

NE 155

**Introduction to Numerical Simulations in
Radiation Transport**

Random Sampling

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MAJOR COMPONENTS OF MC ALGORITHM

- *PDFs: the physical/mathematical system must be described by a set of pdfs.*
- **Random number generator:** a source of random #s uniformly distributed on the unit interval.
- *Sampling rule: prescription for sampling the pdf (given having random #s)*
- **Scoring:** the outcomes must be accumulated/tallied for quantities of interest
- **Error estimation:** an estimate of the statistical error (variance) of the solution
- **Variance Reduction:** methods for reducing the variance and computation time simultaneously
- **Parallelization:** efficient use of computers

OUTLINE

- ① Physics as Probability
- ② Definitions: PDF & CDF
- ③ Motivation & Goal of Random Sampling
- ④ Basic Random Sampling Techniques
 - Direct Discrete Sampling
 - Direct Continuous Sampling
 - Rejection Sampling

Notes derived from Jasmina Vujic and Paul Wilson

LEARNING OBJECTIVES

- ➊ Provide examples of probabilistic representations of physics
- ➋ Distinguish between a PDF and CDF
- ➌ Distinguish between a *discrete* PDF/CDF and a *continuous* PDF/CDF
- ➍ Describe the goal of random sampling
- ➎ Identify and implement the best random sampling technique for a given distribution

PHYSICS AS PROBABILITY

Various physical phenomena can be represented by probability distributions

- Photon emission energy
 - Each possible energy has a different probability (intensity)

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- Photon emission energy
 - Each possible energy has a different probability (intensity)
- Scattering cross-sections
 - Each possible scattering angle has a different probability as a function of the energy
- Transmission through a medium
 - Probability of reaching a particular position depends on the cross-section

PROBABILITY DENSITY FUNCTIONS

All variables, x , have a Probability Density Function (PDF), $p(x)$, with the following characteristics:

Continuous

$$p\{a \leq x \leq b\} = \int_a^b p(x)dx$$

$$p(x) \geq 0$$
$$\int_{-\infty}^{\infty} p(x)dx = 1$$

Discrete

$$p(x = x_k) = p_k \equiv p(x_k)$$
$$k = 1, \dots, N$$

$$p_k \geq 0$$
$$\sum_{k=1}^N p_k = 1$$

CUMULATIVE DISTRIBUTION FUNCTIONS

All PDFs, $p(x)$, have an associated Cumulative Distribution Function (CDF), $P(x)$, with the following properties:

Continuous

$$P\{x' \leq x\} = P(x) = \int_{-\infty}^x p(x') dx'$$

$$P(-\infty) = 0, \quad P(\infty) = 1$$

$$0 \leq P(x) \leq 1$$

$$\frac{dP(x)}{dx} \geq 0$$

Discrete

$$P\{x' \leq x\} = P_k \equiv P(x_k) = \sum_{j=1}^k p_j$$

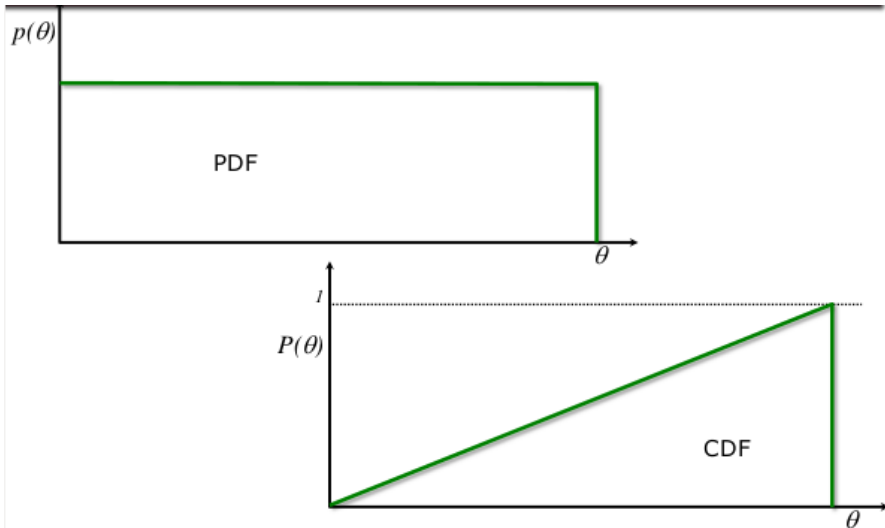
$$k = 1, \dots, N$$

$$P_0 = 0, \quad P_N = 1$$

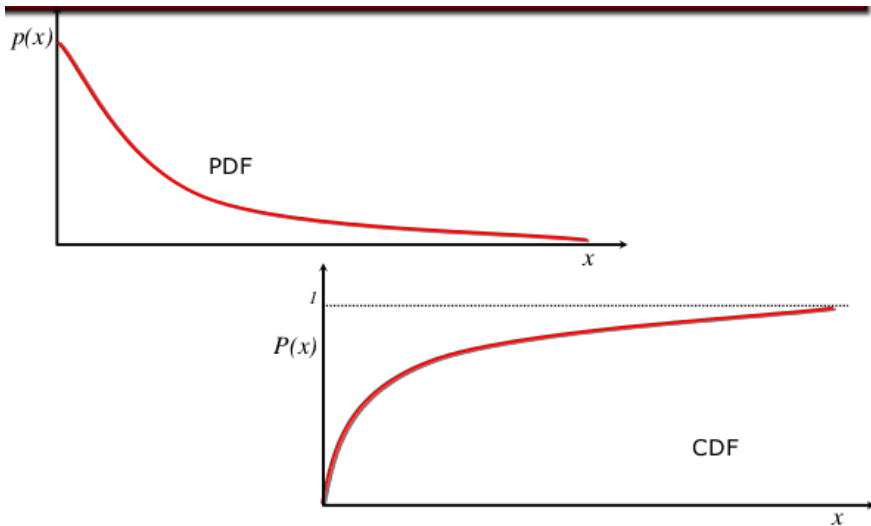
$$0 \leq P_k \leq 1$$

$$P_k \geq P_{k-1}$$

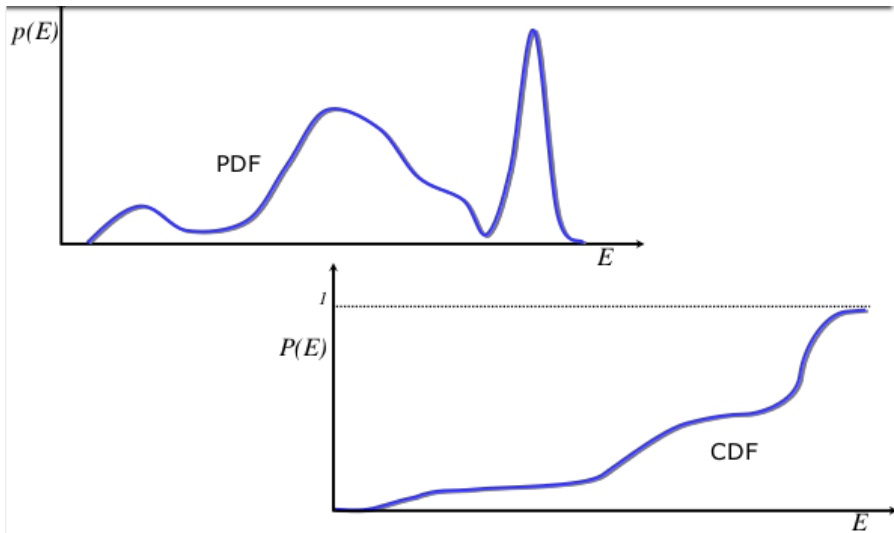
RANDOM SAMPLING BASICS



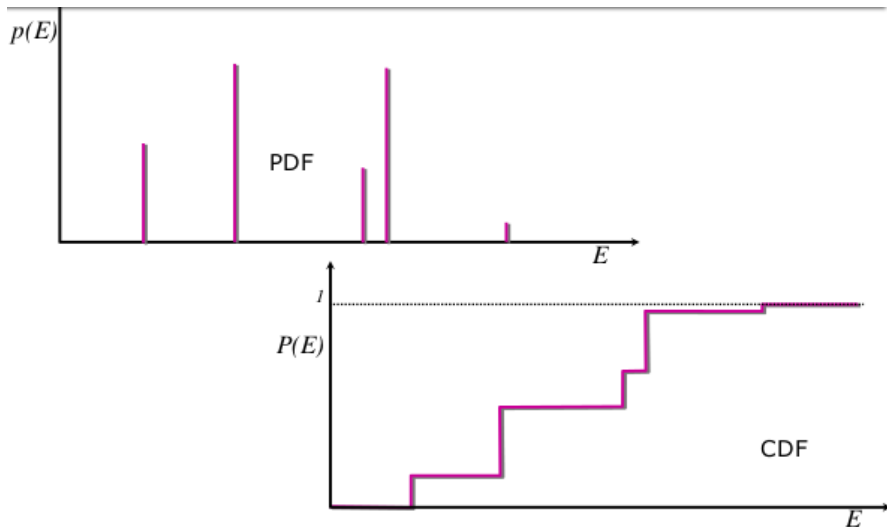
RANDOM SAMPLING BASICS



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RANDOM SAMPLING BASICS



WHY RANDOM SAMPLING

Various physical phenomena can be represented by probabilistic distributions

- The known probability distribution represents the *collective* behavior
- We need to know the behavior at *each* single event
- We need to recreate the collective behavior after many events

RANDOM SAMPLING PURPOSE

Use a random process to select a single value with the following requirements

