BRADLEY University

ECE 302: Probability, Statistics, and Random Processes for EE

Fall 2022

Assignment 5: One Random Variables

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Problem 5.16

5.1 a)

(HH, HH) (2,2)
(HH, HT) (2,1)
(HH, TH) (2,1)
(HH, TT) (2,0)
(HT, HH) (2,1)
(HT, HT) (1,1)
(HT, TH) (1,1)
(HT, TT) (1,0)
(TH, HH) (2,1)
(TH, HT) (1,1)
(TH, TH) (1,1)
(TH, TT) (1,0)
(TT, HH) (2,0)
(TT, HT) (1,0)
(TT, TH) (1,0)
(TT, TT) (0,0)

b.) $Pr[(i,j)] = \frac{1}{16}$

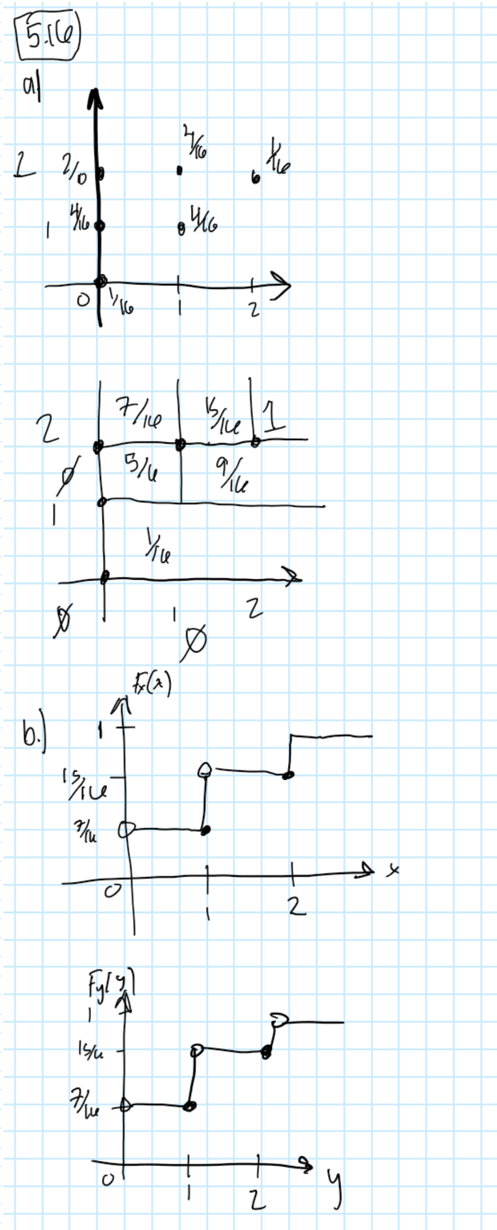
$Pr[(0,0)] = \frac{1}{16}$
$Pr[(1,0)] = \frac{4}{16} = \frac{1}{4}$
$Pr[(1,1)] = \frac{1}{4}$
$Pr[(2,0)] = \frac{1}{8}$
$Pr[(2,1)] = \frac{1}{4}$
$Pr[(2,2)] = \frac{1}{16}$

c.) $Pr[X=Y] = Pr[\{(0,0),(1,1),(2,2)\}] = \frac{1}{16} + \frac{1}{4} + \frac{1}{16}$
$= \frac{3}{8}$

d.) $P[H] = \frac{3}{4}$
$P[T] = 1 - \frac{3}{4} = \frac{1}{4}$

$Pr[(0,0)] = \frac{1}{64}$
$Pr[(1,0)] = \frac{1}{8}$
$Pr[(1,1)] = \frac{3}{16}$
$Pr[(2,0)] = \frac{5}{32}$
$Pr[(2,1)] = \frac{3}{8}$
$Pr[(2,2)] = \frac{9}{64}$

