NE 155, Class 28 and 29, S17
Reactor Kinetics in Zero Dimensions

content provided by: Kathryn Huff

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1 Introduction

In reactors and other fission systems, neutron populations vary over time. This lesson will introduce a method for analyzing this time evolution analytically by neglecting variation of the flux shape. In particular, this lesson will cover:

- delayed neutrons,
- the importance of delayed neutrons for reactor control,
- and the derivation of the point reactor kinetics equations (PRKE).

Additionally, if we have time, this lesson will also cover feedback effects, including:

- the importance of feedbacks
- the form of the PRKEs with simple temperature feedback.

Note that much of this can be found in Duderstadt and Hamilton.

1.1 Transient Analysis

Transient analysis is necessary when the neutron flux varies with time. Commonly studied transient scenarios include normal startup and shutdown of a reactor as well as abnormal scenarios that cause reactivity increases and decreases during otherwise normal operation.

2 Delayed Neutrons

Reactor control relies on a balance of neutrons. When an isotope fissions, it produces neutrons, energy, and fission products. Most of the neutrons emitted due to fission are *prompt*, nearly all released within $10^{-10}s$ of the fission. The average lifetime of prompt neutrons in a thermal reactor is $10^{-4}s$ and in a fast reactor is $10^{-7}s$.

2.1 Delayed Neutron Emission

However, a fraction of the neutrons appear later. Some fission products are unstable and decay within seconds or minutes of the fission. Among those, a few decay by neutron emission. These particular fission products are called "delayed neutron precursors". ^{87}Br, for example, has a half-life of 55.9 seconds and tends to decay by neutron emission.

2.2 Delayed Neutron Precursor Data

Typically, we group delayed neutron precursors into 6 or 8 groups according to their half-lives. Standardized data exist for these calculations, as in Table 1.

j	$t_{1/2}$ $[s]$	λ_j^d $[1/s]$	η_j $[n/f]$	β_j
1	55.72	0.0124	0.00052	0.000215
2	22.72	0.0305	0.00546	0.001424
3	6.22	0.111	0.00310	0.001274
4	2.30	0.301	0.00624	0.002568
5	0.614	1.14	0.00182	0.000748
6	0.230	3.01	0.00066	0.000273

Table 1: Delayed neutron data, ^{235}U thermal fission [**?**].

In the above table:

$$
\begin{aligned}
j &= \text{group index} \\
t_{1/2} &= \text{half life}[s] \\
\lambda_j^d &= \text{decay constant}[1/s] \\
\eta_j &= \text{fission factor}[neutrons/fisson] \\
\beta_j &= \text{delayed neutron fraction} \\
& \quad [\text{neutrons from delayed fission / neutrons from all fission}]
\end{aligned}
$$

These parameters are used to incorporate the contributions of delayed neutrons into transient calculations. Note that in this case the sum of the β_j terms is 0.0065. That means 0.0065 of every 1.0 neutrons coming from fission is delayed.

When we add in delayed neutrons, the average neutron lifetime because approximately 0.1 s, which is much higher than the prompt value of $10^{-6}s$ to $10^{-4}s$.

3 Delayed Neutrons and Reactor Control

These delayed neutrons are critical to controlling the reactor. To capture the reasons why, we will need the following definitions.

$$
\begin{aligned}
\rho =& \text{reactivity}\\
=& \frac{k-1}{k}\\
k =& \text{multiplication factor}\\
& (k<1) \rightarrow \text{negative reactivity}\\
& (k>1) \rightarrow \text{positive reactivity}\\
& (k=1) \rightarrow \text{critical}\\
\beta =& \text{delayed neutron fraction}\\
& (\rho<\beta) \text{ delayed supercriticality}\\
& (\rho>\beta) \text{ prompt supercriticality}\\
l =& \text{mean neutron lifetime}
\end{aligned}
$$

3.1 Units of Reactivity

Note that the units of ρ can be confusing.

Unit	Definition	Example
Δk	actual PRKE units	0.0005
$\%\Delta k$	percent notation of Δk	0.05%
pcm	per cent mille	50pcm
Dollars	$\frac{\Delta k}{\beta}$	$1
Cents	100 cents per dollar	100 cents
Milli-beta	1000 milli-beta per dollar	1000 milli-beta

Table 2: Common units of reactivity.

