[A3] 제3장 2022310853 박연우

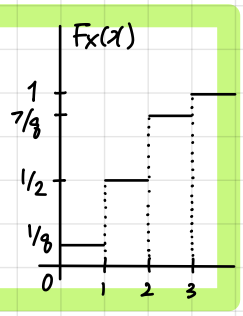
*연습문제 : 2, 4, 6, 7, 9, 11, 12, 13, 15, 16, 17, 18, 20, 22, 25, 27, 28, 31, 34, 36

*컴퓨터연습문제 : 1

연습문제

2. Let X = # of heads

X	P(X)
0	${}_3C_0(\frac{1}{2})^3 = \frac{1}{8}$
1	${}_3C_1(\frac{1}{2})^3 = \frac{3}{8}$
2	${}_3C_2(\frac{1}{2})^3 = \frac{3}{8}$
3	${}_3C_3(\frac{1}{2})^3 = \frac{1}{8}$



4. cdf properties :

1) $\lim_{x\to-\infty} F_X(x) = 0$, $\lim_{x\to\infty} F_X(x) = 1$

2) $F_X(x_0) = \lim_{x\to x_0^+} F_X(x)$

3) monotone non-decreasing function of x

4) $P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

(a) $F_1(x) = (1-e^{-2x})u(x)$ Yes

1) $\lim_{x\to-\infty} F_1(x) = 0$, $\lim_{x\to\infty} F_1(x) = 1$

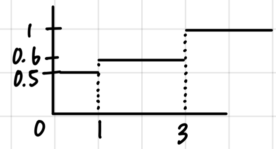
2) $F_1(x_0) = \lim_{x\to x_0^+} F_1(x)$

3) $\frac{dF_1(x)}{dx} = 2e^{-2x} > 0$ for all x 4) ✓

(b) $F_2(x) = 0.5u(x) + 0.1u(x-1) + 0.3u(x-3)$ No

1) $\lim_{x\to-\infty} F_2(x) = 0$ $\lim_{x\to\infty} F_2(x) = 0.9 \ne 1$

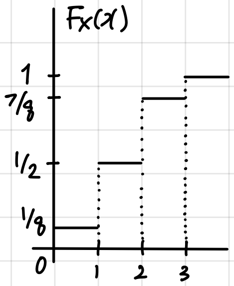
(c) $F_3(x) = 0.5u(x) + 0.1u(x-1) + 0.4u(x-3)$ Yes



1) $\lim_{x\to-\infty} F_3(x) = 0$ $\lim_{x\to\infty} F_3(x) = 1$

2) ✓ 3) see graph 4) ✓

6.



$F_X(x) = \frac{1}{8}u(x) + \frac{3}{8}u(x-1)$
$+ \frac{3}{8}u(x-2) + \frac{1}{8}u(x-3)$

7. $F_X(x) = (1-e^{-x/5})u(x)$

→ continuous r.v라 equal이 붙어도 ok.

(a) $P(5 \le X \le 7)$

$F_X(7) - F_X(5) = (1-e^{-7/5}) - (1-e^{-1}) = 0.12128$

(b) $P(X \ge 3)$

$1 - P(X<3) = 1 - F_X(3) = 1-(1-e^{-3/5}) = 0.54881$

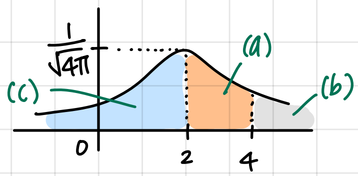
(c) $P(X<3)$

$F_X(3) = 1-e^{-3/5} = 0.45119$

9. $f_X(x) = \frac{e^{-(x-2)^2/4}}{\sqrt{4\pi}}$ $m=2$, $\sigma^2=2$

$F_X = 1 - Q\left(\frac{x-2}{\sqrt{2}}\right)$

where, $Q = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{(x-2)^2}{4}}\frac{}{\sqrt{4\pi}}$



(a) $P(2 \le X \le 4)$

$F_X(4) - F_X(2) = \left(1-Q\left(\frac{4-2}{\sqrt{2}}\right)\right) - \left(1-Q\left(\frac{2-2}{\sqrt{2}}\right)\right)$