$Pr[X=Y] = \frac{11}{32}$



Problem 5.45

$\boxed{5.26}$ $f_{xy}(x,y) = k(x+y)$ for $0 \le x \le 1$, $0 \le y \le 1$

a.) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dy\,dx = 1$

$= \int_0^1\int_0^1 f_{xy}(x,y)\,dx\,dy = 1$

$= \int_0^1\int_0^1 k(x+y)\,dy\,dx = 1$

$= k\int_0^1\int_0^1 (x+y)\,dy\,dx = 1$

$\rightarrow \int_0^1 x\,dx + \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx = \frac{1}{k}$

$\Rightarrow \frac{1}{2}+\frac{1}{2} = \frac{1}{k}$

$\boxed{k=1}$

b.) $F_{xy}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{xy}(s,t)\,dt\,ds = \int_0^x\int_0^y f_{xy}(s,t)\,dt\,ds$

$\Rightarrow \frac{xy(x+y)}{2}$

$F_{xy}(x,y) = \int_0^1\int_0^y (s+t)\,dt\,dt = \frac{y(y+1)}{2}$

$F_{xy}(x,y) = \int_0^x\int_0^1 s\,dt\,ds + \int_0^x\int_0^1 t\,dt\,ds = \frac{x(x+1)}{2}$

$$F_{x,y}(x,y) = \begin{cases} 0 & x<0,\ y<0 \\ \frac{xy(x+y)}{2} & 0\le x\le 1,\ 0\le y\le 1 \\ \frac{x(x+1)}{2} & 0\le x<1,\ y>1 \\ \frac{y(y+1)}{2} & x>1,\ 0\le y<1 \\ 1 & x,y>1 \end{cases}$$

c.) $0 \le x \le 1$

$f_x(x) = \int_0^1 f_{x,y}(x,y)\,dy = x + \frac{1}{2}$

$f_x(x) = \begin{cases} x+\frac{1}{2} & \text{for } 0\le x \le \frac{1}{2} \\ 0 & \text{Otherwise} \end{cases}$

$0 \le y \le 1$

$f_y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dy$

$= \frac{1}{2} + y$

$f_y(y) = \begin{cases} y+\frac{1}{2} & 0\le y\le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

d.) $P_r[X<Y] = \int_0^1\int_x^1 f_{xy}(x,y)\,dy\,dx$

$= \frac{1}{2}$

$P_r[Y<X^2] = \int_0^1\int_0^{x^2} (x+y)\,dy\,dx$

$= \frac{7}{20}$

$P_r[X+Y<.5] = \int_0^{0.5}\int_0^{0.5-x} (x+y)\,dy\,dx$

$= \frac{1}{24} \Rightarrow 1 - \frac{1}{24}$

$P_r[X+Y>.5] = \boxed{\frac{23}{24}}$

$\boxed{5.45}$

$F_{xy}(x,y) = \frac{xy(x+y)}{2}$ $\quad 0\le x\le 1$, $0\le y\le 1$

$F_x(x) = x + \frac{1}{2}$ $\quad 0 \le x \le 1$

$F_y(y) = y + \frac{1}{2}$ $\quad 0 \le y \le 1$

$F_{xy}(x,y) \neq F_x(x)\,F_y(y)$

$\Rightarrow$ X & y are not independent

Problem 5.56

5.56

a) $E[(X+Y)^2] = E[X^2] + 2E[XY] + E[Y^2]$

b) $VAR[X+Y] = E[(X+Y)^2] - E[X+Y]^2$

$= Var[X] + Var[Y] + 2[E[XY] - E[X]E[Y]]$

c) $Var[X+Y] = Var[X] + Var[Y]$ if

$E[XY] = E[X]E[Y]$

only if X & Y are uncorrolelated

Problem 5.59

$\boxed{5.1}$ a)

(HH, HH) (2,2)
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c.) $Pr[X=Y] = Pr[\{(0,0),(1,1),(2,2)\}] = \frac{1}{16} + \frac{1}{4} + \frac{1}{16}$
$= \frac{3}{8}$

d.) $P[H] = 3/4$
$P[T] = 1 - 3/4 = \frac{1}{4}$

$Pr[(0,0)] = \frac{1}{64}$
$Pr[(1,0)] = \frac{1}{8}$
$Pr[(1,1)] = 3/16$
$Pr[(2,0)] = 5/32$
$Pr[(2,1)] = 3/8$
$Pr[(2,2)] = 9/64$

