ECE 302: Probability, Statistics, and Random Processes for EE

Fall 2022

Assignment 3: One Random Variables

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Problem 4.2

Saturday, October 1, 2022 7:29 PM

$\boxed{4.2}$ $S = \{1, 2, 3, 4, 5, 6\}$

x:	0	1	1	2	2	3
y:	1	1	2	2	3	3

$S_x = \{0, 1, 2, 3\}$

$S_y = \{1, 2, 3\}$

$Pr(x=0) = \frac{1}{6}$

$Pr(x=1) = \frac{2}{6} = \frac{1}{3}$

$Pr(x=2) = \frac{2}{6} = \frac{1}{3}$

$Pr(x=3) = \frac{1}{6}$

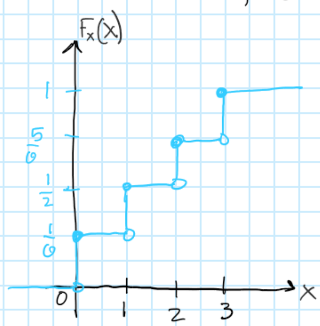
$Pr(y=1) = \frac{2}{6} = \frac{1}{3}$

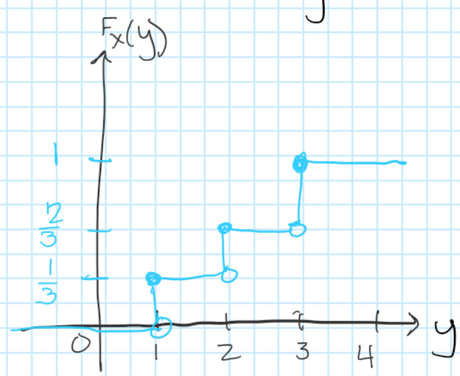
$Pr(y=2) = \frac{1}{3}$

$Pr(y=3) = \frac{1}{3}$

$$F_x(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1/6, & \text{if } 0 \le x < 1 \\ \frac{1}{2}, & \text{if } 1 \le x < 2 \\ 5/6, & \text{if } 2 \le x < 3 \\ 1, & \text{if } x \ge 3 \end{cases}$$

$$F_x(y) = \begin{cases} 0 & \text{if } y \le 1 \\ \frac{1}{3} & \text{if } 1 \le y < 2 \\ 2/3 & \text{if } 2 \le y < 3 \\ 1 & \text{if } y \ge 3 \end{cases}$$





Problem 4.4

$\boxed{4.4}$ \$1 → 8, \$5 → 2

$8+2 = \boxed{10 \text{ total}}$ # of bills

$S_x = \{2, 6, 10\}$

$1+1=2$
$1+5=6$
$5+1=6$
$5+5=10$

$S_y = \{2, 6, 10\}$

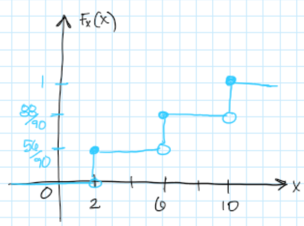
a.) CDF of S_x:

$P_r[x=2] = \frac{8}{10} \times \frac{7}{9} = \frac{56}{90}$

$P_r[x=6] = 2 \cdot \frac{8}{10} \times \frac{2}{9} = \frac{32}{90}$

$P_r[x=10] = \frac{2}{10} \times \frac{1}{9} = \frac{2}{90}$

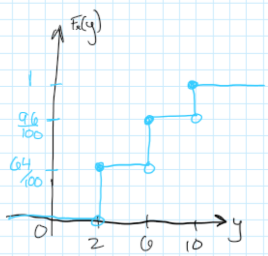
$$F_x(x) = \begin{cases} 0, & \text{if } x<2 \\ 56/90, & \text{if } 2 \le x < 6 \\ 88/90, & \text{if } 6 \le x < 10 \\ 1, & \text{if } x \ge 10 \end{cases}$$