$= Q(0) - Q(\sqrt{2}) = 0.5 - Q(\sqrt{2}) \approx 0.42$

(b) $P(X>4)$

$1 - F_X(4) = 1 - \left(1-Q\left(\frac{4-2}{\sqrt{2}}\right)\right) = Q(\sqrt{2}) \approx 0.08$

(c) $P(X<2) = 0.5$

11. $n=1000$ $p=0.01$

(a)

0 errors : $\binom{1000}{0}(0.01)^0(0.99)^{1000} = 4.317\times10^{-5}$

1 error : $\binom{1000}{1}(0.01)^1(0.99)^{999} = 4.36\times10^{-4}$

2 errors : $\binom{1000}{2}(0.01)^2(0.99)^{998} = 0.0022$

+ 3 errors : $\binom{1000}{3}(0.01)^3(0.99)^{997} = 0.00739$

$= 0.01007265$

4개 이상의 오류가 발생할 확률 : $1-0.01007265 = 0.989$

(b) $\binom{n}{k}p^k(1-p)^{n-k} \approx \frac{(np)^k}{k!}e^{-np} = \frac{10^k}{k!}e^{-10}$

poisson approximation : $np=10$

0 errors : $\frac{10^0}{0!}e^{-10} = 4.54\times10^{-5}$

1 error : $(10^1/1!)e^{-10} = 4.54\times10^{-4}$

2 errors : $(10^2/2!)e^{-10} = 2.27\times10^{-3}$

+ 3 errors : $(10^3/3!)e^{-10} = 7.566\times10^{-3}$

$= 0.0103$

4개 이상의 error가 발생할 확률 : $1-0.0103 = 0.989$

이항분포식 결과 ≈ 포아송 분포식 결과

12. 여자가 태어날 prob = 1/2

여아 3명일 확률: $\binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{4!}{3!\,1!}\cdot \frac{1}{16} = 0.25$

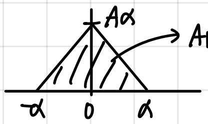
13. (a) $f_1(x) = \frac{A}{(\alpha+x)^2}u(x)$

$\int_0^\infty \frac{A}{(\alpha+x)^2}dx = A\int_0^\infty \frac{1}{(\alpha+x)^2}dx$ let $t = \alpha + x$, $dt = dx$

$= A\int_0^\infty \frac{1}{t^2}dt = A\left(-t^{-1}\Big|_0^\infty\right) = A\left(-\frac{1}{\alpha+x}\Big|_0^\infty\right)$

$= \frac{A}{\alpha} = 1$ $\therefore A = \alpha$

(b) $f_2(x) = \begin{cases} A(\alpha - |x|) & |x| \le \alpha \\ 0 & \text{else}\end{cases}$



Area: $\frac{2\alpha \cdot A\alpha}{2} = A\alpha^2 = 1$ $\therefore A = \frac{1}{\alpha^2}$

15. $100/60 = 5/3$ calls/min $f_W(w) = \lambda e^{-\lambda w}u(w)$

$F_W(w) = 1 - e^{-\lambda w} = 1 - e^{-5/3 w}$

(a) $P(W \ge 1)$

$= 1 - P(W<1) = 1 - F_W(1) = 1 - (1 - e^{-5/3}) = 0.18888$

(b) $P(W \ge 0.5)$

$= 1 - P(W < 0.5) = 1 - F_W(0.5) = 1 - (1 - e^{-5/6}) = 0.4346$

16. (a) $(0.9)^5(0.1) = 0.059049$ → fail to catch 5 times

(b) $(0.9)^{10}(0.1) = 0.034868$ → fail to catch 10 times

17. error prob = 0.01

(a) $\binom{24}{2}(0.01)^2(0.99)^{22}(0.01) = 2.21250 \times 10^{-4}$

(b) $\binom{99}{2}(0.01)^2(0.99)^{97}(0.01) = 0.00182998$

(c) $\binom{499}{2}(0.01)^2(0.99)^{497}(0.01) = 8.413789 \times 10^{-4}$

18. error rate : 0.001 $n = 1000$ sample = 5

hypergeometric r.v.

