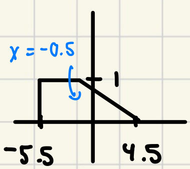


1a ii) $x(-\frac{1}{5}t+\frac{1}{10})$

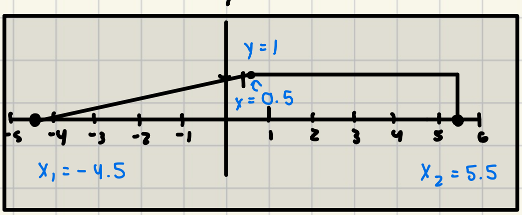
① shift $\frac{1}{10}$ left



② multiply x by 5



③ flip across y axis



(b) (5 points) Express $y(t)$, as shown in the figure below, in terms of $x(t)$.

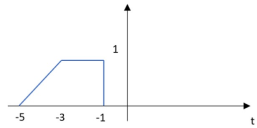


Figure 2: $y(t)$

1b) reversal $a(t) = x(-t)$ reflect over y-axis

stretch $b(t) = a(\frac{1}{2}t)$

shift $y(t) = b(t+3)$ PEMDAS

$$\boxed{y = x(-\frac{1}{2}t + 3)}$$

2. (10 points) **Even and odd signals**

(a) (3 points) Show that the product of two odd signals is even.

(b) (3 points) Show that the product of an even signal and an odd signal is odd.

(c) (4 points) Find the even and the odd components of

$$x(t) = t\sin(t) + t^3\left(\frac{e^t+e^{-t}}{2}\right) + 2024$$

2a) $x_1(t) = -x_1(-t)$ } odd

$x_2(t) = -x_2(-t)$

$y(t) = x_1(t)\,x_2(t)$

$= (-x_1(-t))(-x_2(-t))$

$= x_1(-t)\,x_2(-t)$

$= y(-t)$

2b) $x_1(t) = -x_1(-t)$ odd

$x_2(t) = x_2(-t)$ even

$y(t) = x_1(t)\,x_2(t)$

$= -x_1(-t)\,x_2(-t)$

$= -y(-t)$

2c) $x_o(t) + x_e(t) = x(t)$

$x_o(t) = x(t) - x_e(t)$

$= x_e(-t) - x(-t)$

$x_e(t) = x(t) - x_o(t)$

$= x(t) + x(-t) - x_e(-t)$

$2x_e(t) = x(t) + x(-t)$

$x_e(t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$

$= \frac{1}{2}(x(t) + x(-t))$

$x_o(t) = x(t) - x_e(t)$

$= \frac{1}{2}x(t) - \frac{1}{2}x(-t)$

$= \frac{1}{2}(x(t) - x(-t))$

how do u use a and b?

$x(t) + x(-t)$ for $x_e(t)$

$t\sin(t) + \cancel{t^3\left(\frac{e^t+e^{-t}}{2}\right)} + 2024 +$

$\left(-t\sin(-t) - \cancel{t^3\left(\frac{e^t+e^{-t}}{2}\right)} + 2024\right)$

$= t\sin(t) - t\sin(-t) + 4048$

$\sin(t)$ is odd $\Rightarrow \sin(t) = -\sin(t)$

$t\sin(t) + t\sin(t) + 4048 =$

$2t\sin(t) + 4048$

$x_e(t) = \cancel{\frac{1}{2}}(\cancel{2}t\sin(t) + 4048)$

$\boxed{x_e(t) = t\sin(t) + 2024}$

$x(t) - x(-t)$ for x_o

$t\sin(t) + t^3\left(\frac{e^t+e^{-t}}{2}\right) + 2024$

$-\left(\overset{+}{\cancel{-}}t\sin(-t) + t^3\left(\frac{e^t+e^{-t}}{2}\right)\overset{-}{\cancel{+}}2024\right)$

