Balanced Incomplete Block Designs

Rachel Rivera
Computer Science Department
Loyola Marymount University

January 8, 2015

Abstract

The origins of combinatorial design theory lie within recreational mathematics. With the work of Fisher and Yates in the 1930s, combinatorial design theory began to take on the character of a serious academic subject with its applications to statistical experimentation, tournament scheduling, mathematical biology, algorithm design and analysis, and cryptography.[5] This paper aims to provide an introduction to BIBDs (Balanced Incomplete Block Designs), identify the necessary conditions on v, k, and λ such that (v, k, λ)-BIBDs exist, and discuss the application of BIBDs to the Social Golfer Problem.

1 Introduction

The statistician F. Yates studied subsets of a set subject to certain balance properties in his 1936 paper. [7] In the paper he defined what has become known as *(v, k, λ) balanced incomplete block designs*:

Definition 1. A $\{v, k, \lambda\}$ *balanced incomplete block design* (BIBD) is a collection of k-element subsets (called *blocks*) of a v-element set S ($k <$ v) such that each 2-element subset of S is contained in exactly λ blocks.[7]

A BIBD is called "balanced" because every pair of distinct elements is contained in exactly λ blocks. Since $k < v$, no block can contain all elements of S, hence the designation "incomplete". Yates gave the following example of a (6,3,2)-BIBD in his paper:

Example 1.1. S = $\{a, b, c, d, e, f\}$ with the following 10 3-element blocks: $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, e\}$, $\{a, d, f\}$, $\{a, e, f\}$, $\{b, c, e\}$, $\{b, d, e\}$, $\{b, e, f\}$, $\{c, d, e\}$, and $\{c, d, f\}$.

In this example, every element occurs in the same number of blocks. This property actually holds for all BIBDs, as shown in the following proof. [4]

Proof. Let x be an arbitrary element such that $x \in S$. Suppose x occurs in r blocks. In each of these blocks, x makes a pair with $k - 1$ other elements, so altogether there are $r(k - 1)$ pairs involving x. But, x is paired with the other $v - 1$ elements exactly λ times, so $r(k - 1) = \lambda(v - 1)$. This shows that r is independent of the choice of x, being uniquely determined by v, k, and λ. $\square$

Thus, in a (v, k, λ) design with b blocks, each element occurs in exactly r blocks, and, as noted in the *CRC Handbook of Combinatorial Designs*[4], the following equivalencies hold:

- $\lambda(v-1) = r(k-1)$
- $bk = vr$

The first equivalence was established in the first proof. The second equivalence can be explained by noting that since each of the v elements appear in r blocks, there are vr appearances of elements in blocks. And since each of the b blocks has k elements, $vr = bk$.

Fisher also proved that the following inequality must hold:

- $b \geq$v

When the equality holds, any two blocks have the same number of elements in common. (//TODO: look into this more later)

Proof. Let A be the block incidence matrix of a (v, k, λ)-BIBD with b blocks. The block incidence matrix is a v x b incidence matrix A = (a_{ij}) such that:

$$a_{ij} = \begin{cases} 1 & : \text{ if the } i^{th} \text{ element is in the } j^{th} \text{ block} \\ 0 & : \text{ otherwise} \end{cases}$$

(See Example 1.2)

If:

- A is $v \times b$
- A^T is the transpose[1] of A
- J_v is the all $v \times v$ 1's matrix
- I_v is the $v \times v$ identity matrix[2]

Then:

$$A^T \cdot A = (a_{ij}) = (r - \lambda)\, I_v + \lambda J_v \tag{1}$$

That is:

$$A^T \cdot A = \begin{bmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \cdots & r \end{bmatrix} \tag{2}$$

From Necessary Condition for existence of BIBD[2]:

$$r > \lambda$$

[1]A matrix formed by turning all rows into columns and vice-versa

[2]A square matrix with 1's on the main diagonal and 0's elsewhere

So we can write $r = \lambda + \mu$ for some $\mu > 0$ and so:

$$A^{\intercal} \cdot A = \begin{bmatrix} \lambda+\mu & \lambda & \cdots & \lambda \\ \lambda & \lambda+\mu & \cdots & \lambda \\ \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \cdots & \lambda+\mu \end{bmatrix} \tag{3}$$

This is a combinatorial matrix of order v. So:

$$\det\left(A^{\intercal} \cdot A\right) = \mu^{v-1}\left(\mu + v\lambda\right) \tag{4}$$

$$\det\left(A^{\intercal} \cdot A\right) = \left(r + (v-1)\lambda\right)(r-\lambda)^{v-1} \tag{5}$$

$$\det\left(A^{\intercal} \cdot A\right) = rk\,(r-\lambda)^{v-1} \tag{6}$$

Now since $k < v$ and using the properties of a BIBD, we get that $r > \lambda$. So $\det\left(A^{\intercal}A\right) \neq 0$. Since $A^{\intercal}A$ is a $v \times v$ matrix, then the rank, ρ, of $A^{\intercal}A = v$. Using the facts that $\rho\left(A^{\intercal}A\right) \leq \rho\left(A\right)$ and $\rho\left(A\right) \leq b$, (since A only has b columns), then $v \leq \rho\left(A\right) \leq b$. [1] □

