BRADLEY
University

ECE 302: Probability, Statistics, and Random Processes for EE

Fall 2022

Assignment 2: Discrete Random Variables

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Problem 3.1

$\boxed{3.1}$ $2^4 = 16$ possibilities

a.) $S = \{(HH, HH), (HH, HT), (HH, TH), (HH, TT)$
$(TH, HH), (TH, HT), (TH, TH), (TH, TT)$
$(HT, HH), (HT, HT), (HT, TH), (HT, TT)$
$(TT, HH), (TT, HT), (TT, TH), (TT, TT)\}$

probability of elementary elements can be found with cardinality, $\therefore \frac{1}{16}$

b.) $S_x = ?$ *max # of heads obtained

(HH, HH) → 2
(HH, HT) → 2
(HH, TH) → 2
(HH, TT) → 2
(TH, HH) → 2
(TH, HT) → 1
(TH, TH) → 1
(TH, TT) → 1
(HT, HH) → 2
(HT, HT) → 1
(HT, TH) → 1
(HT, TT) → 1
(TT, HH) → 2
(TT, HT) → 1
(TT, TH) → 1
(TT, TT) → 0

$S_x = \{0, 1, 2\}$

c.) $P_r[X = x] \quad x \in S_x$

$P_r[x=0] = P_r[\{TT, TT\}] = \frac{1}{16}$

$P_r[x=1] = P_r[\{(TH, HT), (TH, TH), (TH, TT), (HT, HT), (HT, TH), (HT, TT), (TT, HT), (TT, TH)\}]$
$= \frac{8}{16} = \frac{1}{2}$

$P_r[x=2] = [\{(TT, HH), (HT, HH), (TH, HH), (HH, TT), (HH, TH), (HH, HT), (HH, HH)\}]$
$= \frac{7}{16}$

$$P_r[X=x] \begin{cases} \frac{1}{16} & x=0 \\ \frac{1}{2} & x=1 \\ \frac{7}{16} & x=2 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3.6

3.6

a.)

P_r	S		S_x
$1/4$	000	$\rightarrow$	$-\log_2(\frac{1}{4}) = 2$
$1/4$	111	$\rightarrow$	$-\log_2(\frac{1}{4}) = 2$
$1/8$	010	$\rightarrow$	$-\log_2(1/8) = 3$
$1/8$	101	$\rightarrow$	$-\log_2(1/8) = 3$
$\frac{1}{16}$	001	$\rightarrow$	$-\log_2(1/16) = 4$
$1/16$	110	$\rightarrow$	$-\log_2(1/16) = 4$
$1/16$	100	$\rightarrow$	$-\log_2(1/16) = 4$
$1/16$	011	$\rightarrow$	$-\log_2(\frac{1}{16}) = 4$

$S_x = \{2, 3, 4\}$

$-\log_2 P_r$

b.) $P_r[X=x]$ for $x \in S_x$

$P_r[X=2] = P_r[\{000, 111\}] = \frac{2}{8} = \frac{1}{4}$

$P_r[X=3] = P_r[\{010, 101\}] = \frac{2}{8} = \frac{1}{4}$

$P_r[X=4] = P_r[\{001, 110, 100, 011\}] = \frac{4}{8} = \frac{1}{2}$

$$P_r[X=x] = \begin{cases} 1/4 & x = 2, 3 \\ \frac{1}{2} & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3.11

3.11

a.) $P_x[x=0] = \frac{1}{16}$

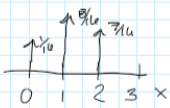
$P_x[x=1] = \frac{8}{16}$

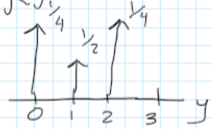
$P_x[x=2] = \frac{7}{16}$

$P_y[y=0] = \frac{1}{4}$

$P_y[y=1] = \frac{1}{2}$

$P_y[y=2] = \frac{1}{4}$





pmf of x is smaller than pmf of y because it has ony one coin + one person tossing it, so less probability.

