[A4] 제 4장

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* 연습문제: 1, 3, 5, 6, 8, 9, 11, 12, 14, 18, 20, 21, 22, 23, 26

* 컴퓨터 실습문제: 1

[연습문제]

1. (a) $F(x,y) = A\,\dfrac{x}{2x+1}\,\dfrac{y}{y+1}\,u(x)u(y)$

$$\lim_{\substack{x\to\infty\\y\to\infty}} A\cdot\frac{x}{2x+1}\cdot\frac{y}{y+1} = 1$$

$$\tfrac{1}{2}\cdot 1\cdot A = 1 \qquad \therefore A = 2$$

(b) 1) $\lim_{\substack{x\to\infty\\y\to\infty}} F(x,y) = 1$

2) $\lim_{\substack{x\to-\infty\\y\to-\infty}} F(x,y) = 0$ bc $u(x)=0$ as $x\to-\infty$

(c) $F_X(x) = \lim_{y\to\infty} F(x,y) = \dfrac{2x}{2x+1}\cdot u(x)$

$F_Y(y) = \lim_{x\to\infty} F(x,y) = \dfrac{2y}{y+1}\cdot u(y)$

(d) $P(-2\le X\le 2,\ 1\le Y\le 5) = F(2,5) - F(2,1)$

$$= 2\left(\frac{2}{5}\cdot\frac{5}{6_3} - \frac{2}{5}\cdot\frac{1}{2}\right) = 4/15$$

3. (a) $f_{XY}(x,y) = \dfrac{\partial F_{XY}(x,y)}{\partial x\partial y} = 2\cdot\dfrac{(2x+1)-2x}{(2x+1)^2}\cdot\dfrac{(y+1)-y}{(y+1)^2}\,u(x)u(y)$

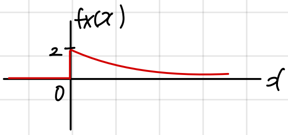
$$\therefore f_{XY}(x,y) = \frac{2}{(2x+1)^2(y+1)^2}\cdot u(x)u(y)$$

(b) 1) marginal pdf of x.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{2u(x)}{(2x+1)^2}\int_0^\infty \frac{1}{(y+1)^2}\,dy$$

$$= \frac{2u(x)}{(2x+1)^2}\left[\frac{-1}{(y+1)}\right]_0^\infty = \frac{2u(x)}{(2x+1)^2}\cdot 1$$

$$\therefore f_X(x) = \frac{2u(x)}{(2x+1)^2}$$

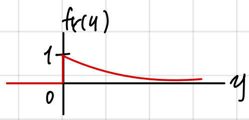


2) Marginal pdf of y

$$f_Y(y) = \frac{2u(y)}{(y+1)^2}\int_0^\infty \frac{1}{(2x+1)^2}\,dx = \frac{2u(y)}{(y+1)^2}\left[\frac{-1}{2(2x+1)}\right]_0^\infty$$

$$= \frac{2u(y)}{(y+1)^2}\cdot\tfrac{1}{2} = \frac{u(y)}{(y+1)^2}$$

$$\therefore f_Y(y) = \frac{u(y)}{(y+1)^2}$$

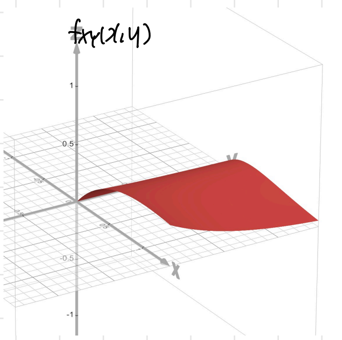


5. (a) $\int_0^\infty\int_0^\infty Axe^{-x(y+1)}\,dy\,dx = \int_0^\infty A\cancel{x}e^{-x(y+1)}\cdot -\frac{1}{\cancel{x}}\Big|_0^\infty dx$

$= \int_0^\infty A\cdot e^{-x(y+1)}\Big|_\infty^0 dx = \int_0^\infty Ae^{-x}\,dx = -Ae^{-x}\Big|_0^\infty$

$= A(0-(-1)) = A \qquad \therefore A = 1$

bc area needs to be 1.



