CISC 3220 Homework Assignment 2

Rachel Friedman

February 9, 2020

Question 1

Use mathematical induction to establish that for every positive integer n:

$$\sum_{i=1}^{n} \frac{1}{(4n+1)(4n-3)} = \frac{n}{4n+1}$$

Base case: $n = 1$

$$L(1) = \sum_{i=1}^{1} \frac{1}{(4(1)+1)(4(1)-3)} = \frac{1}{5}$$

$$R(1) = \frac{1}{4(1)+1} = \frac{1}{5}$$

Evidently, $L(1) = R(1)$

Inductive Step: For every $k \in N^+$, if $p(k)$ is true, then $p(k+1)$ is true.

Let $k \in N^+$. Suppose proposition $p(k)$ is true. Verify that $p(k+1)$ is true:

$$L(k+1) = \sum_{i=1}^{k+1} \frac{1}{(4(k+1)+1)(4(k+1)-3)}$$

$$= \sum_{i=1}^{k} \frac{1}{(4k+1)(4k-3)} + \frac{1}{(4(k+1)+1)(4(k+1)-3)}$$

$$= \frac{k}{4k+1} + \frac{1}{(4k+5)(4k+1)} \qquad \textit{By Inductive Hypothesis}$$

$$= \frac{k(4k+5)}{(4k+5)(4k+1)} + \frac{1}{(4k+5)(4k+1)}$$

$$= \frac{4k^2+5k+1}{(4k+5)(4k+1)} = \frac{(4k+1)(k+1)}{(4k+5)(4k+1)} = \frac{k+1}{4k+5}$$

$$= \frac{k+1}{4(k+1)+1}$$

$$= R(k+1)$$

This establishes that for every $k \in N^+$, if proposition $p(k)$ is true, then proposition $p(k+1)$ is true.

Question 2

Use mathematical induction to establish $7^n - 2^n$ is divisible by 5 for every positive integer n.

Base case: $n = 1$

$$7^1 - 2^1 = 5, \text{which is clearly divisible by 5}$$

Inductive Step: For every $k \in N^+$, if $p(k)$ is true, then $p(k+1)$ is true.

Let $k \in N^+$. Suppose proposition $p(k)$ is true. Verify that $p(k+1)$ is true:

$$7^{k+1} - 2^{k+1}$$
$$= (5+2) \cdot 7^k - 2 \cdot 2^k$$
$$= 5 \cdot 7^k + 2(7^k - 2^k)$$

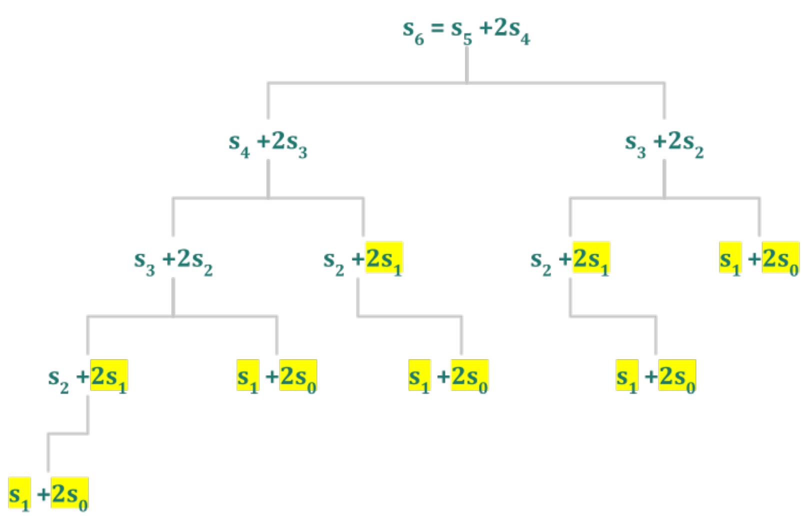
The first term $(5 \cdot 7^k)$ is being multiplied by 5, and is therefore divisible by 5.
The second term $(2(7^k - 2^k))$ is divisible by 5, according to the inductive hypothesis.
Hence, $7^n - 2^n$ is divisible by 5 for every positive integer n.