$Pr[X=Y] = 11/32$

$\boxed{5.59}$

$$E[Y] = 2\left(\frac{7}{16}\right) + 1\left(\frac{8}{16}\right) + 0 = \frac{22}{16}$$

$$E[X] = 0 + 1\left(\frac{8}{16}\right) + 2\left(\frac{1}{16}\right) = \frac{10}{16}$$

$$E[XY] = 0 + 1\left[\frac{4}{16} + 2\left(\frac{4}{16}\right)\right]$$

$$= 1 \rightarrow \text{not orthognal}$$

$$COV(X,Y) = E[XY] - E[Y]E[X]$$

$$= 1 - \frac{22}{16} \cdot \frac{16}{16} = \frac{9}{64}$$

∗ not uncorrelated
∗ not independent

Problem 5.65

$\boxed{5.26}$ $f_{xy}(x,y) = k(x+y)$ for $0 \le x \le 1$, $0 \le y \le 1$

a.) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dy\,dx = 1$

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$= \int_0^1\int_0^1 k(x+y)\,dy\,dx = 1$

$= k\int_0^1\int_0^1 (x+y)\,dy\,dx = 1$

$\rightarrow \int_0^1 x\,dx + \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx = \frac{1}{k}$

$\Rightarrow \frac{1}{2}+\frac{1}{2} = \frac{1}{k}$

$\boxed{k=1}$

b.) $F_{xy}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{xy}(s,t)\,dt\,ds = \int_0^x\int_0^y f_{xy}(s,t)\,dt\,ds$

$\Rightarrow \frac{xy(x+y)}{2}$

$F_{xy}(x,y) = \int_0^1\int_0^y (s+t)\,ds\,dt = \frac{y(y+1)}{2}$

$F_{xy}(x,y) = \int_0^x\int_0^1 s\,dt\,ds + \int_0^x\int_0^1 t\,dt\,ds = \frac{x(x+1)}{2}$

$$F_{x,y}(x,y) = \begin{cases} 0 & x<0,\ y<0 \\ \frac{xy(x+y)}{2} & 0 \le x \le 1,\ 0 \le y \le 1 \\ \frac{x(x+1)}{2} & 0 \le x \le 1,\ y > 1 \\ \frac{y(y+1)}{2} & x > 1,\ 0 \le y \le 1 \\ 1 & x, y > 1 \end{cases}$$

c.) $0 \le x \le 1$

$f_x(x) = \int_0^1 f_{x,y}(x,y)\,dy = x + \frac{1}{2}$

$f_x(x) = \begin{cases} x+\frac{1}{2} & \text{for } 0 \le x \le \frac{1}{2} \\ 0 & \text{Otherwise} \end{cases}$

$0 \le y \le 1$

$f_y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dy$

$= \frac{1}{2} + y$

$f_y(y) = \begin{cases} y+\frac{1}{2} & 0 \le y \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

d.) $P_r[X<Y] = \int_0^1\int_x^1 f_{xy}(x,y)\,dy\,dx$

$= \frac{1}{2}$

$P_r[Y<X^2] = \int_0^1\int_0^{x^2} (x+y)\,dy\,dx$

$= \frac{7}{20}$

$P_r[X+Y < .5] = \int_0^{0.5}\int_0^{0.5-x} (x+y)\,dy\,dx$

$= \frac{1}{24} \Rightarrow 1 - \frac{1}{24}$

$P_r[X+Y > .5] = \boxed{\frac{23}{24}}$

$\boxed{5.65}$

$E[X] = \int_0^1 x(x+\frac{1}{2})\,dx = \frac{7}{12} = E[Y]$

$E[X^2] = E[Y^2] = \int_0^1 x^2(x+\frac{1}{2})\,dx = \frac{5}{12}$

$\text{var}[Y] = \text{var}[X] = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$

$E[XY] = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \frac{1}{3}$

$\rho = \frac{\frac{1}{3} - \left(\frac{7}{12}\right)^2}{\frac{11}{144}} = -\frac{1}{11}$

