3
Why a probabilistic description of turbulence?

3.1 There is something predictable in a turbulent signal

In Chapter 1 we presented some pictures chosen to prompt the study of the symmetries of the Navier–Stokes equation. However important flow visualizations may be, experimental data on turbulence also include a considerable body of quantitative results. Velocimetry, the measurement of the flow velocity (or one component thereof) at a given point as a function of time, is by far the most common way of getting quantitative information. There are many different techniques of velocimetry which we shall not review here.

Let us turn directly to an example. Fig. 3.1(a) shows a one-second signal obtained from a hot-wire probe placed in the very large wind tunnel S1 of ONERA.[1] The signal is the 'streamwise' velocity (component parallel to the mean flow). It is sampled five thousand times per second (5 kHz). The mean flow has been subtracted so that the signal appears to fluctuate around zero.

What strikes us when looking at this signal?

(i) The signal appears highly *disorganized* and presents structures on all scales.
(ii) The signal appears *unpredictable* in its detailed behavior.
(iii) Some properties of the signal are quite *reproducible*.

Regarding item (i), we observe that in contrast to the signal shown in Fig. 2.2 which had only two scales present, the signal shown here displays structures on all scales: the eye directly perceives structures with time-scales of the order of one second, of one-tenth of a second, of one-hundredth of a second, and possibly smaller.

[1] We shall come back later to some of the characteristics of this wind tunnel.

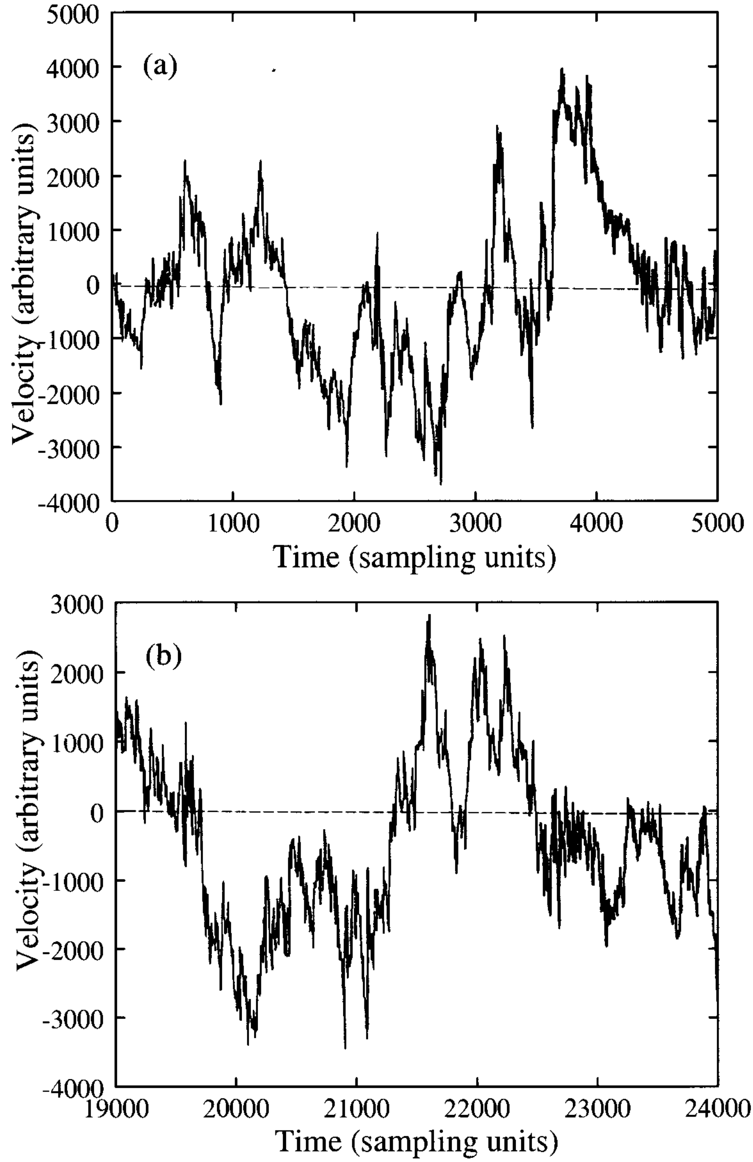


Fig. 3.1. One second of a signal recorded by a hot-wire (sampled at 5 kHz) in the S1 wind tunnel of ONERA (a); same signal, about four seconds later (b). Courtesy Y. Gagne and E. Hopfinger.