$P(X=k) = \frac{\binom{1}{0}\binom{1000-1}{5-0}}{\binom{1000}{5}} = 0.995$

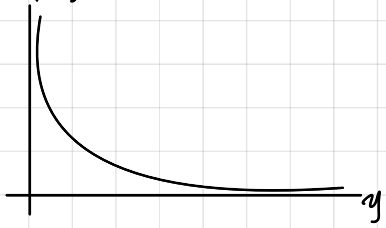
20. $f_X(x) = 3e^{-3x}u(x)$

(a) $Y = X^2$ $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|_{x = g^{-1}(y)}$

$dy/dx = 2x$ $\frac{dx}{dy} = \frac{1}{2x}$ $x = \pm\sqrt{y} \to \sqrt{y}$ bc $x \ge 0$

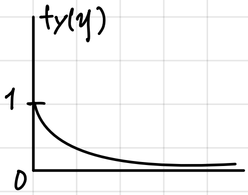
$f_Y(y) = f_X(\sqrt{y})\cdot\frac{1}{2\sqrt{y}} = \frac{3e^{-3\sqrt{y}}}{2\sqrt{y}}\cdot u(\sqrt{y})$, $y \ge 0$

$f_Y(y)$



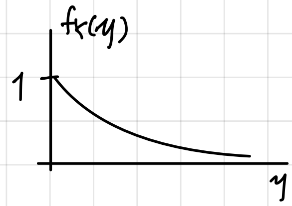
(b) $Y = |X|$

$\left|\frac{dx}{dy}\right| = 1$ $f_Y(y) = f_X(y)\cdot 1 = 3e^{-3y}u(y)$ $y \ge 0$



(c) $Y = Xu(X)$ $u(X)$ 단위계단함수

$\begin{cases} x & (x>0) \to y = x \\ 0 & (x<0) \to y = 0\end{cases}$



$f_Y(y) = f_X(y)\cdot 1 + f_X(0)\cdot 0 = 3e^{-3y}u(y)$

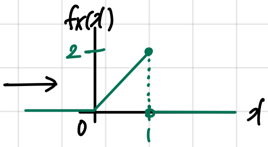
22. $Y = -\ln x \to x = e^{-Y}$, $X \in (0,1]$

$Y \in [0,\infty)$

$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else}\end{cases}$ $\left|\frac{dx}{dy}\right| = \left|-e^{-y}\right| = e^{-y}$, $y \ge 0$

$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = 1\cdot e^{-y} = e^{-y}$ $f_Y(y) = e^{-y}$ $y \ge 0$

25. $f_X(x) = \begin{cases} 0 & x<0 \\ 2x & 0 \le x \le 1 \\ 0 & x>1\end{cases}$ →



mode: pdf의 max ⇒ $x = 1$

mean: $E[X] = \int_0^1 x f_X(x)dx = \int_0^1 x\cdot 2x\,dx = \frac{2}{3}x^3\Big|_0^1 = 2/3$

median: $P(X \le m) = 0.5$

$F_X(x) = \int_0^x 2t\,dt = [t^2]\Big|_0^x = x^2$ $0 \le x \le 1$

$F_X(x) = m^2 = 0.5$

$m = \sqrt{2}/2 = 0.707$

27. 합이 7이 나오는 경우: (1,6) (2,5) (3,4) (4,3) (5,2) (6,1)

→ 총 6가지. $\therefore 6/36 = 1/6 = p$

$E(X) = 1/p = 1/(1/6) = 6$ ∴ 6번

28. $\mu = 0$ $\sigma = 1$ → $\mu' = 2$, $\sigma' = 3$

$E(Y) = aE(X) + b = a\cdot 0 + b = 2$ $\therefore b = 2$

$\sigma_Y = |a|\cdot\sigma_X = |a|\cdot 1 = 3$ $\therefore a = \pm 3$

31. $f_X(x) = 3e^{-3x}u(x) \to \begin{cases} 3e^{-3x} & x \ge 0 \\ 0 & x<0\end{cases}$

