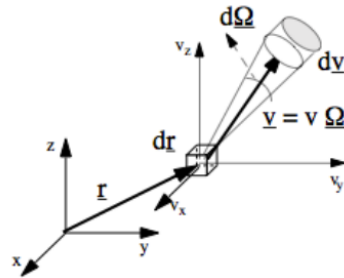
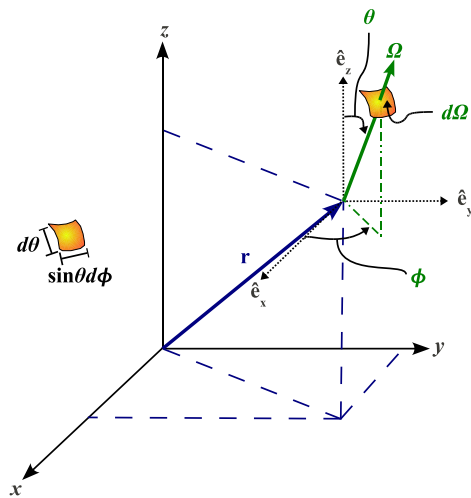


Transport Equation

Largely from Lewis and Miller Chp. 1 [3] and Duderstadt and Hamilton Chp. 4 [1]. Note: Duderstadt and Martin [2] is a very good general reference. It goes through all of this same stuff, but from a slightly more generic point of view (since this applies to any collection of neutral particles).

Definitions

Spatial logistics



- $d\vec{r} = d^3r$ = ordinary volume = $r^2 \sin(\theta) d\theta d\phi dr$
- v = speed (scalar)
- \vec{v} = velocity (vector)
- $d\vec{v} = d^3v$ = velocity volume = $v^2 \sin(\theta') d\theta' d\phi' dv$

- $v = \sqrt{(2E)/m}$ where m is the rest mass of the particle. Thus, we can relate energy and speed.
- $\hat{\Omega}$: unit directional vector in velocity space, $\vec{v} = v\hat{\Omega}$
- $d\hat{\Omega} = \sin(\theta')d\theta'd\varphi' = d^2\Omega$

These are the possible reactions we're generally going to worry about:

total (t): all interactions. We can break total into:

- scattering (s): a neutron interacts with an atom and bounces off either elastically or inelastically.
- absorption (a): a neutron is absorbed by a nucleus. If this happens it might
- fission (f): cause the nucleus to split into two pieces, releasing more neutrons.

Physics terms we will use:

1. **microscopic x-sec** (σ , [cm^2]): measure of the probability that an incident neutron will collide with a specific nucleus; σ_j indicates a specific reaction, e.g. $j = f$ is fission.
2. **macroscopic x-sec** (Σ [cm^{-1}]): measure of the probability per unit path length that an incident neutron will collide with a target

$$\Sigma_j = \sigma_j N ,$$

where N is the atomic density of the target.

3. **double-differential scattering x-sec** ($\sigma_s(E, \hat{\Omega} \rightarrow E', \hat{\Omega}')dE'd\hat{\Omega}'$): measure of the probability that a neutron of energy E and moving in direction $\hat{\Omega}$ scatters off of a specific nucleus into energy range $[E', E' + dE']$ and direction range $[\hat{\Omega}', \hat{\Omega}' + d\hat{\Omega}']$.
4. **fission yield** ($\nu(E)$): average # of neutrons released by a fission induced by a neutron of energy E .
5. **fission spectrum** ($\chi(E)dE$): average # of neutrons produced from fission that are born with energy in $[E, E + dE]$. This is normalized such that

$$\int_0^\infty \chi(E)dE = 1 .$$

6. **particle angular density** ($n(\vec{r}, E, \hat{\Omega}, t)d\vec{r}d\hat{\Omega}dE$): expected number of particles in volume element d^3r at \vec{r} whose energies are in $[E, E + dE]$ and direction of motion is in $[\hat{\Omega}, \hat{\Omega} + d\hat{\Omega}]$ at time t .