$P_r[y=2] = \frac{8}{10} \cdot \frac{8}{10} = 64/100$

$P_r[y=6] = 2 \cdot \frac{8}{10} \cdot \frac{2}{10} = 32/100$

$P_r[y=10] = \frac{2}{10} \cdot \frac{2}{10} = 4/100$



$$F_x(y) = \begin{cases} 0 & \text{if } y<2 \\ 32/100 & \text{if } 2 \le y < 6 \\ 96/100 & \text{if } 6 \le y < 10 \\ 1 & \text{if } y \ge 10 \end{cases}$$

b.) $A = \{x = \$2\}$ $B = \{x < \$7\}$

$C = \{x \ge 6\}$

$P_r[A] : P_r[x = \$2] = \frac{56}{90}$ $P_r[y=\$2] = 64/100$

$P_r[B] : P_r[x < \$7] = 88/90$ $P_r[y < \$7] = 96/100$

$P_r[C] : P_r[x \ge 6] = 1 - 56/90$ $P_r[y \ge 6] = 1 - 64/100$

$= 34/90$ $= 36/100$

Problem 4.11

4.11

a.) $[-1, 2]$

$Pr[U \le x] = 0$

for $a \le x < b$

$Pr[X \le x] = \frac{\ell \text{ of } [a,x]}{\ell \text{ of } [a,b]}$

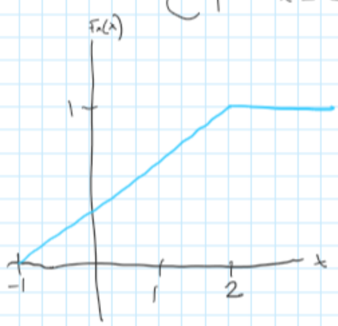
$= \frac{x-a}{a-b}$

for $x \ge b$

$Pr[X \le x] = 1$

$$F_U(x) = \begin{cases} 0 & , \text{if } x < a \\ \frac{x-a}{b-a} & , \text{if } a \le x < b \\ 1 & , x \ge b \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{3} & -1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$



b.) $Pr[X \le 0] = F_X(0) = \frac{1}{3}$

$Pr[|x - .5| < 1] = Pr[-1 < x - \frac{1}{2} < 1]$

$= Pr[-\frac{1}{2} < x < \frac{3}{2}]$

$= \frac{1}{3}(\frac{3}{2} + 1) - \frac{1}{3}(-\frac{1}{2} + 1)$

$= \frac{2}{3}$

$Pr[x \ge -\frac{1}{2}] = 1 - Pr[x \le -\frac{1}{2}] =$

$1 - \frac{1}{3}(-\frac{1}{2} + 1) = \frac{5}{6}$

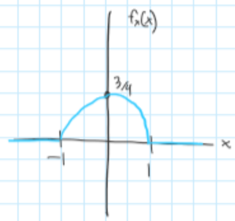
Problem 4.17

4.17

a.) $1 = \int_{1}^{-1} (1-x^2)\,dx \Rightarrow c\left[2-\frac{1}{3}2\right]$

$= \frac{4}{3}c \qquad c=\frac{3}{4}$

$f_x(x) = \frac{3}{4}(1-x^2$



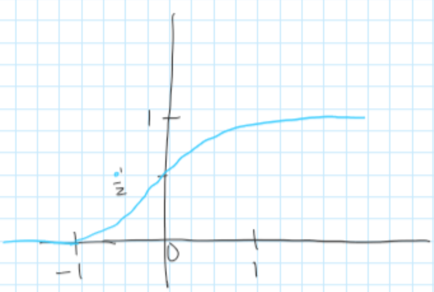
b.) $F_x(x) = P_r[X \le x] = 0$

$P_r(X \le x) = \int_{-\infty}^{x} f_x(t)\,dt$

$= \frac{3}{4}\int_{-1}^{x} 1-t^2\,dt$

$= \frac{3}{4}\left[(x+1) - \frac{x^3+1}{3}\right]$

$$F_x(x) = \begin{cases} 0 & x<-1 \\ \frac{3}{4}\left[(x+1)-\frac{x^3+1}{3}\right] & \text{if } -1\le x<1 \\ 1 & x\ge 1 \end{cases}$$



c.) $P_r[X=0] = 0$

$P_r[0 < x < .5] = \frac{3}{4}\left[\left(\frac{1}{2}+1\right)-\frac{1}{3}\left(\frac{1}{8}+1\right)\right]$

