

Balanced Incomplete Block Designs

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Abstract

The origins of combinatorial design theory lie within recreational mathematics. With the work of Fisher and Yates in the 1930s, combinatorial design theory began to take on the character of a serious academic subject with its applications to statistical experimentation, tournament scheduling, mathematical biology, algorithm design and analysis, and cryptography.[4] This paper aims to provide an introduction to BIBDs (Balanced Incomplete Block Designs), identify the necessary conditions on v , k , and λ such that (v, k, λ) -BIBDs exist, and discuss the application of BIBDs to the Social Golfer Problem.

1 Introduction

The statistician F. Yates studied subsets of a set subject to certain balance properties in his 1936 paper. [6] In the paper he defined what has become known as (v, k, λ) *balanced incomplete block designs*:

Definition 1. A $\{v, k, \lambda\}$ *balanced incomplete block design* (BIBD) is a collection of k -element subsets (called *blocks*) of a v -element set S ($k < v$) such that each 2-element subset of S is contained in exactly λ blocks.[6]

A BIBD is called “balanced” because every pair of distinct elements is contained in exactly λ blocks. Since $k < v$, no block can contain all elements of S , hence the designation “incomplete”. Yates gave the following example of a $(6,3,2)$ -BIBD in his paper:

Example 1.1. $S = \{a, b, c, d, e, f\}$ with the following 10 3-element blocks: $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, e\}$, $\{a, d, f\}$, $\{a, e, f\}$, $\{b, c, e\}$, $\{b, d, e\}$, $\{b, e, f\}$, $\{c, d, e\}$, and $\{c, d, f\}$.

In this example, every element occurs in the same number of blocks. This property actually holds for all BIBDs, as shown in the following proof. [3]

Proof. Let x be an arbitrary element such that $x \in S$. Suppose x occurs in r blocks. In each of these blocks, x makes a pair with $k - 1$ other elements, so altogether there are $r(k - 1)$ pairs involving x . But, x is paired with the other $v - 1$ elements exactly λ times, so $r(k - 1) = \lambda(v - 1)$. This shows that r is independent of the choice of x , being uniquely determined by v , k , and λ . \square

Thus, in a (v, k, λ) design with b blocks, each element occurs in exactly r blocks, and, as noted in the *CRC Handbook of Combinatorial Designs*[3], the following equivalencies hold:

- $\lambda(v - 1) = r(k - 1)$
- $bk = vr$

The first equivalence was established in the first proof. The second equivalence can be explained by noting that since each of the v elements appear in r blocks, there are vr appearances of elements in blocks. And since each of the b blocks has k elements, $vr = bk$.

Fisher also proved that the following inequality must hold:

- $b \geq v$

Proof. Let A be the block incidence matrix of a (v, k, λ) -BIBD with b blocks. The block incidence matrix is a $v \times b$ incidence matrix $A = (a_{ij})$ such that:

$$a_{ij} = \begin{cases} 1 & : \text{if the } i^{th} \text{ element is in the } j^{th} \text{ block} \\ 0 & : \text{otherwise} \end{cases}$$

(See Example 1.2)

If:

- A is $v \times b$
- A^T is the transpose¹ of A
- J_v is the all $v \times v$ 1's matrix
- I_v is the $v \times v$ identity matrix²

Then:

$$A^T \cdot A = (a_{ij}) = (r - \lambda) I_v + \lambda J_v \tag{1}$$

That is:

$$A^T \cdot A = \begin{bmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \cdots & r \end{bmatrix} \tag{2}$$

From Necessary Condition for existence of BIBD[2]:

$$r > \lambda$$

¹A matrix formed by turning all rows into columns and vice-versa

²A square matrix with 1's on the main diagonal and 0's elsewhere

So we can write $r = \lambda + \mu$ for some $\mu > 0$ and so:

$$A^T \cdot A = \begin{bmatrix} \lambda + \mu & \lambda & \cdots & \lambda \\ \lambda & \lambda + \mu & \cdots & \lambda \\ \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \cdots & \lambda + \mu \end{bmatrix} \quad (3)$$

This is a combinatorial matrix of order v . So:

$$\det(A^T \cdot A) = \mu^{v-1} (\mu + v\lambda) \quad (4)$$

$$\det(A^T \cdot A) = (r + (v-1)\lambda)(r - \lambda)^{v-1} \quad (5)$$

$$\det(A^T \cdot A) = rk(r - \lambda)^{v-1} \quad (6)$$

Now since $k < v$ and using the properties of a BIBD, we get that $r > \lambda$. So $\det(A^T A) \neq 0$. Since $A^T A$ is a $v \times v$ matrix, then the rank, ρ , of $A^T A = v$. Using the facts that $\rho(A^T A) \leq \rho(A)$ and $\rho(A) \leq b$, (since A only has b columns), then $v \leq \rho(A) \leq b$. [1] \square

Example 1.2. The incidence matrix, A , of a $(7, 3, 1)$ -BIBD consisting of 7 blocks (numbered 1, 2, 3, 4, 5, 6, and 7 respectively):

$$(1,2,3) \quad (1,4,5) \quad (1,6,7) \quad (2,4,6) \quad (2,5,7) \quad (3,4,7) \quad (3,5,6)$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

2 Resolvable Designs

Definition 2. A BIBD is *resolvable* if its blocks can be partitioned into c classes such that each element of the design occurs in exactly one of the $v/k = b/c$ groups of each class. The classes are called *parallel classes* or *resolution classes*. The partition into classes is called a *resolution*. [5]

Consider the following $(15, 3, 1)$ -design consisting of 35 blocks:

$$\begin{array}{ccccccc} (1,2,3) & (1,4,7) & (1,5,10) & (1,6,15) & (1,8,11) & (1,9,13) & (1,12,14) \\ (4,5,6) & (2,5,8) & (2,6,12) & (2,4,13) & (2,7,14) & (2,11,15) & (2,9,10) \\ (7,8,9) & (3,10,13) & (3,8,15) & (3,7,12) & (3,6,9) & (3,5,14) & (3,4,11) \\ (10,11,12) & (6,11,14) & (4,9,14) & (5,9,11) & (4,10,15) & (4,8,12) & (5,7,15) \\ (13,14,15) & (9,12,15) & (7,11,13) & (8,10,14) & (5,12,13) & (6,7,10) & (6,8,13) \end{array}$$

Each vertical group of 5 blocks is a resolution class.

Theorem 2.1. If D is a resolvable $(v, k, 1)$ design, then a resolution of D into w parallel classes is a solution for the SGP instance $\frac{v}{k}-k-w$. [5]

The v golfers correspond to elements $\{1, 2, 3, \dots, v\}$. The design's blocks correspond to the group of golfers and the w parallel classes are regarded as the weeks. By definition, each 2-element subset of $\{1, \dots, v\}$ is contained in exactly one block. [5]

2.1 Necessary and Sufficient Conditions for Resolvability

3 Steiner Triple Systems

Definition 3. A *Steiner triple system* $\text{STS}(v)$ of order v is a $(v, 3, 1)$ -BIBD.

References

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