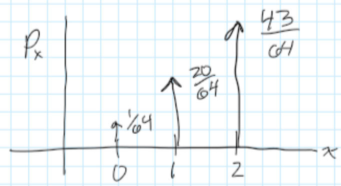
b.) $p = 3/4$ → tails $= 1/4 = q$ $\quad q + p = 1$

heads

$P_r[x=0] = \frac{1}{4}\left(\frac{1}{16}\right) = \frac{1}{64}$

$P_r[x=1] = \frac{1}{16}\left(\frac{1}{2}\right) + \frac{6}{16}\left(\frac{1}{2}\right) + \frac{6}{16}\left(\frac{1}{4}\right) = \frac{20}{64}$

$P_r[x=2] = \frac{1}{16}\left(\frac{1}{4}\right) + \left(\frac{6}{16}\right)\left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)\left(\frac{1}{4}\right) + \frac{9}{16}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{9}{16}\right) = \frac{43}{64}$



Problem 3.14

$\boxed{3.4}$ 4 digits independent

$2^4 = 16$

a.) $S = \{1111, 1110, 1101, 1100,$
$1011, 1010, 1001, 1000$
$0111, 0110, 0101, 0100$
$0011, 0010, 0001, 0000\}$

elementary probability is $\frac{1}{16}$

b.) $S_x = ?$

$0000 \rightarrow 0$
$0001 \rightarrow 1$
$0010 \rightarrow 2$
$0011 \rightarrow 3$
$0100 \rightarrow 4$
$0101 \rightarrow 5$
$0110 \rightarrow 6$
$0111 \rightarrow 7$
$1000 \rightarrow 8$
$1001 \rightarrow 9$
$1010 \rightarrow 10$
$1011 \rightarrow 11$
$1100 \rightarrow 12$
$1110 \rightarrow 14$
$1111 \rightarrow 15$

$S_x = \{1, 2, 3, 4, \ldots, 15\}$

c.) since, all characters are equally likely to be 0 or 1 $\therefore \frac{1}{16}$

$Pr[x=0] = \frac{1}{16}$
$Pr[x=1] = \frac{1}{16}$
$Pr[x=2] = \frac{1}{16}$
$\vdots$
$Pr[x=15] = \frac{1}{16}$

d.) $P_0 + P_1 = 1$

$\frac{1}{3}P_0 = P_1$

$P_0 + \frac{1}{3}P_0 = 1$

$\frac{4}{3}P_0 = 1$

$\boxed{P_0 = \frac{3}{4}}$

$\boxed{P_1 = \frac{1}{4}}$

$0000 \rightarrow \frac{3}{4}\cdot(\frac{1}{2})^3 = \frac{3}{32}$
$0001 \rightarrow \frac{3}{4}\cdot(\frac{1}{2})^3 = \frac{3}{32}$
$0010 \rightarrow \frac{3}{4}(\frac{1}{2})^3 = \frac{3}{32}$
$0011 \rightarrow \frac{3}{4}(\frac{1}{2})^3 = \frac{3}{32}$
$0100 \rightarrow \frac{3}{32}$
$0101 \rightarrow \frac{3}{32}$
$0110 \rightarrow \frac{3}{32}$
$0111 \rightarrow \frac{3}{32}$
$1000 \rightarrow \frac{1}{4}(\frac{1}{2})^3 = \frac{1}{32}$
$1001 \rightarrow \frac{1}{4}(\frac{1}{2})^3 = \frac{1}{32}$
$1010 \rightarrow \frac{1}{32}$
$1011 \rightarrow \frac{1}{32}$
$1100 \rightarrow \frac{1}{32}$
$1110 \rightarrow \frac{1}{32}$
$1111 \rightarrow \frac{1}{32}$

