Studying Kolmogorov flow with persistent homology

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A model for turbulence in 2D: Kolmogorov Flow.

The velocity field $\mathbf{u}(\mathbf{x}, \mathbf{y}, \mathbf{t})$ is given by

$$\frac{\partial \mathbf{u}}{\partial t} + \beta \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} - \alpha \mathbf{u} + \mathbf{f}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0.$$

where:

- ρ is the fluid density
- p is the pressure
- ν is the viscosity
- $\mathbf{f} = \chi \sin(\kappa \mathbf{y}) \hat{\mathbf{x}}$ is the forcing that drives the flow
- β and α are parameters to take into account 3D effects (commonly present in experiments)

Useful to re-write Equation (1) in terms of the z-component of the vorticity field $\omega = (\nabla \times \mathbf{u}) \cdot \hat{\mathbf{k}}$, a scalar field, to get

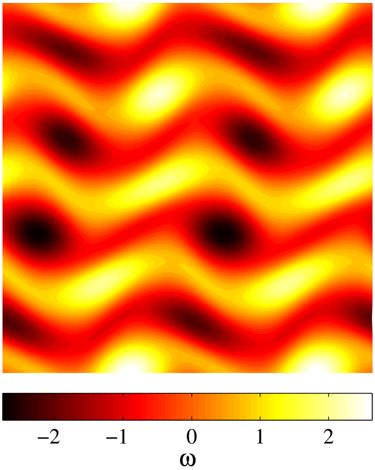
$$\frac{\partial \omega}{\partial t} + \beta \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega - \alpha \omega + \chi \kappa \cos(\kappa y). \qquad (2)$$

For this study, we make the following choices:

- $\beta = 0.83$
- $\nu = 3.26 \times 10^{-6}$ m^2/s
- $\alpha = 0.063$ s^{-1}
- $\rho = 959$ kg/m^3
- $\lambda = 2\pi/\kappa = 0.0254$ m

The strength of the forcing is then parameterized by a dimensionless parameter called the Reynolds number,

$$Re = \sqrt{\frac{\lambda^3 \chi}{8 \nu^2}}.$$


−2
−1
0
1
2
ω

Impose periodic boundary conditions in both the x and y directions:

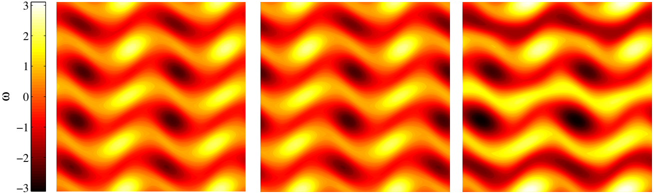
- $\omega(x, y) = \omega(x + L_x, y)$
- $\omega(x, y) = \omega(x, y + L_y)$

where $L_x = 0.085$ m and $L_y = 4\lambda = 0.1016$ m are the dimensions of the domain in the x and y directions, respectively.

Equation (2) with these boundary conditions is invariant under any combination of the following coordinate transformations:

- Translation along x: $\mathcal{T}_{\delta x}(x, y) = (x + \delta x, y)$
- Reflection about the x axis followed by half-period shift along y: $\mathcal{D}(x, y) = (-x, y + \lambda/2)$
- Rotation by π: $\mathcal{R}(x, y) = (-x, -y)$ seen as a rotation by π about the z-axis (vorticity axis)

Consequence: each solution to Equation (2) corresponds to a set of solutions which are dynamically equivalent.



(a) (b) (c)

Figure: One of these things just doesn't belong...

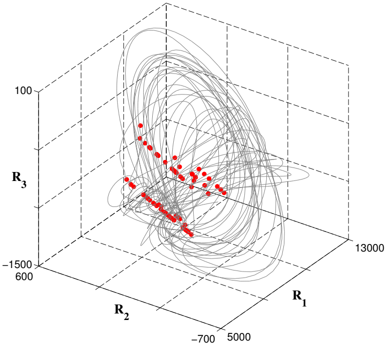
Challenge: identify solutions that are symmetry-related to study relative equilibria (REQ) and relative periodic orbits (RPO).

Two approaches:

- Fourier methods
- Persistent homology

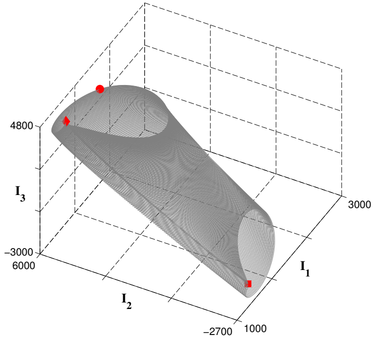
Two examples:

REQ: Weakly turbulent regime at $Re = 26.43$.



Projection onto the real parts of the three dominant Fourier modes. Gray line indicates chaotic evolution of the flow, influenced by the presence of unstable fixed points, indicated in red.