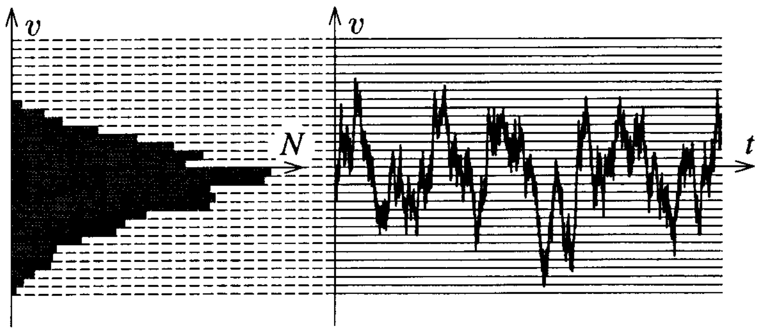


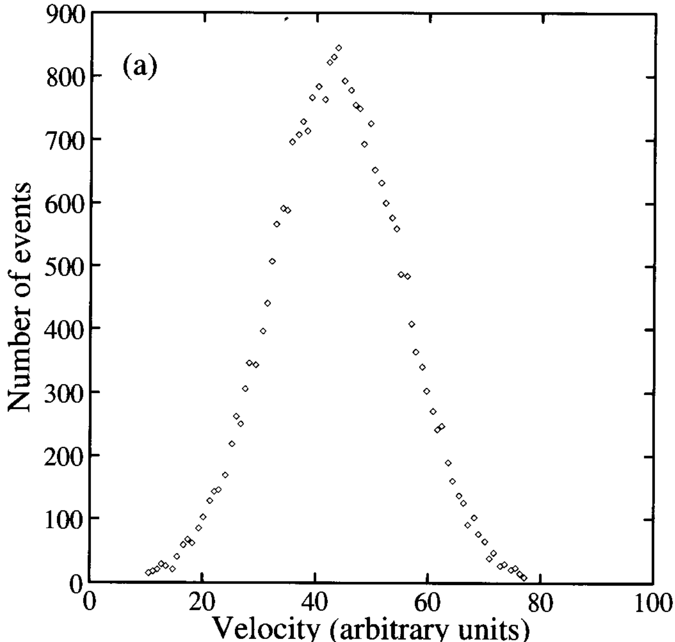
Fig. 3.2. Construction of the histogram of a signal by binning.

Regarding item (ii), let us look at a sample of the same duration taken about four seconds later (Fig. 3.1(b)). The general aspect is the same but all the details are different and could not have been predicted from looking at the previous figure.

Regarding item (iii), one instance of a reproducible property is the *histogram* of the signal. As shown in Fig. 3.2, let us take a finite record of the (discretely sampled) signal and divide the v-axis into a large number N of equal small bins centered around values v_i $(i = 1, \cdots, N)$. The histogram is defined as the function N_i giving the number of times the ith bin is visited. Let us apply this procedure to the signal from the S1 wind tunnel. Fig. 3.3(a) shows the histogram obtained from a record of duration 150 s sampled 5000 times with 100 bins.[2] Fig. 3.3(b) shows the same sort of histogram taken from a record of the same duration but several minutes later. (Several hours would work equally well.) We see that the two histograms are essentially identical.

We can summarize our findings by saying that although the detailed properties of the signal appear not to be predictable, its *statistical properties* are reproducible. Such observations, which have been known for a long time, have induced theoreticians to look for a *probabilistic description* of turbulence (Taylor 1935, 1938). However, we know that the basic equation (Navier–Stokes) is *deterministic*: Although there is no rigorous proof of this, it is widely conjectured that for a given initial condition there is a unique solution for all times. How can *chance* or *chaos* arise in

[2] Why the record has to be much longer than before will become clear in Section 4.4 once we have introduced the concept of integral time scale.



