Simplified P_N Equations for Nonclassical Transport with Isotropic Scattering

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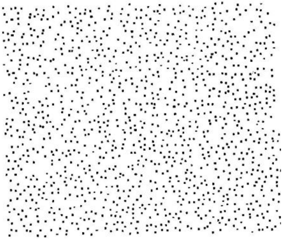
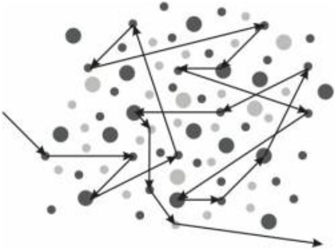
In classical transport

The "colliders" in the background material are, in general, **Poisson-distributed**; that is, their spatial locations are **not** correlated.



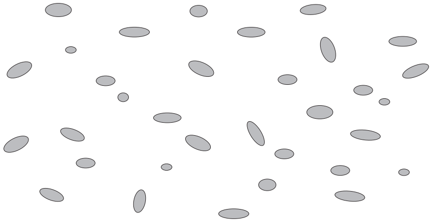


Specifically, this implies that the probability distribution function $p(s)$ for particles' **distances-to-collision** s (free-paths) is given by an exponential:

$$p(s) = \Sigma_t e^{-\Sigma_t s}$$

Nonclassical transport

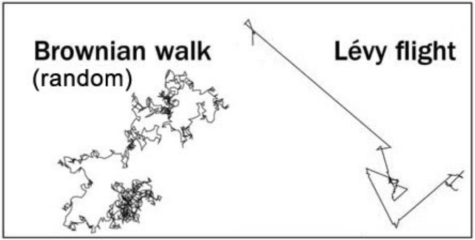
Consider a system consisting of many widely-spaced clumps in which the scattering centers are Poisson distributed, all separated by a "void":



Relatively rare events (streaming between clumps) will significantly affect the particle transport. The free-path distribution $p(s)$ will have a **nonexponential peak** for large s.

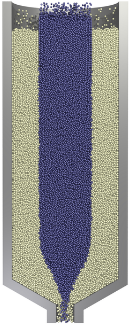
Lévy flights

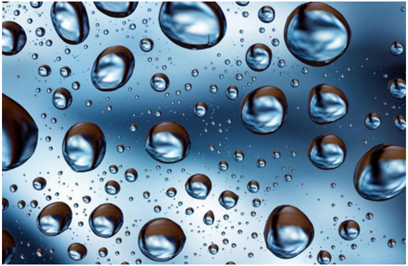
A Lévy flight is a random walk in which the step-lengths have a probability distribution that is heavy-tailed:

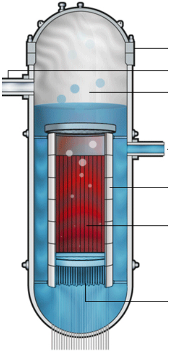


- Lévy glasses
- Astronomy
- Cryptography
- Earthquake data analysis
- Financial mathematics
- Foraging hypothesis

Applications







Path-length distribution

In classical particle transport, the incremental probability dp that a particle will experience a collision while traveling an incremental path length ds is:

$$dp = \Sigma_t \, ds \ ,$$

where the total cross section Σ_t is independent of

$$\boxed{s = \text{the path length traveled since the previous interaction} \ .}$$

Assuming that $\Sigma_t = \Sigma_t(s)$, the conditional distribution function for distance-to-collision is

$$\boxed{p(s) = \Sigma_t(s) e^{-\int_0^s \Sigma_t(s')ds'} \ .}$$

Note: in classical transport, $\Sigma_t(s) = \Sigma_t =$ constant, and

$$p(s) = \Sigma_t e^{-\Sigma_t s} = \text{ exponential} \ .$$

The Nonclassical linear Boltzmann equation

Nonclassical transport

$$\frac{\partial \Psi}{\partial s}(\mathbf{x}, \mathbf{\Omega}, s) + \mathbf{\Omega} \cdot \nabla \Psi(\mathbf{x}, \mathbf{\Omega}, s) + \Sigma_t(s)\Psi(\mathbf{x}, \mathbf{\Omega}, s)$$
$$= \frac{\delta(s)}{4\pi}\left[c \int_{4\pi}\int_0^\infty \Sigma_t(s')\Psi(\mathbf{x}, \mathbf{\Omega}', s')\, ds'\, d\Omega' + Q(\mathbf{x})\right]$$

Classical transport

$$\mathbf{\Omega} \cdot \nabla \Psi(\mathbf{x}, \mathbf{\Omega}) + \Sigma_t \Psi(\mathbf{x}, \mathbf{\Omega}) = \frac{1}{4\pi}\left[c\Sigma_t \int_{4\pi} \Psi(\mathbf{x}, \mathbf{\Omega}') d\Omega' + Q(\mathbf{x})\right]$$

$$\Sigma_t(s) = \frac{p(s)}{1 - \int_0^s p(s')ds'}$$

Thoughts...