Note:

$$n(\vec{r}, E, \hat{\Omega}, t) = \frac{1}{mv} n(\vec{r}, v, \hat{\Omega}, t)$$

$$n(\vec{r}, v, \hat{\Omega}, t) = v^2 n(\vec{r}, \vec{v}, t)$$

$$n(\vec{r}, \vec{v}, t) = \frac{m}{v} n(\vec{r}, E, \hat{\Omega}, t)$$

7. **particle density**: ($N(\vec{r}, E, t)d^3rdE$): expected number of particles in d^3r at \vec{r} whose energies are in $[E, E + dE]$ at time t .

$$N(\vec{r}, E, t)d^3rdE = \int_{4\pi} d\hat{\Omega} n(\vec{r}, E, \hat{\Omega}, t)d^3rdE$$

8. **angular flux**: $\psi(\vec{r}, E, \hat{\Omega}, t) \equiv vn(\vec{r}, E, \hat{\Omega}, t)$.

9. **scalar flux**: $\phi(\vec{r}, E, t) \equiv vN(\vec{r}, E, t)$.

$$= \int_{4\pi} d\hat{\Omega} \psi(\vec{r}, E, \hat{\Omega}, t)$$

10. **interaction rate density**: expected number of j reactions per volume per energy at time t .

$$\int_{4\pi} d\hat{\Omega} \Sigma_j vn(\vec{r}, E, \hat{\Omega}, t) = \Sigma_j \phi(\vec{r}, E, t)$$

11. **angular current density** or partial current: $\vec{j}(\vec{r}, E, \hat{\Omega}, t) = \vec{v}n(\vec{r}, E, \hat{\Omega}, t)$;

$\vec{j}(\vec{r}, E, \hat{\Omega}, t) \cdot \hat{e} dA dE d\hat{\Omega}$ is the expected number of particles crossing dA along unit direction \hat{e} with energy in $[E, E + dE]$ and direction in $[\hat{\Omega}, \hat{\Omega} + d\hat{\Omega}]$ at time t .

12. **net current**: $\vec{J}(\vec{r}, E, t)$ is the net # of particles crossing a unit area per second along a direction normal to that area with energies in $[E, E + dE]$ at time t .

$$\vec{J}(\vec{r}, E, t) = \int_{4\pi} d\hat{\Omega} \hat{\Omega} \psi(\vec{r}, E, \hat{\Omega}, t)$$

Assumptions

1. Particles are point objects ($\lambda = h/(mv)$) is small compared to the atomic diameter): its state is fully described by its location, velocity vector, and a given time. This ignores rotation and quantum effects.
2. Neutral particles travel in straight lines between collisions.
3. Particle-particle collisions are negligible (makes TE linear).
4. Material properties are isotropic (generally valid unless velocities are very low).
5. Material composition is time-independent (generally valid over short time scales).
6. Quantities are expected values: fluctuations about the mean for very low densities are not accounted for.

Derivation

The TE is a detailed balance of the particle population over phase space that is as close to exact as possible.

DRAW a Differential volume picture.

Consider a volume V with surface S . For each point $\vec{r} \in S$, let \hat{e}_S be the outward normal vector.

For a given $\hat{\Omega}$, define S^+ as that part of S for which $\hat{e}_S \cdot \hat{\Omega} > 0$ (outgoing particles) and S^- as that part of S for which $\hat{e}_S \cdot \hat{\Omega} < 0$ (incoming particles).

Then, for this volume V for a fixed E and $\hat{\Omega}$, the general rate equation can be written for particles satisfying $\vec{r} \in V$, energies in $[E, E + dE]$ and direction $[\hat{\Omega}, \hat{\Omega} + d\hat{\Omega}]$ as:

Rate of change of the particle (neutron) population = rate of production - rate of loss

Rate of Change

Recall the definition of n : expected number of particles in volume element d^3r at \vec{r} whose energies are in $[E, E + dE]$ and direction of motion is in $[\hat{\Omega}, \hat{\Omega} + d\hat{\Omega}]$ at time t .

To get the rate of change of particles within the volume, we need to integrate over volume and take the derivative with respect to time:

$$\left[\int_V \frac{\partial}{\partial t} n(\vec{r}, E, \hat{\Omega}, t) d\vec{r} \right] dE d\hat{\Omega}$$

Production Mechanisms

How can we produce neutrons in volume element d^3r at \vec{r} whose energies are in $[E, E + dE]$ and direction of motion is in $[\hat{\Omega}, \hat{\Omega} + d\hat{\Omega}]$ at time t ?

