Name: ____________________

Show your work.

1. Using truth tables, indicate whether each given statement is true or false.

(a) $\big(p \to \neg p\big) \to \big(q \to \neg q\big) \Longrightarrow \big(q \to p\big)$

(a) True

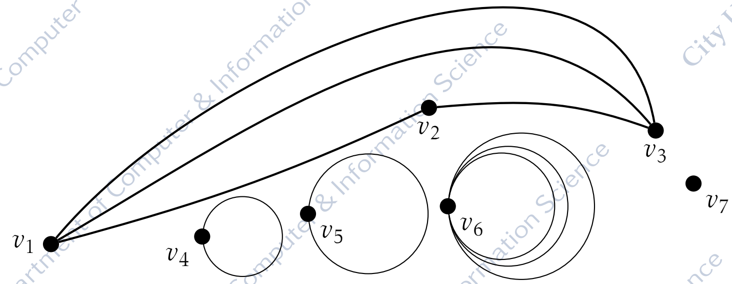
(b) $\big(p \leftrightarrow \neg p\big) \vee \neg\big(q \leftrightarrow \neg q\big) \Longrightarrow \big(\neg p \to p\big)$

(b) False

(c) $\neg p \Longleftrightarrow \Big(\big(\neg p \wedge q\big) \vee \neg\big(\neg p \to q\big)\Big)$

(c) True

2. Consider the pictured graph $\mathbf{G} = \left(V, E, \gamma\right)$:



(a) Supply the adjacency matrix for $\mathbf{G}$.

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Specify the number of paths from vertex v_1 to vertex v_2 of length 2.

(b) 2

(c) Where $\mathbf{M}$ denotes the adjacency matrix for $\mathbf{G}$, specify the value of $\mathbf{M}^7[1,7]$.

(c) 0

(d) Where $\mathbf{M}$ denotes the adjacency matrix for $\mathbf{G}$, specify the value of $\left(\mathbf{M}^T\right)^2[2,3]$.

(d) 2

(e) Specify the property that function γ satisfies.

(e) D

(A) onto and 1-1 (B) onto and not 1-1 (C) 1-1 and not onto (D) neither onto nor 1-1

3. Consider the function $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ such that for every $m, n \in \mathbb{N}$:

$$f(m,n) \quad = \quad 2^m(2n+1) - 1$$

(a) Indicate whether the following statement is true or false.

For some $s \in \mathbb{N}$, there are no $n, m \in \mathbb{N}$ such that $\big((m,n),\, s\big) \in \mathrm{Graph}(f)$.

(a) False

(b) Specify the property that function f satisfies.

(A) onto and 1-1 (B) onto and not 1-1 (C) 1-1 and not onto (D) neither onto nor 1-1

(b) A

(c) Specify whether each given set is finite, countably infinite, or uncountable; if a given set is finite, specify its cardinality.

(i) $\big(\mathbb{R} \times \mathbb{R} \setminus \mathbb{Q} \times \mathbb{Q}\big) \setminus \mathrm{Dom}(f)$.

(i) uncountably infinite

(ii) $\mathbb{R} \times \mathbb{R} \setminus \big(\mathbb{Q} \times \mathbb{Q} \setminus \mathrm{Dom}(f)\big)$

(ii) uncountably infinite

(iii) $f\big(\{0\} \times \mathbb{N}\big)$.

(iii) countably infinite

4. Consider a binary relation S on $\mathbb{R}\times\mathbb{R}$ such that for every $x_1, y_1, x_2, y_2 \in \mathbb{R}$:

$$(x_1, y_1)\,S\,(x_2, y_2) \quad \text{if and only if} \quad \text{either} \quad y_1 < y_2 \quad \text{or} \quad y_1 \le y_2 \ \text{ and } \ x_1 \le x_2$$

Indicate whether each given statement is true or false.

(a) The relation S is transitive.

(a) True

(b) The relation S is symmetric.

(b) False

(c) The relation S is reflexive.

(c) True

5. Consider sets $A = \{0, \beta, a, ?, @, \wedge, \text{U}, \Delta\}$ and $S = \{\$, @, \%, +, !, \#\}$.

(a) How many 3-element subsets of S are there?

(a) $\binom{6}{3} = 20$

(b) How many one-to-one functions are there from A to S?

(b) 0

(c) How many 2-permutations of A are there?

(c) $P(8,2) = 56$

6. An urn has three blue balls and four yellow balls. A set of three balls are to be removed at random from the urn without replacement.

(a) What is the probability that the three balls are all blue?

(a) $\frac{1}{35}$

(b) What is the probability that two of the balls are yellow and one of the balls is blue?

(b) $\frac{3\cdot\binom{4}{2}}{35} = \frac{18}{35}$

7. **Quiz 4 Problem**. Let $p(n)$ express the following proposition:

$$\sum_{m=1}^{n} \frac{1}{(3m+1)(3m-2)} \quad = \quad \frac{n}{3n+1}$$

To prove that proposition $p(n)$ is true for every natural number $n \in \mathbb{P}$ by mathematical induction, carry out the following steps.

(a) Specify the applicable base case.

(a) $p(1)$

(b) In the space below, establish the base case by mathematical argumentation.

Given $n \in \mathbb{P}$, let $\mathrm{L}(n)$ denote the expression on the lefthand side of equation $p(n)$, and let $\mathrm{R}(n)$ denote the expression on the righthand side of equation $p(n)$:

$$\begin{aligned} \mathrm{L}(n) &= \sum_{m=1}^{n} \frac{1}{(3m+1)(3m-2)} \\ \mathrm{R}(n) &= \frac{n}{3n+1}. \end{aligned}$$

Evidently, $\mathrm{L}(1) = 1/4 = \mathrm{R}(1)$, as desired. This establishes the base case.

(c) Specify the applicable inductive step.

(c) For every $k \in \mathbb{P}$, if $p(k)$ is true, then $p(k+1)$ is true.

(d) Specify the inductive hypothesis in the inductive step.

(d) $p(k)$

(e) In the space below, establish the inductive step by mathematical argumentation.

Let $k \in \mathbb{P}$. Suppose proposition $p(k)$ is true — that is, equality $\mathrm{L}(k) = \mathrm{R}(k)$ obtains. To verify that $\mathrm{L}(k+1) = \mathrm{R}(k+1)$, observe:

$$\begin{aligned}
\mathrm{L}\big(k+1\big) &= \sum_{m=1}^{k+1} \frac{1}{(3m+1)(3m-2)} \\
&= \mathrm{L}(k) + \frac{1}{(3(k+1)+1)(3(k+1)-2)} \\
&= \mathrm{R}(k) + \frac{1}{(3(k+1)+1)(3(k+1)-2)} \\
&= \frac{k}{3k+1} + \frac{1}{(3k+4)(3k+1)} \\
&= \frac{k(3k+4)+1}{(3k+4)(3k+1)} \\
&= \frac{3k^2+4k+1}{(3k+4)(3k+1)} \\
&= \frac{(k+1)(3k+1)}{(3k+4)(3k+1)} \\
&= \frac{k+1}{3k+4} \\
&= \frac{k+1}{3(k+1)+1} \\
&= \mathrm{R}\big(k+1\big).
\end{aligned}$$

This establishes that for every $k \in \mathbb{P}$, if proposition $p(k)$ is true, then proposition $p(k+1)$ is true.

Scratch.