- Nonclassical transport requires one to know $\Sigma_t(s)$ (or $p(s)$), which is not easy to obtain
- Nonclassical *diffusion* is simpler: it only requires the first and second moments of $p(s)$ to be known
- Can we extend this result to obtain more accurate diffusion approximations? Maybe using something similar to the SP_N approach?

P_N Equations

Consider the planar (slab) geometry P_N equations: for $l' = 0, 1, \ldots,$ we have

$$\left(\frac{l'+1}{2l'+1}\right)\frac{d}{dx}\phi_{l'+1}(x) + \left(\frac{l'}{2l'+1}\right)\frac{d}{dx}\phi_{l'-1}(x) + \Sigma_t(x)\phi_{l'} = \Sigma_{sl'}(x)\phi_{l'}(x) + s_{l'}(x)\,,$$

with

$$\phi_{-1} = 0 \quad \text{and} \quad \phi_{N+1} = 0 \quad \left(\text{or} \quad \frac{d}{dx}\phi_{N+1} = 0\right).$$

The classical simplified P_N equations (SP_N) can be obtained from the equation above in a heuristic way.

"Heuristic" Derivation of SP$_N$ Equations

First, for odd values of l', $\phi_{l'}$ is replaced by a vector:

$$\phi_{l'} \to \vec{\phi}_{l'} = (\phi^x_{l'}, \phi^y_{l'}, \phi^z_{l'})^t \,.$$

Then, in the even l' equations the derivative in x is replaced by a divergence:

$$\frac{d}{dx} \to \nabla\cdot\,;$$

and in the odd l' equations the x derivative is changed to a gradient:

$$\frac{d}{dx} \to \nabla$$

"Heuristic" Derivation of SP_N Equations

This allows us to write the first-order form of the SP_N equations as

$$\nabla \cdot \vec{\phi}_1 + \Sigma_a \phi_0 = s_0 \,,$$

$$\left(\frac{l'+1}{2l'+1}\right) \nabla \phi_{l'+1} + \left(\frac{l'}{2l'+1}\right) \nabla \phi_{l'-1} + \Sigma_t \vec{\phi}_{l'} = \Sigma_{sl'} \vec{\phi}_{l'} + s_{l'} \text{ , for odd } l',$$

$$\left(\frac{l'+1}{2l'+1}\right) \nabla \cdot \vec{\phi}_{l'+1} + \left(\frac{l'}{2l'+1}\right) \nabla \cdot \vec{\phi}_{l'-1} + \Sigma_t \phi_{l'} = \Sigma_{sl'} \phi_{l'} + s_{l'} \text{ , for even } l' > 0.$$

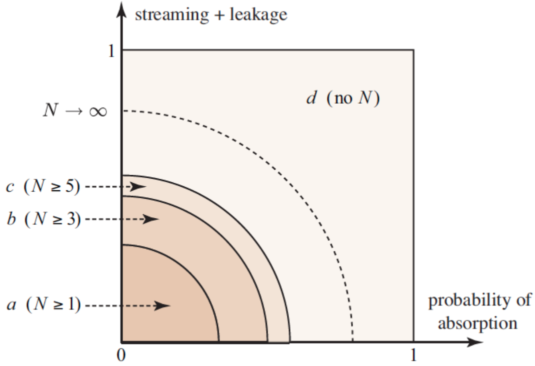
We can get rid of the odd moments and rewrite them in their second-order form by using the relation

$$\vec{\phi}_{l'} = -\frac{1}{\Sigma_t - \Sigma_{sl'}} \left(\frac{l'}{2l'+1} \nabla \phi_{l'-1} + \frac{l'+1}{2l'+1} \nabla \phi_{l'+1} \right) .$$

Why SP_N?

1. Mathematical structure is simpler: SP_N are elliptic; P_N are hyperbolic
2. The SP_N equations can be understood as a "super" diffusion theory
3. The structure of the SP_N equations is that of a coupled system of diffusion equations
4. Much simpler than P_N in multidimensional problems (with fewer equations)
5. Simpler code implementation: just use/adapt a diffusion code!!
6. The SP_N equations contain more "transport physics" than the diffusion equations.

Qualitative Behavior of SP$_N$ Equations



The amounts of absorption and streaming/leakage are indicated on arbitrary scales ranging from 0 to 1.