- Each sample should be independent from other samples
- The PDF formed from a large number of samples should converge to the initial PDF
- Recover the full resolution of the initial PDF

SAMPLING TECHNIQUES

Random sampling uses uniformly distributed random variables to choose a value for a variable according to its probability density function

- *Basic* sampling techniques
 - Direct discrete sampling
 - Continuous discrete sampling
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- *Advanced* sampling techniques
 - Histogram
 - Piecewise linear
 - Alias sampling
 - Advanced continuous PDFs

UNIFORMLY-DISTRIBUTED RANDOM VARIABLE

- Standard notation
 - Single random variable: ξ
 - Pair of random variables: (ξ, η)
- PDF for random variables:

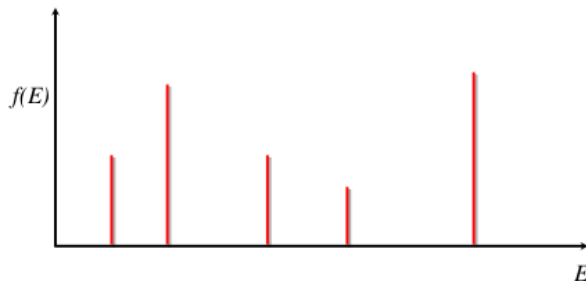
$$p(\xi) = \begin{cases} 1 & 0 \leq \xi < 1 \\ 0 & \text{otherwise} \end{cases}$$



DIRECT DISCRETE SAMPLING

Sampling Procedure

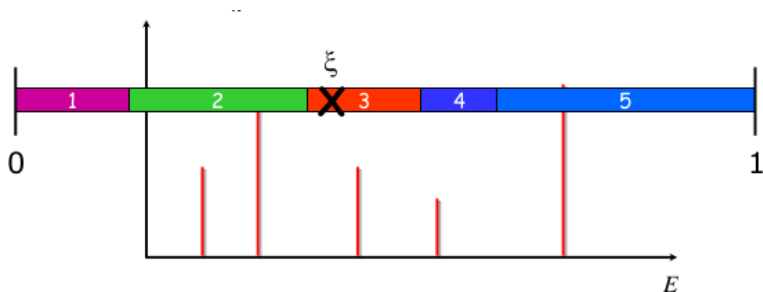
- Generate ξ
- Determine k such that $P_{k-1} \leq \xi \leq P_k$
- Return $x = x_k$