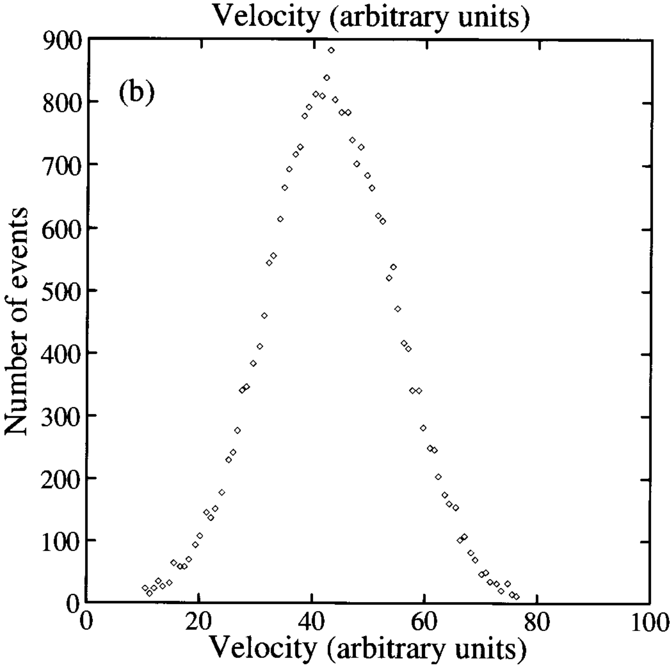


Fig. 3.3. Histogram for same signal as in Fig. 3.1(a), sampled 5000 times over a time-span of 150 seconds (a); same histogram, a few minutes later (b).

a purely deterministic context?[3] To give some insight into this question in a context which keeps a flavor of the Navier–Stokes equation, we shall now discuss a toy model.

3.2 A model for deterministic chaos

In this section we shall study the following discrete map:

$$v_{t+1} = 1 - 2v_t^2, \qquad v_0 = \varpi, \quad t = 0, 1, 2, \ldots. \tag{3.1}$$

Here, v_t is a real number between -1 and $+1$ and the time t is discrete. The tth iterate starting from the initial value ϖ is denoted by $v(t, \varpi)$. The set of iterates of a given initial value is known as its *orbit*. The map (3.1) is an instance of the logistic map $v \mapsto v - av^2$. Here, we could as well call it the *poor man's Navier–Stokes equation.* Let us indeed rewrite the map in a way paralleling the Navier–Stokes equation written directly underneath:

$$\left.\begin{array}{rclcccc} v_{t+1} - v_t & = & -2v_t^2 & - & v_t & + & 1 \\ \partial_t \boldsymbol{v} & = & -(\boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p) & + & \nu \nabla^2 \boldsymbol{v} & + & \boldsymbol{f}. \end{array}\right\} \tag{3.2}$$

Written in this way, our logistic map has the equivalent of the nonlinear term, the viscous term and the force term. Of course, the simple map has no spatial structure whatsoever.

We now define:

$$G : \; v_t \mapsto v_{t+1}, \tag{3.3}$$

and

$$G_\tau \equiv G^\tau : \; v_t \mapsto v_{t+\tau}. \tag{3.4}$$

Thus,

$$v_t(\varpi) = G_t \, \varpi. \tag{3.5}$$

With the poor man's Navier–Stokes equation we can repeat the same sort of experiment as performed with the wind-tunnel data. We choose an arbitrary initial condition ϖ (between -1 and $+1$) and iterate many times, say 5000. From these iterates we can then construct the histogram which is shown in Fig. 3.4. If we repeat the process with 5 000 consecutive iterates taken much later (say iterate numbers 20 000–25 000), we obtain again essentially the same histogram.

The reason we chose the particular map defined by (3.1) is that it

[3] 'Chance' (*Le hasard*) is the word used by Henri Poincaré in the introduction to his 'Calcul des probabilités'; nowadays, in deterministic situations, we say 'chaos'.