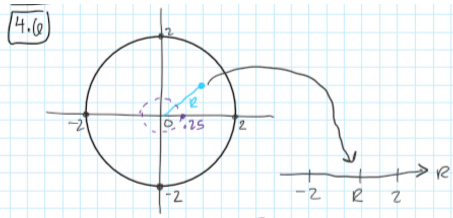
$= \frac{11}{32}$

$P_r\left[\left|1-\frac{1}{2}\right| < \frac{1}{4}\right] = P_r\left[\frac{1}{4} < x < \frac{3}{4}\right]$

$= 0.2734$

Problem 4.19

4.6



a.) $S = \{(x,y) : x^2+y^2 < 2^2\}$

$r = \sqrt{x^2+y^2}$

$S_e = \{r : 0 \le r \le 2\}$

b.) $S \rightarrow S_R$ also mapped here

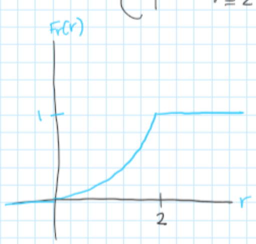
$f((x,y)) = \sqrt{x^2+y^2}$, for all $(x,y) \in S$

c.) $P_r[A] = P_r[R \le .25] = \frac{\text{area } A}{\text{area } S}$

$= \frac{\pi(.25)^2}{\pi(2)^2} \approx 0.015625 = \frac{1}{64}$

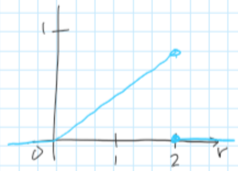
d.) $P(R \le r) = \frac{\pi \cdot r^2}{\pi(2)^2} = \frac{r^2}{4}$

$$F_R(r) = \begin{cases} 0 & r < 0 \\ \frac{r^2}{4} & 0 \le r < 2 \\ 1 & r \ge 2 \end{cases}$$



4.19

a.) $f_R(r) = \frac{d}{dr} F_R(r) = 2\left(\frac{r}{2}\right)\left(\frac{1}{2}\right) = \frac{r}{2} \quad 0 \le r \le 2$



$f_R(r) \begin{cases} \frac{r}{2} & , 0 \le r < 2 \\ 0 & \text{elsewhere} \end{cases}$

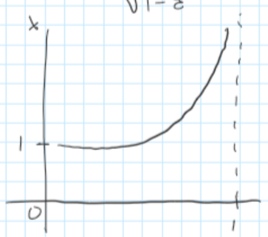
b.) $\{R > 0.25\} = A$

$P_r[A] = \int_{.25}^{\infty} \frac{r}{2}\, dr = \frac{63}{64}$

Problem 4.21

4.8

a.) $x = \frac{1}{\sqrt{1-y}}$

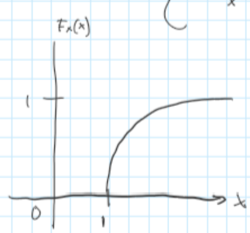


b.) $P_r[X(y) \le x] = P_r\left[\frac{1}{\sqrt{1-y}} \le x\right] = P_r\left[\frac{1}{1-y} \le x^2\right]$

$P_r = \left[\frac{1}{x^2} \le 1-y\right] = P_r\left[y \le 1-\frac{1}{x^2}\right]$

$= 1-\frac{1}{x^2}$

$$F_X(x) = \begin{cases} 0 & \text{if } x<1 \\ 1-\frac{1}{x^2} & \text{if } x \ge 1 \end{cases}$$



c.) $P_r[X \ge 1] = 1 - F_X(1)$

$= 1-(1-\frac{1}{1^2}) = 1$

$P_r[5 \le x \le 7] = F_X(7) - F_X(5) = (1-\frac{1}{49}) - (1-\frac{1}{25})$

$= \frac{1}{25} - \frac{1}{49} \approx .01959$

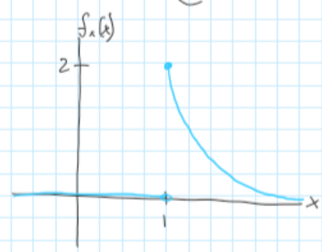
$P_r[X \le 20] = 1 - \frac{1}{400} = 0.9975$