3.2 Thought Experiment

Reactor power behaves as:

$$l = \text{mean generation time} \tag{1}$$
$$n(t+l) = n(t) + l\frac{dn}{dt} = kn(t) \tag{2}$$

such that

$$\frac{dn}{dt} = \left(\frac{k-1}{l}\right) n(t) \tag{3}$$

which gives

$$n(t) = n_0 e^{\frac{(k-1)t}{l}} \tag{4}$$

characterized by the time constant

$$T = \text{reactor period} \tag{5}$$
$$= \frac{l}{k-1} . \tag{6}$$

In a universe without delayed neutrons, the mean neutron lifetime (l) would be the prompt neutron lifetime, l_p. Noting that the prompt neutron lifetime is about $2 \times 10^{-5}s$, take a moment to think about the implications of this. What would it be like to try to control a reactor like that?

Exercise *If a control rod were moved to introduce an excess reactivity of* $0.0005\Delta k$*, what would the power be one second later?*

(a) with l = 0.1s
(b) with l = 1 $\times$ 10^{-4}s

Here's what it looks like when something goes prompt supercritical on purpose: `https://www.youtube.com/watch?v=6I3JKYdGWTE`

4 Derivation

It's clear that the delayed neutrons are important. So, how do we include them in our model of neutrons in a reactor?

4.1 The Diffusion Equation

Let's begin by observing the steady-state diffusion equation.

$$
\begin{aligned}
-\nabla D(\vec{r},E)\nabla\phi(\vec{r},E)+&\Sigma_t(\vec{r},E)\phi(\vec{r},E) = \\
&\int_0^\infty \Sigma_s(\vec{r},E'\to E)\phi(\vec{r},E')dE' \\
&+\sum_j \chi^j(E)\int_0^\infty \upsilon\Sigma_f^j(\vec{r},E')\phi(\vec{r},E')dE' + Q(\vec{r},E) \qquad (7)
\end{aligned}
$$

Review this equation. Note how, to incorporate delayed neutrons, the $\chi(E)$ fission spectrum must be properly weighted with prompt and delayed contributions. To incorporate delayed neutrons explicitly (rather than implicitly as above), it is necessary to begin with the time-dependent, one-speed diffusion equation, and include a source contribution from delayed neutrons. That contribution depends on the concentration of delayed neutron precursors.

$$
\begin{aligned}
C_i(r,t)d^3r \equiv\ &\text{the expected number of fictitious precursors of the ith} \qquad (8)\\
&\text{kind about r in } d^3r \text{ that will always decay by neutron emission.}
\end{aligned}
$$

The delayed neutron precursor concentrations obey the equation,

$$
\begin{aligned}
\frac{\partial\hat{C}_i(t,r)}{\partial t} &= \beta_i\nu\Sigma_f(r,t)\phi(r,t) - \lambda_i\hat{C}_i(r,t)\ , \qquad (9)\\
&\text{where } i\in[1,6]. \qquad (10)
\end{aligned}
$$

β_i and λ_i depend on the incident neutron energy, fissionable isotope, and precursor group. In this way, the linearized Boltzmann transport equation has seven dimensions. Taking delayed neutrons into account, the one speed neutron diffusion equation can be written,

$$
\frac{1}{v}\frac{\partial\phi(r,t)}{\partial t} - \nabla D(r,t)\nabla\phi(r,t) + \ \Sigma_a(r,t)\phi(r,t) = (1-\beta)\nu\Sigma_F(r,t)\phi(r,t) + \ \sum_{i=1}^{6}\lambda_i\hat{C}_i(r,t). \qquad (11)
$$

Transient analysis methods seek to solve this equation for changes in the parameters caused by changing reactor conditions. Additional PDEs are added to this calculation to capture the dependence of temperature on heat conduction and the effect of fluid flow.

