# Welcome

# To

# **MKS TUTORIALS**

# By

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**Topics covered under playlist of Laplace Transform:** Definition, Transform of Elementary Functions, Properties of Laplace Transform, Transform of Derivatives and Integrals, Multiplication by t^n, Division by t, Evaluation of Integrals, Inverse Laplace Transform, Convolution Theorem, Unit Step Function, Unit Impulse Functions, Periodic Function, Application to Solution of Ordinary Differential Equations

**Complete playlist of Laplace Transform (in hindi)**
**-** https://www.youtube.com/playlist?list=PLhSp9OSVmeyIYLVvSJ8m6KvVwJs7M9QBm

**Complete playlist of Laplace Transform (in english)**
**-** https://www.youtube.com/playlist?list=PL0d0PH4hlFOfVXmPh3CcqWm_mpbf_9cfy

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# LAPLACE TRANSFORM

Let $f(t)$ be a function of 't' defined for all positive values of t, then the laplace transform of $f(t)$, denoted by $L\{f(t)\}$ is defined by

$$L\{f(t)\} = \int_0^{\infty} e^{-st} \cdot f(t)\, dt$$

provided the integral exist.

where, $s \rightarrow$ parameter which may be real or complex number.

## FORMULAE :

1. $L\{1\} = \frac{1}{s}$
2. $L\{e^{at}\} = \frac{1}{s-a}$
3. $L\{t^n\} = \frac{\lfloor n}{s^{n+1}}$
4. $L\{\sin at\} = \frac{a}{s^2+a^2}$
5. $L\{\cos at\} = \frac{s}{s^2+a^2}$
6. $L\{\sinh at\} = \frac{a}{s^2-a^2}$
7. $L\{\cosh at\} = \frac{s}{s^2-a^2}$

1. To prove: $L\{1\} = \frac{1}{s}$

$\because L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$

Here, $f(t) = 1$

$\therefore L\{1\} = \int_0^\infty e^{-st} \cdot 1 \cdot dt = \left| \frac{e^{-st}}{-s} \right|_0^\infty = -\frac{1}{s}\left(e^{-\infty} - e^{0}\right)$

$= -\frac{1}{s}\left(\frac{1}{e^\infty} - 1\right) = \frac{1}{s}$ Proved

$\Rightarrow L\{1\} = \frac{1}{s}$

2. To prove: $L\{e^{at}\} = \frac{1}{s-a}$

$\because L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$

Here, $f(t) = e^{at}$

$\therefore L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\,dt$

$= \int_0^\infty e^{-(s-a)t}\,dt = \left| \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty$

$= -\frac{1}{s-a}\left(e^{-\infty} - e^{0}\right) = -\frac{1}{s-a}\left(\frac{1}{e^\infty} - 1\right) = \frac{1}{s-a}$

$\Rightarrow L\{e^{at}\} = \frac{1}{s-a}$ Hence proved

3. To prove: $L\{t^n\} = \dfrac{\underline{|n}}{s^{n+1}}$

$\because L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$

Here, $f(t) = t^n$.

$\therefore L\{t^n\} = \int_0^\infty e^{-st} \cdot t^n\, dt$

Let $st = u \Rightarrow t = \dfrac{u}{s}$

$s\,dt = du \Rightarrow dt = \dfrac{1}{s}\, du$.

$\because t = 0 \Rightarrow u = 0$

$t = \infty \Rightarrow u = \infty$

$\therefore L\{t^n\} = L\left\{\left(\dfrac{u}{s}\right)^n\right\} = \int_0^\infty e^{-u} \cdot \left(\dfrac{u}{s}\right)^n \cdot \dfrac{1}{s}\, du$.

$= \int_0^\infty \dfrac{1}{s^{n+1}} \cdot e^{-u} \cdot u^n\, du = \dfrac{1}{s^{n+1}} \int_0^\infty e^{-u} \cdot u^{(n+1)-1} \cdot du$.

$= \dfrac{1}{s^{n+1}} \cdot \Gamma(n+1)$

$= \dfrac{\underline{|n}}{s^{n+1}}$

Proved

Gamma function

$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$.