4.21

a.) $1 \le x < \infty$:

$f_X(x) = \frac{d}{dx}\left(1-\frac{1}{x^2}\right) = \frac{2}{x^3}$

$$f_X(x) = \begin{cases} 2/x^3 & x \ge 1 \\ 0 & \text{, elsewhere} \end{cases}$$



b.) $X \ge a$ & $a<1$:

$P_r[X \ge a] = \int_1^\infty \frac{2}{x^3}\,dx = 1$

$a \ge 1$:

$P_r[X \ge a] = \int_a^\infty \frac{2}{x^3}\,dx$

$= \frac{1}{a^2}$

$$P_r[X \ge a] = \begin{cases} 1 & \text{if } a<1 \\ \frac{1}{a^2} & \text{if } a \ge 1 \end{cases}$$

if $P_r\{X \ge 2a\}$:

$a < 1/2$

$P_r[X \ge 2a] = \int_1^\infty \frac{2}{x^3}\,dx$

$= 1$

$P_r[X \ge 2a] = \int_{2a}^\infty \frac{2}{x^3}\,dx$

$= \frac{1}{4a^2}$

$$P_r[X \ge 2a] \begin{cases} 1 & \text{if } a<1/2 \\ \frac{1}{4a^2} & \text{if } a \ge 1/2 \end{cases}$$

Problem 4.23

4.23

$f_x(x) = \frac{d}{dx}\left[F_x(x)\right]$

for $x>0$

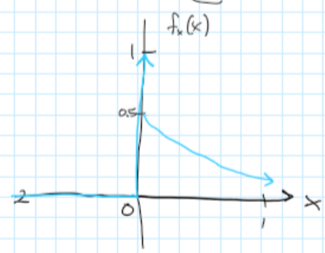
$f_x(x) = \frac{d}{dx}\left[1 - \frac{1}{4}e^{-2x}\right] = \frac{1}{2}e^{-2x}$

@ $x=0$

$f_x(x) = \frac{3}{4}\delta(x)$

a.)

$$f_x(x) = \begin{cases} \frac{3}{4}\delta(x), & x=0 \\ \frac{1}{2}e^{-2x}, & x>0 \\ 0, & \text{elsewhere} \end{cases}$$

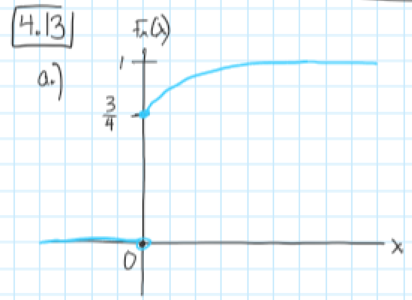


b.) $Pr[X=0] = 3/4$

$Pr[X>8] = \int_8^\infty f_x(x)\,dx = \frac{1}{2}\int_8^\infty e^{-2x}\,dx = \frac{1}{4}e^{-16}$

4.13

a.)