Example 1.2. The incidence matrix, A, of a (7, 3, 1)-BIBD consisting of 7 blocks (numbered 1, 2, 3, 4, 5, 6, and 7 respectively):

(1,2,3) (1,4,5) (1,6,7) (2,4,6) (2,5,7) (3,4,7) (3,5,6)

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

2 Resolvable Designs

Definition 2. A BIBD is *resolvable* if its blocks can be partitioned into c classes such that each element of the design occurs in exactly one of the $v/k = b/c$ groups of each class. The classes are called *parallel classes* or *resolution classes*. The partition into classess is called a *resolution*. [6]

Consider the following (15, 3, 1)-design consisting of 35 blocks:

(1,2,3)	(1,4,7)	(1,5,10)	(1,6,15)	(1,8,11)	(1,9,13)	(1,12,14)
(4,5,6)	(2,5,8)	(2,6,12)	(2,4,13)	(2,7,14)	(2,11,15)	(2,9,10)
(7,8,9)	(3,10,13)	(3,8,15)	(3,7,12)	(3,6,9)	(3,5,14)	(3,4,11)
(10,11,12)	(6,11,14)	(4,9,14)	(5,9,11)	(4,10,15)	(4,8,12)	(5,7,15)
(13,14,15)	(9,12,15)	(7,11,13)	(8,10,14)	(5,12,13)	(6,7,10)	(6,8,13)

Each vertical group of 5 blocks is a resolution class.

Theorem 2.1. If D is a resolvable (v, k, 1) design, then a resolution of D into w parallel classes is a solution for the SGP instance $\frac{v}{k}$–k–w. [6]

The v golfers correspond to elements $\{1, 2, 3, ..., v\}$. The design's blocks correspond to the group of golfers and the w parallel classes are regarded as the weeks. By definition, each 2-element subset of $\{1, ..., v\}$ is contained in exactly one block. [6]

2.1 Necessary and Sufficient Conditions for Resolvability

It has been shown in [3] that for a *resolvable* balanced incomplete block design, Fisher's inequality can be replaced by the more stringent inequality:

$$v \geq v + r - 1 \tag{7}$$

When the equality holds, then two blocks belonging to different classes have the same number of elements in common. Balanced incomplete block designs with this property may be called *affine* resolvable. For a resolvable balanced incomplete block design, the number of elements common to two blocks of different classes has to be k^2/v. (This number must be *integral*)

Proof. Consider a bibd with parameters v, b, r, k, λ. If it is resolvable then $v = nk$ and $b = nr$ for some integer n (recap of previously stated definition). The b blocks are divisible into r classes or n blocks each such that the blocks of a given class give a complete replication (i.e. all the elements occur exactly once among the blocks of a given class). Let the blocks belonging to the ith class (C_i) be

$$C_{i,1}, C_{i,2}, C_{i,3}, ...C_{i,n} (i = 0, 1, ...r - 1) \tag{8}$$

Let us take any particular block $b_{0,1}$ of class C_0 and let $l_{i,j}$ be the number of elements common to the block $b_{0,1}$ and the block $b_{i,j}$ of the class C_i, for i = 1, 2, ..., r - 1, j = 1, 2, ..., n. (//TODO: figure out this proof for the number of elements common to two blocks of different classes) □

3 Steiner Triple Systems

Definition 3. A *Steiner triple system* STS(v) of order v is a (v, 3, 1)-BIBD.

References

[1] Fisher's inequality. `https://www.proofwiki.org/wiki/Fisher%27s_Inequality`.

[2] Necessary condition for existence of bibd. `https://www.proofwiki.org/wiki/Necessary_Condition_for_Existence_of_BIBD`.

[3] R. C. Bose. On the construction of balanced incomplete block designs. *Annals of Eugenics*, pages 353–399.

[4] C. H. Colbourn and J. H. Dinitz. The crc handbook of combinatorial designs. CRC Press, 1996.

[5] Douglas R. Stinson. Combinatorial designs: Constructions and analysis. *SIGACT News*, 39(4):17–21, November 2008.

[6] Markus Triska. Solution methods for the social golfer problem. Vienna University of Technology, 2008.

[7] F. Yates. Incomplete randomized blocks. In *Annals of Eugenics*, pages 121–140, 1936.