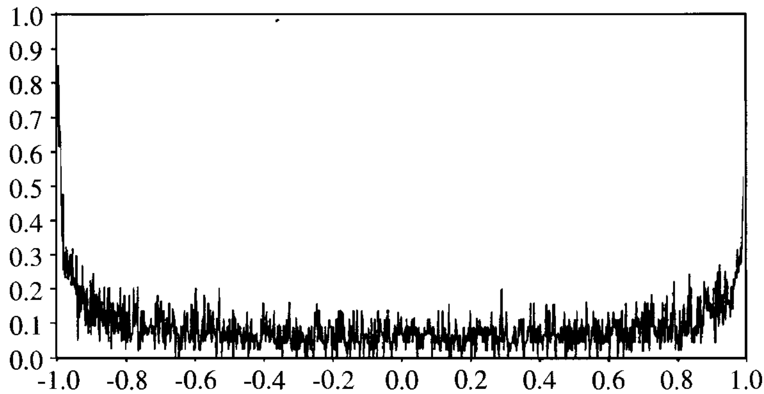


Fig. 3.4. Normalized histogram of the values of v obtained by iterating (3.1) (Ruelle 1989).

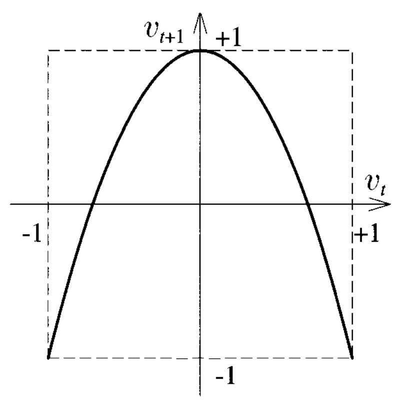


Fig. 3.5. The graph of the 'poor man's Navier–Stokes' map (3.1).

is now possible to understand the reason for this reproducibility and thereby get an insight into the behavior of a large class of nonlinear deterministic systems.

First, we shall relate the map (3.1) to a simpler map. In Fig. 3.5 we have drawn the graph of the map. Observe that it falls within a square of side two, centered at the origin. Let us make the following change of variable:

$$v_t = \sin\left(\pi x_t - \frac{\pi}{2}\right), \qquad 0 \le x_t \le 1, \tag{3.6}$$

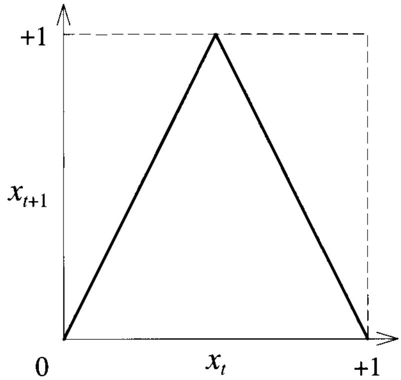


Fig. 3.6. The tent map (3.8).

and similarly:

$$v_{t+1} = \sin\left(\pi x_{t+1} - \frac{\pi}{2}\right). \tag{3.7}$$

An elementary calculation then gives

$$x_{t+1} = \begin{cases} 2x_t & \text{for } 0 \leq x_t \leq \frac{1}{2} \\ 2 - 2x_t & \text{for } \frac{1}{2} \leq x_t \leq 1. \end{cases} \tag{3.8}$$

We shall denote by B the map $x_t \mapsto x_{t+1}$. This is known as the *tent map*, because of the shape of its graph (shown in Fig. 3.6). Thus the map (3.1) and the tent map are *conjugate*: if we know how to iterate one of them, the iterates of the other are readily obtained from (3.6).