RPO: Flow exhibits a steady relative periodic orbit at $Re = 25.43$.



Projection onto the imaginary parts of the three dominant Fourier modes. Gray line indicates the evolution of an RPO.

General idea:

- Project the scalar fields to the space of persistence diagrams
- Study the dynamics in the space of persistence diagrams

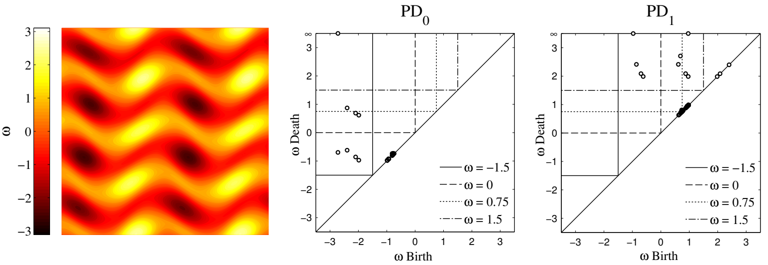
Q: Why persistent homology?

A: Invariance under coordinate transformations.

Step 0. Generate the data.

- **REQ:** Choose initial solutions from chaotic trajectory and use Newton's method to solve for fixed points. Get 67 fixed points.
- **RPO:** Numerically integrate the equation and sample the trajectory at equally-spaced time points. Get 500 samples.

Step 1. Project each scalar field (image) to a vector of persistence diagrams.



(a) (b) (c)

Figure: (a) Vorticity plot and persistence diagrams for (b) H_0 and (c) H_1. (Diagram for H_2 not shown.)

Video.

Step 2. Choose a metric (e.g. d_B, d_{W_p}) and generate the distance matrix D_{ij} corresponding to the projected set of scalar fields in the space of persistence diagrams.

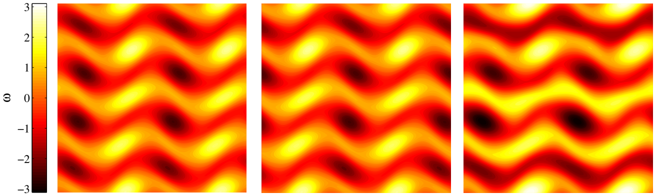
Bottleneck Distance

$$d_B(\mathrm{PD}, \mathrm{PD}') = \max_k \inf_{\gamma\colon \mathrm{PD}_k \to \mathrm{PD}'_k} \sup_{p \in \mathrm{PD}_k} \|p - \gamma(p)\|_\infty,$$

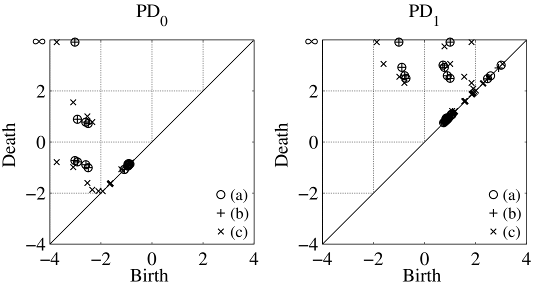
where $\|(a_0, b_0) - (a_1, b_1)\|_\infty := \max\{|a_0 - a_1|, |b_0 - b_1|\}$ and γ ranges over all bijections between persistence points.

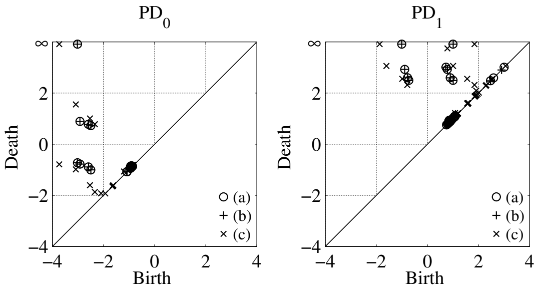
Degree-p Wasserstein Distance

$$d_{W^p}(\mathrm{PD}, \mathrm{PD}') = \left[\sum_k \inf_{\gamma\colon \mathrm{PD}_k \to \mathrm{PD}'_k} \sum_{p \in \mathrm{PD}_k} \|p - \gamma(p)\|_\infty^p\right]^{1/p}.$$



(a) (b) (c)





	d_B	d_{W^2}	d_{W^1}
$(\mathrm{PD}^a, \mathrm{PD}^b)$	0.01	0.049	0.497
$(\mathrm{PD}^a, \mathrm{PD}^c)$	0.864	2.648	12.35

Table: Distances between selected persistence diagrams (rounded to 3 decimal places).