DIRECT DISCRETE SAMPLING

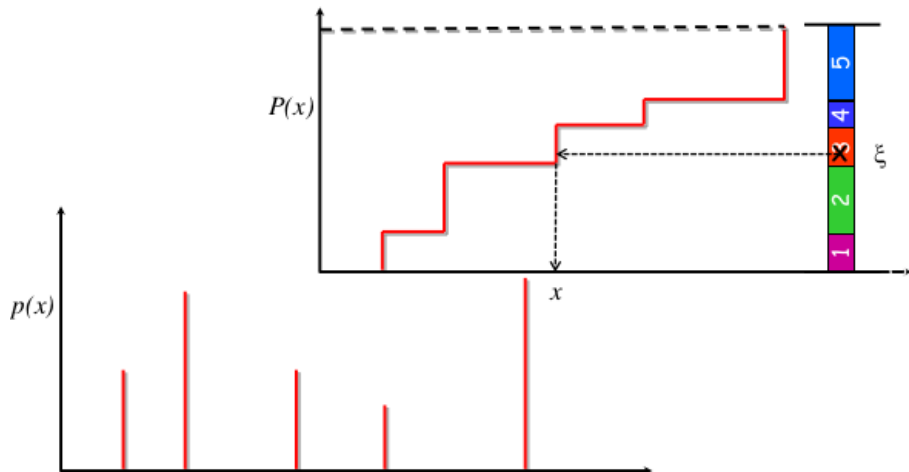
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DIRECT DISCRETE SAMPLING

Consider the CDF



DIRECT DISCRETE SAMPLING

- Requires a table search on P_k
 - Linear search requires $O(N)$ time
 - Binary search requires $O(\log_2 N)$ time

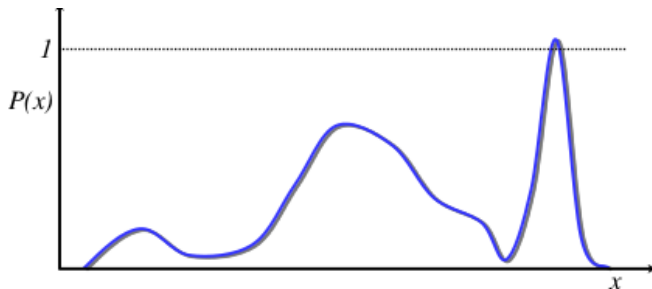
DIRECT DISCRETE SAMPLING

- Requires a table search on P_k
 - Linear search requires $O(N)$ time
 - Binary search requires $O(\log_2 N)$ time
- Special case: Uniform discrete PDF
 - $p_k = 1/N$
 - $P_k = k/N$
 - $k = \lfloor 1 + N\xi \rfloor$ (floor function)

DIRECT CONTINUOUS SAMPLING

- Can only be used if CDF can be inverted
- Direct solution of $P(x) = \xi$
- Sampling Procedure:

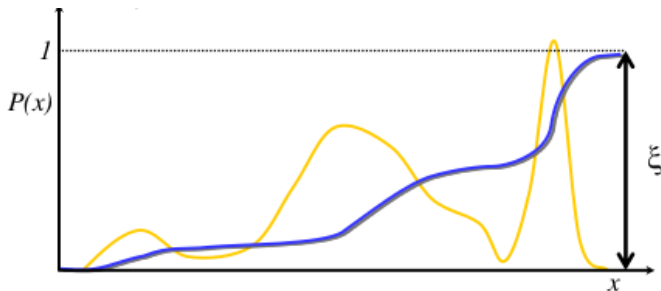
Generate ξ , Determine $x = P^{-1}(\xi)$



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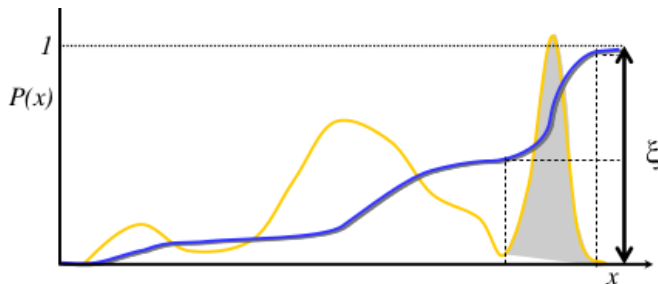
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Generate ξ , Determine $x = P^{-1}(\xi)$



DIRECT CONTINUOUS SAMPLING

- Advantages:
 - Straightforward math & coding
- Disadvantages:
 - Can involve computationally slow functions
 - Not always possible to invert $P(x)$

NORMALIZATION

- Random sampling depends on **shape** and not on ~~magnitude~~
- Normalization for formal definition of PDF/CDF required