Actually, it is quite easy to iterate the tent map. For this we use the binary decomposition of real numbers between 0 and 1. Let α_1, α_2,... denote binary digits taking the values 0 or 1 and let N denote the *negation* which interchanges 0 and 1. It is then a simple exercise to check that if x has the binary decomposition

$$x = 0.\alpha_1\alpha_2\alpha_3\ldots = \alpha_1 2^{-1} + \alpha_2 2^{-2} + \alpha_3 2^{-3} + \ldots, \tag{3.9}$$

then its image Bx by the tent map has the decomposition

$$Bx = 0.(N^{\alpha_1}\alpha_2)(N^{\alpha_1}\alpha_3)(N^{\alpha_1}\alpha_4)\ldots. \tag{3.10}$$

This relation is easily iterated to give

$$\left.\begin{aligned} B^t x &= 0.(P\alpha_{t+1})(P\alpha_{t+2})(P\alpha_{t+3})\ldots, \\ P &= N^{\alpha_1+\alpha_2+\cdots+\alpha_t}. \end{aligned}\right\} \tag{3.11}$$

An immediate consequence of (3.11) is *the sensitivity to the initial conditions.* Two initial conditions which differ in a minute way (say, beyond the nth significant bit) will, after iterations, separate very quickly. Indeed, at each iteration, the discrepancy is shifted left and thus grows by a factor 2. An example will make this clear. Consider the following two initial conditions:

$$x_0 = 0.10011101001011010\ldots, \tag{3.12}$$

$$x_0' = 0.10011101001111001\ldots, \tag{3.13}$$

which differ only beyond the tenth significant bit, i.e. by about $2^{-10} \simeq 10^{-3}$. After ten iterations, they become

$$x_{10} = 0.0100101\ldots, \tag{3.14}$$

$$x_{10}' = 0.0000110\ldots. \tag{3.15}$$

The orbits have now completely separated. It is this sensitivity to initial conditions which is often loosely referred to as *chaos.*

Another important property of the tent map is the existence of an *invariant measure.* Suppose that we select x_0 at random in the interval $[0, 1]$ with a uniform distribution; then all the iterates will also have a uniform distribution.

It is obviously enough to prove this assertion for the first iterate $x = Bx_0$. The statement that x_0 is uniformly distributed is tantamount to

$$\mathrm{Prob}\{x_0 \in [a, b]\} = b - a, \qquad \forall\ 0 \le a \le b \le 1, \tag{3.16}$$

where $\mathrm{Prob}\{\cdot\}$ denotes the probability of an event. In other words the probability measure is just the Lebesgue measure dx_0. To find how this probability measure transforms under the tent map B we must use the relation

$$\mathrm{Prob}\{Bx_0 \in [a, b]\} = \mathrm{Prob}\{x_0 \in B^{-1}[a, b]\}, \tag{3.17}$$

which expresses the conservation of probability. In (3.17) $B^{-1}[a, b]$ denotes the preimage under the tent map of the interval $[a, b]$, i.e. the set of points which are mapped into $[a, b]$. To understand this preimage it is useful to draw a picture (Fig. 3.7). It is seen that the preimage of $[a, b]$ is made of two disjoint intervals, each half the length of the original interval. This immediately implies the invariance of the uniform measure. In other words, the Lebesgue measure is an invariant measure for the tent map.

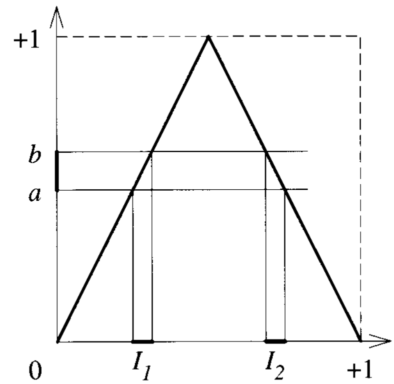


Fig. 3.7. Construction of the preimages of an interval $[a, b]$ for the tent map.

At this point, the reader may have the feeling that probabilities were introduced through the back door. Given a purely deterministic system such as the tent map, why should we decide to resort to a probabilistic description with the initial value x_0 *selected at random*? The answer is that it (almost surely) does not matter if the initial value is deterministic or random. Indeed there is an important result called Birkhoff's ergodic theorem which states the following (roughly).[4]

Ergodic theorem for the tent map. *Let* $f(x)$ *be an integrable function defined in the interval* $[0, 1]$. *For almost all* x_0

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{t=T} f\left(B^t x_0\right) = \int_0^1 f(x)dx. \tag{3.18}$$

The ergodic theorem states that the 'time average' of $f(\cdot)$ along the orbit of almost all initial x_0 is equal to its *ensemble average* calculated with the invariant measure (the uniform measure). Thus the deterministic tent map behaves in an essentially probabilistic way.

