Balanced Incomplete Block Designs

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Abstract

The origins of combinatorial design theory lie within recreational mathematics. With the work of Fisher and Yates in the 1930s, combinatorial design theory began to take on the character of a serious academic subject with its applications to statistical experimentation, tournament scheduling, mathematical biology, algorithm design and analysis, and cryptography.[2] This paper aims to provide an introduction to BIBDs (Balanced Incomplete Block Designs), identify the necessary conditions on v, k, and λ such that (v, k, λ)-BIBDs exist, and discuss the application of BIBDs to the Social Golfer Problem.

1 Introduction

The statistician F. Yates studied subsets of a set subject to certain balance properties in his 1936 paper. [4] In the paper he defined what has become known as *(v, k, λ) balanced incomplete block designs*:

Definition 1. A $\{v, k, \lambda\}$ *balanced incomplete block design* (BIBD) is a collection of k-element subsets (called *blocks*) of a v-element set S ($k <$ v) such that each 2-element subset of S is contained in exactly λ blocks.[4]

A BIBD is called "balanced" because every pair of distinct elements is contained in exactly λ blocks. Since $k < v$, no block can contain all elements of S, hence the designation "incomplete". Yates gave the following example of a (6,3,2)-BIBD in his paper:

Example 1.1. S $= \{a, b, c, d, e, f\}$ with the following 10 3-element blocks: $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, e\}$, $\{a, d, f\}$, $\{a, e, f\}$, $\{b, c, e\}$, $\{b, d, e\}$, $\{b, e, f\}$, $\{c, d, e\}$, and $\{c, d, f\}$.

In this example, every element occurs in the same number of blocks. This property actually holds for all BIBDs, as shown in the following proof. [1]

Proof. Let x be an arbitrary element such that $x \in S$. Suppose x occurs in r blocks. In each of these blocks, x makes a pair with $k - 1$ other elements, so altogether there are $r(k - 1)$ pairs involving x. But, x is paired with the other $v - 1$ elements exactly λ times, so $r(k - 1) = \lambda(v - 1)$. This shows that r is independent of the choice of x, being uniquely determined by v, k, and λ. $\square$

Thus, in a (v, k, λ) design with b blocks, each element occurs in exactly r blocks, and, as noted in the *CRC Handbook of Combinatorial Designs*[1], the following equivalencies hold:

1. $\lambda(v-1) = r(k-1)$
2. $bk = vr$

The first equivalence was established in the first proof. The second equivalence can be explained by noting that since each of the v elements appear in r blocks, there are vr appearances of elements in blocks. And since each of the b blocks has k elements, $vr = bk$.

2 Resolvable Designs

Definition 2. A BIBD is *resolvable* if its blocks can be partitioned into c classes such that each element of the design occurs in exactly one of the $v/k = b/c$ groups of each class. The classes are called *parallel classes* or *resolution classes*. The partition into classess is called a *resolution*. [3]

Consider the following (15, 3, 1)-design consisting of 35 blocks:

(1,2,3)	(1,4,7)	(1,5,10)	(1,6,15)	(1,8,11)	(1,9,13)	(1,12,14)
(4,5,6)	(2,5,8)	(2,6,12)	(2,4,13)	(2,7,14)	(2,11,15)	(2,9,10)
(7,8,9)	(3,10,13)	(3,8,15)	(3,7,12)	(3,6,9)	(3,5,14)	(3,4,11)
(10,11,12)	(6,11,14)	(4,9,14)	(5,9,11)	(4,10,15)	(4,8,12)	(5,7,15)
(13,14,15)	(9,12,15)	(7,11,13)	(8,10,14)	(5,12,13)	(6,7,10)	(6,8,13)

Each vertical group of 5 blocks is a resolution class.

Theorem 2.1. If D is a resolvable $(v,\ k,\ 1)$ design, then a resolution of D into w parallel classes is a solution for the SGP instance $\frac{v}{k}$–k–w.

3 Steiner Triple Systems

Definition 3. A *Steiner triple system* STS(v) of order v is a (v, 3, 1)-BIBD.

References

[1] C. H. Colbourn and J. H. Dinitz. The crc handbook of combinatorial designs. CRC Press, 1996.

[2] Douglas R. Stinson. Combinatorial designs: Constructions and analysis. *SIGACT News*, 39(4):17–21, November 2008.

[3] Markus Triska. Solution methods for the social golfer problem. Vienna University of Technology, 2008.

[4] F. Yates. Incomplete randomized blocks. In *Annals of Eugenics*, pages 121–140, 1936.