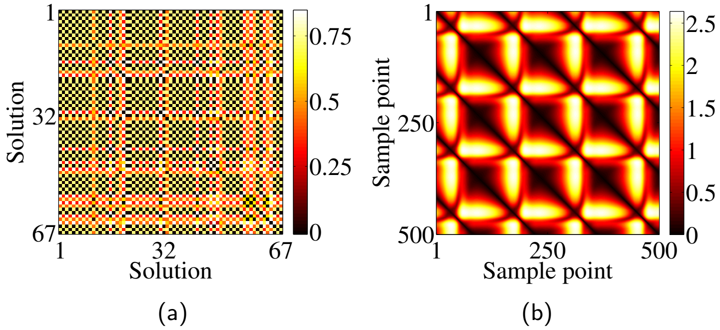


Figure: Distance matrices using d_B for (a) fixed-point solutions and (b) samples taken from stable RPO.

Step 3. Generate a filtration of Vietoris-Rips complexes from D_{ij}.

Vietoris-Rips Complex

Given a point cloud $X = \{x_0, ..., x_N\}$ in a metric space with distance function d, the Vietoris-Rips complex at scale θ, *denoted $\mathcal{R}(X, \theta)$, is the simplicial complex defined by the collection of simplicies*

$$\{x_{n_0}, ..., x_{n_k}\} \in \mathcal{R}(X, \theta)$$

if and only if

$$d(x_{n_i}, x_{n_j}) \leq 2\theta, \text{ for all } i, j \in \{0, 1, 2, ..., k\}.$$

Definition only relies on the distance matrix! Filtration sweeps through the scales θ. Video.

A detailed look at our computational framework so far...

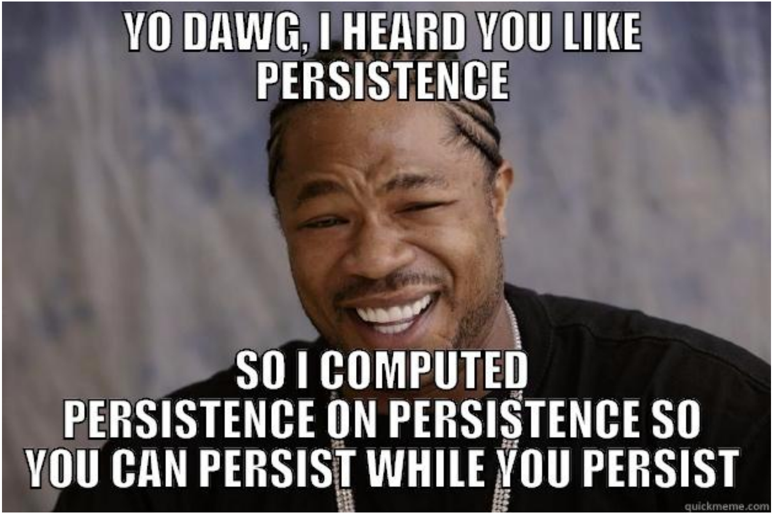
Pick a metric!

↓

Mathematical object:	Scalar field	Point cloud
Input data:	Bitmap image	Distance matrix
Complex structure:	Cubical complex	Vietoris-Rips complex
Filtration:	Sublevelset filtration	Distance filtration
Output data:	Persistence diagrams	Persistence diagrams

↑

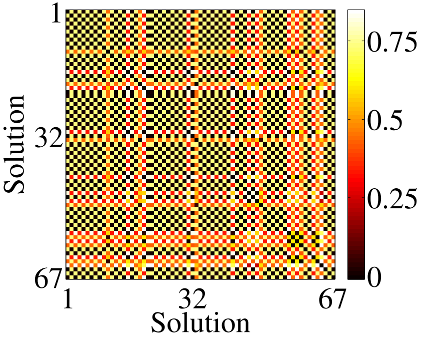
Collection of persistence diagram vectors!

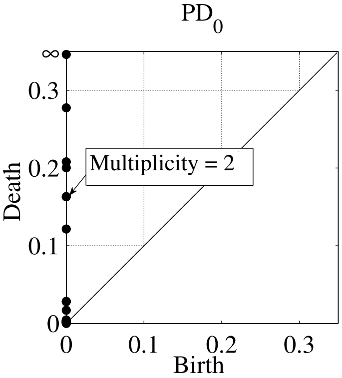

YO DAWG, I HEARD YOU LIKE PERSISTENCE
SO I COMPUTED PERSISTENCE ON PERSISTENCE SO YOU CAN PERSIST WHILE YOU PERSIST
quickmeme.com

Step 4. Compute the persistence diagrams corresponding to the Vietoris-Rips filtration of the point cloud in the space of persistence diagrams.

