Derive the matrix expressions each reflecting boundary in the 2-D diffusion equation formulation derived with the finite volume method and describe the associated equations.
In 2-D:

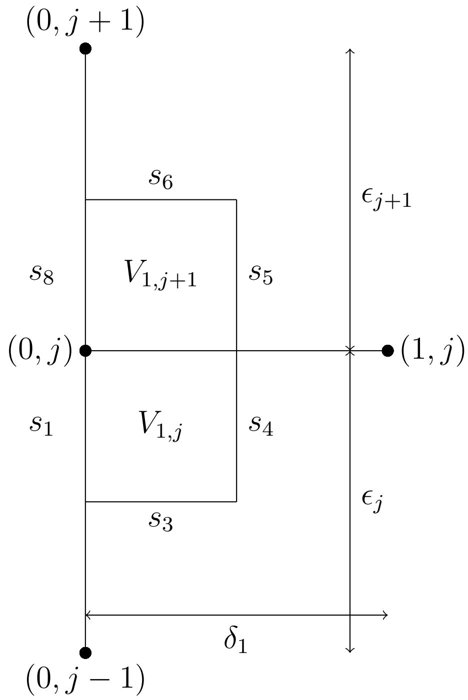
$$-\frac{\partial}{\partial x}D(x,y)\frac{\partial}{\partial x}\phi(x,y) - \frac{\partial}{\partial y}D(x,y)\frac{\partial}{\partial y}\phi(x,y) + \Sigma_a(x,y)\phi(x,y) = S(x,y)$$

with $x \in [0,a]$ and $y \in [0,b]$.

LEFT

The left side boundary condition is:

$$\frac{\partial}{\partial x}\phi(x,y)\big|_{x=0} = 0$$



To implement the boundary condition, we integrate in the first half-cell, so from $x = 0$ to $x = \delta_1/2$.
Recall that when we considered the streaming term we converted the volume integral into a surface integral:

$$\begin{aligned} &-\int_V d\vec{r}\left[\nabla\cdot\big(D(\vec{r})\nabla\phi(\vec{r})\big)\right] \\ &= -\int_S d\vec{S}\,D(\vec{r})\frac{\partial}{\partial \hat{n}}\phi(\vec{r}) \end{aligned}$$

and we defined the partial derivative w.r.t. direction on each surface. We now have two fewer surfaces (no s_2 or s_7), and we apply the zero current boundary condition to s_1 and s_8.

$$\begin{aligned} \frac{\partial}{\partial \hat{n}}\phi(\vec{r}) &= \frac{\phi_{0,j-1} - \phi_{0,j}}{\epsilon_j} && \text{on } S_3 \\ &= \frac{\phi_{0,j+1} - \phi_{0,j}}{\epsilon_{j+1}} && \text{on } S_6 \\ &= \frac{\phi_{1,j} - \phi_{0,j}}{\delta_1} && \text{on } S_4\,, S_5 \\ &= 0 && \text{on } S_1\,, S_8 \end{aligned}$$

We then use the midpoint rule for the integration and integrate along each surface. Recall that the physics values are cell-centered while the flux is edge-centered. The two terms that are different are for the top and bottom of the cell:

$$
\begin{aligned}
-\int_{S_3} d\vec{S}\, D(\vec{r})\frac{\partial}{\partial \hat{n}}\phi(\vec{r}) &= \frac{\phi_{0,j}-\phi_{0,j-1}}{\epsilon_j}\left(\frac{D_{1,j}\delta_1}{2}\right)\\
-\int_{S_6} d\vec{S}\, D(\vec{r})\frac{\partial}{\partial \hat{n}}\phi(\vec{r}) &= \frac{\phi_{0,j+1}-\phi_{0,j}}{\epsilon_{j+1}}\left(\frac{D_{1,j+1}\delta_1}{2}\right)\\
S_1 + S_8 &= 0\,,\\
S_4 + S_5 &= \frac{\phi_{1,j}-\phi_{0,j}}{\delta_1}\left(\frac{D_{1,j}\epsilon_j + D_{1,j+1}\epsilon_{j+1}}{2}\right)\,.
\end{aligned}
$$

The absorption and streaming terms now become:

$$
\begin{aligned}
\int\int dxdy\, \Sigma_a(x,y)\phi(x,y) &= \boxed{\phi_{0,j}\left(\Sigma_{a,1,j}V_{1,j} + \Sigma_{a,1,j+1}V_{1,j+1}\right) \equiv \Sigma^*_{a,0j}}\,,\\
\int\int dxdy\ \ S(x,y) &= \boxed{S_{1,j}V_{1,j} + S_{1,j+1}V_{1,j+1} \equiv S^*_{0j}}\,.
\end{aligned}
$$

Collecting all of the terms and separating them, we get a 4-point difference equation for $i = 0$; $j = 1, \ldots, m-1$:

$$
\boxed{a^{0j}_{1,j}\phi_{1,j} + a^{*,0j}_{0,j-1}\phi_{0,j-1} + a^{*,0j}_{0,j+1}\phi_{0,j+1} + a^{*,0j}_{0,j}\phi_{0,j} = S^*_{0j}}
$$

