CISC 3220 Homework Chapter 3

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Problem 3-4

Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures.

Question a

$f(n) = \mathcal{O}(g(n))$ implies $g(n) = \mathcal{O}(f(n))$.

False. Counterexample:

Let $f(n)$ be n.

Let $g(n) = n^2$.

There exists a constant c, $\forall n \geq n_0$ such that $n \leq cn^2$ but there does not exist a constant $c, \forall n \geq n_0$ such that $n^2 \leq cn$.

Thus, $n = \mathcal{O}(n^2)$ but $n^2 \neq \mathcal{O}(n)$.

Question b

$f(n) + g(n) = \Theta(min(f(n), g(n)))$.

$f(n) + g(n) = \Theta(min(f(n), g(n)))$ implies that there exists a constant c_1 and a constant c_2, for all $n \geq n_0$ such that:

$c_1 \leq (min(f(n), g(n)) \leq f(n) + g(n) \leq c_2 \cdot min((f(n), g(n))$

False. Counterexample:

Let $f(n) = n$ and $g(n) = 1$

There is no c_2 such that $n + 1 \leq c_2 \cdot min(n, 1)$

So, $f(n) + g(n) \neq \Theta(min(f(n), g(n)))$

Question c

$f(n) = \mathcal{O}(g(n))$ implies $lg(f(n)) = \mathcal{O}(lg(g(n)))$, where $lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n.

True:

We need to prove that $lg(f(n) = \mathcal{O}(lg(g(n)))$, or in other words $lg(f(n)) \leq d \cdot lg(g(n))$.

The first equation, as given:

$f(n) = \mathcal{O}(g(n))$ implies that $f(n) \leq cg(n)$.

Taking the log of both sides, we get: $lg(f(n) \leq lg(c) + lg(g(n))$

Now, set $d \cdot lg(g(n)) = lg(c) + lg(g(n))$

Divide both sides by $lg(g(n))$, we have $d = \frac{lg(c)}{lg(g)} + 1$

Plugging this value of d into the equation, we have: $lg(f(n)) \leq \left(\frac{lg(c)}{lg(g(n))} + 1\right) \cdot lg(g(n))$

Since $lg(g(n)) \geq 1,\ lg(c) \geq \frac{lg(c)}{lg(g(n))}$, so clearly $lg(f(n)) \leq (lg(c) + 1) \cdot lg(g(n))$

And so, $lg(f(n)) \leq d \cdot lg(g(n))$, thus proving that $lg(f(n)) = \mathcal{O}(lg(g(n)))$.

Question d

$f(n) = \mathcal{O}(g(n))$ implies $2^{f(n)} = \mathcal{O}(2^{g(n)})$.

False. Counterexample:

Let $f(n) = 2n$ and $g(n) = n$.

So $2^{2n} = \mathcal{O}(2^n)$ implies that $2^{2n} \leq c \cdot 2^n$

$$2^{2n} \leq c \cdot 2^n = (2^2)^n \leq c \cdot 2^n = 4^n \leq c \cdot 2^n$$

The value of c is dependent on n, so there is no value of c such that $c \geq 2n$ for all $n \geq n_0$. Hence, this statement is false.

Question e

$f(n) = \mathcal{O}((f(n))^2)$.

False.

$f(n) = \mathcal{O}((f(n))^2)$ implies that $f(n) \leq c(f(n))^2$.

But when $0 \leq f(n) \leq 1$, then there is no value of c such that $f(n) \leq c \cdot (f(n))^2, \forall n \geq n_0$.

Hence, this statement is false.

Question f

$f(n) = \mathcal{O}(g(n))$ implies $g(n) = \Omega(f(n))$.

True.

$f(n) = \mathcal{O}(g(n))$ implies that there exists a constant c such that $f(n) \leq cg(n), \forall n \geq n_0$.

Divide both sides by c: $\frac{1}{c}f(n) \leq g(n)$

Thus, $g(n) \geq \frac{1}{c}f(n)$, which means that $g(n) = \Omega(f(n))$.

Question g

$f(n) = \Theta(f(n/2))$.

$f(n) = \Theta(f(n/2))$ implies that there exists two constants such that:

$c_1 \cdot f(n/2) \leq f(n) \leq c_2 \cdot f(n/2)$

False. Counterexample:

Let $f(n) = 2^n$

There is no c such that: $2^n \leq c_2 \cdot 2^{n/2}$, for all of $n \geq n_0$. So this statement is false.

Question h

$f(n) + o(f(n)) = \Theta(f(n))$

True:

Let $g(n) = o(f(n))$

This implies that $g(n) < f(n)$

$f(n) + g(n) = \Theta(f(n))$ implies that there exists a constant c_1 and a constant c_2 such that $c_1 f(n) \leq f(n) + g(n) \leq c_2 f(n)$, for all $n \geq n_0$

Let $c_1 = 1$ and let $c_2 = 2$

It is true that $f(n) \leq f(n) + g(n) \leq 2(f(n))$

Thus, it is true that $f(n) + o(f(n)) = \Theta(f(n))$.

Class Exercise

Prove $f(n) = \mathcal{O}(n^2)$, when $f(n) = an + b$, for $a > 0$.

$f(n) = \mathcal{O}(n^2)$ implies that there is a constant $c, \forall n \geq n_0$ such that $f(n) \leq cn^2$.

We need to prove that $an + b \leq cn^2$.

Let $n = 1$ and $c = a + b^2$.

So we have: $\forall n \geq 1,\ an + b \leq (a + b^2) \cdot n^2$

$$= an + b \leq an^2 + b^2 n^2$$

Thus proving that $f(n) = \mathcal{O}(n^2)$, when $f(n) = an + b$.