Here, $x = u$

Also, $\Gamma n = \underline{|n-1}$

$\Rightarrow \Gamma(n+1) = \underline{|n}$

4. To prove: $L\{\sin at\} = \frac{a}{s^2+a^2}$

$\because L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$

Here, $f(t) = \sin at$

$\therefore L\{\sin at\} = \int_0^\infty e^{-st} \cdot \sin at\, dt$

Use formula: $\int e^{at} \cdot \sin bt\, dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$

$= \left| \frac{e^{-st}}{s^2+a^2}(-s \cdot \sin at - a\cos at) \right|_0^\infty$

$= 0 - \frac{e^0}{s^2+a^2}(0-a) = \frac{a}{s^2+a^2}$ Proved

5. To prove: $L\{\cos at\} = \frac{s}{s^2+a^2}$

$\because L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$

Here, $f(t) = \cos at$

$\therefore L\{\cos at\} = \int_0^\infty e^{-st} \cdot \cos at\, dt$

Use formula: $\int e^{at} \cdot \cos bt\, dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$

$= \left| \frac{e^{-st}}{s^2+a^2}(-s \cdot \cos at + a\sin at) \right|_0^\infty$

$= 0 - \frac{e^0}{s^2+a^2}(-s+0) = \frac{s}{s^2+a^2}$ Proved

6. To prove: $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$

$\because L\{f(t)\} = \int_0^\infty e^{-st}\cdot f(t)\,dt$

Here, $f(t) = \sinh at = \dfrac{e^{at}-e^{-at}}{2}$

$\therefore L\{\sinh at\} = \int_0^\infty e^{-st}\left(\dfrac{e^{at}-e^{-at}}{2}\right)dt$

$= \dfrac{1}{2}\left[\int_0^\infty e^{-st}\cdot e^{at}dt - \int_0^\infty e^{-st}\cdot e^{-at}dt\right]$

$= \dfrac{1}{2}\left[\int_0^\infty e^{-(s-a)t}dt - \int_0^\infty e^{-(s+a)t}dt\right] = \dfrac{1}{2}\left[\left|\dfrac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty - \left|\dfrac{e^{-(s+a)t}}{-(s+a)}\right|_0^\infty\right]$

$= \dfrac{1}{2}\left[-\dfrac{1}{s-a}\left(e^{-\infty}-e^0\right) + \dfrac{1}{s+a}\left(e^{-\infty}-e^0\right)\right] = \dfrac{1}{2}\left[\dfrac{1}{s-a} - \dfrac{1}{s+a}\right]$

$= \dfrac{1}{2}\left(\dfrac{s+a-s+a}{s^2-a^2}\right) = \dfrac{2a}{2(s^2-a^2)} = \dfrac{a}{s^2-a^2}$ Proved

7. To prove: $L\{\cosh at\} = \dfrac{s}{s^2-a^2}$

$\because L\{f(t)\} = \int_0^\infty e^{-st}\cdot f(t)\,dt$

Here, $f(t) = \cosh at = \dfrac{e^{at}+e^{-at}}{2}$

$\therefore L\{\cosh at\} = \int_0^\infty e^{-st}\left(\dfrac{e^{at}+e^{-at}}{2}\right)dt = \dfrac{1}{2}\int_0^\infty \left\{e^{-(s-a)t} + e^{-(s+a)t}\right\}dt$

$= \dfrac{1}{2}\left|\dfrac{e^{-(s-a)t}}{-(s-a)} + \dfrac{e^{-(s+a)t}}{-(s+a)}\right|_0^\infty = \dfrac{1}{2}\left[\dfrac{-1}{s-a}\left(e^{-\infty}-e^0\right) - \dfrac{1}{s+a}\left(e^{-\infty}-e^0\right)\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{s-a} + \dfrac{1}{s+a}\right] = \dfrac{1}{2}\left(\dfrac{s+a+s-a}{s^2-a^2}\right) = \dfrac{1}{2}\cdot\dfrac{2s}{s^2-a^2}$

$= \dfrac{s}{s^2-a^2}$ Proved

## Properties :

1. **Shifting property** : If $L\{f(t)\} = \bar{f}(s)$

then $L\{e^{at} \cdot f(t)\} = \bar{f}(s-a)$

2. **Division property** : If $L\{f(t)\} = \bar{f}(s)$

then $L\{\frac{1}{t} \cdot f(t)\} = \int_s^{\infty} \bar{f}(s)\, ds$.