Asymptotic Analysis

Let us write the nonclassical Boltzmann equation in the mathematically equivalent form

$$\frac{\partial \Psi}{\partial s}(s) + \boldsymbol{\Omega} \cdot \nabla \Psi(s) + \Sigma_t(s)\Psi(s) = 0, \qquad s > 0,$$
$$\Psi(0) = \frac{1}{4\pi}\left[\int_{4\pi}\int_0^\infty c\Sigma_t(s')\Psi(\boldsymbol{x}, \boldsymbol{\Omega}', s')ds'd\Omega' + Q(\boldsymbol{x})\right].$$

Defining $0 < \varepsilon \ll 1$, we perform the following scaling:

$$\Sigma_t(s) = \varepsilon^{-1}\Sigma_t(s/\varepsilon)$$
$$c = 1 - \varepsilon^2\kappa$$
$$Q(\boldsymbol{x}) = \varepsilon q(\boldsymbol{x})$$

where κ and q are $O(1)$.
Under this scaling,

$$\langle s^m \rangle = \varepsilon^m \int_0^\infty s^m \Sigma_t(s) e^{-\int_0^s \Sigma_t(s')ds'} ds = \varepsilon^m \langle s^m \rangle_\varepsilon,$$

where $\langle s^m \rangle_\varepsilon$ is $O(1)$.

Asymptotic Analysis

Next, we define

$$\psi(\boldsymbol{x}, \boldsymbol{\Omega}, s) \equiv \frac{\varepsilon \left\langle s \right\rangle_\epsilon}{e^{-\int_0^s \Sigma_t(s')ds'}} \Psi(\boldsymbol{x}, \boldsymbol{\Omega}, \varepsilon s).$$

This satisfies

$$\frac{\partial \psi}{\partial s}(s) + \varepsilon \boldsymbol{\Omega} \cdot \nabla \psi(s) = 0, \qquad s > 0,$$
$$\psi(0) = \frac{1}{4\pi} \left[\int_{4\pi} \int_0^\infty (1 - \varepsilon^2 \kappa) p(s') \psi(\boldsymbol{x}, \boldsymbol{\Omega}', s') ds' d\Omega' + \varepsilon^2 \left\langle s \right\rangle_\epsilon q(\boldsymbol{x}) \right],$$

and the classical scalar flux can be written as

$$\Phi(\boldsymbol{x}) = \int_{4\pi} \int_0^\infty \psi(\boldsymbol{x}, \boldsymbol{\Omega}, s) \frac{e^{-\int_0^s \Sigma_t(s')ds'}}{\left\langle s \right\rangle_\epsilon} ds d\Omega.$$

We now integrate the first equation over $0 < s' < s$.

Asymptotic Analysis

Using the "initial condition" in s, we obtain

$$\left(I + \varepsilon \boldsymbol{\Omega} \cdot \nabla \int_0^s (\cdot) ds \right) \psi = \frac{1}{4\pi} \left[\int_0^\infty (1 - \varepsilon^2 \kappa) p(s') \varphi(\boldsymbol{x}, s') ds' + \varepsilon^2 \langle s \rangle_\epsilon q \right],$$

where

$$\varphi(\boldsymbol{x}, s) = \int_{4\pi} \psi(\boldsymbol{x}, \boldsymbol{\Omega}, s) d\Omega.$$

Inverting the operator on the left-hand side of the above equation and expanding it in a power series, we obtain

$$\psi = \left(\sum_{n=0}^{\infty} (-\varepsilon)^n \left(\boldsymbol{\Omega} \cdot \nabla \int_0^s (\cdot) ds \right)^n \right) \times \left[\int_0^\infty \frac{1 - \varepsilon^2 \kappa}{4\pi} p(s') \varphi(\boldsymbol{x}, s') ds' + \varepsilon^2 \langle s \rangle_\epsilon \frac{q}{4\pi} \right].$$

Asymptotic Analysis

Next we will need the identity

$$\frac{1}{4\pi}\int_{4\pi}\left(\boldsymbol{\Omega}\cdot\nabla\int_0^s(\cdot)ds\right)^n d\Omega = \frac{1+(-1)^n}{2}\frac{3^{n/2}}{n+1}\mathcal{B}^{n/2},$$

for $n = 0, 1, 2, \dots$, where

$$\mathcal{B} = \nabla_0\left(\int_0^s(\cdot)ds\right)^2,$$
$$\nabla_0 = \frac{1}{3}\nabla^2.$$

Integrating the nonclassical angluar flux over the unit sphere we obtain

$$\varphi = \left(\sum_{n=0}^{\infty}\frac{\varepsilon^{2n}}{2n+1}(3\mathcal{B})^n\right)\left[\int_0^{\infty}(1-\varepsilon^2\kappa)p(s')\varphi(\boldsymbol{x},s')ds' + \varepsilon^2\langle s\rangle_\epsilon q\right].$$

Asymptotic Analysis

Inverting the operator on the right-hand side and once again expanding it in a power series, we get

$$\left(I - \varepsilon^2 \mathcal{B} - \frac{4\varepsilon^4}{5}\mathcal{B}^2 - \frac{44\varepsilon^6}{35}\mathcal{B}^3 + O(\varepsilon^8) \right)\varphi = \int_0^\infty (1 - \varepsilon^2 \kappa) p(s')\varphi(\boldsymbol{x}, s')ds' + \varepsilon^2 \langle s \rangle_\epsilon q.$$