$= t\sin(t) + t\sin(-t) + t^3(e^t + e^{-t})$

$t\,\cancel{\sin(t) - t\sin(t)} + t^3(e^t + e^{-t})$

$\boxed{y_o(t) = t^3(e^t + e^{-t})}$

3. (22 points) **Periodic signals**

(a) (12 points) For each of the following signals, determine whether it is periodic or not. If the signal is periodic, determine its fundamental frequency.

i. $x(t) = 24 + 50\sin(60\pi t)$

ii. $x(t) = 10\cos^2(\frac{\pi}{2}t)$

iii. $x(t) = \sin(5\pi t) + \cos(16\pi^2 t)$

iv.

$$x(t) = \begin{cases} 24 + 50\sin(60\pi t) & t < 0 \\ 10\cos^2(10\pi t) & t \geq 0 \end{cases} \quad (1)$$

(b) (5 points) Suppose $x(t)$ is odd and periodic with period T_0. What is the value of $x(T_0)$?

(c) (1 point) What is the effect of time shifting on the fundamental frequency of a signal?

(d) (4 points) If $x(t)$ is periodic, is $x(5t+2)$ periodic?

3a) i yes, it is a sinusoid

$60 = 2\pi f$

$f = \boxed{30}$

ii $x(t) = 10\cos^2(\frac{\pi}{2}t)$

trig identity $\cos^2 x = \frac{1+\cos 2x}{2}$

$x(t) = \cancel{10}\,5\left(\frac{1+\cos(\pi t)}{\cancel{2}}\right)$

$x(t) = 5 + \cos(\pi t)$

$\pi = 2\pi f$

$\boxed{f = \frac{1}{2}}$

iii use sum/product rule of 2 periodic functions

$\sin(5\pi t)$
$5\pi = 2\pi f$
$f = \frac{5}{2}, T_1 = \frac{2}{5}$

$\cos(16\pi^2 t)$
$16\pi^2 = 2\pi f$
$8\pi = f, T = \frac{1}{8\pi}$

check: (k_1 and k_2 have to be integers)

$k_1 T_1 = k_2 T_2$

$k_1 \left(\frac{2}{5}\right) = k_2 \left(\frac{1}{8\pi}\right)$

if $\frac{T_1}{T_2} = \frac{k_2}{k_1}$ where $\frac{k_2}{k_1}$ is rational

$\frac{2/5}{1/8\pi} = \frac{k_2}{k_1} = \frac{2}{5} \cdot 8\pi = \frac{16\pi}{5}$

not rational

$\boxed{\text{no}}$

iv. $\boxed{\text{no}}$ the periods for the pieces found in i and ii are different

furthermore, there is a discontinuity at $t=0$ that is non-periodic given $\sin(t)$ and $\cos(t)$ are continuous

$24 + 50\sin(0) = 24 + 0 = 24$
$10\cos^2(0) = 10(1) = 10$
$24 \neq 10$

3b) $x(t + T_0) = x(t)$ def. of periodic

$x_0(t) = -x_0(-t)$ x_0

simplify by plugging $t=0$

$x_0(0) = -x_0(-0) = -x(0)$

only if $x(0) = 0$

$x(0 + T_0) = x(0) = 0$

$\boxed{x(T_0) = 0}$

3c) $\boxed{\text{no effect}}$ time shifting

shifts the entire graph but the peaks and troughs stay the same distance apart from each other

3d) $\boxed{\text{yes}}$ $x(5t+2)$ will still be periodic but only the period will change; it will be $\frac{1}{5}$th the period of $x(t)$ and $+2$ does not change the period from 3c

4. (21 points) **Energy and power signals**

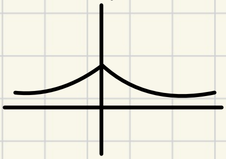
(a) (15 points) Determine whether the following signals are energy or power signals. If the signal is an energy signal, determine its energy. If the signal is a power signal, determine its power.

i. $x(t) = e^{-|t|}$

ii. $x(t) = \begin{cases} \frac{1}{\sqrt{t}}, & \text{if } t \geq 1 \\ 0, & \text{otherwise} \end{cases}$

iii. $x(t) = \begin{cases} 1 + e^{-t}, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$

