Principle of Simple Induction

$$\underbrace{P(0) \wedge \left(\forall n,\ P(n) \Rightarrow P(n+1) \right)}_{\text{what we prove}} \Rightarrow \underbrace{\forall n,\ P(n)}_{\text{what we conclude}}$$

Example: What amounts of money n that can be made up using only 3¢ and 7¢ coins?

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
✓	✗	✗	✓	✗	✗	✓	✓	✗	✓	✓	✗	✓	✓	✓	✓

16	17	18	19...
✓	✓	✓	✓

Intuition: 0¢ - 11¢: case varies
starting at 12¢: always yes

$$\left.\begin{array}{l} 12 \xrightarrow{+3¢} 15 \rightarrow 18 \ \ldots \\ 13 \rightarrow 16 \rightarrow 19 \ \ldots \\ 14 \rightarrow 17 \rightarrow 20 \ \ldots \end{array}\right\} \text{Induction!}$$

Formalize

Predicate:

- For each $n \in \mathbb{N}$, let $P(n)$ be:

$\cancel{\forall n \in \mathbb{N}},\ \exists t, s \in \mathbb{N},\ n = 3t + 7s$

Cannot quantify n inside, or else it becomes a dummy variable

WTP: $\forall n \geq 12, P(n)$

$(\forall n \in \mathbb{N}, n \geq 12 \Rightarrow P(n))$

Inductive Structure:

$P(12) \wedge$
$P(13) \wedge$
$P(14) \wedge$ $\forall n \geq 15, P(n-3) \Rightarrow P(n)$

$\forall n \geq 12, P(n) \Rightarrow P(n+3)$

} equivalent statements! (See prep 2)

Homework: write the proof

Formalize as code:

```
# pre(n): n ∈ ℕ and n ≥ 12                          (pre-condition)
# Return (t, s) such that Post(n, t, s): t, s ∈ ℕ ∧ n = 3t + 7s
                                                    (post-condition)
def change(n):
    if n == 12: return (4, 0)
    if n == 13: return (2, 1)
    if n == 14: return (0, 2)
    # else (n ≥ 15)
    (t, s) = change(n - 3)
    return t + 1, s
```

Recursion	Induction
• structure ↳ no ∞ loop ↳ no crashing	◦ structure ↳ simple vs. complete
◦ correctness (postcond.)	◦ predicate (result)

```
# Pre(n): n ∈ ℕ  ∧  n ≥ 12
# return t, s ∈ ℕ  s.t.  n = 3t + 7s        Post(n, t, s)

def change(n):

    if n == 12:  return (4,0)
    if n == 13:  return (2,1)
    if n == 14:  return (0,2)

    # else     KNOWN:  n ≥ 15
    # NEEDED: ◦ Pre(n-3):  n-3 ∈ ℕ ✓  and  n-3 ≥ 12 ✓
              ◦ n-3 < n ✓  smaller problem

    (t,s) = change(n-3)

    # "Recursive Hypothesis":  Post(n-3, t, s) :  n-3 = 3t + 7s

    return (t+1, s)

    # t + 1, s   satisfies   Post(n, t+1, s)
```

If we meet these conditions, then we know our recursive structure is sound

⇕

no ∞ loop, no crashing

Formalizing as proof:

$\forall n \in \mathbb{N},\ n \geq 12 \Rightarrow \exists t, s \in \mathbb{N},\ n = 3t + 7s$

$\equiv \quad \forall n \geq 12,\ \underbrace{\exists t, s \in \mathbb{N},\ n = 3t + 7s}_{P(n)}$

Pf.

Algorithm starts w/ n that meets Pre(n)

Let $n \in \mathbb{N}$ with $n \geq 12$.

Two equally good ways to prove

- Case $n = 12$: Then, $12 = 3 \cdot 4 + 7 \cdot 0$
- Case $n = 13$: Let $t = 2$, $s = 1$. Then, $t, s \in \mathbb{N}$ and $3t + 7s = 3 \cdot 2 + 7 \cdot 1 = 6 + 7 = 13$
- Case $n = 14$: (Similarly)
- Case $n \geq 15$:

Assume $P(n-3)$. (Inductive hypothesis) This is the justification in algorithm.

(Why? $n - 3 \in \mathbb{N}$ $\because$ $n \in \mathbb{N}$
$n - 3 \geq 12$ $\because$ $n \geq 15$
$n - 3 < n$)

Let $t, s \in \mathbb{N}$ witness $P(n-3)$, so $n - 3 = 3t + 7s$.

So, $n = 3t + 3 + 7s = 3(t+1) + 7s$

so, $t+1, s \in \mathbb{N}$ and witness $P(n)$