b.) $Pr[X\le 2] = 1 - \frac{1}{4}e^{-2(2)} \approx 0.995$

$Pr[X=0] = 1 - \frac{1}{4}e^{-2(0)} = 0.75$

$Pr[2<X<6] = Pr[X\le 6] - Pr[X\le 2]$

$= 1 - \frac{1}{4}e^{-2(6)} - 1 + \frac{1}{4}e^{-2(2)}$

≈ 0.005

$Pr[X>10] = 1 - Pr[X\le 10]$

$= 1 - \left(1 - \frac{1}{4}e^{-2(10)}\right)$

$= 5.15\times 10^{-10}$

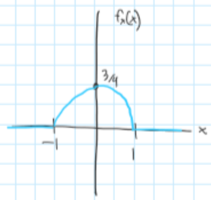
Problem 4.39

$\boxed{4.17}$

a.) $1 = \int_{1}^{-1} (1-x^2)dx \Rightarrow c[2-\frac{1}{3}2]$

$= \frac{4}{3}c$ $\quad c=\frac{3}{4}$

$f_x(x) = \frac{3}{4}(1-x^2$



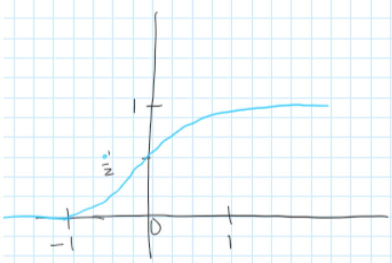
b.) $F_x(x) = Pr[X \le x] = 0$

$P_r(X \le x) = \int_{-\infty}^{x} f_x(t)dt$

$= \frac{3}{4}\int_{1}^{x} 1-t^2 \, dt$

$= \frac{3}{4}[(x+1) - \frac{x^3+1}{3}]$

$$F_x(x) = \begin{cases} 0 & x<-1 \\ \frac{3}{4}[(x+1) - \frac{x^3+1}{3}] & \text{if } -1 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$



c.) $Pr[X=0] = 0$

$Pr[0 < x < .5] = \frac{3}{4}[(\frac{1}{2}+1) - \frac{1}{3}(\frac{1}{8}+1)]$

$= \frac{11}{32}$

$P[|1-\frac{1}{2}| < \frac{1}{4}] = Pr[\frac{1}{4} < x < \frac{3}{4}]$

$= 0.2734$

$\boxed{4.39}$

$$f_x(x) = \begin{cases} \frac{3}{4}(1-x^2), & \text{if } -1 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$E[g(y)] = \int_{-\infty}^{\infty} g(y) \cdot f_y(y)\, dy$$

$$E[X] = \frac{3}{4}\int_{-1}^{1} x \cdot (1-x^2)\, dx$$

$$= \frac{3}{4}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{3}{4} \times 0 = 0$$

mean

$$E[X^2] = \frac{3}{4}\int_{-1}^{1} x^2 \cdot (1-x^2)dx$$

$$= \frac{3}{4}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1}$$

$$= \frac{1}{5}$$

variance:

$E(x^2) - E(x) = \frac{1}{5} - 0^2 = \frac{1}{5}$

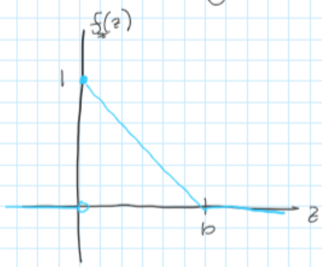
Problem 4.42

4.20

$0 \le z < b$

a.) $\frac{d}{dz}\left[1 - \frac{(b-z)^2}{b^2}\right] = \frac{2(b-z)}{b^2} = f_z(0)$

$$f_z(z) = \begin{cases} \frac{2(b-z)}{b^2}, & 0 \le z < b \\ 0, & \text{elsewhere} \end{cases}$$



b.) $P_r\{z > b/3\} = \int_{b/3}^{\infty} f_z(z)\,dz$

$= \int_{b/3}^{\infty} \frac{2(b-z)}{b^2}\,dz$

$= \frac{4}{9}$

4.42

$E[z] = \frac{2}{b^2}\int_0^b z\,(b-z)\,dz$

$= \frac{2}{b^2}\left[\frac{bz^2}{2} - \frac{z^3}{3}\right]_0^b$

$E[z] = \frac{2}{b^2} \times \frac{b^3}{6} = \frac{b}{3}$ mean

$E[z^2] = \frac{2}{b^2}\int_0^b z^2(b-z)\,dz$

$= \frac{2}{b^2}\left[\frac{b^4}{3} - \frac{b^4}{4}\right]$

$= \frac{b^2}{6}$

variance:

$E[z^2] - (E[z])^2 = \frac{b^2}{6} - \frac{b^2}{9}$

$V[z] = \frac{b^2}{18}$

Problem 4.46

4.46

Gaussian Random Variable

PDF:

$$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$E[Z] = \int_{-\infty}^{\infty} x\,\phi(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x\left(e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(y+\mu)\,e^{\frac{-y^2}{2\sigma^2}}\,dy \qquad x = y+\mu$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\left[\int y e^{\frac{-y^2}{2\sigma^2}} dy + \mu\int_{-\infty}^{\infty} e^{\frac{-y^2}{2\sigma^2}} dy\right]$$

$$= \underbrace{\cancel{\frac{\sigma^2}{\sqrt{2\pi}\sigma}\left[e^{\frac{-y^2}{2\sigma^2}}\right]_{-\infty}^{\infty}}}_{0} + m = \underbrace{m}_{\text{mean}}$$

$$V[Z] = E[(Z-E(Z))^2] = [(Z-\mu)^2]$$

$$\therefore E[(Z-\mu)^2] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

$y = x - m$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2\sigma^2}}dy$$

$u = y$

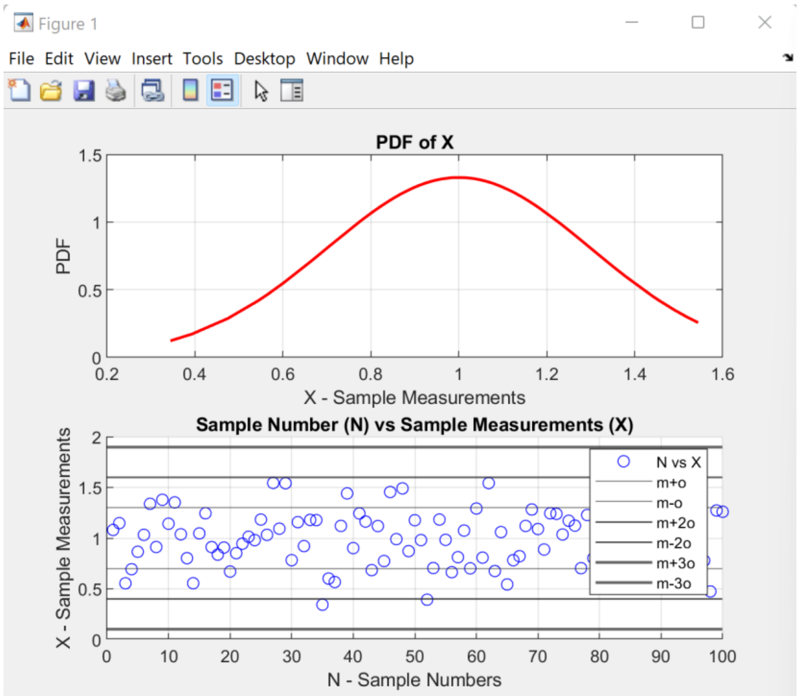
$dv = e^{\frac{-y^2}{2\sigma^2}}$

$$= \frac{1}{\sqrt{2\pi}\sigma}\left(-\sigma^2 y e^{\frac{-y^2}{2\sigma^2}}\right)\Big|_{-\infty}^{\infty} + \sigma\int_{-\infty}^{\infty} e^{\frac{-y^2}{2\sigma^2}}dy$$