The solution of this equation is

$$\varphi(\boldsymbol{x}, s) = \left(I + \varepsilon^2 \frac{s^2}{2!}\nabla_0 + \frac{9\varepsilon^4}{5}\frac{s^4}{4!}\nabla_0^2 + \frac{27\varepsilon^6}{7}\frac{s^6}{6!}\nabla_0^3 + O(\varepsilon^8) \right)\phi(\boldsymbol{x}),$$

where

$$\phi(\boldsymbol{x}) = \sum_{n=0}^{\infty} \varepsilon^2 \phi_{2n}(\boldsymbol{x}),$$

with $\phi_{2n}(\boldsymbol{x})$ undetermined at this point.

Asymptotic Analysis

We now multiply φ by $e^{-\int_0^s \Sigma_t(s')ds'}/\langle s\rangle_\epsilon$ and operate by $\int_0^\infty(\cdot)ds$. We obtain an expression for the scalar flux:

$$\Phi(\boldsymbol{x}) = \left(I + \varepsilon^2 \frac{\langle s^3\rangle_\epsilon}{3!\langle s\rangle_\epsilon}\nabla_0 + \frac{9\varepsilon^4}{5}\frac{\langle s^5\rangle_\epsilon}{5!\langle s\rangle_\epsilon}\nabla_0^2 + \frac{27\varepsilon^6}{7}\frac{\langle s^7\rangle_\varepsilon}{7!\langle s\rangle_\epsilon}\nabla_0^3 + O(\varepsilon^8)\right)\phi(\boldsymbol{x}).$$

Moreover, we can write

$$\int_0^\infty p(s)\varphi(\boldsymbol{x},s)ds = \left(\sum_{n=0}^\infty \varepsilon^{2n}U_n\nabla_0^n\right)\Phi(\boldsymbol{x}),$$

with $U_0 = 1$; $U_1 = \frac{\langle s^2\rangle_\epsilon}{2!} - \frac{\langle s^3\rangle_\epsilon}{3!\langle s\rangle_\epsilon}$;

$$U_2 = \frac{9}{5}\left[\frac{\langle s^4\rangle_\epsilon}{4!} - \frac{\langle s^5\rangle_\epsilon}{5!\langle s\rangle_\epsilon}\right] - \frac{\langle s^3\rangle_\epsilon}{3!\langle s\rangle_\epsilon}U_1;$$

$$U_3 = \frac{27}{7}\left[\frac{\langle s^6\rangle_\epsilon}{6!} - \frac{\langle s^7\rangle_\varepsilon}{7!\langle s\rangle_\epsilon}\right] - \frac{9}{5}\frac{\langle s^5\rangle_\epsilon}{5!\langle s\rangle_\epsilon}U_1 - \frac{\langle s^3\rangle_\epsilon}{3!\langle s\rangle_\epsilon}U_2;$$

Asymptotic Analysis

We can now rewrite the whole equation as

$$\left(\sum_{n=0}^{\infty}\varepsilon^{2n}V_n\nabla_0^n\right)\Phi(\boldsymbol{x}) = (1-\varepsilon^2\kappa)\left(\sum_{n=0}^{\infty}\varepsilon^{2n}U_n\nabla_0^n\right)\Phi(\boldsymbol{x}) + \varepsilon^2\langle s\rangle_\epsilon q(\boldsymbol{x}),$$

where

$$V_0 = 1;$$

$$V_1 = -\frac{\langle s^3\rangle_\epsilon}{3!\langle s\rangle_\epsilon}V_0;$$

$$V_2 = -\frac{9}{5}\frac{\langle s^5\rangle_\epsilon}{5!\langle s\rangle_\epsilon}V_0 - \frac{\langle s^3\rangle_\epsilon}{3!\langle s\rangle_\epsilon}V_1;$$

$$V_3 = -\frac{27}{7}\frac{\langle s^7\rangle_\varepsilon}{7!\langle s\rangle_\epsilon}V_0 - \frac{9}{5}\frac{\langle s^5\rangle_\epsilon}{5!\langle s\rangle_\epsilon}V_1 - \frac{\langle s^3\rangle_\epsilon}{3!\langle s\rangle_\epsilon}V_2;$$

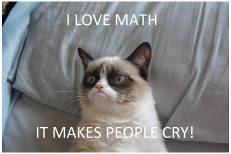
$$\ldots$$

Asymptotic Analysis

Finally, we rearrange the terms and get

$$\boxed{\left(\sum_{n=0}^{\infty}\varepsilon^{2n}\left[W_{n+1}\nabla_0^{n+1}+\kappa U_n\nabla_0^n\right]\right)\Phi(\boldsymbol{x})=\langle s\rangle_\epsilon q(\boldsymbol{x}),}$$

where $W_n = V_n - U_n$.



