Name: ______________________________

Show your work.

1. Consider the alphabet $\Sigma = \left\{ \mathbf{1}, \mathbf{0} \right\}$. Let $c : \mathbb{N} \longrightarrow \mathbb{N}$ be the function such that the value $c(n)$ for any given $n \in \mathbb{N}$ is the number of Σ-words of length n wherein the string **10** fails to appear.

   (a) Specify the indicated values.

   (i) $c(0)$

   (i) 1

   (ii) $c(1)$

   (ii) 2

   (iii) $c(2)$

   (iii) 3

   (iv) $c(19)$

   (iv) 20

   Observe that for each $n \in \mathbb{N}$:

$$\begin{aligned} c(n) &= \left| \bigcup_{m=0}^{n} \{\mathbf{0}\}^m \times \{\mathbf{1}\}^{n-m} \right| \\ &= \sum_{m=0}^{n} \left| \{\mathbf{0}\}^m \times \{\mathbf{1}\}^{n-m} \right| \\ &= n + 1. \end{aligned}$$

   (b) Let $p(n)$ express the following proposition:

$$\sum_{i=0}^{n} c(i) = \frac{(n+1)(n+2)}{2}.$$

   To prove that proposition $p(n)$ is true for every natural number $n \in \mathbb{N}$ by mathematical induction, carry out the following steps.

   (i) Specify the applicable base case.

   (i) $p(0)$

(ii) In the space below, establish the base case by mathematical argumentation.

Given $n \in \mathbb{N}$, let $\mathrm{L}(n)$ denote the expression on the lefthand side of equation $p(n)$, and let $\mathrm{R}(n)$ denote the expression on the righthand side of equation $p(n)$:

$$\begin{aligned} \mathrm{L}(n) &= \sum_{i=0}^{n} \mathrm{c}(i) \\ \mathrm{R}(n) &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Evidently, $\mathrm{L}(0) = 1 = \mathrm{R}(0)$, as desired.

(iii) Specify the applicable inductive step.

(iii) For every $k \in \mathbb{N}$, if $p(k)$ is true, then $p(k+1)$ is true.

(iv) Identify the inductive hypothesis in the given inductive step.

(iv) $p(k)$

(v) In the space below, establish the inductive step by mathematical argumentation.

Let $k \in \mathbb{N}$. Suppose proposition $p(k)$ is true — that is, equality $\mathrm{L}(k) = \mathrm{R}(k)$ obtains. To verify that $\mathrm{L}(k+1) = \mathrm{R}(k+1)$, observe that since $c(k+1) = k+2$, it follows:

$$\begin{aligned} \mathrm{L}\big(k+1\big) &= \sum_{i=0}^{k+1} \mathrm{c}(i) \\ &= \mathrm{L}(k) + c(k+1) \\ &= \mathrm{R}(k) + \big(k+2\big) \\ &= \frac{(k+1)(k+2)}{2} + \big(k+2\big) \\ &= (k+2)\left(\frac{k+1}{2} + 1\right) \\ &= (k+2)\frac{(k+1)+2}{2} \\ &= \frac{\big((k+1)+1\big)\big((k+1)+2\big)}{2} \\ &= \mathrm{R}\big(k+1\big). \end{aligned}$$

This establishes that for every $k \in \mathbb{N}$, if proposition $p(k)$ is true, then proposition $p(k+1)$ is true.

2. Consider the function $\Phi : \mathbb{N} \longrightarrow \mathbb{N}$ for which $\Phi(n) = \min\left(n!, \frac{(2n)!}{2^n}\right)$ for each $n \in \mathbb{N}$. Let $p(n)$ express the following proposition:

$$\Phi(n) \quad = \quad n!$$

To prove that proposition $p(n)$ is true for every natural number $n \in \mathbb{N}$ by mathematical induction, carry out the following steps.

(a) Specify the applicable base case.

(a) $p(0)$

(b) In the space below, establish the base case by mathematical argumentation.

Observe:

$$\begin{aligned} \Phi(0) &= \min\left(0!, \frac{(2 \cdot 0)!}{2^0}\right) \\ &= \min\left(0!, 0!\right) \\ &= 0!. \end{aligned}$$

This establishes the base case.

(c) Specify the applicable inductive step.

(c) For every $k \in \mathbb{N}$, if $p(k)$ is true, then $p(k+1)$ is true.

(d) Identify the inductive hypothesis in the given inductive step.

(d) $p(k)$

1

(e) In the space below, establish the inductive step by mathematical argumentation.

Let $k \in \mathbb{N}$. Suppose proposition $p(k)$ is true — that is, equality $\Phi(k) = k!$ obtains. Then $k! \leq \frac{(2k)!}{2^k}$. Now observe:

$$\begin{aligned} (k+1)! &= (k+1)k! \\ &\leq \frac{(k+1)(2k)!}{2^k} \\ &\leq \frac{(k+1)(2k+1)(2k)!}{2^k} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2(k+1)(2k+1)(2k)!}{2^{k+1}} \\
&= \frac{(2k+2)(2k+1)(2k)!}{2^{k+1}} \\
&= \frac{(2k+2)!}{2^{k+1}} \\
&= \frac{\big(2(k+1)\big)!}{2^{k+1}}.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\Phi\big(k+1\big) &= \min\left((k+1)!, \frac{\big(2\cdot(k+1)\big)!}{2^{k+1}}\right) \\
&= (k+1)!
\end{aligned}
$$

This establishes that for every $k \in \mathbb{N}$, if proposition $p(k)$ is true, then proposition $p(k+1)$ is true, as desired.