1. Inscattering (from some other energy and/or angle into our energy and angle),
2. Fission neutrons, or
3. Fixed/interior sources.

1) Scattering into $[E, E + dE]$ and $[\hat{\Omega}, \hat{\Omega} + d\hat{\Omega}]$

This is the definition of the double differential scattering cross section:

$$\left[\int_V d^3r \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(E', \hat{\Omega}' \rightarrow E, \hat{\Omega}) v' n(\vec{r}, E', \hat{\Omega}', t) \right] dE d\hat{\Omega}$$

2) Expected rate of neutron production by fission

Note: fission neutrons are isotropic, thus they are produced at $\frac{1}{4\pi}$ per steradian. This means the fraction within $[\hat{\Omega}, \hat{\Omega} + d\hat{\Omega}]$ is $\frac{d\hat{\Omega}}{4\pi}$ (fyi: including this normalization differs among textbooks and research papers).

Also, recall that $\chi(E)dE$ is the fraction of neutrons born into $[E, E + dE]$. Thus

$$\frac{\chi(E)}{4\pi} \left[\int_V d^3r \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \nu(E') \Sigma_f(E') v' n(\vec{r}, E', \hat{\Omega}', t) \right] dE d\hat{\Omega}$$

3) Production from a fixed source

Sources are fully specified by a function reminiscent of the n definition: $s(\vec{r}, E, \hat{\Omega}, t)$ s.t.

$s(\vec{r}, E, \hat{\Omega}, t) d^3r dE d\hat{\Omega} \equiv$ the expected number of particles that are produced at time t inside volume d^3r at \vec{r} with energy $[E, E + dE]$ and direction $[\hat{\Omega}, \hat{\Omega} + d\hat{\Omega}]$.

Rate of production of such particles in V is

$$\left[\int_V d^3r s(\vec{r}, E, \hat{\Omega}, t) \right] dE d\hat{\Omega}$$

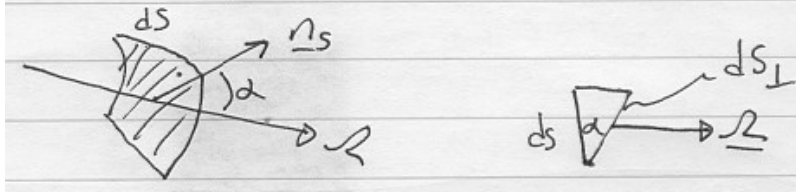
Loss Mechanisms

How can we lose neutrons from volume element d^3r at \vec{r} whose energies are in $[E, E + dE]$ and direction of motion is in $[\hat{\Omega}, \hat{\Omega} + d\hat{\Omega}]$ at time t ?

4. Neutrons can collide and exit the phase space (any collision will change its state) or
 5. Neutrons can stream into other locations and/or directions of motion (leakage).
- 4) Total interaction: any collision can lead to (a) absorption or (b) a change in E or $\hat{\Omega}$ or both.

$$\left[\int_V d^3r \Sigma_t(E) v n(\vec{r}, E, \hat{\Omega}, t) \right] dE d\hat{\Omega}$$

5) Net leakage out of phase space



Use the definition of $\vec{j} \rightarrow$ the expected number of particles crossing dS along \hat{e}_S with energy $[E, E + dE]$ and direction $[\hat{\Omega}, \hat{\Omega} + d\hat{\Omega}]$ at time t

$$= \vec{j}(\vec{r}_s, E, \hat{\Omega}, t) \cdot d\vec{S} dE d\hat{\Omega} ,$$

where \vec{r}_s is a point on the surface and $d\vec{S} = \hat{e}_S dS$.