3. **Transforms of integral** : If $L\{f(t)\} = \bar{f}(s)$

then $L\{\int_0^t f(t)\,dt\} = \frac{1}{s}\bar{f}(s)$

4. **Multiplication property** : If $L\{f(t)\} = \bar{f}(s)$

then $L\{t^n \cdot f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\right]$

5. **Laplace transform of a derivative** :

If $L\{f(t)\} = \bar{f}(s)$

then, $L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0)$
$- s^{n-3} f''(0) - s^{n-4} f'''(0) - \cdots$

Que ① Find the laplace transform of

(i) $f(t) = t\,e^{-4t}\sin 3t$ (ii) $f(t) = e^{-t}\int_0^t \frac{\sin t}{t}\,dt$

Sol.ⁿ (i) $f(t) = t\,e^{-4t}\sin 3t$

$L\{\sin 3t\} = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}$

Using multiplication property,

$L\{t\cdot\sin 3t\} = (-1)^1\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = (-3)\frac{d}{ds}(s^2+9)^{-1}$

$= (-3)(-1)(s^2+9)^{-2}(2s) = \frac{6s}{(s^2+9)^2}$

Using shifting property,

$L\{e^{-4t}\cdot t\cdot\sin 3t\} = \frac{6(s+4)}{[(s+4)^2+9]^2} = \frac{6(s+4)}{(s^2+8s+25)^2}$ Ans

(ii) $f(t) = e^{-t}\int_0^t \frac{\sin t}{t}\,dt$

$L\{\sin t\} = \frac{1}{s^2+1}$

Using division property,

$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \left|\tan^{-1}s\right|_s^\infty = \tan^{-1}\infty - \tan^{-1}s$

$= \pi/2 - \tan^{-1}s = \cot^{-1}s.$

Using transform of integral property,

$L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}s$

Using shifting property,

$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s+1}\cot^{-1}(s+1)$ Ans

Que(2) Find the LT of

(i) $e^{-3t}(2\cos 5t - 3\sin 5t)$

(ii) $e^{2t}\cos^2 t$

(iii) $e^{4t}\cdot\sin 2t\cos t$

Sol.$^n$ (i) $f(t) = e^{-3t}(2\cos 5t - 3\sin 5t)$

$\because L\{2\cos 5t - 3\sin 5t\} = 2L\{\cos 5t\} - 3L\{\sin 5t\}$

$$= 2\cdot\frac{s}{s^2+5^2} - 3\cdot\frac{5}{s^2+5^2} = \frac{2s-15}{s^2+25}$$

Using shifting property.

$$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = \frac{2(s+3)-15}{(s+3)^2+25} = \frac{2s-9}{s^2+6s+34}.$$

Ans

(ii) $f(t) = e^{2t}\cdot\cos^2 t$

$\because L\{\cos^2 t\} = \frac{1}{2}L\{1+\cos 2t\} = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+2^2}\right)$

$\cos 2t = 1 - 2\sin^2 t = 2\cos^2 t - 1$

$\Rightarrow \cos^2 t = \frac{1+\cos 2t}{2}$

Using shifting property.

$$L\{e^{2t}\cdot\cos^2 t\} = \frac{1}{2}\left[\frac{1}{s-2} + \frac{s-2}{(s-2)^2+4}\right]$$ Ans

(iii) $f(t) = e^{4t}\cdot\sin 2t\cos t$

$\because L\{\sin 2t\cdot\cos t\} = \frac{1}{2}L\{2\sin 2t\cdot\cos t\}$

$$= \frac{1}{2}L\{\sin 3t + \sin t\} = \frac{1}{2}\left[\frac{3}{s^2+3^2} + \frac{1}{s^2+1}\right]$$

Using shifting property,

$$L\{e^{4t}\sin 2t\cos t\} = \frac{1}{2}\left[\frac{3}{(s-4)^2+9} + \frac{1}{(s-4)^2+1}\right]$$