(b) $f_X(x) = \int_0^\infty xe^{-x(y+1)}u(x)\,dy = xe^{-x(y+1)}\cdot -\frac{1}{x}\Big|_0^\infty\cdot u(x)$

$f_Y(y) = \int_0^\infty xe^{-x(y+1)}u(y)\cdot dx = \underset{\cdot u(y)}{e^{-x(y+1)}}\left(-\frac{x}{y+1} - \frac{1}{(y+1)^2}\right)\Big|_0^\infty$

$= -e^{-0}\left(-\frac{1}{(y+1)^2}\right) = \frac{u(y)}{(y+1)^2}$

$\cdot u(y)$

$$f_X(x) = e^{-x}\cdot u(x) \qquad f_Y(y) = \frac{1}{(y+1)^2}\cdot u(y)$$

6. (a) 나: $f_X(x) = \begin{cases} 1/15 & 30\le x\le 40 \\ 0 & \text{otherwise.}\end{cases}$

친구: $f_Y(y) = \begin{cases} 1/20 & 20\le y\le 40 \\ 0 & \text{otherwise}\end{cases}$

$f_{XY}(x,y) = \begin{cases} 1/300 & 30\le x\le 45,\ 20\le y\le 40 \\ 0 & \text{otherwise}\end{cases}$

$$\int_{y=30}^{40}\int_x^{45} 1/300\,dy\,dx = \int_{30}^{40}\frac{1}{300}x\Big|_x^{45}dx$$

$$= \int_{30}^{40}\frac{45}{300} - \frac{x^2}{300}\,dx = \frac{45}{300}x - \frac{x^3}{900}\Big|_{30}^{40}$$

$$= \frac{45}{300}\cdot 40 - \frac{40^3}{900} - \frac{45}{300}\cdot 30 + \frac{30^3}{900} = 1/6$$

$$\therefore P(X>Y) = 1 - 1/6 = 5/6$$

8. $f_X(x) = e^{-2|x-1000|} = \frac{1}{2b}e^{-\frac{|x-M|}{b}} \qquad b = 1/2,\ M = 1000$

1) $P(999<X<1001) = \int_{999}^{1001} e^{-2|x-1000|}dx = \int_{999}^{1000} e^{2x-2000}dx + \int_{1000}^{1001} e^{-2x+2000}dx$

$= \frac{1}{2}e^{2x-2000}\Big|_{999}^{1000} + -\frac{1}{2}e^{-2x+2000}\Big|_{1000}^{1001} = \frac{1}{2}(e^0-e^{-2}) - \frac{1}{2}(e^{-2}-1)$

$= 1-e^{-2} = 0.864$

$f_{X|B}(x|B) = \dfrac{f_X(x)}{P(B)} = 1.1574\cdot e^{-2|x-1000|}$

$P(999.5<X<1000.5\,|\,B) = 1.1574\int_{999.5}^{1000.5} e^{-2|x-1000|}dx$

$= 1.1574\left(\int_{999.5}^{1000} e^{2x-2000}dx + \int_{1000}^{1000.5} e^{-2x+2000}dx\right)$

$= 1.1574\left(\frac{1}{2}e^{2x-2000}\Big|_{999.5}^{1000} - \frac{1}{2}e^{-2x+2000}\Big|_{1000}^{1000.5}\right)$

$= 1.1574\left(\frac{1}{2}(e-e^{-1}) - \frac{1}{2}(e^{-1}-1)\right) = 1.1574(1-1/e) = 0.7316$

9. $f_{X|S}(x|S) = \dfrac{e^{-(x-S)/8}}{\sqrt{8\pi}}$

$P_{miss} = \int_{-\infty}^{S/2}\dfrac{e^{-(x-S)^2/8}}{\sqrt{8\pi}}dx \overset{u=\frac{x-S}{2},\ \frac{1}{2}dx=du}{=} \int_{-\infty}^{-S/4}\dfrac{e^{-u^2/2}}{\sqrt{2\pi}}du = Q(S/4)$

