CISC 3220 Homework Assignment 2

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Question 1

Use mathematical induction to establish that for every positive integer n:

$$\sum_{i=1}^{n} \frac{1}{(4i+1)(4i-3)} = \frac{n}{4n+1}$$

Base case: $n = 1$

$$L(1) = \sum_{i=1}^{1} \frac{1}{(4(1)+1)(4(1)-3)} = \frac{1}{5}$$

$$R(1) = \frac{1}{4(1)+1} = \frac{1}{5}$$

Evidently, $L(1) = R(1)$

Inductive Step: For every $k \in N^+$, if $L(k) = R(k)$, then $L(k+1) = R(k+1)$.

Let $k \in N^+$. Suppose $L(k) = R(k)$.

$$L(k+1) = \sum_{i=1}^{k+1} \frac{1}{(4(k+1)+1)(4(k+1)-3)}$$

$$= \sum_{i=1}^{k} \frac{1}{(4k+1)(4k-3)} + \frac{1}{(4(k+1)+1)(4(k+1)-3)}$$

$$= \frac{k}{4k+1} + \frac{1}{(4k+5)(4k+1)} \qquad \textit{by Inductive Hypothesis}$$

$$= \frac{k(4k+5)}{(4k+5)(4k+1)} + \frac{1}{(4k+5)(4k+1)}$$

$$= \frac{4k^2+5k+1}{(4k+5)(4k+1)} = \frac{(4k+1)(k+1)}{(4k+5)(4k+1)} = \frac{k+1}{4k+5}$$

$$= \frac{k+1}{4(k+1)+1}$$

$$= R(k+1)$$

This establishes that for every $k \in N^+$, if $L(k) = R(k)$, then $L(k+1) = R(k+1)$.

Question 2

Use mathematical induction to establish $7^n - 2^n$ is divisible by 5 for every positive integer n.

Base case: $n = 1$

$$7^1 - 2^1 = 5, \text{which is clearly divisible by 5}$$

Inductive Step: For every $k \in N^+$, if $p(k)$ is true, then $p(k+1)$ is true.

Let $k \in N^+$. Suppose proposition $p(k)$ is true. Verify that $p(k+1)$ is true:

$$\begin{aligned}
&7^{k+1} - 2^{k+1} \\
&= (5+2) \cdot 7^k - 2 \cdot 2^k \\
&= 5 \cdot 7^k + 2(7^k - 2^k)
\end{aligned}$$

The first term $(5 \cdot 7^k)$ is being multiplied by 5, and is therefore divisible by 5.
The second term $(2(7^k - 2^k))$ is divisible by 5, according to the inductive hypothesis.
Hence, $7^n - 2^n$ is divisible by 5 for every positive integer n.

Question 3

Consider the sequence $(\mathrm{S}_n)_{n=0}^{\infty}$ for which $s_0 = s_1 = 1$ and for every integer $n \geq 2$:

$$s_n = s_{n-1} + 2s_{n-2}.$$

a) Calculate the value of s_6 :

$$\begin{aligned}
&s(0) = 1\\
&s(1) = 1\\
&s(2) = s(1) + 2s(0) = 1 + 2(1) = 3\\
&s(3) = s(2) + 2s(1) = 3 + 2(1) = 5\\
&s(4) = s(3) + 2s(2) = 5 + 2(3) = 11\\
&s(5) = s(4) + 2s(3) = 11 + 2(5) = 21\\
&s(6) = s(5) + 2s(4) = 21 + 2(11) = 43
\end{aligned}$$

b) Prove that all terms of the sequence $(\mathrm{S}_n)_{n=0}^{\infty}$ are odd.

Base cases: $S(0) = 1$, which is odd.
$S(1) = 1$, which is odd.

Inductive Hypothesis: $S(i)$ is odd for $0 \leq i \leq k$, for $k \geq 1$

Inductive Steps: If $S(i)$ is odd, for $0 \leq i \leq k$, for $k \geq 1$, then $S(k+1)$ is odd.

$S(k+1) = S(k) + 2 \cdot S(k-1)$

We know that $S(k)$ is odd, by the inductive hypothesis.

We know that $S(k-1)$ is odd, by the inductive hypothesis.

Therefore, $S(k+1) = (2p+1) + 2 \cdot (2q+1)$, for $p, q \in Z$.

$= (2p + 4q + 2) + 1$

Hence, $S(k+1)$ is odd.

This establishes that for every $n \in N^+$, if $S(n-1)$ is odd, and if $s(n-2)$ is odd, then $S(_n)$ is odd.

Question 4

Consider the sequence $(\mathrm{t}_n)_{n=0}^{\infty}$ for which $t_0 = 2$ and $t_1 = 1$ and for every integer $n \geq 2$:

$$t_n = t_{n-1} + t_{n-2}.$$

a) Calculate the value of t_6.
$t_6 = 18$.

$$\begin{aligned}
&t(0) = 2\\
&t(1) = 1\\
&t(2) = t(1) + t(0) = 1 + 2 = 3\\
&t(3) = t(2) + t(1) = 3 + 1 = 4\\
&t(4) = t(3) + t(2) = 4 + 3 = 7\\
&t(5) = t(4) + t(3) = 7 + 4 = 11\\
&t(6) = t(5) + t(4) = 11 + 7 = 18
\end{aligned}$$

b. Give an explicit formula for t_n:

$t_n = t_{n-1} + t_{n-2}$

Rewrite as: $t_n - t_{n-1} - t_{n-2} = 0$

Characteristic equation: $t^2 - t - 1 = 0$

Characteristic roots: $\dfrac{1 \pm \sqrt{1-4(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$.

$$t_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$t_0 = a\left(\frac{1+\sqrt{5}}{2}\right)^0 + b\left(\frac{1-\sqrt{5}}{2}\right)^0 = a + b = 2$$

$$t_1 = a\left(\frac{1+\sqrt{5}}{2}\right)^1 + b\left(\frac{1-\sqrt{5}}{2}\right)^1 = \frac{a+b}{2} + \frac{(a-b)\sqrt{5}}{2} = 1$$

Since a+b=2, the first term of the last equation $\dfrac{a+b}{2}$ is equal to $\dfrac{2}{2} = 1$.

And so the second term $\dfrac{(a-b)\sqrt{5}}{2}$ must be equal to 0.

Solving $a + b = 2$ when $a - b = 0$ results in $a = 1$ and $b = 1$.

Thus, $t_n = \left(\dfrac{1+\sqrt{5}}{2}\right)^n + \left(\dfrac{1-\sqrt{5}}{2}\right)^n$