Thus, the total leakage out of V is

$$\left[\int_S \vec{j}(\vec{r}_s, E, \hat{\Omega}, t) \cdot d\vec{S} \right] dE d\hat{\Omega} .$$

We can use divergence theorem to rewrite this w.r.t. V (rather than S):

$$\int_S \hat{e}_S \cdot \vec{F}(\vec{r}) dS = \int_V \nabla \cdot \vec{F}(\vec{r}) dV$$

This gives

$$\left[\int_V d^3r \nabla \cdot \underbrace{(\vec{j}(\vec{r}_s, E, \hat{\Omega}, t))}_{=\vec{v}n=v\hat{\Omega}n} \right] dE d\hat{\Omega}$$

And we can use this identity:

$$\hat{\Omega} \cdot (\nabla f) = \nabla \cdot \hat{\Omega} f \text{ because } \hat{\Omega} \text{ is not a function.}$$

$$\nabla \cdot \hat{\Omega} f = f \underbrace{(\nabla \cdot \hat{\Omega})}_0 + \hat{\Omega} \cdot (\nabla f)$$

$$\therefore \left[\int_V d^3r \hat{\Omega} \cdot \nabla (vn(\vec{r}_s, E, \hat{\Omega}, t)) \right] dE d\hat{\Omega}$$

Note: $\hat{\Omega} \cdot \nabla$ represents the derivative along the direction of motion.

All Together Now

The balance of neutrons: rate of change - production + loss = 0.

Suppressing dependencies to save space for the moment

$$\int_V d^3r \left[\frac{\partial n}{\partial t} - \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(E', \hat{\Omega}' \rightarrow E, \hat{\Omega}) v' n' - \frac{\chi(E)}{4\pi} \int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \nu(E') \Sigma_f(E') v' n' - s + \Sigma_t v n + \hat{\Omega} \cdot \nabla v n \right] = 0$$

We note that since the volume was arbitrarily chosen, the integral will only vanish if the integrand is zero

$$\int_{\text{any } V} d^3r f(\vec{r}) = 0 \rightarrow f(\vec{r}) = 0.$$

Now we have a balance relation that we can rearrange, substituting $\psi(\vec{r}, E, \hat{\Omega}, t) = vn(\vec{r}, E, \hat{\Omega}, t)$,

to get what we usually call the Boltzmann Equation for neutron transport

$$\frac{1}{v} \frac{\partial \psi(\vec{r}, E, \hat{\Omega}, t)}{\partial t} + \hat{\Omega} \cdot \nabla \psi(\vec{r}, E, \hat{\Omega}, t) + \Sigma_t \psi(\vec{r}, E, \hat{\Omega}, t) =$$

$$\int_{4\pi} d\hat{\Omega}' \int_0^\infty dE' \Sigma_s(E', \hat{\Omega}' \rightarrow E, \hat{\Omega}) \psi(\vec{r}, E', \hat{\Omega}', t) +$$

$$\frac{\chi(E)}{4\pi} \int_0^\infty dE' \nu(E') \Sigma_f(E') \int_{4\pi} d\hat{\Omega}' \psi(\vec{r}, E', \hat{\Omega}', t) + s(\vec{r}, E, \hat{\Omega}, t)$$

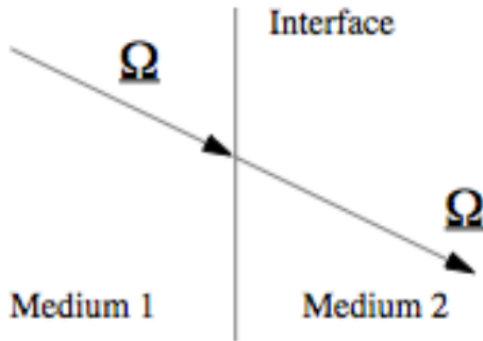
Initial and Boundary Conditions

1. Initial Condition: we start with some initial “known” state:

$$\psi(\vec{r}, E, \hat{\Omega}, 0) = \psi_0(\vec{r}, E, \hat{\Omega})$$

for the problem domain. Note, the initial flux can be a functional expression.

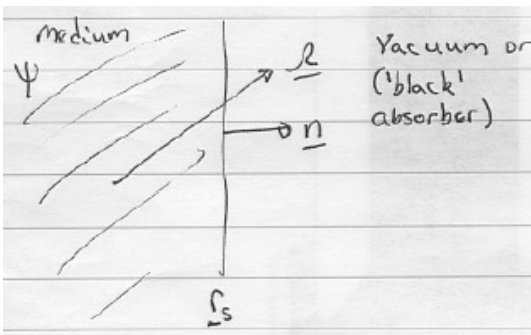
2. Interface Condition: the angular flux must be continuous along $\hat{\Omega}$ at all points, including material interfaces.



$$\psi_1(\vec{r}_S, E, \hat{\Omega}, t) = \psi_2(\vec{r}_S, E, \hat{\Omega}, t)$$

$\forall E$ and $\hat{\Omega}$.