Problem 5.66

$\boxed{5.27}$

a.) $\int_0^1\int_0^1 kx(1-x)y\,dy\,dx = 1$

$\Rightarrow \frac{1}{4}-\frac{1}{6}=\frac{1}{k}$

$\boxed{k=12}$

b.) $F_{XY}(x,y) = \int_0^x\int_0^y 12s(1-s)t\,dt\,ds$

$= 12\left[\frac{x^2y^2}{4} - \frac{x^3y^2}{12}\right]$

$= x^2(3-2x)y^2$

$F_{XY}(1,y) = \int_0^1\int_0^y 12s(1-s)t\,dt\,ds$

$= y^2$

$F_{XY}(x,1) = \int_0^x\int_0^1 12s(1-s)t\,dt\,ds$

$= x^2(3-2x)$

$$F_{XY}(x,y) = \begin{cases} 0 & x<0,\ y<0 \\ x^2(3-2x)y^2 & 0<x<1, 0<y<1 \\ x^2(3-2x) & 0<x<1,\ y\geq 1 \\ y^2 & x\geq 1, 0<y<1 \\ 1 & x,y\geq 1 \end{cases}$$

c.) $f_X(x) = \int_0^1 12x(1-x)y\,dy = 6x(1-x)$

$$f_X(x) = \begin{cases} 6x(1-x) & 0\leq x\leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$f_Y(y) = \int_0^1 12x(1-x)y\,dx$

$= 2y$

$$f_Y(y) = \begin{cases} 2y & 0\leq y\leq 1 \\ 0 & \text{therwise} \end{cases}$$

d.) $P[Y<X^{1/2}] = \int_0^1\int_0^{\sqrt{x}} 12x(1-x)y\,dy\,dx$

$= \frac{1}{2}$

$P[X<Y] = \int_0^1\int_x^1 12x(1-x)y\,dy\,dx$

$= \frac{7}{10}$

$\boxed{5.66}$

$E[XY] = \int_0^1\int_0^1 12x(1-x)y\,y\,x\,dy\,dx$

$= \frac{1}{3}$

$E[X] = \int_0^1 6x(1-x)x\,dx$

$= \frac{1}{2}$

$\text{cov} = E[XY - E[X]E[Y]]$

$\text{cov}(x,y) = \frac{1}{3} - \frac{1}{2}\cdot\frac{2}{3} = 0$

$E[Y] = \int_0^1 2y\cdot y\,dy = \frac{2}{3}$

not orthogonal

but yes it is

indypendent + uncorelated

Problem 5.105

5.105

a.) $\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$B^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = B^{-1} \begin{bmatrix} w \\ z \end{bmatrix}$$

$$x = (w+z)\frac{1}{2}$$

$$y = \frac{1}{2}(w-z)$$

$$f_{wz}(w,z) = f_{xy}\left(\frac{w+z}{2}, \frac{w-z}{2}\right)$$

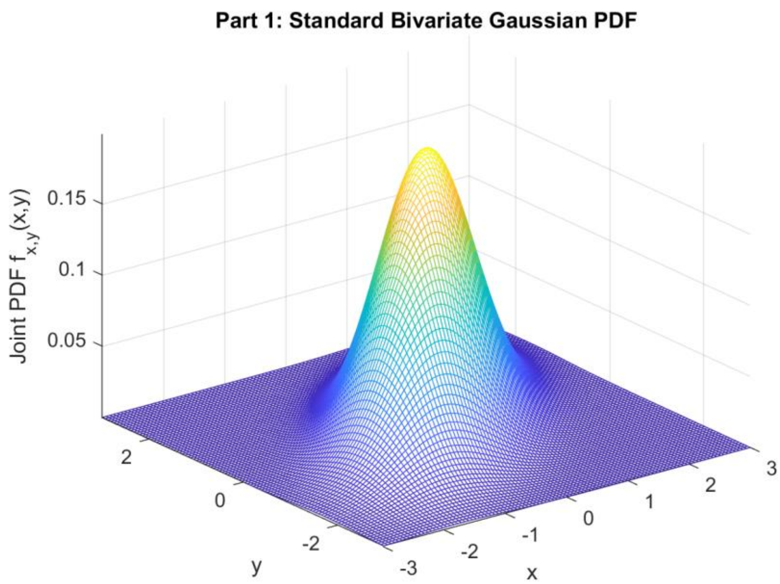
b.) $f_{wz}(w,z) = e^{-\left(\frac{w+z}{2}\right)} e^{\frac{-(w-z)}{2}} = e^{-w}$ for $w > 0$