$M_X(jv) = \int_{-\infty}^\infty f_X(x)e^{jvx}dx = \int_0^\infty 3e^{-3x}\cdot e^{jvx}dx + 0$

$= \int_0^\infty 3e^{(jv-3)x}dx = \frac{3}{jv-3}e^{(jv-3)x}\Big|_0^\infty$

$= \frac{-3}{jv-3}$

$\frac{dM_X(jv)}{dv}\Big|_{v=0} = -\frac{-3}{(jv-3)^2}\cdot j\Big|_{v=0} = \frac{3j}{9} = j/3$

$f_X(x) = \lambda e^{-\lambda x}$ $E(X) = 1/\lambda = 1/3$ $Var(X) = 1/\lambda^2 = 1/9$

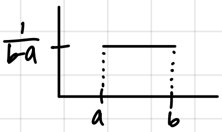
or $-j\cdot j/3 = 1/3$

$M_X(jv) = \frac{3}{3-jv}$, mean : 1/3, var: 1/9

34. (a) $P(|X-\mu_X| \ge k\sigma) \le 1/k^2$ $\quad k=2$ $\quad$ chebyshev's inequality

$P(|X-\mu_X| \ge 2\sigma) \le 1/4$

(b) $f_X(x) = \begin{cases} 1/(b-a), & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$



$$E(X^2) = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\cdot\frac{1}{3}x^3\Big|_a^b = \frac{1}{3}\,\frac{b^3-a^3}{b-a}$$

$$= \frac{1}{3}\cdot\frac{(b-a)(b^2+ab+a^2)}{b-a} = \frac{1}{3}\cdot\frac{a^2+ab+b^2}{b-a}$$

$$\sigma_X^2 = E[(X-\mu_X)^2]$$

$$= E(X^2 - 2X\mu_X + \mu_X^2) = E(X^2) - \mu_X^2$$

$$= \frac{1}{3}\cdot\frac{a^2+ab+b^2}{b-a} - \frac{(a+b)^2}{4} = \frac{(a-b)^2}{12}$$

$$\to \sigma_X = \frac{-(a-b)}{\sqrt{12}}$$

Let $X \sim \text{Uniform}(0,1) \to \mu = \frac{0+1}{2} = 0.5$

$\sigma = \frac{-(0-1)}{\sqrt{12}} = 0.2887$

$2\sigma = 0.577$

$$P(|X-0.5| \ge 0.577) = \underbrace{P(X \le 0.5-0.577)}_{0} + \underbrace{P(X \ge 0.5+0.557)}_{0}$$

$\therefore P(|X-\mu| \ge 2\sigma) = 0$

(a)에서 구한 값보다 작다.

(c) $f_X(x) = \alpha e^{-\alpha x}u(x)$

$$P(|X-\mu_X| \ge 2\sigma_X) = 1 - \int_{\mu_X-2\sigma_X}^{\mu_X+2\sigma_X} \alpha e^{-\alpha x}\,dx$$

$$= 1-(-e^{-\alpha x})\Big|_{\mu_X-2\sigma_X}^{\mu_X+2\sigma_X} \qquad \mu_X : \int_0^\infty \alpha x e^{-\alpha x}\,dx$$

$$= \alpha\left(\left(-\frac{x}{\alpha}e^{-\alpha x}\right)\Big|_0^\infty + \frac{1}{\alpha}\int_0^\infty e^{-\alpha x}\,dx\right)$$

$$= \int_0^\infty e^{-\alpha x}\,dx = -\frac{1}{\alpha}e^{-\alpha x}\Big|_0^\infty = 1/\alpha$$

$$\sigma_X^2 = E(X^2) - \mu_X^2 = E(X^2) - \frac{1}{\alpha^2}$$

$$E(X^2) = \int_0^\infty \alpha x^2 e^{-\alpha x}\,dx = \cancel{\alpha}\left(\cancel{+}\int_0^\infty \cancel{+}\frac{2x}{\cancel{\alpha}}e^{-\alpha x}\right) = \frac{2}{\alpha^2}$$

$$\to \sigma_X = \frac{1}{\alpha}$$

If $X \sim \text{Exponential}(\overset{=\alpha}{\lambda})$ and $\lambda = 1 \to \mu = \frac{1}{\lambda} = 1$

$\sigma = \frac{1}{\lambda} = 1$

$2\sigma = 2$

$P(|X-\mu| \ge 2)$

$$= P(X \le \mu-2) + P(X \ge \mu+2) = \underbrace{P(X \le -1)}_{0 \text{ since } X \ge 0} + P(X \ge 3)$$

$$P(X \ge 3) = 1 - F(3) = 1-(1-e^{-3}) = e^{-3} = 0.0498$$

$0.0498 < 0.25$

(a)에서 구한 값보다 작다.