5 The Point Reactor Kinetics Equations

One common method to evaluate transient scenarios is through reduction of dimensions. If we assume a separation of variables solution to (11), we arrive at:

$$\phi(r,t) = vn(t)\psi_1(r) \tag{12}$$
$$\hat{C}_i(r,t) = C_i(t)\psi_1(r) \tag{13}$$

where ψ_1 is the fundamental mode solution of

$$\nabla^2\psi_n + B_g^2\psi_n = 0. \tag{14}$$

ASIDE: B^2 is called buckling, which we can think of two ways [2]

$$B^2 = \frac{1}{D}\big(\frac{1}{k}\Sigma_f - \Sigma_a\big)$$

or

$$= \frac{k_\infty - 1}{L^2}$$

in either case

$$\nabla^2\phi + B^2\phi = 0 \ .$$

Using this separation of variables solution reduces the spatial complexity of the reactor to a single point. Inserting (12) and (13) into (11) gives the Point Reactor Kinetics Equations (PRKE).

$$\frac{dn(t)}{dt} = \frac{\rho(t) - \beta}{\Lambda}n(t) + \sum_{i=1}^{6}\lambda_i C_i(t) \tag{15}$$
$$\frac{dC_i(t)}{dt} = \frac{\beta_i}{\Lambda}n(t) - \lambda_i C_i(t) \tag{16}$$

where

$$
\begin{aligned}
i &\in [1,6] \\
n &= \text{neutron population} \\
\beta &= \text{fraction of neutrons that are delayed} \\
\lambda_i &= \text{effective decay constant of the ith precursor}[\frac{1}{s}] \\
C_i(t) &= \text{delayed neutron concentration due to the ith precursor} \\
\Lambda &= \text{effective neutron lifetime} \\
&\equiv (v\nu\Sigma_F)^{-1} \\
\rho(t) &= \text{reactivity} \\
&\equiv \frac{k(t)-1}{k(t)} \\
&\equiv \frac{\nu\Sigma_F - \Sigma_a(1+L^2B_g^2)}{\nu\Sigma_F}
\end{aligned}
$$

and

$$
k \equiv \frac{\nu\Sigma_F/\Sigma_a}{1+L^2B_g^2}.
$$

The PRKEs allow a nuclear engineer to remove the spatial aspects of the reactor from consideration to simplify the analysis of this initial value problem. In particular, the assumptions that have gone into this set of equations include:

- there is no external neutron source
- One energy group ($\phi(\vec{r}, E, t) = \phi(\vec{r}, t)$)
- Separation of variables ($\phi(\vec{r}, t) = S(t)\psi(\vec{r})$)
- β_i and λ_i are constant over the transient
- There is no feedback (e.g. flux is low, feedbacks are slow/weak, $\rho(t)$ is known)

Exercise: *What initial conditions need to be defined in order to solve this initial value problem (eqn.* (16)*?)*

6 Analytical Solution of the PRKE

(Chapter 6 of Duderstadt and Hamilton is a good reference here as well)

$$\rho(0) = 0 \tag{17}$$
$$P(0) = P_0 \tag{18}$$
$$C(0) = \frac{\beta}{\lambda\Lambda} P_0 \tag{19}$$
$$\rho(t > 0) = \rho_0 \tag{20}$$

Assume solutions:

$$P(t) = P e^{wt} \tag{21}$$
$$C(t) = C e^{wt} \tag{22}$$

Such that:

$$\omega P = \frac{\rho_0 - \beta}{\Lambda} P + \lambda C \tag{23}$$
$$\omega C = \frac{\beta}{\Lambda} P - \lambda C \tag{24}$$

This system can be written:

$$\begin{bmatrix} \omega - \frac{\rho_0 - \beta}{\Lambda} & \lambda \\ -\frac{\beta}{\Lambda} & \omega + \lambda \end{bmatrix} \begin{bmatrix} P \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A solution exists only if the determinant is 0:

$$\Lambda \omega^2 + (\lambda\Lambda + \beta - \rho_0)\,\omega - \rho_0 \Lambda = 0 \tag{25}$$