c.) $f_{wz}(w,z) = \dfrac{k^2 x_m^{2k}}{\left(\frac{w^2-z^2}{4}\right)^{k+1}}$

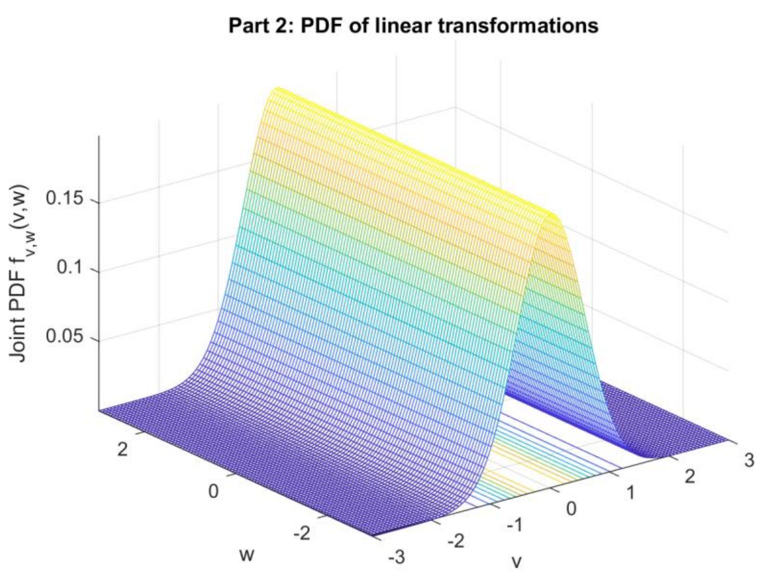
$x > x_m$

Computer Experiments

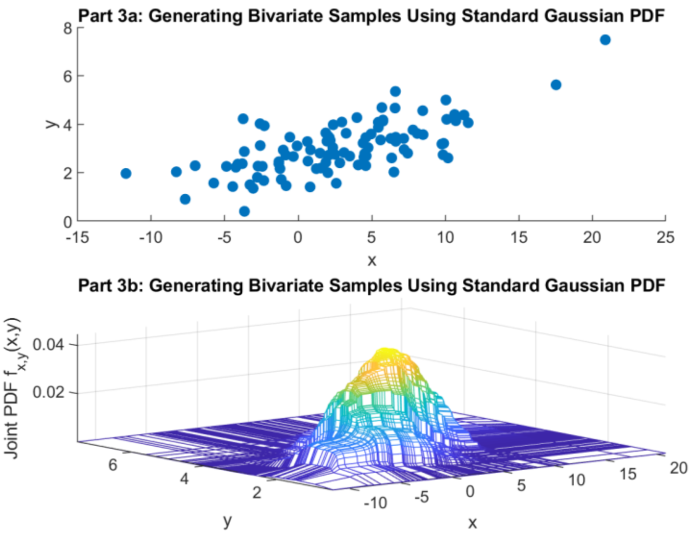
1.