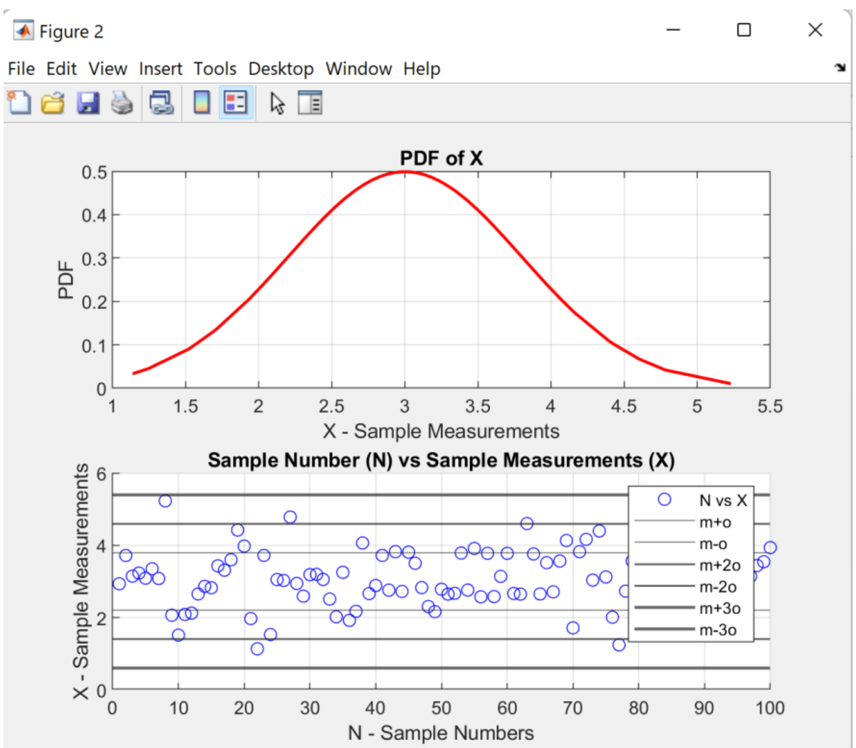
$$= \sigma^2$$

Computer Experiments

1.



2.



MATLAB Solutions

```
%Gottschalk, Rachel ECE 302: Assignment #3
close all;
clear all;
clc;


n = 100; % max of sample number
xs = 1:n; % sample number array

%%%%%%%%%%%%%%% Part 1 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fig1=1; % figure 1 variable
m = 1; % mean
o = 0.3; % standard deviation
x1 = (m + o * randn(n,1))'; % generates random sample measurements x
x11=sort(x1); % sorts x1 from smaller to larger values

a(m,o,x11,fig1); % passes mean, standard deviation, sorted x, and fig1 to a function
b(x1,xs,o,m,fig1); % passes unsorted x, sample number array, standard deviation, mean and
fig1 to b function


%%%%%%%%%%%%%%% Part 2 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fig2=2; % figure 2 variable
m = 3; % mean
o = 0.8; % standard deviation
x2 = (m + o * randn(n,1))'; % generates random sample measurements x
x22 = sort(x2); % sorts x1 from smaller to larger values

a(m,o,x22,fig2); % passes mean, standard deviation, sorted x, and fig1 to a function
b(x2,xs,o,m,fig2); % passes unsorted x, sample number array, standard deviation, mean and
fig1 to b function




%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Functions %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function fx = a(m,o,x,fig)
    for i = 1:length(x) % finds length of x and makes array from 1 to that length and
iterates through that array
        fx(i) = (1/(o*sqrt(2*pi)))*exp(-0.5*(((x(i)-m)^2)/(o^2))); % gaussian fx(x) eq
    end
    %plots figure 1 with the first subplot
    figure(fig)
    subplot(2,1,1)
    plot(x,fx, 'r',"LineWidth", 1.5)
    title('PDF of X')
    xlabel('RV (X)')
    ylabel('PDF')
    grid on
    hold on
end
```

```
function y = b(x,xs,o,m,fig)
    % makes confidence interval lines
    y=m+o;
    y1=m-o;
    y2=m+2*o;
    y3=m-2*o;
    y4=m+3*o;
    y5=m-(3*o);

    % plots scatter of N vs X and has confidence intervals on same subplot
    figure(fig)
    subplot(2,1,2)
    scatter(xs,x, 'b')
    hold on
    yline(y)
    hold on
    yline(y1)
    hold on
    yline(y2,LineWidth=1)
    hold on
    yline(y3',LineWidth=1)
    hold on
    yline(y4,LineWidth=1.5)
    hold on
    yline(y5,LineWidth=1.5)
    hold on
    title('Sample Number (N) vs Sample Measurements (X)')
    xlabel('N')
    ylabel('X')
    legend('N vs X','m+o','m-o','m+2o','m-2o','m+3o','m-3o')
    grid on

end
```