For the proof of the ergodic theorem we refer the reader to textbooks on ergodic theory such as Halmos (1956). A feeling for why the theorem holds can be gotten as follows. Since (3.18) is linear in $f(\cdot)$, we can without loss of generality suppose that f is equal to 1 in a small interval $I = [a, b]$ and zero outside. The l.h.s. of (3.18) is then just the average fraction of the time the orbit of x_0 visits the interval I. The points x_0 such

[4] Since a reader interested in turbulence may not necessarily be familiar with measure-theoretic jargon, we shall generally water down our statements using, e.g., 'integrable' instead of 'measurable'. We apologize for this to the more mathematically minded reader.

that their tth iterates fall into I are obtained by iterating the preimage construction of Fig. 3.7. By trying a few more iterates, the reader will see that the resulting 2^t tiny disjoint intervals appear to be spread in an increasingly uniform way over the interval $[0, 1]$. Thus the average time the orbit spends in I is just its length.

The restriction 'almost all x_0' is to be taken seriously. Actually, suppose we take $x_0 = 0$, which is a fixed point of the tent map, then the l.h.s. of (3.18) tends to $f(0)$ as $T \to \infty$, a value which is usually not equal to the r.h.s..

As an illustration of the ergodic theorem, we take $f(x) = x^n$. We then have (for almost all x_0):

$$\lim_{T\to\infty} \frac{1}{T} \sum_{0}^{T} (x_t)^n = \int_0^1 x^n dx = \frac{1}{n+1}. \tag{3.19}$$

We are now in a position to return to the original poor man's Navier–Stokes equation, the map (3.1). Since it is conjugated to the tent map by the transformation

$$x \mapsto \sin\left(\pi x - \frac{\pi}{2}\right), \tag{3.20}$$

it will also possess an invariant measure, which is the image of the uniform measure by (3.20). An elementary calculation leads to the invariant measure $P(v)dv$ with the *probability density function* (p.d.f.) $P(v)$ given by

$$P(v) = \frac{1}{\pi\sqrt{1 - v^2}}. \tag{3.21}$$

Except for a normalization factor (the integral of a p.d.f. is 1), this is the same as the histogram shown in Fig. 3.4.

3.3 Dynamical systems

Birkhoff's theorem, which allows us to replace time averages over one orbit by ensemble averages, is valid in a much broader context than the two maps considered in Section 3.2. The appropriate framework is that of *dynamical systems*. We here introduce the following definition:

Definition. *A dynamical system is a quadruplet* $(\Omega, \mathscr{A}, P, G_t)$. *The set* Ω *is called the* probability space.[5] $\mathscr{A}$ *is a family of subsets*[6] *of* Ω. P, *the*

[5] In this and the next chapter Ω has its standard probabilistic meaning; elsewhere Ω will denote the enstrophy.

[6] Actually, a σ-algebra but, as we stated before, we do not intend to go into measure-theoretic fine points.

probability measure, *maps $\mathscr{A}$ to the real numbers between 0 and 1 and satisfies*

$$P(A) \geq 0 \quad \forall A \in \mathscr{A}, \qquad P\left(\cup_i A_i\right) = \sum_i P(A_i), \qquad P(\Omega) = 1, \tag{3.22}$$

where A_i is any enumerable set of disjoint sets $\in \mathscr{A}$. The time-shifts, G_t, *are a family of operators depending on a variable $t \geq 0$ which can be either continuous or discrete. The G_ts satisfy the semi-group property*

$$G_0 = I, \qquad G_t G_{t'} = G_{t+t'} \tag{3.23}$$

and conserve the probability:

$$P\left(G_t^{-1} A\right) = P(A), \qquad \forall t \geq 0, \quad \forall A \in \mathscr{A}. \tag{3.24}$$

The special case of the tent map corresponds to the following choices: $\Omega = [0, 1]$, P is the Lebesgue measure dx and G_t is the tth iterate of the tent map.