- $S=1 \to Q(1/4) = 0.39$
- $S=2 \to Q(1/2) = 0.309$
- $S=4 \to Q(1) = 0.1587$
- $S=8 \to Q(2) = 0.0228$

$P_{FA} = \int_{S/2}^{\infty}\dfrac{e^{-x^2/8}}{\sqrt{8\pi}}dx \overset{u=x/2}{=} \int_{S/4}^{\infty}\dfrac{e^{-u^2/2}}{\sqrt{2\pi}}du = Q(S/4)$

→ P_{miss}와 동일.

11. $f_{XY}(x,y) = f_X(x) f_Y(y)$ $\quad \left(\begin{array}{l} f_X(x) = e^{-x}\cdot u(x) \\ f_Y(y) = \frac{1}{(y+1)^2}\cdot u(y) \end{array} \right)$

$xe^{-x(y+1)} \neq e^{-x}\cdot \frac{1}{(y+1)^2}$ ($u(x)$, $u(y)$ 생략)

$\Rightarrow$ Not independent

12. $f_{XY}(x,y) = e^{-2(|x|+|y|)}$

x & y are independent : $f_{XY}(x,y) = f_X(x)\cdot f_Y(y)$

$f_X(x) = e^{-2|x|}$ $\quad f_Y(y) = e^{-2|y|}$

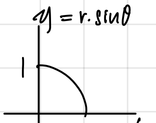
(a) $E[\cos X \sin Y] = \underset{\text{even func}}{E(\cos X)}\,\underset{\text{odd func.}}{E(\sin Y)} = 0$ $\quad \int \sin x = 0$ so $E(\sin X) = 0$

(b) $E[\cos(X)+\sin(Y)] = E[\cos X + 0] = \int_{-\infty}^{\infty} \cos x\, e^{-2|x|}dx$

$= 2\cdot\frac{2}{2^2+1^2} = 4/5$

14. (a) $f_{XY}(x,y) = \begin{cases} c & x^2+y^2\le 1,\ x>0,\ y>0 \\ 0 & \text{그외} \end{cases}$

$y = r\sin\theta$



1 $\quad x = r\cos\theta$

$\int_0^{\pi/2}\int_0^1 c\, r\, dr\, d\theta = \int_0^{\pi/2} \frac{c}{2}\, d\theta = \frac{c\pi}{4} = 1$

$\therefore c = \frac{4}{\pi}$

(b) $E(XY) = \frac{4}{\pi}\iint xy\, dx dy = \frac{4}{\pi}\int_0^{\pi/2}\int_0^1 \cos\theta\sin\theta\, r^3 dr d\theta$

$= \frac{4}{\pi}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta \cdot \int_0^1 r^3 dr = \frac{4}{\pi}\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{2\pi}$

$E(X^2) = \frac{4}{\pi}\iint x^2\, dx\, dy = \frac{\pi}{4}\int_0^{\pi/2}\int_0^1 r^2\cos^2\theta\, r dr d\theta$

$= \frac{4}{\pi}\int_0^{\pi/2}\cos^2\theta d\theta \int_0^1 r^3 dr = \frac{4}{\pi}\cdot\frac{\pi}{4}\cdot\frac{1}{4} = 1/4$

$E(Y^2) = \frac{4}{\pi}\iint y^2 dx dy = \frac{4}{\pi}\int_0^{\pi/2}\int_0^1 r^2\cos^2\theta\, r\, dr d\theta$

$= \frac{4}{\pi}\int_0^{\pi/2}\cos^2\theta\, d\theta \int_0^1 r^3 dr = \frac{4}{\pi}\cdot\frac{\pi}{4}\cdot\frac{1}{4} = 1/4.$

(c) $C_{XY} = E(XY) - E(X)E(Y) = \frac{1}{2\pi} - \left(\frac{2}{\pi}\right)^2 \neq 0$

$\therefore$ correlated

18. $\left(\begin{array}{ll} \mu_X = 6 & \sigma_X = 2 \\ \mu_Y = -3 & \sigma_Y = 4 \end{array} \right) \longrightarrow r = 0$; uncorrelated.