Using the quadratic formula gives the solution:

$$\omega = \frac{1}{2\Lambda} \left[-(\beta - \rho_0 + \lambda\Lambda) \pm \sqrt{(\beta - \rho_0 + \lambda\Lambda)^2 + 4\lambda\Lambda\rho_0} \right]$$

thus, if we assume $\lambda\Lambda << \beta$ and $|\rho_0| << \beta$ then :

$$\omega_1 \approx \frac{\lambda \rho_o}{\beta - \rho_0} \qquad \omega_2 \approx \frac{\beta - \rho_o}{\Lambda}$$

which we then use for our general solutions

$$P(t) = P_1 e^{\omega_1 t} + P_2 e^{\omega_2 t}$$
$$C(t) = C_1 e^{\omega_1 t} + C_2 e^{\omega_2 t}$$

7 Coupling to Thermal-Hydraulic Feedbacks

In addition to modeling the neutronic properties of a nuclear reactor, the PRKE can be modified to include the thermal-hydraulic feedback effects that the power transient will induce.

Exercise: *Can you think of some sources of feedback in a reactor?*

7.1 Coupled Multiphysics

Typically, transient analyses in reactors seek to characterize the relationship between neutron population and temperature, which are coupled together by reactivity. That is, any change in power of a reactor can be related to a quantity known as the reactivity, ρ, which characterizes the offset of the nuclear reactor from criticality. In all power reactors, the scalar flux of neutrons is what determines the reactor's power. The reactor power, in turn, affects the temperature. Reactivity feedback comes from the temperature dependence of geometry, material densities, the neutron spectrum, and microscopic cross sections [1].

Nuclear reactors operate in a state of criticality, which implies a steady state neutron flux (and hence power). Should the reactor deviate from criticality, a power transient will result. The magnitude and duration of the power transient will be dependent upon the length and strength of the deviation. In an Accident Transient Without Scram (ATWS), only intrinsic negative feedback responses within a reactor design are relied upon to prevent an uncontrollable power excursion. Since some feedback may be positive while other feedback is negative, an analysis of the balance must be addressed. Additionally, because of the effect of delayed neutrons, the timescales of fluid flow, heat conduction, thermal transport, or other phenomena, both positive and negative feedback may be delayed. Thus, a transient assessment of the time evolution of the neutron population within the core is essential to capture power-level instabilities resulting when reactivity feedback that is out-of-phase with the neutron population [3].

7.2 Coupled PRKE

When coupled to other physics, the PRKE are a set of stiff, nonlinear ordinary differential equations. For a reactor in which the only reactivity feedback comes from the fuel and the coolant:

$$
\frac{d}{dt}\begin{bmatrix} p \\ \zeta_1 \\ . \\ . \\ . \\ \zeta_j \\ . \\ . \\ . \\ \zeta_J \\ T_{fuel} \\ T_{cool} \end{bmatrix} = \begin{bmatrix} \frac{\rho(t,T^{fuel},T_{cool})-\beta}{\Lambda}p + \sum_{j=1}^{j=J}\lambda_j\zeta_j \\ \frac{\beta_1}{\Lambda}p - \lambda_1\zeta_1 \\ . \\ . \\ . \\ \frac{\beta_j}{\Lambda}p - \lambda_j\zeta_j \\ . \\ . \\ . \\ \frac{\beta_J}{\Lambda}p - \lambda_J\zeta_J \\ f_{fuel}(p, c_p^{fuel}, T_{fuel}, T_{cool}, \cdots) \\ f_{cool}(c_p^{cool}, T_{fuel}, T_{cool}, h, A_{fuel}, A_{flow}\cdots) \end{bmatrix} \tag{26}
$$

Equation 26 shows a generalized set of PRKE where variables include the normalized power, p, the delayed neutron precursor concentrations ζ_j, and the core average fuel and coolant temperatures T_{fuel} and T_{cool}. Additional equations quantifying other phenomena can add complexity to this suite of PDEs.

References

[1] George I. Bell and Samuel Glasstone. *Nuclear Reactor Theory*. Van Nostrand Reinhold Company, New York, 1970.

[2] John R. Lamarsh and Anthony John Baratta. *Introduction to nuclear engineering*. Addison-Wesley Massachusetts, 3 edition, 2001.

[3] Weston M. Stacey. *Nuclear reactor physics*. Wiley. com, 2007.