In the general framework of dynamical systems, *Birkhoff's ergodic theorem* states (roughly) the following. The basic hypothesis is that the only sets in $\mathscr{A}$ which are globally invariant under the time shifts G_t are those of measure zero and one (e.g., the empty set or the entire set Ω).[7] It then follows that for any integrable function f defined on Ω and for almost all $\varpi' \in \Omega$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f\left(G_t \varpi'\right) dt = \int_\Omega f(\varpi)\, dP \equiv \langle f \rangle. \tag{3.25}$$

Eq. (3.25) is seen to be a generalization of (3.18).

3.4 The Navier–Stokes equation as a dynamical system

We can now return to the flow of an incompressible fluid governed by the Navier–Stokes equation and formulate it as a dynamical system. The equation is written as

$$\left.\begin{aligned} \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} &= -\nabla p + \nu \nabla^2 \boldsymbol{v} + \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{v} &= 0, \\ \boldsymbol{v}_0 \equiv \boldsymbol{v}(t = 0) &= \varpi \quad \text{(plus boundary conditions).} \end{aligned}\right\} \tag{3.26}$$

The initial condition, denoted ϖ, is chosen in a suitable space $\mathscr{H}$ of functions satisfying the boundary conditions and the incompressibility

[7] This assumption is known as 'metric transitivity' and may be very hard to prove for a given dynamical system.

constraint.[8] The force $\boldsymbol{f}$ is assumed independent of the time.[9] The space Ω is now simply the space $\mathscr{H}$ of all possible initial conditions ϖ. The time-shift G_t is the map

$$G_t : \varpi \mapsto \boldsymbol{v}(t). \tag{3.27}$$

(Since $\boldsymbol{f}$ does not depend on the time, G_t also maps $\boldsymbol{v}(s)$ into $\boldsymbol{v}(s+t)$.) As for P, it is a probability measure on Ω, invariant under the time shift.

The existence of the time-shift G_t and an invariant measure P are, in general, only conjectures. In three dimensions, we do not even have a theorem guaranteeing the existence and the uniqueness of the solution to the Navier–Stokes equation. The existence of invariant measures is an even harder problem (see Ruelle 1989; Vishik and Fursikov 1988). For chaotic systems, rigorous proofs are available only for very simple finite-dimensional models. This is perhaps the place to warn the reader that the poor man's Navier–Stokes equation (3.1) is indeed a poor model. It is pathological in at least two ways.

First its invariant measure fills all of the available space $[0, 1]$. In contrast, it is typical for dissipative systems in finite dimensions to have their invariant measure concentrated on an *attractor* with zero Lebesgue measure and with a fractal structure (see. e.g., Ruelle 1989, 1991). A well-known instance is the Hénon (1976) map $(x, y) \mapsto (y + 1 - ax^2, bx)$.

Second, it is typical for dissipative dynamical systems to have more than one attractor and therefore more than one invariant measure.[10] Each attractor has an associated basin. The statistical properties of the solution will then depend on to which basin the initial condition belongs. Thus, not only may the detailed behavior of orbits be unpredictable (because of the sensitivity to the initial conditions), but even their *statistical properties may be unpredictable*, insofar as it may be impossible to determine to which basin the initial condition belongs. Translated into meteorological vocabulary, this is equivalent to stating that not only the weather but also the climate may be unpredictable.

To conclude this chapter, we observe that at the present stage of development of the theory of dynamical systems there has been little quantitative impact on the understanding of high Reynolds number flow. We shall come back to such matters in Section 9.4. For the moment,

8 At this point there is no need to restrict ourselves to periodic boundary conditions.

9 The formalism can be readily extended to periodic time-dependence.

10 Similarly, observe that it is typical for a solid resting on a table to have more than one stable equilibrium position.

the (partial) understanding of chaos in deterministic systems gives us confidence that a *probabilistic description* of turbulence is justified.[11] In the next chapter we shall review some of the basic tools of probability theory.

[11] This statement does not in any way imply that it would be justified to describe turbulence with a finite number of averaged quantities.