36. $f_V(v) = \begin{cases} \frac{v}{\sigma^2}e^{-v^2/2\sigma^2}, & v > 0 \\ 0, & v < 0 \end{cases}$ $\qquad U \sim \text{Uniform}(0,1)$

$$F_V(v) = \int_0^v \frac{t}{\sigma^2}e^{-t^2/2\sigma^2}\,dt \qquad \text{let } \frac{t^2}{2\sigma^2} = a \qquad \frac{t}{\sigma^2}dt = da$$

$$= \int_0^{\frac{v^2}{2\sigma^2}} e^{-a}\,da = -e^{-a}\Big|_0^{\frac{v^2}{2\sigma^2}} = -e^{-\frac{v^2}{2\sigma^2}} + 1 \qquad v \ge 0$$

$$U = 1 - e^{-v^2/2\sigma^2}$$

$$-\frac{v^2}{2\sigma^2} = \ln(1-U)$$

$$v^2 = -2\sigma^2\ln(1-U)$$

$\therefore v = \sqrt{-2\sigma^2\ln(1-U)}$, where $U \sim \text{Uniform}(0,1)$

or $v = \sqrt{-2\sigma^2\ln(U)}$ since $1-U \sim \text{Uniform}(0,1)$ as well

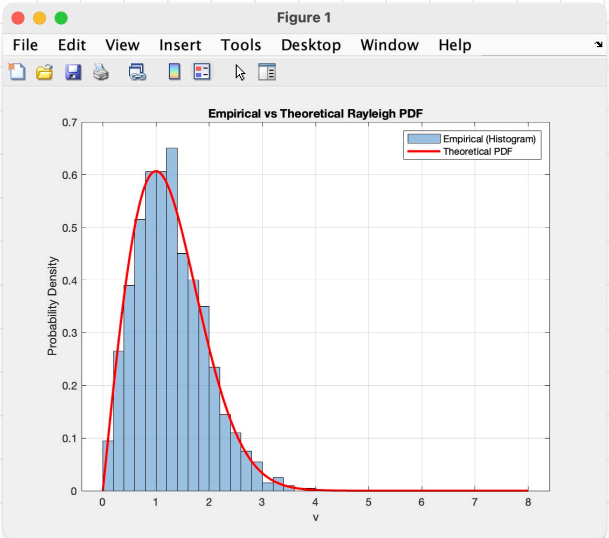
컴퓨터 실습문제

1.

Editor – /Users/rachelie/Documents/MATLAB/a3_1.m

a1_1.m | a1_2.m | a1_3.m | a2_1.m | a2_2.m | a3_1.m

```
sigma = 1;
n = 1000;

% generate uniform~(0,1) random variables
U = rand(n, 1);

% transform to rayleigh using inverse transform
V = sqrt(-2 * sigma^2 * log(1 - U));

% plot histogram
binWidth = 0.2;
edges = 0:binWidth:8;
histogram(V, edges, 'Normalization', 'pdf', 'FaceColor', [0.4 0.6 0.8])
hold on

% plot theoretical rayleigh
v = linspace(0, 8, 1000);
pdf_rayleigh = (v ./ sigma^2) .* exp(-v.^2 / (2 * sigma^2));
plot(v, pdf_rayleigh, 'r', 'LineWidth', 2)

% labels
title('Empirical vs Theoretical Rayleigh PDF')
xlabel('v')
ylabel('Probability Density')
legend('Empirical (Histogram)', 'Theoretical PDF')
grid on
```



실측데이터와 theoretical rayleigh pdf가 거의 완벽하게 일치함.