- If we discard the terms of $O(\varepsilon^{2n})$ in this equation, we obtain a partial differential equation for $\Phi(\boldsymbol{x})$ of order $2n$
- We will use this approach to explicitly derive the nonclassical SP$_1$, SP$_2$, and SP$_3$ equations
- Higher-order equations can be derived by continuing to follow the same procedure

Nonclassical Diffusion (SP_1)

Discarding the terms of $O(\varepsilon^2)$ and reverting to the original unscaled parameters, we obtain

$$-\frac{1}{6}\frac{\langle s^2\rangle}{\langle s\rangle}\nabla^2\Phi(\boldsymbol{x}) + \frac{1-c}{\langle s\rangle}\Phi(\boldsymbol{x}) = Q(\boldsymbol{x}),$$

which is the nonclassical diffusion equation.
If the free-path distribution $p(s)$ is an exponential, $\langle s^m\rangle = m!\Sigma_t^{-m}$ and this equation reduces to the classical diffusion equation

$$-\frac{1}{3\Sigma_t}\nabla^2\Phi(\boldsymbol{x}) + \Sigma_a\Phi(\boldsymbol{x}) = Q(\boldsymbol{x}).$$

Nonclassical SP_2

Discarding the terms of $O(\varepsilon^4)$ and reverting to the original unscaled parameters, we obtain

$$-\frac{1}{6}\frac{\langle s^2\rangle}{\langle s\rangle}\nabla^2\Big[\Phi(\boldsymbol{x})+\lambda_1\left[(1-c)\Phi(\boldsymbol{x})-\langle s\rangle Q(\boldsymbol{x})\right]\Big]+ \\ \frac{1-c}{\langle s\rangle}\left[1-\beta_1(1-c)\right]\Phi(\boldsymbol{x})=\left[1-\beta_1(1-c)\right]Q(\boldsymbol{x}),$$

with $\lambda_1=\frac{3}{10}\frac{\langle s^4\rangle}{\langle s^2\rangle^2}-\frac{1}{3}\frac{\langle s^3\rangle}{\langle s\rangle\langle s^2\rangle}$ and $\beta_1=\frac{1}{3}\frac{\langle s^3\rangle}{\langle s\rangle\langle s^2\rangle}-1$.

If the free-path distribution $p(s)$ is an exponential, $\lambda_1=\frac{4}{5}$, $\beta_1=0$, and this equation reduces to the classical SP_2 equation

$$-\frac{1}{3\Sigma_t}\nabla^2\left[\Phi(\boldsymbol{x})+\frac{4}{5}\frac{\Sigma_a\Phi(\boldsymbol{x})-Q(\boldsymbol{x})}{\Sigma_t}\right]+\Sigma_a\Phi(\boldsymbol{x})=Q(\boldsymbol{x}).$$

Nonclassical SP_3

Discarding the terms of $O(\varepsilon^6)$ and reverting to the original unscaled parameters, we obtain

$$-\frac{1}{6}\frac{\langle s^2\rangle}{\langle s\rangle}\nabla^2\Big[\big[1+\beta_1(1-c)\big]\Phi(\boldsymbol{x})+2\nu(\boldsymbol{x})\Big]+\frac{1-c}{\langle s\rangle}\Phi(\boldsymbol{x})=Q(\boldsymbol{x}),$$

$$-\frac{1}{6}\frac{\langle s^2\rangle}{\langle s\rangle}\nabla^2\left[\frac{\lambda_1}{2}\Phi(\boldsymbol{x})+\lambda_2\nu(\boldsymbol{x})\right]+\frac{1-\beta_2(1-c)}{\langle s\rangle}\nu(\boldsymbol{x})=0,$$

with

$$\lambda_2=\frac{1}{10\langle s^2\rangle\langle s^3\rangle-9\langle s\rangle\langle s^4\rangle}\left[\frac{9}{5}\langle s^5\rangle-\frac{27}{21}\frac{\langle s\rangle\langle s^6\rangle}{\langle s^2\rangle}+3\frac{\langle s^3\rangle\langle s^4\rangle}{\langle s^2\rangle}-\frac{10}{3}\frac{\langle s^3\rangle^2}{\langle s\rangle}\right],$$

$$\beta_2=\frac{1}{10\langle s^2\rangle_\epsilon\langle s^3\rangle_\epsilon-9\langle s\rangle_\epsilon\langle s^4\rangle_\epsilon}\left[\frac{10}{3}\frac{\langle s^3\rangle_\epsilon^2}{\langle s\rangle_\epsilon}-\frac{9}{5}\langle s^5\rangle_\epsilon\right]-1.$$