$f_{XY}(x,y) = \frac{1}{2\pi\cdot 2\cdot 4} e^{-\frac{(x-6)^2}{2\cdot 2^2} - \frac{(y+3)^2}{2\cdot 4^2}} = \frac{1}{16\pi} e^{-\frac{(x-6)^2}{8} - \frac{(y+3)^2}{32}}$

20. $E(X) = E(Y) = 0$ $\quad \sigma_X^2 = \sigma_Y^2 = \sigma^2$

$\frac{X}{\sigma}, \frac{Y}{\sigma} \sim N(0,1) \rightarrow \left(\frac{X}{\sigma}\right)^2 + \left(\frac{Y}{\sigma}\right)^2 \sim \chi^2(2)$

$\frac{Z}{\sigma^2} \sim \chi^2(2) \Rightarrow Z \sim \sigma^2\cdot\chi^2(2)$

$\chi^2(2) \sim Exp(\lambda = 1/2) \Rightarrow f_W(w) = 1/2\, e^{-w/2}$, $w>0$

$Z = \sigma^2 W \Rightarrow f_Z(z) = \frac{1}{\sigma^2}\cdot 1/2\, e^{-z/2\sigma^2} = \frac{1}{2\sigma^2} e^{-z/2\sigma^2}$ $\quad z>0$

$\therefore f_Z(z) = \frac{1}{2\sigma^2} e^{-\frac{z}{2\sigma^2}}$

21. $f_Z(z) = \int_{-\infty}^{\infty} f_{XY}\left(\frac{z}{y}, y\right)\frac{dy}{|y|}$

if $Z = XY$, $f_Z(z) = \iint f_{XY}(x,y)\,\delta(z - xy)\,dx dy$

y is fixed, $x = \frac{z}{y}$ $\Rightarrow \delta(z - xy) = \delta(x - z/y)\cdot\frac{1}{|y|}$

$f_Z(z) = \int_{-\infty}^{\infty}\underbrace{\int_{-\infty}^{\infty} f_{XY}(x,y)\cdot\delta(x - z/y)\cdot\frac{1}{|y|}dx}_{=0 \text{ unless } x = \frac{z}{y}}\cdot dy$

$= \int_{-\infty}^{\infty} f_{XY}\left(\frac{z}{y}, y\right)\cdot\frac{1}{|y|}dy$

22. $\begin{bmatrix} U \\ V \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{=A}\begin{bmatrix} X \\ Y \end{bmatrix}$ $\qquad \mu_{UV} = A\mu_{XY} = 0$, $\Sigma_{UV} = A\Sigma_{XY}A^T$

$f_{UV}(u,v) = \frac{1}{2\pi|\Sigma_{UV}|^{1/2}} e^{-1/2[u\ v]\Sigma_{UV}^{-1}\begin{bmatrix} u \\ v \end{bmatrix}}$ $\qquad$ let $\Sigma = \begin{bmatrix} \sigma_U^2 & \rho\sigma_U\sigma_V \\ \rho\sigma_U\sigma_V & \sigma_V^2 \end{bmatrix}$

$[u\ v]\Sigma^{-1}\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sigma_U^2\sigma_V^2(1-\rho^2)}\cdot[u^2\sigma_V^2 - 2uv\rho\sigma_U\sigma_V + v^2\sigma_U^2]$ $\qquad \Sigma^{-1} = \frac{1}{\det(\Sigma)}\begin{bmatrix} \sigma_V^2 & -\rho\sigma_U\sigma_V \\ -\rho\sigma_U\sigma_V & \sigma_U^2 \end{bmatrix}$