Ans

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Que ③ Find the LT of

(i) $t^3 e^{-3t}$ (ii) $t e^{-t} \sin 3t$

Sol.ⁿ (i) $f(t) = t^3 e^{-3t}$

$\because L\{e^{-3t}\} = \frac{1}{s+3}$

Using multiplication property,

$$L\{t^3 \cdot e^{-3t}\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) = -\frac{d^3}{ds^3}(s+3)^{-1}$$

$$= -(-1)\frac{d^2}{ds^2}(s+3)^{-2} = \frac{d^2}{ds^2}(s+3)^{-2} = (-2)\frac{d}{ds}(s+3)^{-3}$$

$$= (-2)(-3)(s+3)^{-4} = 6(s+3)^{-4} = \frac{6}{(s+3)^4} \quad \text{Ans}$$

(ii) $f(t) = t\, e^{-t} \sin 3t$

$\because L\{\sin 3t\} = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}$

Using multiplication property,

$$L\{t^1 \sin 3t\} = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right) = (-3)\frac{d}{ds}(s^2+9)^{-1}$$

$$= (-3)(-1)(s^2+9)^{-2}(2s) = 6s(s^2+9)^{-2} = \frac{6s}{(s^2+9)^2}$$

Using shifting property,

$$L\{e^{-t} \cdot t \sin 3t\} = \frac{6(s+1)}{[(s+1)^2+9]^2} = \frac{6(s+1)}{(s^2+2s+10)^2} \quad \text{Ans}$$

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**Que(1)** Find the LT of

(i) $\dfrac{\cos at - \cos bt}{t}$ (ii) $(\sin t - \cos t)^2$

**Sol.ⁿ** (i) $\because L\{\cos at - \cos bt\} = \left(\dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}\right)$

Using division property,

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right) ds = \frac{1}{2}\left|\log(s^2+a^2) - \log(s^2+b^2)\right|_s^\infty$$

$$= \frac{1}{2}\left|\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right|_s^\infty = \frac{1}{2}\left[\lim_{s\to\infty} \log\left(\frac{s^2+a^2}{s^2+b^2}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} \log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right) - \log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\left[\log(1) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$= \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2} \quad \text{Ans}$$

(ii) $f(t) = (\sin t - \cos t)^2 = (\sin^2 t + \cos^2 t - 2\sin t\cos t)$
$= 1 - \sin 2t$

$$\therefore L\{f(t)\} = L\{(\sin t - \cos t)^2\} = L\{1\} - L\{\sin 2t\}$$

$$= \frac{1}{s} - \frac{2}{s^2+2^2} = \frac{1}{s} - \frac{2}{s^2+4} \quad \text{Ans}$$

Que 5) Find the laplace transform of

(i) $f(t) = 1 + 2\sqrt{t} + \frac{3}{\sqrt{t}}$ (ii) $f(t) = \cos^3 2t$

Sol$^n$ (i) $L\{f(t)\} = L\{1\} + 2L\{t^{1/2}\} + 3L\{t^{-1/2}\}$

$$= \frac{1}{s} + 2\frac{\lfloor 1/2}{s^{1/2+1}} + 3\frac{\lfloor -1/2}{s^{-1/2+1}}$$

$$= \frac{1}{s} + 2\frac{\Gamma(1/2+1)}{s^{3/2}} + \frac{3\,\Gamma(-1/2+1)}{s^{1/2}}$$

$$= \frac{1}{s} + \frac{2\,\Gamma(3/2)}{s^{3/2}} + \frac{3\,\Gamma(1/2)}{s^{1/2}}$$

$$= \frac{1}{s} + 2\cdot\frac{\frac{1}{2}\cdot\sqrt{\pi}}{s^{3/2}} + 3\cdot\frac{\sqrt{\pi}}{s^{1/2}}$$

$$= \frac{1}{s} + \frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} \text{ Ans}$$

$\lfloor n = \Gamma(n+1)$

$\Gamma 1/2 = \sqrt{\pi}$

$\Gamma 3/2 = \frac{1}{2} \times \sqrt{\pi}$

$\Gamma 5/2 = \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$

(ii) $f(t) = \cos^3 2t = \frac{1}{4}\cos 6t + \frac{3}{4}\cos 2t$

$\therefore L\{\cos^3 2t\} = \frac{1}{4}L\{\cos 6t\} + \frac{3}{4}L\{\cos 2t\}$

$$= \frac{1}{4}\left(\frac{s}{s^2+6^2}\right) + \frac{3}{4}\left(\frac{s}{s^2+2^2}\right)$$

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right] \text{ Ans}$$

$\cos 3t = 4\cos^3 t - 3\cos t$

$\cos^3 t = \frac{\cos 3t + 3\cos t}{4}$

$\cos^3 2t = \frac{\cos 6t + 3\cos 2t}{4}$

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**Que 6** Find the laplace transform of