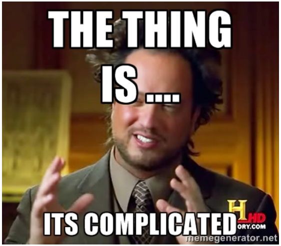
Nonclassical SP_3

If the free-path distribution $p(s)$ is an exponential, $\lambda_2 = \frac{11}{7}$, $\beta_2 = 0$, and these equations reduce to the classical SP_3 equations

$$-\frac{1}{3\Sigma_t}\nabla^2\left[\Phi(\boldsymbol{x}) + 2\nu(\boldsymbol{x})\right] + \Sigma_a\Phi(\boldsymbol{x}) = Q(\boldsymbol{x}),$$

$$-\frac{1}{3\Sigma_t}\nabla^2\left[\frac{2}{5}\Phi(\boldsymbol{x}) + \frac{11}{7}\nu(\boldsymbol{x})\right] + \Sigma_t\nu(\boldsymbol{x}) = 0.$$

Regarding Boundary Conditions...



- Nonclassical transport boundary conditions are not yet well-defined for the "backward" nonclassical equation
- The asymptotic analysis for the classical SP_N equations does not yield boundary conditions... and neither does the present one

Solution: manipulate the nonclassical SP_N equations into a classical form and use classical (Marshak) boundary conditions.

Nonclassical Diffusion with Vacuum Boundaries

The nonclassical diffusion equation is

$$-\frac{1}{6}\frac{\langle s^2\rangle}{\langle s\rangle}\nabla^2\Phi(\boldsymbol{x}) + \frac{1-c}{\langle s\rangle}\Phi(\boldsymbol{x}) = Q(\boldsymbol{x}).$$

We define

$$\widehat{\Sigma}_t = 2\frac{\langle s\rangle}{\langle s^2\rangle}, \qquad \widehat{\Sigma}_a = \frac{1-c}{\langle s\rangle},$$

and rewrite it as a classical diffusion equation, for which we use Marshak boundary conditions:

$$-\frac{1}{3\widehat{\Sigma}_t}\nabla^2\Phi(\boldsymbol{x}) + \widehat{\Sigma}_a\Phi(\boldsymbol{x}) = Q(\boldsymbol{x}),$$

$$\text{B.C.:} \qquad \frac{1}{2}\Phi(\boldsymbol{x}) - \frac{1}{3\widehat{\Sigma}_t}\vec{\boldsymbol{n}}\cdot\nabla\Phi(\boldsymbol{x}) = 0.$$

Nonclassical SP_2 with Vacuum Boundaries

$$-\frac{1}{3\widehat{\Sigma}_t}\nabla^2\widehat{\Phi}(\boldsymbol{x}) + \widehat{\Sigma}_a\widehat{\Phi}(\boldsymbol{x}) = \widehat{Q}(\boldsymbol{x}),$$

$$\text{B.C.:} \qquad \frac{1}{2}\widehat{\Phi}(\boldsymbol{x}) - \frac{1}{3\widehat{\Sigma}_t}\vec{\boldsymbol{n}}\cdot\nabla\widehat{\Phi}(\boldsymbol{x}) = 0,$$

with

$$\widehat{\Sigma}_t = 2\frac{\langle s\rangle}{\langle s^2\rangle}, \qquad \widehat{\Sigma}_a = \frac{(1-c)}{\langle s\rangle}\frac{1-\beta_1(1-c)}{1+\lambda_1(1-c)},$$

$$\widehat{Q}(\boldsymbol{x}) = \frac{1-\beta_1(1-c)}{1+\lambda_1(1-c)}Q(\boldsymbol{x}), \qquad \Phi(\boldsymbol{x}) = \frac{\widehat{\Phi}(\boldsymbol{x}) + \lambda_1\langle s\rangle Q(\boldsymbol{x})}{1+\lambda_1(1-c)}.$$

Nonclassical SP_3 with Vacuum Boundaries

$$-\frac{1}{3\widehat{\Sigma}_t}\nabla^2\left[\Phi(\boldsymbol{x})+2\widehat{\Phi}_2(\boldsymbol{x})\right]+\widehat{\Sigma}_a\Phi(\boldsymbol{x})=\widehat{Q}(\boldsymbol{x}),$$

$$-\frac{1}{3\widehat{\Sigma}_t}\nabla^2\left[\frac{2}{5}\Phi(\boldsymbol{x})+\left(\frac{4}{5}+\frac{27\widehat{\Sigma}_t}{35\widehat{\Sigma}_3}\right)\widehat{\Phi}_2(\boldsymbol{x})\right]+\widehat{\Sigma}_2\widehat{\Phi}_2(\boldsymbol{x})=0,$$