$\det\Sigma = \sigma_U^2\sigma_V^2(1-\rho^2)$

$\therefore\ 1/2[u\ v]\Sigma^{-1}\begin{bmatrix} u \\ v \end{bmatrix} = Au^2 + Buv + Cv^2$

where $A = \frac{1}{2(1-\rho^2)}\cdot\frac{1}{\sigma_V^2}$, $B = -\frac{\rho}{(1-\rho)^2}\cdot\frac{1}{\sigma_U\sigma_V}$, $C = \frac{1}{2(1-\rho^2)}\cdot\frac{1}{\sigma_V^2}$

(U,V) are linear combinations of jointly Gaussian $(X,Y) \longrightarrow$ Therefore, (U,V) are jointly Gaussian.

4.19 is a conditional joint distribution, while this problem is a linear transformation of the entire joint system. Both cases preserve Gaussianity.

23. $X_i \sim U(0.99, 1.01)$ km $\quad N=10 \quad \sigma = 0.01$ km

$\mu = \frac{a+b}{2} = 1$ $\qquad \sigma^2 = \frac{(b-a)^2}{12} = \frac{0.02^2}{12}$

$S = \sum_{i=1}^{10} x_i \Rightarrow \mu_S = \mu = 9$ $\quad \sigma_S^2 = 9\cdot\sigma^2 = 3\cdot\frac{0.02^2}{4}$

$Z = \frac{S-9}{\sigma_S} \sim N(0,1)$

$P(8.8 \le S \le 9.2) = P\left(\frac{8.8-9}{0.01732} \le Z \le \frac{9.2-9}{0.01732}\right)$

$= \int_{-2\sqrt{10}}^{\sqrt{10}} \frac{e^{-z^2/2}}{2\pi}dz = 1 - \underbrace{2Q(2\sqrt{10})}_{\approx 0.} = 1$

26. (a) $\sigma_{X_1}^2 = 2$, $\sigma_{X_2}^2 = 5$, $\sigma_{X_3}^2 = 3$

$\rho_{12} = 0.2$, $\rho_{13} = 0.1$, $\rho_{23} = 0.4$

$\left(\begin{array}{l} C_{12} = 0.2\sqrt{10} = 0.6325 \\ C_{13} = 0.1\sqrt{6} = 0.2449 \\ C_{23} = 0.4\sqrt{15} = 1.5492 \end{array} \right.$

$Var(Z) = 2+5+3+2(0.6325+0.2449+1.5492) = 14.85$

(b) $W = 5^2\cdot 2 + 2^2\cdot 5 + 1^2\cdot 3 + 2\cdot 5\cdot 2\cdot 0.6325 + 2\cdot 5\cdot 1\cdot 0.2449$

$+ 2\cdot 2\cdot 1\cdot 1.5492 = 50+20+3+20\cdot 0.6325 + 10\cdot 0.2449$

$+ 4\cdot 1.5492 = 94.3$

[컴퓨터 연습문제]

1. (a) $R=\sqrt{-2\ln U}$ $\quad \Theta = 2\pi V \quad \longrightarrow \quad X = R\cos\Theta \quad Y = R\sin\Theta$

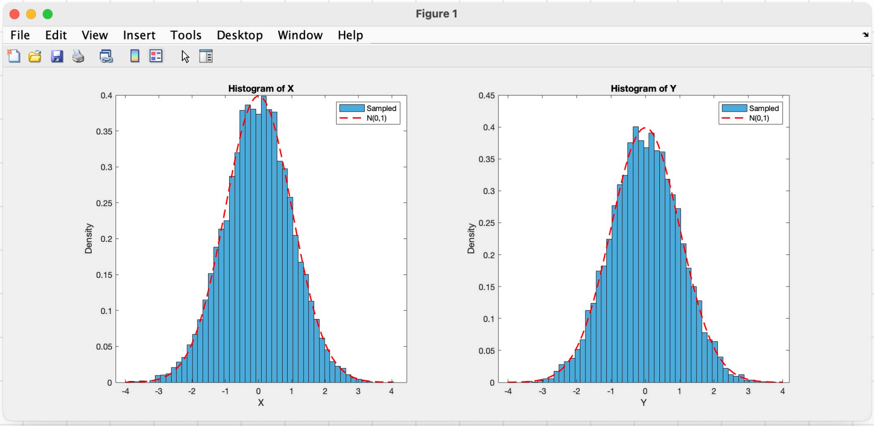
$x = r\cos\theta,\ y = r\sin\theta \quad \Rightarrow r = \sqrt{x^2+y^2} \qquad J = r$