(i) $f(t) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$ (ii) $f(t) = \sinh^2 3t$

**Sol.ⁿ** (i) $f(t) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3 = \left(t^{1/2} - t^{-1/2}\right)^3$

$= t^{3/2} - t^{-3/2} - 3\,t^{1/2} \cdot t^{-1/2}\left(t^{1/2} - t^{-1/2}\right)$

$= t^{3/2} - t^{-3/2} - 3t^{1/2} + 3t^{-1/2}$

$\left\{ \begin{array}{l} (a-b)^3 \\ = a^3 - b^3 \\ \quad - 3ab(a-b) \end{array} \right.$

$L\{f(t)\} = L\{t^{3/2}\} - L\{t^{-3/2}\} - 3L\{t^{1/2}\} + 3L\{t^{-1/2}\}$

$= \frac{\lfloor 3/2}{s^{3/2+1}} - \frac{\lfloor -3/2}{s^{-3/2+1}} - 3 \cdot \frac{\lfloor 1/2}{s^{1/2+1}} + 3 \cdot \frac{\lfloor -1/2}{s^{-1/2+1}}$

$= \frac{\Gamma(3/2+1)}{s^{5/2}} - \frac{\Gamma(-3/2+1)}{s^{-1/2}} - \frac{3\,\Gamma(1/2+1)}{s^{3/2}} + 3\,\frac{\Gamma(-1/2+1)}{s^{1/2}}$

$= \frac{\Gamma(5/2)}{s^{5/2}} - \frac{\Gamma(-1/2)}{s^{-1/2}} - \frac{3\,\Gamma(3/2)}{s^{3/2}} + 3\,\frac{\Gamma(1/2)}{s^{1/2}}$

$= \frac{\frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}}{s^{5/2}} - \frac{(-2\sqrt{\pi})}{s^{-1/2}} - \frac{3 \times \frac{1}{2} \times \sqrt{\pi}}{s^{3/2}} + \frac{3 \times \sqrt{\pi}}{s^{1/2}}$

$= \frac{3}{4}\,\frac{\sqrt{\pi}}{s^{5/2}} + 2\sqrt{\pi s} - \frac{3}{2}\,\frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{\sqrt{s}}$ Ans

(ii) $f(t) = \sinh^2 3t = \frac{\cosh 6t - 1}{2}$

$\therefore L\{f(t)\} = \frac{1}{2}\,L\{\cosh 6t - 1\}$

$= \frac{1}{2}\left[\frac{s}{s^2 - 6^2} - \frac{1}{s}\right]$

$= \frac{1}{2}\left[\frac{s}{s^2 - 36} - \frac{1}{s}\right]$ Ans

$\left\{ \begin{array}{l} \cosh 2t = 1 + 2\sinh^2 t \\ \sinh^2 t = \frac{\cosh 2t - 1}{2} \\ \sinh^2 3t = \frac{\cosh 6t - 1}{2} \end{array} \right.$

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Que (7) Find the laplace Transform of $\left(\frac{1-\cos t}{t^2}\right)$

Sol.$^n$ $L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$

By division property,

$$L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds = \int_s^\infty \left[\frac{1}{s} - \frac{2s}{2(s^2+1)}\right] ds$$

$$= \left|\log s - \frac{1}{2}\log(s^2+1)\right|_s^\infty = \left|\log s - \log(s^2+1)^{1/2}\right|_s^\infty$$

$$= \left|\log \frac{s}{(s^2+1)^{1/2}}\right|_s^\infty = \lim_{s\to\infty} \log \frac{s}{s\left(1+\frac{1}{s^2}\right)^{1/2}} - \log\frac{s}{(s^2+1)^{1/2}}$$

$$= \cancelto{0}{\log 1} - \log\frac{s}{(s^2+1)^{1/2}} = \log\frac{(s^2+1)^{1/2}}{s} = \log\left(\frac{s^2+1}{s^2}\right)^{1/2}$$

$$= \frac{1}{2}\log\left(1+\frac{1}{s^2}\right) = \frac{1}{2}\log\left(1+s^{-2}\right)$$