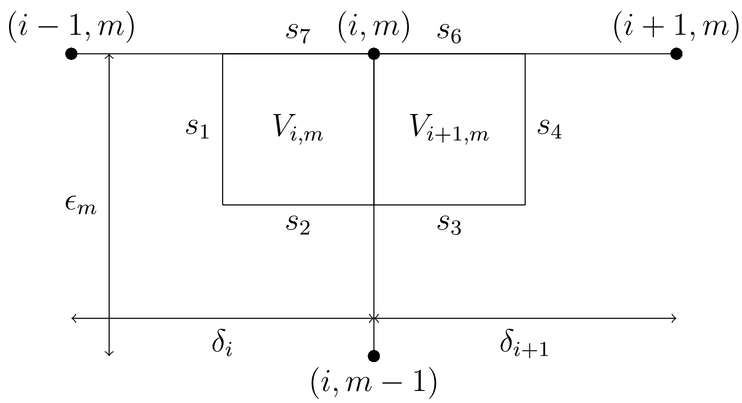
Recall: the lower index is the cell to which you are coupling, and the upper index is which cell you are in. The coefficients how change to become

$$
\begin{aligned}
a^{0j}_{1,j} &= -\frac{D_{1,j}\epsilon_j + D_{1,j+1}\epsilon_{j+1}}{2\delta_1}\\
a^{*,0j}_{0,j-1} &= -\frac{D_{1,j}\delta_1}{2\epsilon_j}\\
a^{*,0j}_{0,j+1} &= -\frac{D_{1,j+1}\delta_1}{2\epsilon_{j+1}}\\
a^{*,0j}_{0,j} &= \Sigma^*_{a,0j} - \left(a^{0j}_{1,j} + a^{*,0j}_{0,j-1} + a^{*,0j}_{0,j+1}\right)\,.
\end{aligned}
$$

TOP

The top boundary condition is:

$$\frac{\partial}{\partial y}\phi(x,y)\Big|_{y=b} = 0$$



To implement the boundary condition, we integrate in the last half-cell, so from $y = b - \epsilon_m/2$ to $y = b$.
Recall that when we considered the streaming term we converted the volume integral into a surface integral:

$$\begin{aligned}
&-\int_V d\vec{r}\,\left[\nabla\cdot\big(D(\vec{r})\nabla\phi(\vec{r})\big)\right] \\
&= -\int_S d\vec{S}\,D(\vec{r})\frac{\partial}{\partial\hat{n}}\phi(\vec{r})
\end{aligned}$$

and we defined the partial derivative w.r.t. direction on each surface. We now have two fewer surfaces (no s_8 or s_5), and we apply the zero current boundary condition to s_7 and s_6.

$$\begin{aligned}
\frac{\partial}{\partial\hat{n}}\phi(\vec{r}) &= \frac{\phi_{i-1,m}-\phi_{i,m}}{\delta_i} && \text{on } S_1 \\
&= \frac{\phi_{i+1,m}-\phi_{i,m}}{\delta_{i+1}} && \text{on } S_4 \\
&= \frac{\phi_{i,m-1}-\phi_{i,m}}{\epsilon_m} && \text{on } S_2\,,S_3 \\
&= 0 && \text{on } S_7\,,S_6
\end{aligned}$$

We then use the midpoint rule for the integration and integrate along each surface. Recall that the physics values are cell-centered while the flux is edge-centered. The two terms that are different are for the left and right of the cell:

$$
\begin{aligned}
-\int_{S_1} d\vec{S}\, D(\vec{r})\frac{\partial}{\partial \hat{n}}\phi(\vec{r}) &= \frac{\phi_{i,m}-\phi_{i-1,m}}{\delta_i}\left(\frac{D_{i,m}\epsilon_m}{2}\right) \\
-\int_{S_4} d\vec{S}\, D(\vec{r})\frac{\partial}{\partial \hat{n}}\phi(\vec{r}) &= \frac{\phi_{i+1,m}-\phi_{i,m}}{\delta_{i+1}}\left(\frac{D_{i+1,m}\epsilon_m}{2}\right)
\end{aligned}
$$

The absorption and streaming terms now become:

$$
\begin{aligned}
\int\int dxdy\, \Sigma_a(x,y)\phi(x,y) &= \boxed{\phi_{n,j}\left(\Sigma_{a,i,m}V_{i,m}+\Sigma_{a,i+1,m}V_{i+1,m}\right) \equiv \Sigma^*_{a,im}}\,, \\
\int\int dxdy\;\; S(x,y) &= \boxed{S_{i,m}V_{i,m}+S_{i+1,m}V_{i+1,m} \equiv S^*_{im}}\,.
\end{aligned}
$$

Collecting all of the terms and separating them, we get a 4-point difference equation for $i = 1, \ldots, n-1$, $j = m$:

$$
\boxed{a^{*,im}_{i-1,m}\phi_{i-1,m} + a^{*,im}_{i,m-1}\phi_{i,m-1} + a^{*,im}_{i+1,m}\phi_{i+1,m} + a^{*,im}_{i,m}\,\phi_{i,m} = S^*_{im}}
$$

Recall: the lower index is the cell to which you are coupling, and the upper index is which cell you are in. The coefficients how change to become

$$
\begin{aligned}
a^{*,im}_{i-1,m} &= -\frac{D_{i,m}\epsilon_m}{2\delta_i} \\
a^{*,im}_{i,m-1} &= -\frac{D_{i,m}\delta_i + D_{i+1,m}\delta_{i+1}}{2\epsilon_m} \\
a^{*,im}_{i+1,m} &= -\frac{D_{i+1,m}\epsilon_m}{2\delta_{i+1}} \\
a^{*,im}_{n,j} &= \Sigma_{a,im} - \left(a^{*,im}_{i-1,m} + a^{*,im}_{i,m-1} + a^{*,im}_{i+1,m}\right)\,.
\end{aligned}
$$