DEPARTMENT OF MATHEMATICS
University of Toronto

MAT 354F Problems 4

Due Thursday, November 17, 2022 via Crowdmark

1. Map the interior of *Cassini's oval* $|z^2 - a^2| < r^2$, where $0 < a < r$, conformally onto the unit disk $|w| < 1$, so that the axes of symmetry are preserved. (A solution is sketched in Cartan, Problem 7, p. 208.)

2. What are the different values of
$$\int_0^1 \frac{dz}{1+z^2}$$
when all possible paths of integration are considered?

3. (a) Let γ be a piecewise C^1 curve, and let $\overline{\gamma}$ be its image under the mapping $z \mapsto \overline{z}$ (symmetry in the real axis). Show that, if $f(z)$ is continuous on γ, then $z \mapsto \overline{f(\overline{z})}$ is continuous on $\overline{\gamma}$, and
$$\overline{\int_\gamma f(z)dz} = \int_{\overline{\gamma}} \overline{f(\overline{z})}\, dz\,.$$
(b) Show that, in particular, if γ is the unit circle described in the positive sense, then
$$\overline{\int_\gamma f(z)dz} \;=\; -\int_\gamma \overline{f(z)}\,\frac{dz}{z^2}\,.$$

4. Suppose that $f(z)$ is holomorphic. Let γ be a large positively oriented circle enclosing the points $\zeta = 0$ and $\zeta = z$. Show that
$$f(z) \;-\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\cdot\frac{z^n}{\zeta^n}d\zeta\,,$$
is a polynomial $g(z)$ of degree $n-1$, such that
$$g^{(k)}(0) \;=\; f^{(k)}(0)\,,\quad k \;=\; 0,\dots,n-1\,.$$

5. Suppose that $f(z)$ is an entire function (i.e., holomorphic in $\mathbb{C}$). What can be said about $f(z)$ in each of the following cases? Why?
   (a) $|f(z)| < 1 + |z|^{1/2}$, for all $z \in \mathbb{C}$.
   (b) $|f(z)| < 1 + |z|^n$, for all $z \in \mathbb{C}$.

6. Suppose that $f(z)$ is holomorphic in a disk $|z| < R$, and that $|f(z)| \leq M$, for $|z| < R$. Prove that, if $|z| \leq r < R$, then
$$|f^{(n)}(z)| \leq \frac{Mn!}{(R-r)^n}.$$

7. (a) Show that a holomorphic function in $\mathbb{C}$ with a pole at infinity is a polynomial.
   (b) Show that a meromorphic function on the Riemann sphere is necessarily rational.