$\boxed{3.14}$

$$P_r[X \geq 8] = \sum_{k=8}^{15} P_k = \frac{1}{16} \times 8 = \frac{1}{2}$$

$$P_r[Y \geq 8] = \sum_{k=8}^{15} P_k = 8 \times \frac{3}{32} = \frac{3}{4}$$

$P_r[X \geq 8]$ is larger than $P_r[Y \geq 8]$

Problem 3.19

$\boxed{3.19}$ 0 or 1

$p = \frac{1}{10}$ $n = 5$ trials

≥ 3 bits

binomial$(5, \frac{1}{10})$

$$P_r[x \geq 3] = \binom{5}{3}\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^2 + \binom{5}{4}\left(\frac{1}{10}\right)^4\left(\frac{9}{10}\right)^1 + \binom{5}{5}\left(\frac{1}{10}\right)^5$$

$= 0.00856$

Problem 3.22

$\boxed{3.12}$ $S_x = \{1, 2, 3, 4\}$

a.) $P_k = \dfrac{P_1}{k} \quad k \in S_x$

$1 = \sum_{k=1}^{4} P_k = \sum_{k=1}^{4} \frac{P_1}{k} = P_1 \sum_{k=1}^{4} \frac{1}{k} = P_1\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$ ← pmf

$1 = P_1\left(\frac{25}{12}\right)$

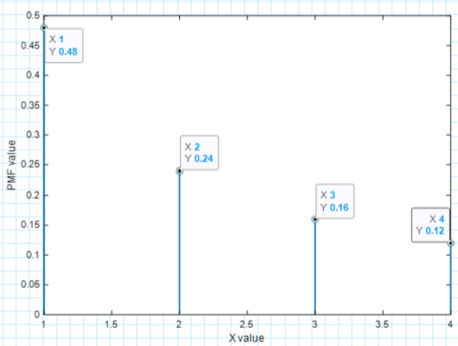
$P_1 = \frac{12}{25}$

$P_2 = \dfrac{\frac{12}{25}}{2} = \frac{6}{25}$

$P_3 = \dfrac{\frac{12}{25}}{3} = \frac{4}{25}$

$P_4 = \dfrac{\frac{12}{25}}{4} = \frac{3}{25}$

$$P_x(x) = \begin{cases} \frac{12}{25} & x = 1 \\ \frac{6}{25} & x = 2 \\ 4/25 & x = 3 \\ 3/25 & x = 4 \end{cases}$$



b.) ? $P_{k+1} = \dfrac{P_k}{K} \quad k = 2, 3, 4$

$P_2 = \frac{P_1}{1}$

$P_3 = \frac{P_2}{2} = \frac{P_1}{2} \times \frac{1}{2} = \frac{P_1}{4}$

$P_4 = \frac{P_3}{3} = \frac{P_1}{4} \times \frac{1}{2} = \frac{P_1}{8}$

$\sum_{k \in S_x} P_k = 1$

$= P_1 + \frac{P_1}{2} + \frac{P_1}{4} + \frac{P_1}{8} = 1$

$= P_1\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = 1$

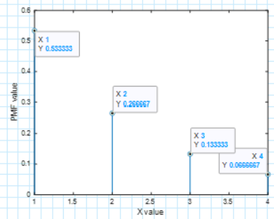
$P_1\left(\frac{15}{8}\right) = 1 \Rightarrow P_1 = \frac{8}{15}$

$P_2 = \frac{4}{15}$

$P_3 = \frac{2}{15}$

$P_4 = \frac{1}{15}$

$$P_x(x) = \begin{cases} \frac{8}{15} & x = 1 \\ 4/15 & x = 2 \\ 2/15 & x = 3 \\ 1/15 & x = 4 \end{cases}$$



c.) $k = 2, 3, 4$

$P_{k+1} = \frac{P_k}{2^k}$

$P_2 = \frac{P_1}{2}$

$P_3 = \frac{P_2}{4} = \frac{P_1}{8}$

$P_4 = \frac{P_1}{64}$

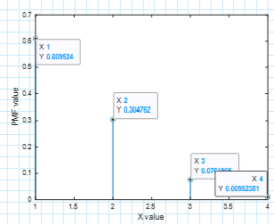
$\Rightarrow \sum_{k \in S_x} P_k = 1 = P_1\left(1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{64}\right) = 1$