4 a) i graph will look similar to



energy signal

break into piecewise function

it is an energy signal

$$E_x = \lim_{T\to\infty} \int_{-T}^{T} |(x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} e^{-2|t|} dt = \int_{-\infty}^{0} e^{2t} + \int_{0}^{\infty} e^{-2t} dt$$

$$= \frac{1}{2} e^{2t} \Big|_{-\infty}^{0} + \left(-\frac{1}{2} e^{-2t}\right)\Big|_{0}^{\infty} =$$

$$\frac{1}{2} - 0 + 0 + \frac{1}{2} = 1 \qquad \boxed{E_x = 1}$$

ii $\quad E_x = \int_{-\infty}^{1} \cancel{0\,dt} + \int_{1}^{\infty} \left|\frac{1}{\sqrt{t}}\right|^2 dt$

$$= \int_{1}^{\infty} \frac{1}{t} dt = \ln|t| \Big|_{1}^{\infty} = \underline{\infty}$$

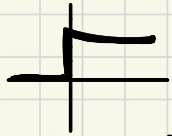
power signal : $\lim_{t\to\infty}$ in terms of T

$$\lim_{T\to\infty} \frac{\ln t}{2t} \quad \text{L'Hopital} \overset{\lim_{T\to\infty}}{\Rightarrow} \frac{1/t}{2} = \lim_{T\to\infty} \frac{2}{t}$$

$$\underline{P_x = 0}$$

it is neither power or energy signal

iii



$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{0} \cancel{0\,dt} +$$

$$\int_{0}^{\infty} |1 + e^{-t}|^2 dt = \int_{0}^{\infty} 1 + 2e^{-t} + e^{-2t} d$$

$$= t - 2e^{-t} - \frac{1}{2} e^{-2t} \Big|_{0}^{\infty} = \infty$$

power signal

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt =$$

$$\lim_{T\to\infty} \frac{t - \cancelto{0}{2e^{-2t}} - \cancelto{0}{\frac{1}{2} e^{-2t}}}{2t} = \frac{1}{2}$$

$\boxed{P_x = \frac{1}{2}}$ it is a power signal

(b) (3 points) Show that if $x(t)$ is odd, then

for all / every

$$\int_{-\tau}^{\tau} x(t)dt = 0, \forall \tau > 0$$

$$x_o(t) = -x_o(-t)$$

$$\int_{-T}^{0} x(t) + \int_{0}^{T} x(t)$$

let $s = -t,\ t = -s,\ dt = -ds$

$$-\int_{-T}^{0} x(-s)\, ds = \int_{-T}^{0} x(s)\, ds$$

$$= \underbrace{-\int_{0}^{T} x(s)\, ds = -\int_{0}^{T} x(t)\, dt}_{\text{because dummy variable}}$$

$$-\int_{0}^{T} x(t)dt + \int_{0}^{T} x(t) = 0$$

(c) (3 points) Show that the energy of a real signal $x(t)$ is the sum of the energy of its even component $x_e(t)$ and that of its odd component $x_o(t)$, i.e., $E_x = E_{x_e} + E_{x_o}$.
Hint: Use properties derived in 2(b) and 4(b).

4c) $E_x = E_{xe} + E_{xo}$

$$E_x = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$|x(t)|^2 = |x_e(t) + x_o(t)|^2 =$$

$$(x_e(t))^2 + (x_o(t))^2 + 2\underbrace{x_e(t)x_o(t)}_{\text{this is odd from 2b}}$$

$$E_x = \lim_{T\to\infty}\left(\int_{-T}^{T}(x_e(t))^2\,dt + \int_{-T}^{T}(x_o(t))^2\,dt + \cancel{\int_{-T}^{T} 2x_e(t)x_o(t)\,dt}\right)$$

integral of odd function = 0

$$E_x = \lim_{T\to\infty}\int_{-T}^{T} x_e(t)^2\,dt + \lim_{T\to\infty}\int_{-T}^{T} x_o(t)^2\,dt$$