B.C.: $$\frac{1}{2}\Phi(\boldsymbol{x})-\frac{1}{3\widehat{\Sigma}_t}\vec{\boldsymbol{n}}\cdot\nabla\Phi(\boldsymbol{x})-\frac{2}{3\widehat{\Sigma}_t}\vec{\boldsymbol{n}}\cdot\nabla\widehat{\Phi}_2(\boldsymbol{x})+\frac{5}{8}\widehat{\Phi}_2(\boldsymbol{x})=0,$$

B.C.: $$-\frac{1}{8}\Phi(\boldsymbol{x})+\frac{5}{8}\widehat{\Phi}_2(\boldsymbol{x})-\frac{3}{7\widehat{\Sigma}_3}\vec{\boldsymbol{n}}\cdot\nabla\widehat{\Phi}_2(\boldsymbol{x})=0,$$

with

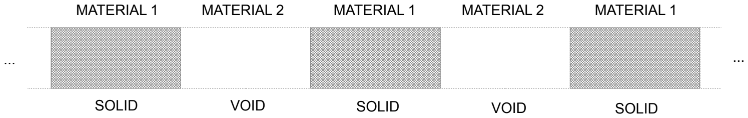
$$\widehat{\Phi}_2(\boldsymbol{x})=\frac{\nu(\boldsymbol{x})}{1+\beta_1(1-c)},\qquad \widehat{\Sigma}_t=2\frac{\langle s\rangle}{\langle s^2\rangle},$$

$$\widehat{\Sigma}_a=\frac{(1-c)}{\langle s\rangle}\frac{1}{1+\beta_1(1-c)},\qquad \widehat{Q}(\boldsymbol{x})=\frac{Q(\boldsymbol{x})}{1+\beta_1(1-c)},$$

$$\widehat{\Sigma}_2=\frac{4\left[1+\beta_1(1-c)\right]\left[1-\beta_2(1-c)\right]}{5\lambda_1\langle s\rangle},\qquad \widehat{\Sigma}_3=\frac{27}{28}\frac{\lambda_1\widehat{\Sigma}_t}{\lambda_2\left[1+\beta_1(1-c)\right]-\lambda_1}.$$

1-D random periodic media

We consider a 1-D physical system consisting of alternating layers of solid and void, periodically arranged:



- layers of material 1 and 2 have thicknesses ℓ_1 and ℓ_2, respectively; (period $\ell = \ell_1 + \ell_2$)
- the origin ($x = 0$) is *randomly placed* in the periodic system (this is equivalent to randomly placing the system in the infinite line $-\infty < x < \infty$)
- the probability P_i of finding material i in a given point x is $\ell_i/(\ell_1 + \ell_2)$

The Path-length distribution function

★ $\ell_1 < \ell_2$:

$$p(\mu,s) = \begin{cases} \frac{\Sigma_{t1}}{\ell_1}(n\ell + \ell_1 - s|\mu|)e^{-\Sigma_{t1}(s - n\ell_2/|\mu|)}, & \text{if } n\ell \le s|\mu| \le n\ell + \ell_1 \\ 0, & \text{if } n\ell + \ell_1 \le s|\mu| \le n\ell + \ell_2 \\ \frac{\Sigma_{t1}}{\ell_1}(s|\mu| - n\ell + \ell_2)e^{-\Sigma_{t1}[s-(n+1)\ell_2/|\mu|)]}, & \text{if } n\ell + \ell_2 \le s|\mu| \le (n+1)\ell \end{cases}$$

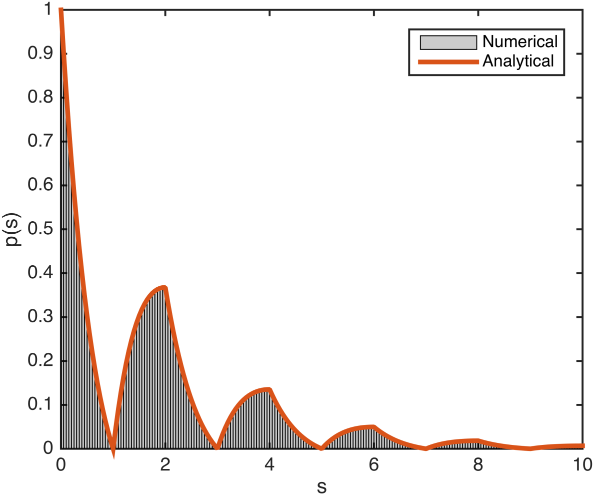
★ $\ell_1 = \ell_2$:

$$p(\mu,s) = \begin{cases} \frac{\Sigma_{t1}}{\ell_1}(n\ell + \ell_1 - s|\mu|)e^{-\Sigma_{t1}(s - n\ell_2/|\mu|)}, & \text{if } n\ell \le s|\mu| \le n\ell + \ell_1 \\ \frac{\Sigma_{t1}}{\ell_1}(s|\mu| - n\ell + \ell_2)e^{-\Sigma_{t1}[s-(n+1)\ell_2/|\mu|)]}, & \text{if } n\ell + \ell_2 \le s|\mu| \le (n+1)\ell \end{cases}$$