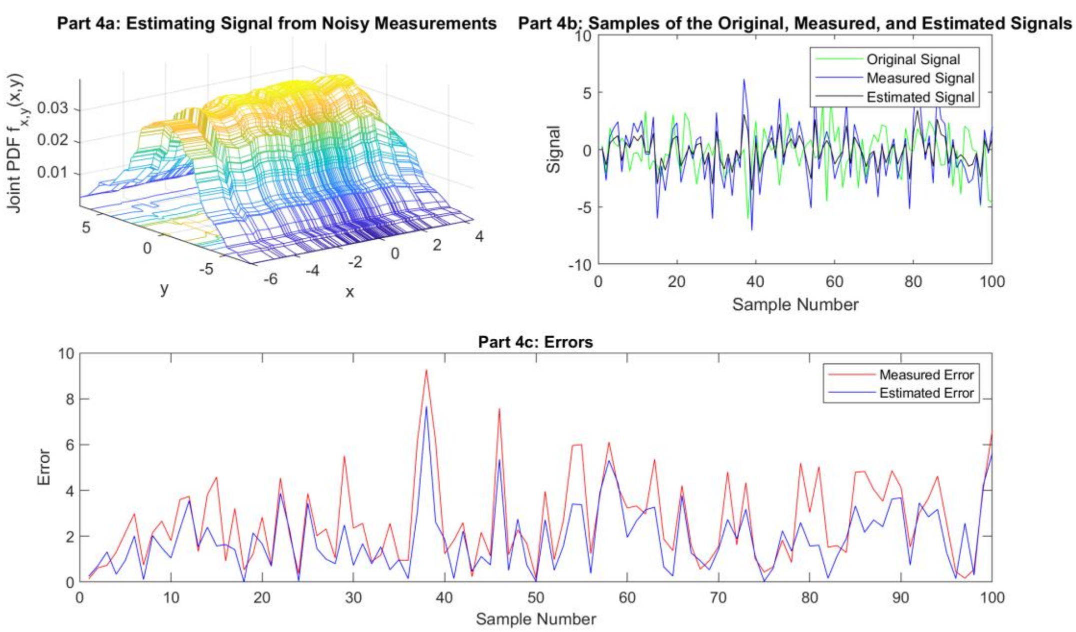
2.



3.



4.




```
s1 =

   1.0691e+03


s2 =

  564.9875
```

d.