$$= E_{x_e(t)} + E_{x_o(t)}$$

5. (16 points) **Euler's formula and complex numbers**

(a) (8 points) Use Euler's formula to prove the following identities:

i. $\cos^2(\theta) + \sin^2(\theta) = 1$

ii. $\cos(\theta+\psi) + \cos(\theta-\psi) = 2\cos(\theta)\cos(\psi)$

5a) i. $e^{j\emptyset} = \cos(\emptyset) + j\sin(\emptyset)$

consider complex conjugate $e^{-j\emptyset} = \cos(\emptyset) - j\sin(\emptyset)$

$$e^{j\emptyset}e^{-j\emptyset} = \cos^2(\emptyset) - j^2\sin^2(\emptyset)$$

$$e^{j\emptyset - j\emptyset} = \cos^2(\emptyset) + \sin^2(\emptyset)$$

$$e^0 = \cos^2(\emptyset) + \sin^2(\emptyset)$$

$$1 = \cos^2(\emptyset) + \sin^2(\emptyset)$$

ii. from lecture: $\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$

$$\cos(\theta+\psi) = \frac{1}{2}\left(e^{j(\theta+\psi)} + e^{-j(\theta+\psi)}\right)$$

$$+ \ \cos(\theta-\psi) = \frac{1}{2}\left(e^{j(\theta-\psi)} + e^{-j(\theta-\psi)}\right)$$

$$\frac{1}{2}\left(e^{j\theta}e^{j\psi} + e^{-j\theta}e^{-j\psi}\right) +$$

$$\frac{1}{2}\left(e^{j\theta}e^{-j\psi} + e^{-j\theta}e^{j\psi}\right) =$$

$$\frac{1}{2}e^{j\theta}\left(e^{j\psi} + e^{-j\psi}\right) + \frac{1}{2}e^{-j\theta}\left(e^{j\psi - j\psi}\right)$$

$$= \left(e^{j\psi} + e^{-j\psi}\right)\left(\frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}\right)$$

$$2(\cos\psi)(\cos\theta)$$

(b) (8 points) $x(t) = (5+\sqrt{2}j)e^{j(t+2)}$ and $y(t) = 1/(2-j)$.

i. Compute the real and imaginary parts of $x(t)$ and $y(t)$.

ii. Compute the magnitude and phase of $x(t)$ and $y(t)$.

b) i. $r = 5 + \sqrt{2}j$

$$e^{j(t+2)} = \cos(t+2) + j\sin(t+2)$$

$$x(t) = (5+\sqrt{2}j)(\cos(t+2) + j\sin(t+2))$$

$$= 5\cos(t+2) + 5j\sin(t+2) +$$

$$\sqrt{2}j\cos(t+2) - \sqrt{2}\sin(t+2)$$

$$\boxed{x(t) = \underbrace{5\cos(t+2) - \sqrt{2}\sin(t+2)}_{\text{real}} + \underbrace{(5\sin(t+2) + \sqrt{2}\cos(t+2))j}_{\text{imaginary}}}$$

$$y(t) = \frac{1}{2-j} \cdot \frac{2+j}{2-j} = \frac{2+j}{4-j^2} = \frac{2+j}{5}$$

$$\boxed{y(t) = \frac{2}{5} + \frac{1}{5}j}$$ imaginary

ii.

convert $5+\sqrt{2}j$ into polar

$$r = \sqrt{25+2} = \sqrt{27} = 3\sqrt{3}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{2}}{5}\right)$$

$$5+\sqrt{2}j = 3\sqrt{3}\,e^{j(\tan^{-1}\frac{\sqrt{2}}{5})}$$

multiply

$$3\sqrt{3}\,e^{j(\tan^{-1}\frac{\sqrt{2}}{5})}\,e^{j(t+2)}$$

$$\boxed{\underbrace{3\sqrt{3}}_{\text{magnitude}}\,e^{j\overbrace{(t+2+\tan^{-1}\frac{\sqrt{2}}{5})}^{\text{phase}}}}$$

$$r = \sqrt{\frac{4}{25} + \frac{1}{25}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