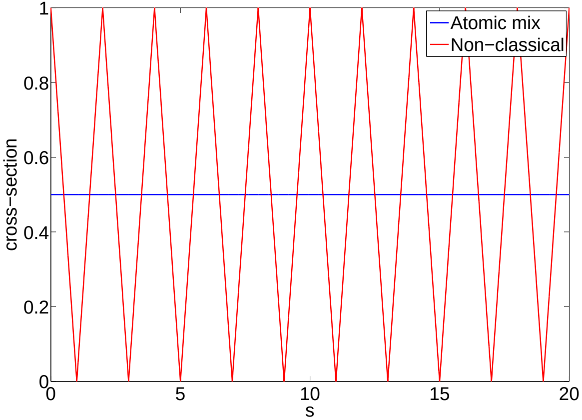
★ $\ell_1 > \ell_2$:

$$p(\mu,s) = \begin{cases} \frac{\Sigma_{t1}}{\ell_1}(n\ell + \ell_1 - s|\mu|)e^{-\Sigma_{t1}(s - n\ell_2/|\mu|)}, & \text{if } n\ell \le s|\mu| \le n\ell + \ell_2 \\ \frac{\Sigma_{t1}}{\ell_1}[(n\ell + \ell_2 - s|\mu|)(1 - e^{\Sigma_{t1}\ell_2/|\mu|}) \\ \qquad + \ell_1 - \ell_2]e^{-\Sigma_{t1}(s - n\ell_2/|\mu|)} \quad , & \text{if } n\ell + \ell_2 \le s|\mu| \le n\ell + \ell_1 \\ \frac{\Sigma_{t1}}{\ell_1}(s|\mu| - n\ell + \ell_2)e^{-\Sigma_{t1}[s-(n+1)\ell_2/|\mu|)]}, & \text{if } n\ell + \ell_1 \le s|\mu| \le (n+1)\ell \end{cases}$$

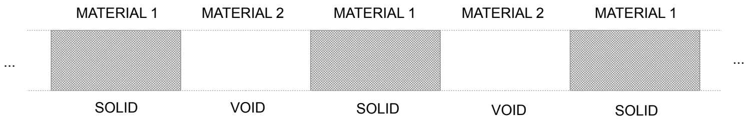
$p(\mu = 1, s)$ for $\Sigma_{t1} = 1$, with $\ell_1 = \ell_2 = 1$



$p(\mu = 1, s)$ for $\Sigma_{t1} = 1$, with $\ell_1 = \ell_2 = 1$

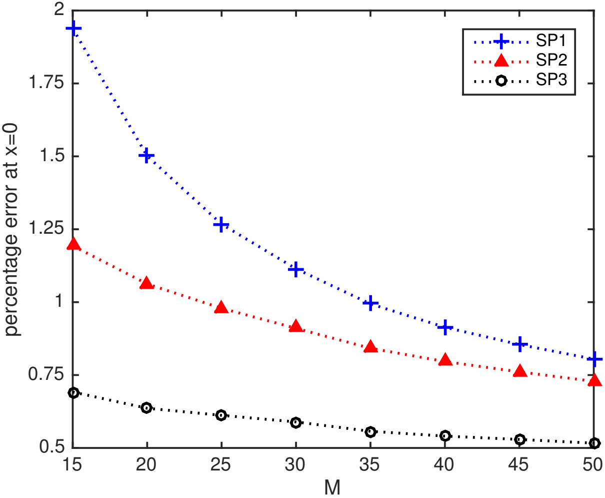


1-D diffusive system



- Slab geometry
- Isotropic scattering
- Vacuum boundaries
- Isotropic source
- We define $M = \varepsilon^{-1}$, and
    - $\ell_1 = \ell_2 = 0.5 \implies \ell = 1$
    - $-M \leq x \leq M \implies \ell M = O(\varepsilon^{-1})$
    - $\Sigma_{t1} = 1 = O(1)$
    - $1 - c = 0.1 \times M^{-2} = O(\varepsilon^2)$
    - $Q_1 = 0.2 \times M^{-2} = O(\varepsilon^2)$

Error Estimates at $x = 0$



Discussion

1. The nonclassical SP_N equations provide more accurate diffusion approximations to nonclassical transport
2. However, they require all the moments of the $p(s)$ up to $2N$ to exist
3. They can be manipulated into a set of *classical* SP_N equations with modified parameters
4. Therefore, they can be implemented in already existing SP_N codes

Immediate things to do:

- Perform a complete analysis of these results in nonclassical multi-dimensional systems
- Extend the analysis to angular dependent free-path distributions $p(\boldsymbol{\Omega}, s)$
- Extend the analysis to anisotropic scattering

Thank you for your attention!!!

Questions?