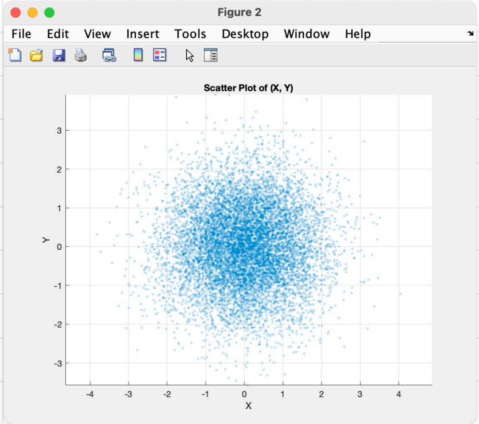
since $U \sim U(0,1)$ $\quad f_R(r) = re^{-r^2/2} \quad r \geq 0$ (Rayleigh distr.)

$f_{R\Theta}(r,\theta) = f_R(r)\cdot f_\Theta(\theta) = re^{-r^2/2}\cdot \frac{1}{2\pi} \qquad r\geq 0,\ \theta\in[0,2\pi]$

$f_{XY}(x,y) = f_{R\Theta}(r,\theta)\,J^{-1} = \left(re^{-r^2/2}\cdot\frac{1}{2\pi}\right)\cdot\frac{1}{r} = \frac{1}{2\pi}e^{-r^2/2} = \frac{1}{2\pi}e^{-(x^2+y^2)/2} \quad \Rightarrow X, Y \sim N(0,1)$, independent.

(b)





Histograms of both variables show the characteristic of a bell-shaped curve of the standard normal distribution. Furthermore, the scatter plot of the generated (X, Y) pairs shows a circularly symmetric distr. centered at the origin. Absence of any visible correlation in the scatter cloud further supports the conclusion that X and Y are both independent and Gaussian distributed.

Editor – /Users/rachelie/Documents/MATLAB/a4_1.m *

a1_1.m | a1_2.m | a1_3.m | a2_1.m | a2_2.m | a3_1.m | a4_1.m *

```
N = 10000;

U = rand(N, 1);
V = rand(N, 1);

R = sqrt(-2 * log(U));
X = R .* cos(2 * pi * V);
Y = R .* sin(2 * pi * V);

x_vals = linspace(-4, 4, 100);
std_normal_pdf = (1 / sqrt(2 * pi)) * exp(-x_vals.^2 / 2);

figure;
subplot(1,2,1);
histogram(X, 50, 'Normalization', 'pdf');
hold on;
plot(x_vals, std_normal_pdf, 'r--', 'LineWidth', 1.5);
title('Histogram of X');
xlabel('X'); ylabel('Density');
legend('Sampled', 'N(0,1)');

subplot(1,2,2);
histogram(Y, 50, 'Normalization', 'pdf');
hold on;
plot(x_vals, std_normal_pdf, 'r--', 'LineWidth', 1.5);
title('Histogram of Y');
xlabel('Y'); ylabel('Density');
legend('Sampled', 'N(0,1)');

figure;
scatter(X, Y, 10, 'filled', 'MarkerFaceAlpha', 0.2);
axis equal;
xlabel('X'); ylabel('Y');
title('Scatter Plot of (X, Y)');
grid on;
```

Command Window