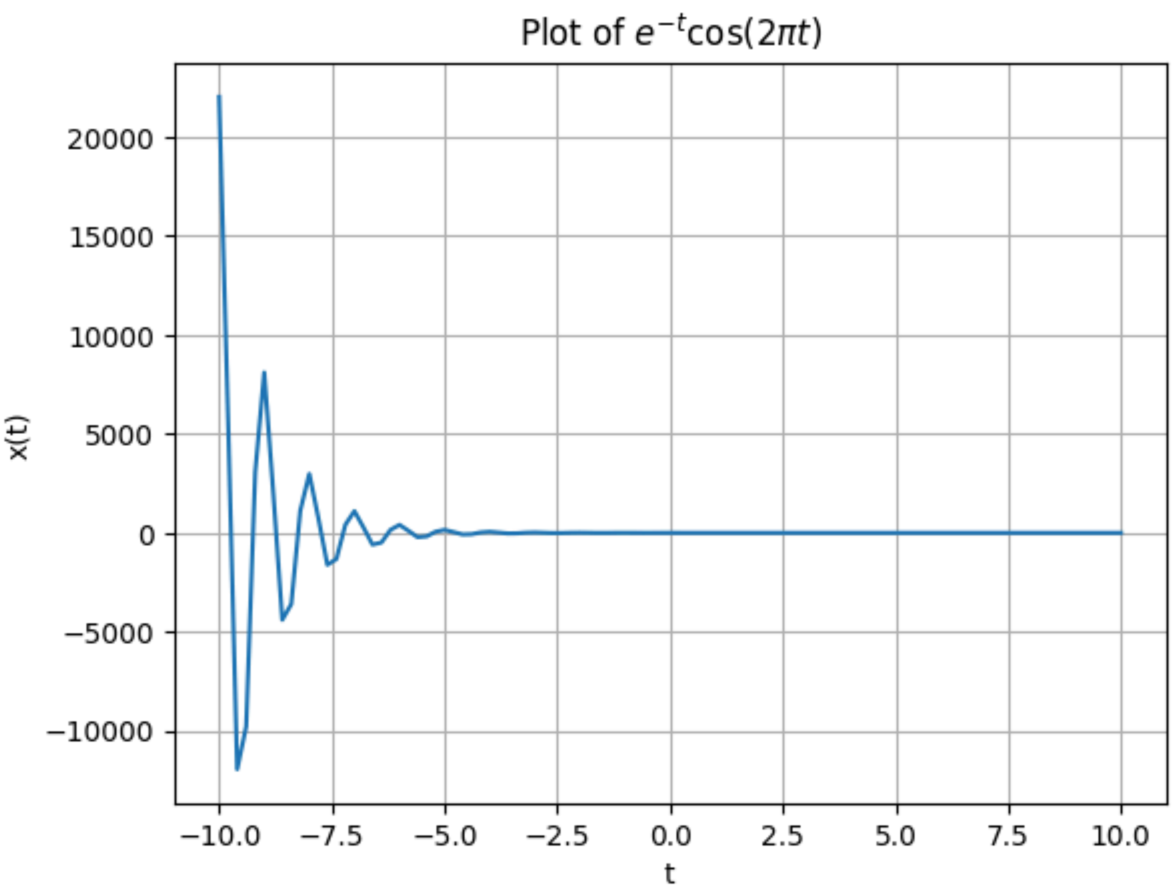
$$\tan\theta = \frac{y}{x} = \frac{1/5}{2/5} = \frac{1}{2}$$

$$\arctan\left(\frac{1}{2}\right)$$

$$\boxed{y(t) = \underbrace{\frac{\sqrt{5}}{5}}_{\text{magnitude}}\,e^{j\overbrace{(\tan^{-1}(\frac{1}{2}))}^{\text{phase}}}}$$