Again using division property,

$$L\left\{\frac{1-\cos t}{t^2}\right\} = \frac{1}{2}\int_s^\infty \underset{\text{I}}{\log(1+s^{-2})}\,.\underset{\text{II}}{ds}$$

$$= \frac{1}{2}\left[\log(1+s^{-2})\int ds - \int\left\{\frac{d}{ds}\log(1+s^{-2})\int ds\right\} ds\right]_s^\infty$$

$$= \frac{1}{2}\left[s\log(1+s^{-2}) - \int \frac{1}{1+s^{-2}}\cdot(-2)s^{-3}\cdot s\, ds\right]_s^\infty$$

$$= \frac{1}{2}\left[s\log(1+s^{-2}) + 2\int \frac{s^{-2}}{1+s^{-2}}\, ds\right]_s^\infty$$

$$= \frac{1}{2}\left[s\log(1+s^{-2}) + 2\int \frac{s^{-2}}{s^{-2}\left(\frac{1}{s^{-2}}+1\right)}\, ds\right]_s^\infty$$

$$= \frac{1}{2}\left[s\log(1+s^{-2}) + 2\int \frac{ds}{1+s^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[s\log(1+s^{-2}) + 2\tan^{-1}s\right]_s^\infty$$

$$= \frac{1}{2}\left[s\log(1+s^{-2}) + 2\tan^{-1}s\right]_s^{\infty}$$

$$= \frac{1}{2}\left[0 + 2\tan^{-1}\infty - s\log(1+s^{-2}) - 2\tan^{-1}s\right]$$

$$= \frac{1}{2}\left[2\times\frac{\pi}{2} - 2\tan^{-1}s - s\log(1+s^{-2})\right]$$

$$= \frac{1}{2}\left[2\left(\frac{\pi}{2} - \tan^{-1}s\right) - s\log(1+s^{-2})\right]$$

$$= \frac{1}{2}\left[2\cot^{-1}s - s\log(1+s^{-2})\right]$$

$$= \cot^{-1}s - \frac{1}{2}s\log\left(1+\frac{1}{s^2}\right) \quad \text{Ans}$$

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**Que (8)** Find the laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi. \end{cases}$$

**Sol.^n** By definition of laplace transform,

$$L\{f(t)\} = \int_0^{\infty} e^{-st}\cdot f(t)\,dt$$

$$= \int_0^{\pi} e^{-st}\cdot \sin t\,dt + \int_{\pi}^{\infty} e^{-st}\cdot 0\cdot dt$$

Use formula: $\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$

$$= \left|\frac{e^{-st}}{s^2+1}(-s\cdot\sin t - \cos t)\right|_0^{\pi}$$

$$= \frac{e^{-s\pi}}{s^2+1}(-s\cdot\sin\pi - \cos\pi) - \frac{e^0}{s^2+1}(-s\cdot\sin 0 - \cos 0)$$

$$= \frac{1}{s^2+1}\left[e^{-s\pi}(-)(-1) + 1\right] = \frac{1+e^{-s\pi}}{s^2+1} \quad \text{Ans}$$

15 **Que (9)** Evaluate:

$$\int_0^\infty t \cdot e^{-3t} \cdot \sin t \, dt$$

{Evaluation of Integrals by Laplace Transform}

**Sol.^n** $\because L\{\sin t\} = \frac{1}{s^2+1}$

Using multiplication property,

$$L\{t^1 \cdot \sin t\} = (-1)^1 \frac{d^1}{ds^1}\left(\frac{1}{s^2+1}\right) = -\frac{d}{ds}(s^2+1)^{-1}$$

$$= -(-1)(s^2+1)^{-2}(2s) = \frac{2s}{(s^2+1)^2}$$

Using shifting property,

$$L\{e^{-3t} \cdot t \sin t\} = \frac{2(s+3)}{[(s+3)^2+1]^2}$$

Now, by definition of Laplace transform,

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t) \, dt$$

$$L\{e^{-3t} \cdot t \sin t\} = \int_0^\infty e^{-st} \cdot t e^{-3t} \cdot \sin t \, dt = \frac{2(s+3)}{[(s+3)^2+1]^2}$$

$\therefore$ Put $s=0$,

$$\int_0^\infty t e^{-3t} \sin t \, dt = \frac{2(0+3)}{[(0+3)^2+1]^2} = \frac{6}{100} = \frac{3}{50}$$

Ans

Que (10) Evaluate: $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt$

or, Prove that: $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt = \log\frac{b}{a}$