$P_1 = \frac{64}{105}$

$P_2 = \frac{32}{105}$

$P_3 = \frac{8}{105}$

$P_4 = 1/105$



d.) $S_x = \{1, 2, 3, \ldots\} \quad k \geq 1$

$P_k = \frac{P_1}{k}$

$\sum_{k \in S_x} P_k = 1 = \sum_{k=1}^{\infty} \frac{P_1}{k}$

$P_{k+1} = \frac{P_k}{2} \Rightarrow \frac{P_{k+1}}{P_k} = \frac{1}{2} \quad \frac{P_k}{P_{k-1}} + \frac{P_{k+1}}{P_k} \cdots = \frac{1}{2} \times \frac{1}{2} \cdots$

$\therefore \frac{P_k}{P_1} = \frac{1}{2^{k-1}} \Rightarrow P_k = \frac{P_1}{2^{k-1}}$

geometric

$\Rightarrow \sum_{k=1}^{\infty} \frac{P_1}{2^{k-1}} \Rightarrow P_1 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = P_1 \frac{1}{1 - 1/2} = 1$

$P_1 = \frac{1}{2}$

$1 = \left(1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{1+2} + \left(\frac{1}{2}\right)^{1+2+3} + \ldots\right)$

$1 = P_1 \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{\frac{k(k-1)}{2}}$

* convergent b/c geometric series

$P_k = \dfrac{P_1}{2^{\frac{k(k-1)}{2}}}$

$$P(x = k) = \begin{cases} \dfrac{P_1}{2^{\frac{k(k-1)}{2}}} & k = \{1, 2, \ldots\} \\ 0 & k = 0 \end{cases}$$

3.22

a.) $E[X] = 1\left(\frac{12}{25}\right) + 2\left(\frac{6}{25}\right) + 3\left(\frac{4}{25}\right) + 4\left(\frac{3}{25}\right) = \frac{48}{25} = 1.92$

$VAR[X] = \sum_{x \in S_X} (x - \mu_x)^2 p_x(x) = E[X^2] - \mu_x^2$

$E[X^2] = 1\left(\frac{12}{25}\right) + 4\left(\frac{6}{25}\right) + 9\left(\frac{4}{25}\right) + 16\left(\frac{3}{25}\right) = \frac{120}{25}$

$VAR[X] = \frac{120}{25} - 1.92^2 = \boxed{1.114}$

b.) $E[X] = 1\left(\frac{8}{15}\right) + 2\left(\frac{4}{15}\right) + 3\left(\frac{2}{15}\right) + 4\left(\frac{1}{15}\right) = \frac{26}{15}$

$E[X^2] = 1\left(\frac{8}{15}\right) + \frac{16}{15} + \frac{18}{15} + \frac{16}{15} = \frac{58}{15}$

$VAR[X] = \frac{58}{15} - \left(\frac{26}{15}\right)^2 \approx 0.862$

c.) $E[X] = 1\left(\frac{64}{105}\right) + 4\left(\frac{32}{105}\right) + \left(\frac{8}{105}\right)9 + 16\left(\frac{1}{105}\right) = \frac{280}{105}$

$E[X^2] = 1\left(\frac{64}{105}\right) + 9\left(\frac{8}{105}\right) + 16\left(\frac{1}{105}\right) = \frac{280}{105}$

$VAR[X] \approx \frac{280}{105} - \left(\frac{156}{105}\right)^2 \approx 0.459$

Problem 3.26

$\boxed{3.9}$?