3. Fixed Condition: you can specify incoming flux



$$\psi(\vec{r}_S, E, \hat{\Omega}, t) = \psi_{IN}(\vec{r}_S, E, \hat{\Omega}, t) \text{ for}$$

$\vec{e} \cdot \hat{\Omega} < 0$: specifying incoming neutrons.

Note, the incoming flux can be a functional expression; it can also be zero.

This is equivalent to specifying the incoming partial current,

$$\vec{j}^-(\vec{r}_S, E, t) = \int_{\vec{e} \cdot \hat{\Omega} < 0} d\hat{\Omega} (\vec{e} \cdot \hat{\Omega}) \psi(\vec{r}_S, E, \hat{\Omega}, t).$$

4. Reflective Condition: there is mirror symmetry at some surface:

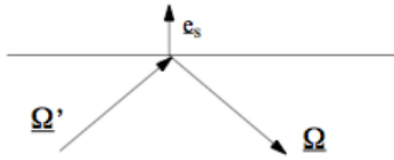


FIGURE XII.7. Mirror reflection

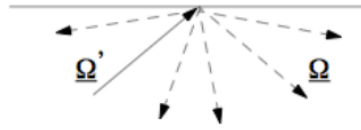
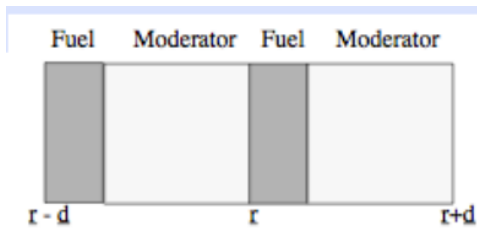


FIGURE XII.8. Isotropic reflective boundary condition

$$\psi(\hat{\Omega}_{IN}) = \psi(\hat{\Omega}_{OUT}) .$$

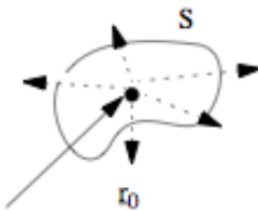
5. Periodic Condition: you know there is a repetition in the system



$$\psi(\vec{r}_S, E, \hat{\Omega}, t) = \psi(\vec{r}_S \pm \vec{p}, E, \hat{\Omega}, t) .$$

6. Finiteness Condition: to be physically valid we need to meet the condition $0 < \psi(\vec{r}, E, \hat{\Omega}, t) < \infty$, with the possible exception of point sources,
7. which we handle with the Source Condition: localized sources are introduced as mathematical singularities at the location of the source.

For a source $s(\vec{r}_0, E, \hat{\Omega}, t)$:



$$\lim_{\vec{r} \rightarrow \vec{r}_0} \int_S dS \vec{e} \cdot \hat{\Omega} \psi(\vec{r}, E, \hat{\Omega}, t) = s(\vec{r}_0, E, \hat{\Omega}, t) ,$$

$$s(\vec{r}, E, \hat{\Omega}, t) = s(\vec{r}_0, E, \hat{\Omega}, t) \delta(\vec{r} - \vec{r}_0) .$$

Simplified Forms

One Speed

Assume all particles are at the same speed, so we no longer need energy dependence.

$$\frac{1}{v} \frac{\partial \psi(\vec{r}, \hat{\Omega}, t)}{\partial t} + \hat{\Omega} \cdot \nabla \psi(\vec{r}, \hat{\Omega}, t) + \Sigma_t \psi(\vec{r}, \hat{\Omega}, t) = \int_{4\pi} d\hat{\Omega}' \Sigma_s(\hat{\Omega}' \rightarrow \hat{\Omega}) \psi(\vec{r}, \hat{\Omega}', t) + \frac{\nu \Sigma_f}{4\pi} \int_{4\pi} d\hat{\Omega}' \psi(\vec{r}, \hat{\Omega}', t) + s(\vec{r}, \hat{\Omega}, t)$$