Sol.$^n$ $L\{e^{-at}-e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$

Using division property,

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$= \left|\log(s+a)-\log(s+b)\right|_s^\infty$$

$$= \left|\log\left(\frac{s+a}{s+b}\right)\right|_s^\infty = \lim_{s\to\infty}\log\left(\frac{s+a}{s+b}\right) - \log\left(\frac{s+a}{s+b}\right)$$

$$= \lim_{s\to\infty}\log\frac{s(1+a/s)}{s(1+b/s)} - \log\left(\frac{s+a}{s+b}\right)$$

$$= \log 1 + \log\left(\frac{s+b}{s+a}\right) = \log\left(\frac{s+b}{s+a}\right)$$

By definition of Laplace transform,

$$L\{f(t)\} = \int_0^\infty e^{-st}\cdot f(t)\,dt$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_0^\infty e^{-st}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \log\frac{s+b}{s+a}$$

Put $s=0$

$$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt = \log\frac{b}{a}$$

Hence proved

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# INVERSE LAPLACE TRANSFORM

## FORMULAE:

1. $L^{-1}\left\{\frac{1}{s}\right\} = 1$

2. $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

3. $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\lfloor n-1}$

4. $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

5. $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

6. $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

7. $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

## Properties of ILT:

If $L^{-1}\left\{\bar{f}(s)\right\} = f(t)$

1. $L^{-1}\left\{\bar{f}(s-a)\right\} = e^{at} f(t)$ → Shifting property

2. $L^{-1}\left\{\int_s^\infty \bar{f}(s)\,ds\right\} = \frac{1}{t} f(t)$ → Integral property

3. $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$ → Division property

4. $L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = -t\,f(t)$ → Derivative property

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Que (1) Evaluate: $L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}$

Sol.ⁿ $\frac{1}{s(s+1)(s+2)} = \frac{1}{s}\cdot\frac{1}{(s+1)(s+2)} = \bar{f}(s)$

Using partial fraction,

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2)+B(s+1)}{(s+1)(s+2)}$$

$\Rightarrow \quad 1 = A(s+2) + B(s+1)$

Put $s=-1$, $\Rightarrow 1 = A+0 \Rightarrow A=1$

Put $s=-2$, $\Rightarrow 1 = 0+B(-1) \Rightarrow B=-1$

$$\therefore \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Taking ILT on both sides,

$$L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-t} - e^{-2t}$$

Using division rule,

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s+1)(s+2)}\right\} = \int_0^t \left(e^{-t} - e^{-2t}\right)dt$$

$$= \left|\frac{e^{-t}}{-1} - \frac{e^{-2t}}{-2}\right|_0^t = -e^{-t} + 1 + \frac{e^{-2t}}{2} - \frac{1}{2}$$

$$= \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2} \quad \text{Ans}$$

Que ② Evaluate: $L^{-1}\left\{\dfrac{4s+5}{(s-1)^2(s+2)}\right\}$

Sol$^n$ Given: $\bar{f}(s) = \dfrac{4s+5}{(s-1)^2(s+2)} = \dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{s+2}$

$$\Rightarrow \frac{4s+5}{(s-1)^2(s+2)} = \frac{A(s-1)(s+2) + B(s+2) + C(s-1)^2}{(s-1)^2(s+2)}$$

$$\Rightarrow 4s+5 = A(s^2+2s-s-2) + B(s+2) + C(s^2-2s+1)$$

$$\Rightarrow 0s^2+4s+5 = A(s^2+s-2) + B(s+2) + C(s^2-2s+1)$$

On comparing the co-efficients of

(i) $s^2$, $\quad 0 = A+C \Rightarrow A = -C$

(ii) $s$, $\quad 4 = A+B-2C \Rightarrow 4 = -C+B-2C \Rightarrow 4 = B-3C$ —(I)

(iii) constants, $\quad 5 = -2A+2B+C = 2C+2B+C$

$\Rightarrow 5 = 2B+3C$ —(II)

from (I) & (II), $\quad B-3C = 4$

$\quad 2B+3C = 5$

$\quad 3B = 9 \Rightarrow \underline{B=3}$

from (I), $B-3C=4 \Rightarrow B-4 = 3C \Rightarrow \underline{C = -1/3}$

$\Rightarrow A = 1/3$

$$\therefore \bar{f}(s) = \frac{4s+5}{(s-1)^2(s+2)} = \frac{1}{3}\left(\frac{1}{s-1}\right) + 3\left(\frac{1}{s-1}\right)^2 - \frac{1}{3}\left(\frac{1}{s+2}\right)$$