a.) $k = \#$ of tails $0 \le k \le n$

$y = n - k - k = n - 2k$

* # of heads in n-k and the difference

$\therefore S_y = \{-n, -n+2, \ldots, n-2, n\}$

b.) $P_r[Y=0] = P_r[n = 2k] = P_r[\frac{n}{2} = k]$ *n must be even

c.) $P_r[Y=m] = P_r[n-2k=m] = P_r[m = \frac{n-k}{2}]$

* for n-k even

$\sum y P_r(y=y)$

$= y_1 P_1 + y_2 P_2$

$= (0 \times \frac{n}{2}) + (k \times \frac{n-k}{2})$

$= 0 + \frac{k(n-k)}{2}$

$\boxed{\therefore E(Y) = \frac{k(n-k)}{2}}$

E[Y] is the difference of the number of heads and tails

$VAR(Y) = E[Y^2] - (E[Y])^2$

$E[Y^2] = \sum y^2 p(y=y$

$= y_1^2 P_1 + y_2^2 P_2$

$= \cancel{(0^2 \times \frac{n}{2})}^{0} + (k^2 \times (\frac{n-k}{2})$

$= \frac{k^2(n-k)}{2}$

$\therefore E[Y^2] = \frac{k^2(n-k)}{2}$

$(E[Y^2])^2 = \frac{k^2(n-k)^2}{4}$

$VAR[Y] = \frac{k^2(n-k)}{2} - \frac{k^2(n-k)^2}{4}$

$\boxed{= \frac{k^2(n-k)}{2}\left[1 - \frac{(n-k)}{2}\right]}$

$$E[Y] = \sum_{k=-5}^{5} kP[Y=k] = -5\left(\frac{1}{36}\right) - 4\left(\frac{2}{36}\right) - 3\left(\frac{3}{36}\right) - 2\left(\frac{4}{36}\right)$$

$$-1\left(\frac{5}{36}\right) + 0 + \frac{5}{36} + 2\left(\frac{4}{36}\right) + 3\left(\frac{3}{36}\right) + 4\left(\frac{2}{36}\right) + 5\left(\frac{1}{36}\right)$$

$$= 0$$

$$VAR[Y] = 1\left(\frac{10}{36}\right) + 4\left(\frac{8}{36}\right) + 9\left(\frac{6}{36}\right) + 16\left(\frac{4}{36}\right) + 25\left(\frac{1}{36}\right)$$

$$= \frac{185}{36}$$

$$E[Y] = \sum_{i} i\, p_Y(i)$$

$$= \sum_{y=} i\, p_Y[Y=i] =$$

$$Var[Y] = E[Y^2]$$

$$= \sum_{i=} i^2 p_Y(i)$$

$$= \sum_{i} i^2 p_Y(i)$$

Problem 3.34

3.34

$M = 2^m$ ← casion dollar

a.) $Y = 2^x$ ← gambler dollars

$2^x \leq 2^m$

$Y \leq M$

∴ # of tosses, x should be less than or equal to M dollars

b.) $E[Y] = \sum_{k=1}^{m} 2^k \left(\frac{1}{2}\right)^k$

$= \sum_{k=1}^{m} 1 = m$

c.) the player should not pay anything more than m dollars, else he will bear a loss.

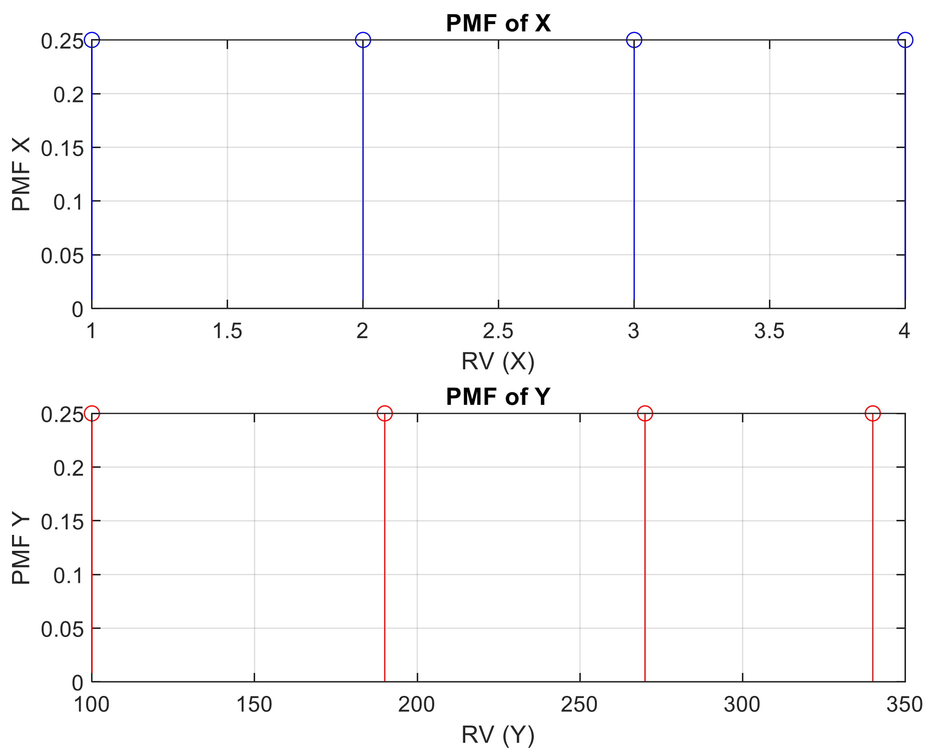
∴ to win, player must play for less than m dollars