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$$G(\infty) = \frac{1}{\lambda}$$

$$p(t) = \lambda g(t) = \lambda e^{-\lambda t}dt, \quad t > 0$$

$$P(t) = \int_{-\infty}^t p(t')dt' = \int_0^t \lambda g(t')dt' = [e^{-\lambda t'}]_0^t = 1 - e^{-\lambda t}$$

$$P(\infty) = 1$$

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$$p(x) = \frac{g(x)}{G(\infty)} = \frac{C}{C(b - a)} = \frac{1}{b - a} \quad a \leq x < b$$

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$$P(x) = \int_{-\infty}^x p(x')dx' = \frac{1}{b - a} \int_a^x dx' = \frac{x - a}{b - a}$$

$$x = P^{-1}(\xi) = \xi(b - a) + a$$

SIMPLE LINE, SLOPE = m

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$$P(x) = \int_{-\infty}^x p(x')dx' = \int_0^x 2x'dx' = [x'^2]_0^x = x^2$$

$$x = P^{-1}(\xi) = \sqrt{\xi} \quad \text{Independent of } m$$

SHIFTED LINE

$$g(x)dx = m(x - a) \quad a \leq x < b$$

$$G(x) = \int_{-\infty}^x g(x')dx' = \int_a^x m(x' - a)dx' = \frac{m}{2} [(x' - a)^2]_a^x = \frac{m}{2}(x - a)^2$$

$$G(\infty) = G(1) = \frac{m}{2}(b - a)^2$$

$$p(x) = \frac{m(x - a)}{\frac{m}{2}(b - a)^2} = 2\frac{x - a}{(b - a)^2} \quad a \leq x < b$$

$$P(x) = \int_{-\infty}^x p(x')dx' = \frac{1}{(b - a)^2} \int_a^x 2(x' - a)dx' = \frac{(x - a)^2}{(b - a)^2}$$

$$x = P^{-1}(\xi) = \sqrt{\xi}(b - a) + a \quad \text{Independent of } m$$

REJECTION SAMPLING

- Many CDFs cannot be inverted
 - e.g. Klien-Nishina cross-section

REJECTION SAMPLING

- Many CDFs cannot be inverted
 - e.g. Klien-Nishina cross-section
- Use an approach that is more graphical
 - Select a point in a 2-D domain
 - Determine whether that point is above or below the PDF
 - Keep those that are below
 - Start over if above

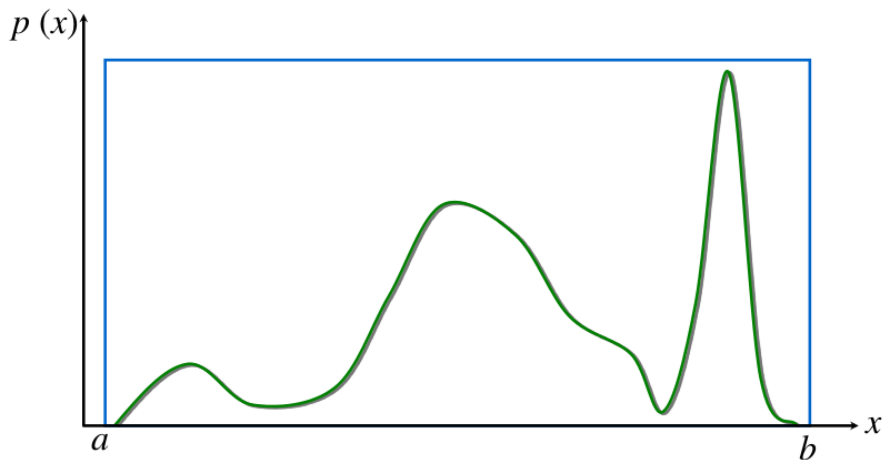
REJECTION SAMPLING

- Select a bounding function, $g(x)$, such that
 - $g(x) \geq p(x)$ for all x
 - $g(x)$ is easy to sample
- Simplest choice is $g(x) = C$
- May not be best choice