MATLAB Solutions

```
%Gottschalk, Rachel ECE 302: Assignment #5
close all;
clear all;
clc;


%%%%%%%%%%%%%%%%%%%% Part 1 %%%%%%%%%%%%%%%%%%%%%%
m = 100;
x = linspace(-3,3,m);
y = linspace(-3,3,m);

fxy=biPdf(x,y,0,0,1,1,0.6); % sends to function

%plots part 1 with mechc
figure(1)
meshc(x,y,fxy)
title("Part 1: Standard Bivariate Gaussian PDF")
xlabel('x');
ylabel('y');
zlabel('Joint PDF f_x_,_y(x,y)');
grid on;


%%%%%%%%%%%%%%%%%%%% Part 2 %%%%%%%%%%%%%%%%%%%%%%
v = linspace(-3,3,m);
w = linspace(-3,3,m);
M=[v;w];
A =  1/sqrt(2)*[1 1; -1 1];   %    A =  [1/sqrt(2)  1/sqrt(2)]
                              %         [-1/sqrt(2) 1/sqrt(2)]
% calculates transform
t = A^-1*M;
x = t(1,:);
y = t(2,:);

% sends to function and then divides by determinate
fvw = biPdf(x,y,0,0,1,1,0.6);
fvw = fvw/det(A);

%plots figure 2
figure(2)
meshc(v,w,fvw)
title('Part 2: PDF of linear transformations')
xlabel('v');
ylabel('w');
zlabel('Joint PDF f_v_,_w(v,w)')


%%%%%%%%%%%%%%%%%%%% Part 3 %%%%%%%%%%%%%%%%%%%%%%
mx = 3;
ox = 5^2;
my = 3;
```

```
oy = 1^2;
p = 0.7;

% generates random numbers and then makes A matrix
z = randn(2,m);
A = [sqrt(ox) 0; p*sqrt(oy) sqrt((1-p^2)*oy)];
M = [mx;my];
X = M + A*z;
x = X(1,:);
y = X(2,:);

% sorts x and y
x1 = sort(x);
y1 = sort(y);

%sends to function to calculate buvariate PDF
fxy = bivariatePdf(x1,y1,mx,ox,my,oy,p);

% plots figure 3 with all parts on one figure
figure(3)
subplot(2,1,1)
scatter(x,y,'filled')
title('Part 3a: Generating Bivariate Samples Using Standard Gaussian PDF');
xlabel('x');
ylabel('y');
subplot(2,1,2)
meshc(x1,y1,fxy)
title('Part 3b: Generating Bivariate Samples Using Standard Gaussian PDF');
xlabel('x');
ylabel('y');
zlabel('Joint PDF f_x_,_y(x,y)')


%%%%%%%%%%%%%%%%%%%% Part 4 %%%%%%%%%%%%%%%%%%%%%%
% intial values
mx = 0;
sigx = 2;
ox=4;
mn = 0;
sign = 2;
on=4;
my = 0;
sigy=sqrt(ox+on);
oy=sigy^2;
p = 0;

% generates normal distrubuted numbers
x = normrnd(0,sigx,1,m);
y = normrnd(0,sigy,1,m);

% stores origional numbers
x0=x;
y0=y;
```

```
% sort x and y and make A matrix
x=sort(x);
y=sort(y);
A = [1 0; 1 1];

% makes function to calculate x and n matrixes
t=A\[x;y];
x = t(1,:);
n = t(2,:);

% sends to function
fxy = bivariatePdf(x,n,mx,ox,mn,on,p);
fxy = fxy/det(A);

% plots all parts of 4 on one figure
figure(4)
subplot(2,2,1)
meshc(x,y,fxy)
title('Part 4a: Estimating Signal from Noisy Measurements')
xlabel('x');
ylabel('y');
zlabel('Joint PDF f_x_,_y(x,y)')

% generated x and n
tran = A\[x0;y0];
x = tran(1,:);
n = tran(2, :);

% stores origin variables
x1 = x;
n1 = n;

y1 = y0;

% created estimated number
x_estimated = (ox/(ox+on))*y1;

% plots part 4b
vals = 1:1:m;
subplot(2,2,2)
plot(vals, x1,'g')
hold on;
plot(vals, y1,'b-')
hold on;
plot(vals,x_estimated,"k-")
hold on;
title('Part 4b: Samples of the Original, Measured, and Estimated Signals')
xlabel('Sample Number'); ylabel('Signal');
legend('Original Signal','Measured Signal','Estimated Signal');

% generated errors
error_M = abs(y1-x1);
```

```
error_E = abs(x_estimated - x1);

plots part 4c
subplot(2,1,2)
plot(vals,error_M,"r")
hold on;
plot(vals, error_E, "b")
title("Part 4c: Errors")
xlabel('Sample Number'); ylabel('Error')
legend("Measured Error", "Estimated Error")


% part d - calulates sum and prints them in command window
s1 = sum((y1-x1).^2)
s2 = sum((x_estimated-x1).^2)



%%%%%%%%%%%%%%%%%%%% Functions %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function fxy = biPdf(x,y,mx,my,ox,oy,p)

    % calculates the bivariate PDF with 1st formula
    for i = 1:length(x)
        for j = 1:length(y)
            fxy(i,j) = 1/((ox*ox)*sqrt(1-p^2)*sqrt((2*pi)^2))*exp(-.5*(((x(i)-mx)/ox)^2-
(2*p*(x(i)-mx)/ox)*((y(j)-my)/oy)+(y(j)-my/oy)^2)/(1-p^2));
        end
    end

end


function f_xy = bivariatePdf(x,y,mu1,var1,mu2,var2,corr_coeff)
    cov = [var1,  corr_coeff*sqrt(var1)*sqrt(var2); % Calculate covariance matrix
           corr_coeff*sqrt(var1)*sqrt(var2)   var2];

    mu = [mu1;mu2]; % Mean matrix

    f_xy = zeros(length(x),length(y));

    for i=1:length(x) % Nested for loop to calculate bivariate guassian pdf
        for j=1:length(y)
            xVec = [x(i); y(j)];
            f_xy(i,j) = (1/(2*pi*sqrt(det(cov))))*exp(-0.5*(xVec-mu)'*cov^-1*(xVec-mu));
        end
    end
end
```