Computer Experiments

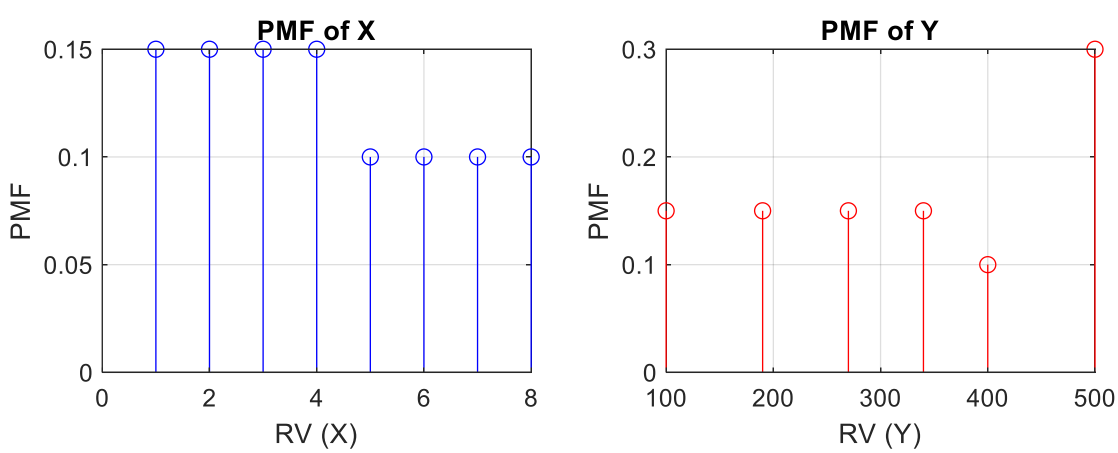
1.

```
100
```

2.



3.



4.

```
 190   270   340   340     0   340     0   400     0   190
```

5.

```
   325

   32.5000

    3.2500
```

As m increases, the average cost decreases.