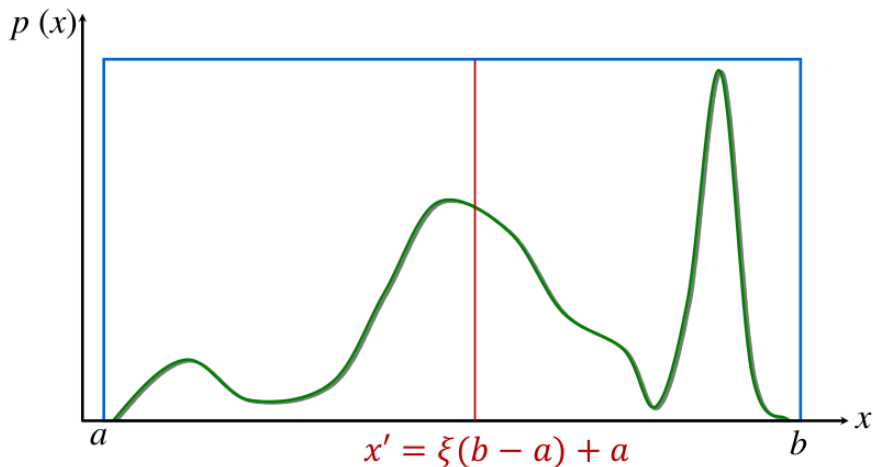
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- Generate pair of random variables, (ξ, η)
 - $x' = G^{-1}(\xi)$
 - If $\eta < p(x')/g(x')$, accept x'
 - Else, reject x'

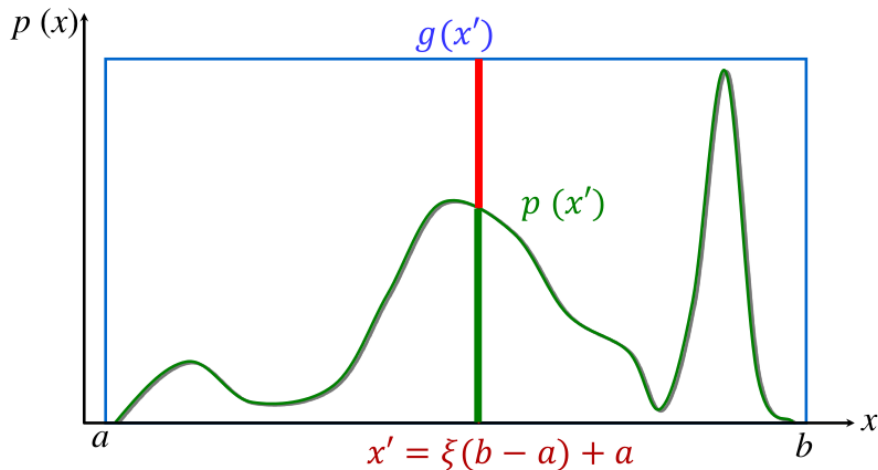
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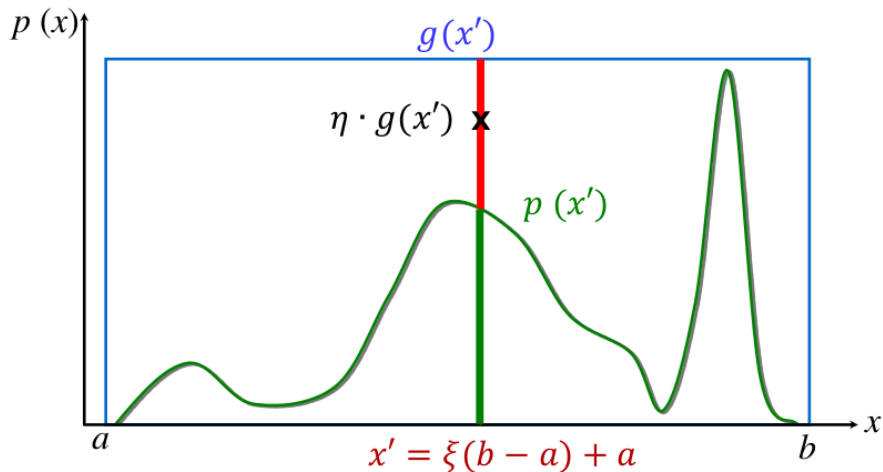
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REJECTION SAMPLING

- Advantages
 - Computationally simple
 - Always works

REJECTION SAMPLING

- Advantages
 - Computationally simple
 - Always works
- Disadvantages
 - Will be inefficient if shapes of $g(x)$ and $p(x)$ are not similar

$$\text{Efficiency} = \frac{\int p(x)dx}{\int g(x)dx}$$

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- We can create PDFs and from those generate CDFs
- These can be either continuous or discrete
- We learned some basic ways to use random numbers to *sample* from these distributions to **simulate physics**