In [3]:
```
import numpy as np
import matplotlib.pyplot as plt
t = np.arange(-10, 10.2, 0.2)

y = np.exp(-t) * np.cos(2*np.pi * t)

plt.plot(t, y)
plt.xlabel('t')
plt.ylabel('x(t)')
plt.title('Plot of $e^{-t} \cos(2\pi t)$')
plt.grid(True)
display(plt)
```

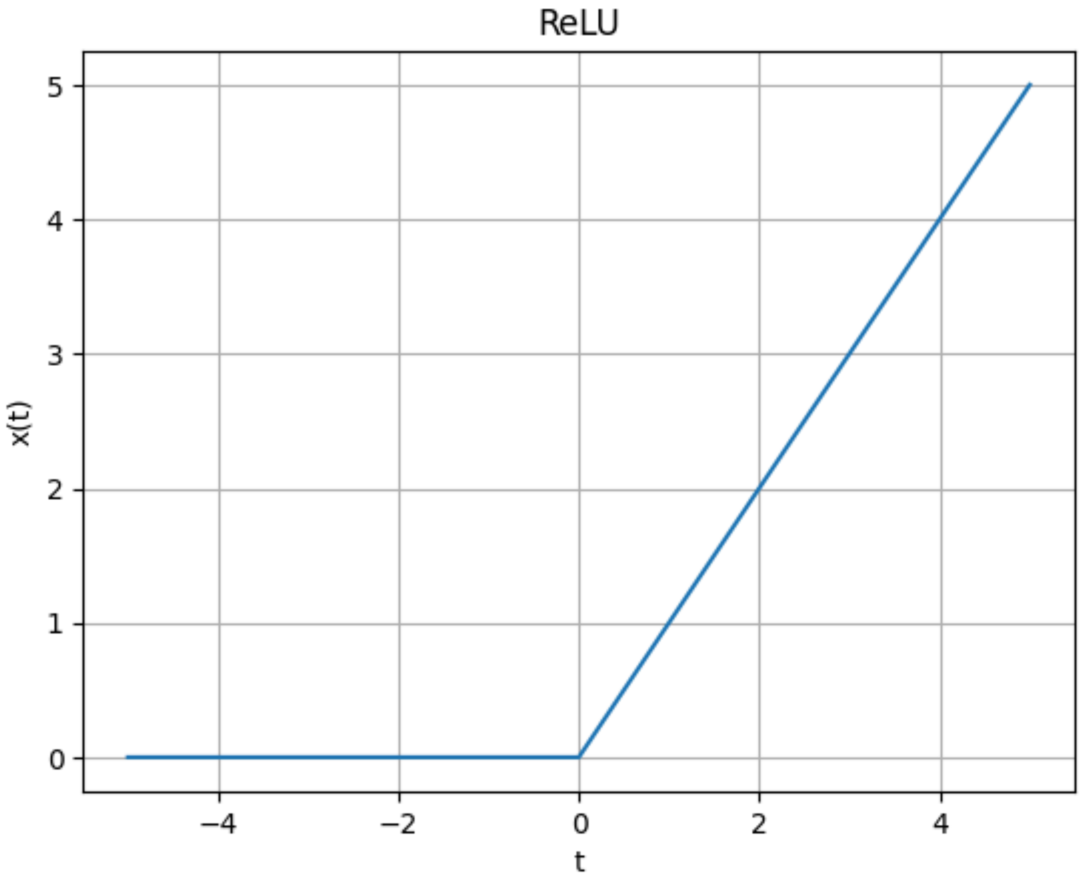
```
<module 'matplotlib.pyplot' from '/usr/local/lib/python3.10/dist-packages/ma
tplotlib/pyplot.py'>
```



In [6]:
```
def relu(t):
  if t < 0:
    return 0
  else:
    return t

y = []
time = np.arange(-5, 5.1, 0.1)
for x in time:
  y.append(relu(x))

plt.plot(time, y)
```