One Speed, One Dimensional

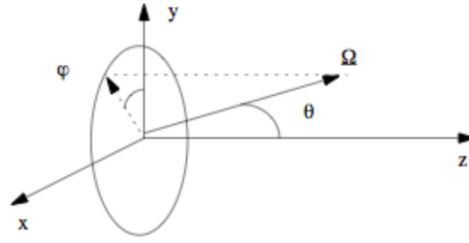
We'd like to simplify even farther by only worrying about one dimension, so we'll get rid of y and z .

$$\vec{r} = (x, y, z) \quad d\hat{\Omega} = \sin(\theta) d\theta d\varphi = d\mu d\varphi$$

$$\text{Note } \mu = \cos(\theta) \text{ so } d\mu = \sin(\theta) d\theta$$

$$\Omega_x = \cos(\theta) = \mu$$

(in the image mentally swap y and z to get the notation we are using)



$$\frac{1}{v} \frac{\partial \psi(x, \hat{\Omega}, t)}{\partial t} + \left(\Omega_x \frac{\partial}{\partial x} + \cancel{\Omega_y \frac{\partial}{\partial y}} + \cancel{\Omega_z \frac{\partial}{\partial z}} \right) \psi(x, \hat{\Omega}, t) + \Sigma_t \psi(x, \hat{\Omega}, t) = \int_{4\pi} d\hat{\Omega}' \Sigma_s(\hat{\Omega}' \rightarrow \hat{\Omega}) \psi(x, \hat{\Omega}', t) + \frac{\nu \Sigma_f}{4\pi} \int_{4\pi} d\hat{\Omega}' \psi(x, \hat{\Omega}', t) + s(x, \hat{\Omega}, t)$$

Using this idea we can write things more cleanly:

$$\frac{1}{v_0} \frac{\partial \psi(x, \hat{\Omega}, t)}{\partial t} + \mu \frac{\partial}{\partial x} \psi(x, \hat{\Omega}, t) + \Sigma_t \psi(x, \hat{\Omega}, t) = \int_0^{2\pi} d\phi \int_{4\pi} d\hat{\Omega}' \Sigma_s(\hat{\Omega}' \cdot \hat{\Omega}) \frac{\psi(x, \mu', t)}{2\pi} + \frac{1}{4\pi} \nu \Sigma_f \int_0^{2\pi} d\phi \psi(x, \mu, t) + s(x, \mu, t)$$

If scattering is also isotropic:

$$\Sigma_s(\hat{\Omega}' \cdot \hat{\Omega}) = \frac{\Sigma_s}{4\pi}$$

And we get the 1-group, 1-D, isotropic neutron TE:

$$\frac{1}{v_0} \frac{\partial \psi(x, \mu, t)}{\partial t} + \mu \frac{\partial}{\partial x} \psi(x, \mu, t) + \Sigma_t \psi(x, \mu, t) = \frac{1}{2} \Sigma_s \int_{-1}^1 d\mu' \psi(x, \mu', t) + \frac{1}{2} (\nu \Sigma_f \phi(x, t) + S(x, t))$$

Steady State

If we get rid of time dependence.

$$\mu \frac{\partial}{\partial x} \psi(x, \mu) + \Sigma_t \psi(x, \mu) = \frac{1}{2} \Sigma_s \int_{-1}^1 d\mu' \psi(x, \mu') + \frac{1}{2} (\nu \Sigma_f \phi(x) + S(x))$$

And very finally - we can have the unrealistic case of **purely absorbing media**:

$$\mu \frac{\partial}{\partial x} \psi(x, \mu) + \Sigma_a \psi(x, \mu) = \frac{1}{2} (\nu \Sigma_f \phi(x) + S(x))$$

References

- [1] James J. Duderstadt and Louis J. Hamilton. *Nuclear Reactor Analysis*. John Wiley & Sons, Inc., New York, 1 edition, 1976.
- [2] James J. Duderstadt and William R. Martin. *Transport Theory*. John Wiley & Sons, Inc., New York, 1979.
- [3] E. E. Lewis and Jr. W. F. Miller. *Computational Methods of Neutron Transport*. American Nuclear Society, La Grange Park, IL, 1 edition, 1993.