MATLAB Solutions

```
%Gottschalk, Rachel ECE 302: Assignment #2
close all;
clear all;
clc;

%%%%%%%%%%%%%%%% 1 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% RV X = Number of pounds rounded
% RV Y = Charge in Cents for sending 1 package --- use function y=g(x)
% $1 = 1pound, $0.90 = 2pound, $0.80 = 3pounds, $0.70 = 4pounds,
% $0.60 = 5pounds, $5.00 6pounds<=X<=10pounds, X>10 = will not accept4

x = 10*rand; % select rand numbers for x
disp(g(1)); % pass 5 through the function to show that it is grabbing the x values and then
plugging them into g(x)

%%%%%%%%%%%%%%%% 2 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Sx = [1 2 3 4]; % initialize range of x
Sy = [100 190 270 340]; % initialize range of y

% loop going each of the Sx and Sy to calculate the PMF of them
for i = 1:length(Sx)
    %%%Px(x) x-g(x)
    % Discrete Uniform RV - X = 1,2,3,4 - so use 1/(l-k+1)
    px(i) = 1/(length(Sx)-Sx(1)+1);
    % Discrete Uniform RV - Y = 1, 1.9, 2.7, 3.4 - so use 1/(l-k+1)
    py(i) = 1/(length(Sy)-1+1);
end

% plotting the PMG of Px and Py
figure(1)
subplot(2,1,1)
stem(Sx,px,'b')
title('PMF of X')
xlabel('RV (X)')
ylabel('PMF X')
hold on;
grid on;
subplot(2,1,2)
stem(Sy,py,'r')
title('PMF of Y')
xlabel('RV (Y)')
ylabel('PMF Y')
grid on;
hold on;

Ey=0;
% loop through Sy to calulate expected values
for i = 1:length(Sy)
    Ey = Ey + Sy(i)*py(i);
end
```

```
%%%%%%%%%%%%%%%% 3 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Sx3 = 1:8; % intializing range of RV x to be from 1 to 8
Sy3 = [100 190 270 340 400 500]; % initializing range of RV y

py=(zeros(size(Sy3))); % setting values (number is the amount of Sy3) of py to all zeros and
Ey=0; % initializing expected value of y to 0

py5=0;
% looping through all of the values in the range of Sx3
for i = 1:length(Sx3)
    if 0<i && i<5 % checking to see if value of Sx3 is greater than 0 and less than 5
        px(i)= 0.15;
        py(i) = px(i);
    elseif 5<i && i<9 % checks if i is greater than 5 and less than 9
        px(i)=0.1;
        py5=py5+0.1;
    else % if i = 5
        px(i) = 0.1;
        py(i) = px(i);
    end
   py(6)=py5;
end

Ey = (Sy3(1) + Sy3(2) + Sy3(3) + Sy3(4))*px(1) + Sy3(5)*px(6)+Sy3(6)*py(6); % expected value
of y

figure(2)
subplot(2,2,1)
stem(Sx3,px,'b')
title('PMF of X')
xlabel('RV (X)')
ylabel('PMF')
hold on;
grid on;
subplot(2,2,2)
stem(Sy3,py,'r')
title('PMF of Y')
xlabel('RV (Y)')
ylabel('PMF')
hold on;
grid on;

%%%%%%%%%%%%%%%% 4 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp(shipweight8(10));

%%%%%%%%%%%%%%%% 5 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp(avg(10));
disp(avg(100));
disp(avg(1000));


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%% 5 - Function %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
function y=avg(m)
    s = cumsum([100 90 80 70 60 50]); % array of cumulative sum starting at the beginning of
the first array
    y2 = [s 500 500 500]; % places s in array y2
    y1=sum(y2); % sums the y2 vector
    y=y1/m; % divides y1 by m to get the average
end

%%%%%%%%%%%%%%%% 4 - Function %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [y,x] = shipweight8(m)
    rx = [1 1 1 2 2 2 3 3 3 4 4 4 5 5 6 6 7 7 8 8]; % takes in account that 3/20 is the same
probility and then we normalize
    x=zeros(1,m); % puts zeros in x for all m
    y=zeros(1,m); % puts zeros in y for all m

    for n = 1:m
        x(n)=rx(randi(length(rx))); % picks randi number of the length of rx and grabs that
correspodning number in rx
        y(n)=g(x(n)); % sends value of x(n) to g(x) function and returns corresponding y
value
    end

% in class:
%     sx = 1:8;
%     p = [0.15 0.15 0.15 .15 0.1 0.1 0.1 0.1];
%
%     c =cumsum([0,p(:).']);
%     c = c/c(end); % take last value of index and divide entire cummlative sum
%
%     randNo = rand(1,m);
%     [N,i]=histc(randNo,c);
%     x=sx(i); % map to v values
end


%%%%%%%%%%%%%%%% 1 - Function %%%%%%%%%%%%%%%%%%

function y = g(x)
    x = ceil(x);

    if x > 10
        y = NaN;
    end

    switch x
        case 1
            y = 100;
        case 2
            y = 190;
        case 3
            y = 270;
        case 4
            y = 340;
```

```
        case 5
            y = 400;
        case 5<x && x<11
            y= 500;
        otherwise
            y= 0;
            disp("Pounds not Accepted");

    end
end
```