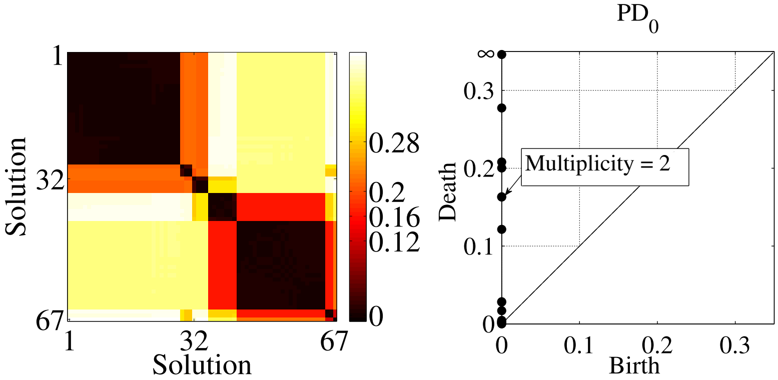
Step 5. Analyze the results.

REQ: Clustering symmetry-related equilibria.



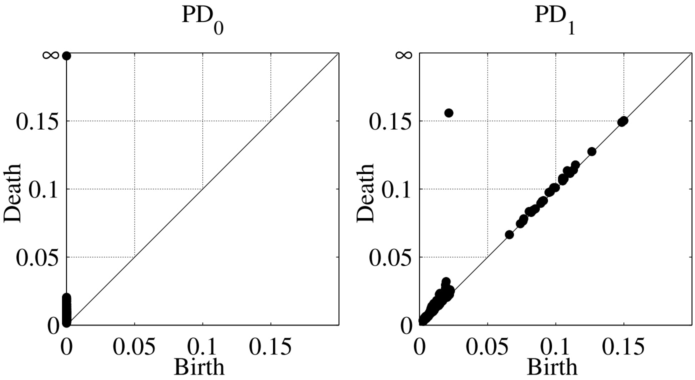


REQ: Clustering symmetry-related equilibria.



Using Fourier method confirms seven distinct clusters of solutions after identifying symmetry-related solutions.

RPO: Studying a relative periodic orbit. Video.



For a closer look, visit arXiv.org

Analysis of Kolmogorov Flow and Rayleigh-Bénard Convection using Persistent Homology

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Abstract

We use persistent homology to build a quantitative understanding of large complex systems that are driven far-from-equilibrium; in particular, we analyze image time series of flow field patterns from numerical simulations of two important problems in fluid dynamics: Kolmogorov flow and Rayleigh-Bénard convection. For each image we compute a persistence diagram to yield a reduced description of the flow field; by applying different metrics to the space of persistence diagrams, we relate characteristic features in persistence diagrams to the geometry of the corresponding flow patterns. We also examine the dynamics of the flow patterns by a second application of persistent homology to the time series of persistence diagrams. We demonstrate that persistent homology provides an effective method both for quotienting out symmetries in families of solutions and for identifying multiscale recurrent dynamics. Our approach is quite general and it is anticipated to be applicable to a broad range of open problems exhibiting complex spatio-temporal behavior.

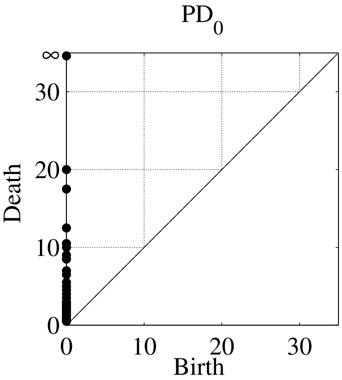
1. Introduction

We introduce new mathematical techniques for analyzing complex spatiotemporal nonlinear dynamics and demonstrate their efficacy in problems from two differ-

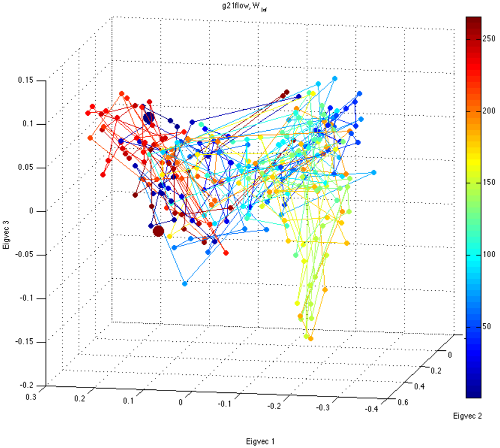
lifespan $\theta_d - \theta_b > 0$ indicates the prominence of the feature. In particular, features with long lifespans are considered important and features with short lifespans are often associated with noise. Thus, the persistence diagram is a highly simplified representation of the field generating the

What about more complicated dynamics?

Rayleigh-Bénard Convection: Almost-periodic orbit. Video.



Rayleigh-Bénard Convection: Spiral-defect chaos.



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  - Balachandra Suri
  - Michael F. Schatz
- Virginia Tech
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- Shaun Harker (Subsampling/Cluster-delegator)
- Miro Kramár (Diffusion Map projection)
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