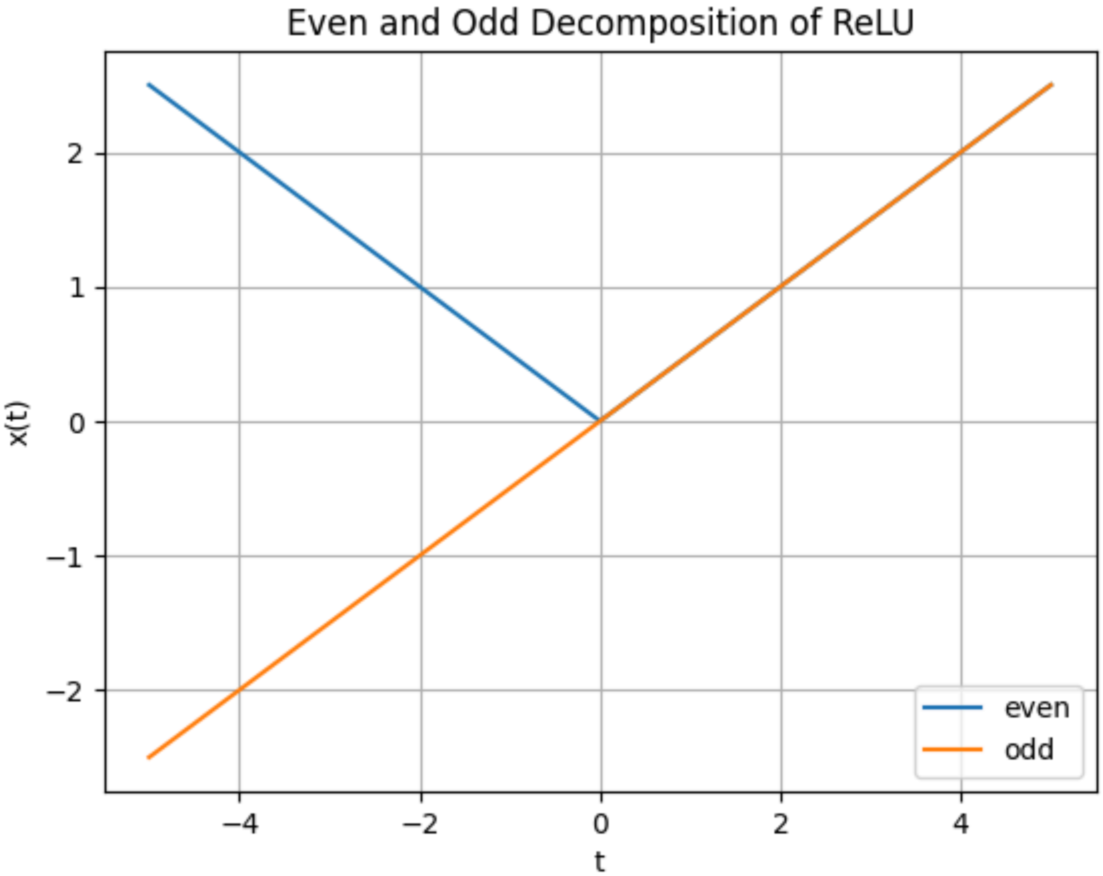
```
plt.xlabel('t')
plt.ylabel('x(t)')
plt.title('ReLU')
plt.grid(True)
display(plt)
```

```
<module 'matplotlib.pyplot' from '/usr/local/lib/python3.10/dist-packages/matplotlib/pyplot.py'>
```



In [9]:
```
def even(t, f):
  return 0.5 * (f(t) + f(-t))

def odd(t, f):
  return 0.5 * (f(t) - f(-t))

t = np.arange(-5, 5 + 0.1, 0.1)
y_even = [even(tt, relu) for tt in t]
plt.plot(t, y_even, label = 'even')
y_odd = [odd(tt, relu) for tt in t]
plt.plot(t, y_odd, label = 'odd')

plt.xlabel('t')
plt.ylabel('x(t)')
plt.title("Even and Odd Decomposition of ReLU")
plt.grid(True)
plt.legend()

display(plt)
```

```
<module 'matplotlib.pyplot' from '/usr/local/lib/python3.10/dist-packages/matplotlib/pyplot.py'>
```




```
In [ ]:
```