Question 3

Consider the sequence $(\mathrm{S}_n)_{n=0}^{\infty}$ for which $s_0 = s_1 = 1$ and for every integer $n \geq 2$:

$$s_n = s_{n-1} + 2s_{n-2}.$$

a) Calculate the value of s_6.
$s_6 = 21$. See diagram of recursive calls below:



b) Prove that all terms of the sequence $(\mathrm{S}_n)_{n=0}^{\infty}$ are odd.

Since $s_0 = 1$, this recurrence relation begins with the number 1 (which is odd) and then adds an even amount $(2s_{n-2})$ to that number. Adding an even number to an odd number results in an odd number. Therefore, the result will be odd. Adding an even number to that odd number will also be odd. And so on and so forth...

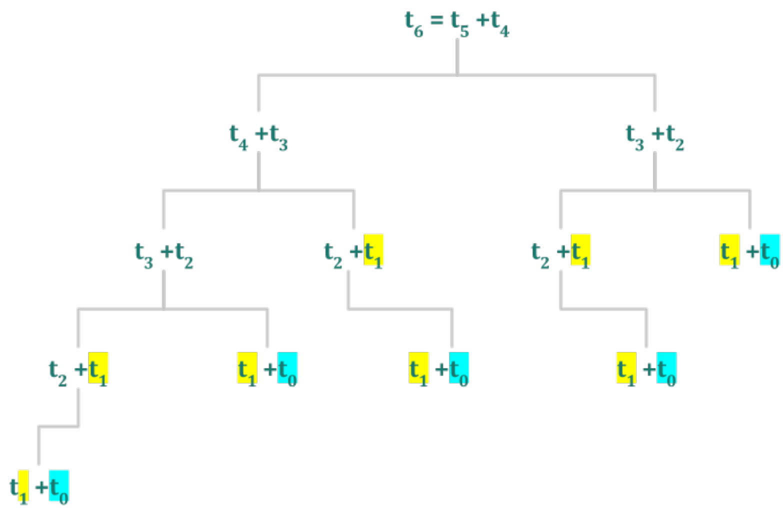
Question 4

Consider the sequence $(t_n)_{n=0}^{\infty}$ for which $t_0 = 2$ and $t_1 = 1$ and for every integer $n \geq 2$:

$$t_n = t_{n-1} + t_{n-2}.$$

a) Calculate the value of t_6.

$t_6 = 18$. See diagram of recursive calls below:



b. Give an explicit formula for t_n:

$t_n = t_{n-1} + t_{n-2}$

Rewrite as: $t_n - t_{n-1} - t_{n-2} = 0$

Characteristic equation: $t^2 - t - 1 = 0$

Characteristic roots: $\dfrac{1 \pm \sqrt{1-4(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$.

$$t_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$t_0 = a\left(\frac{1+\sqrt{5}}{2}\right)^0 + b\left(\frac{1-\sqrt{5}}{2}\right)^0 = a + b = 2$$

$$t_1 = a\left(\frac{1+\sqrt{5}}{2}\right)^1 + b\left(\frac{1-\sqrt{5}}{2}\right)^1 = \frac{a+b}{2} + \frac{(a-b)\sqrt{5}}{2} = 1$$

Since a+b=2, the first term of the last equation $\dfrac{a+b}{2}$ is equal to $\dfrac{2}{2} = 1$.

And so the second term $\dfrac{(a-b)\sqrt{5}}{2}$ must be equal to 0.

Solving $a + b = 2$ when $a - b = 0$ results in $a = 1$ and $b = 1$.

Thus, $t_n = \left(\dfrac{1+\sqrt{5}}{2}\right)^n + \left(\dfrac{1-\sqrt{5}}{2}\right)^n$