Taking ILT on both sides,

$$L^{-1}\{\bar{f}(s)\} = \frac{1}{3}L^{-1}\left\{\frac{1}{s-1}\right\} + 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{1}{3}e^t + 3e^t L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{3}e^{-2t}$$

$$= \frac{1}{3}e^t + 3e^t \cdot \frac{t^1}{\lfloor 1} - \frac{1}{3}e^{-2t} = \frac{1}{3}e^t + 3te^t - \frac{1}{3}e^{-2t}$$

Ans

Que(3) Solve: $L^{-1}\left\{\frac{s+2}{s^2(s+1)(s-2)}\right\}$

Solⁿ Solving $\frac{s+2}{(s+1)(s-2)}$ by partial fraction.

$$\frac{s+2}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} = \frac{A(s-2)+B(s+1)}{(s+1)(s-2)}$$

$$\Rightarrow s+2 = A(s-2) + B(s+1)$$

Put $s=2$, $B = 4/3$

Put $s=-1$, $A = -1/3$

$$\therefore \frac{s+2}{(s+1)(s-2)} = -\frac{1}{3}\left(\frac{1}{s+1}\right) + \frac{4}{3}\left(\frac{1}{s-2}\right)$$

Taking ILT on both sides,

$$L^{-1}\left\{\frac{s+2}{(s+1)(s-2)}\right\} = -\frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{4}{3}L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= -\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t}$$

Now, $L^{-1}\left\{\frac{1}{s}\cdot\frac{s+2}{(s+1)(s-2)}\right\} = \int_0^t\left(\frac{4}{3}e^{2t} - \frac{1}{3}e^{-t}\right)dt$ } Division property

$$= \left|\frac{4}{3}\frac{e^{2t}}{2} - \frac{1}{3}\frac{e^{-t}}{(-1)}\right|_0^t = \left|\frac{2}{3}e^{2t} + \frac{e^{-t}}{3}\right|_0^t$$

$$= \frac{2}{3}e^{2t} + \frac{e^{-t}}{3} - \frac{2}{3} - \frac{1}{3} = \frac{2}{3}e^{2t} + \frac{e^{-t}}{3} - 1$$

Again using division property,

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s}\cdot\frac{s+2}{(s+1)(s-2)}\right\} = \int_0^t\left(\frac{2}{3}e^{2t} + \frac{e^{-t}}{3} - 1\right)dt$$

$$= \left|\frac{2}{3}\frac{e^{2t}}{2} + \frac{1}{3}\frac{e^{-t}}{(-1)} - t\right|_0^t = \left|\frac{e^{2t}}{3} - \frac{e^{-t}}{3} - t\right|_0^t$$

$$= \frac{e^{2t}}{3} - \frac{e^{-t}}{3} - t - \frac{e^0}{3} + \frac{e^0}{3} + 0 = \frac{e^{2t}}{3} - \frac{e^{-t}}{3} - t$$

Ans

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**Que (4)** Evaluate: $L^{-1}\left\{\dfrac{s}{(s+3)^2+4}\right\}$

**Sol$^n$**

$$L^{-1}\left\{\frac{s}{(s+3)^2+4}\right\} = L^{-1}\left\{\frac{s+3-3}{(s+3)^2+4}\right\}$$

$$= L^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\} - 3L^{-1}\left\{\frac{1}{(s+3)^2+4}\right\}$$

$$= e^{-3t}L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - 3e^{-3t}L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

(Using shifting property)

$$= e^{-3t}\cos 2t - 3e^{-3t}\cdot\frac{1}{2}\sin 2t$$

$$= e^{-3t}\left(\cos 2t - \frac{3}{2}\sin 2t\right) \quad \text{Ans}$$

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**Que (5)** Evaluate: $L^{-1}\left\{\dfrac{3(s^2-2)^2}{2s^5}\right\}$

**Sol$^n$** $\bar{f}(s) = \dfrac{3(s^2-2)^2}{2s^5} = 3\left(\dfrac{s^4-4s^2+4}{2s^5}\right) = \dfrac{3}{2}\left[\dfrac{1}{s} - \dfrac{4}{s^3} + \dfrac{4}{s^5}\right]$

Taking ILT on both sides,

$$L^{-1}\{\bar{f}(s)\} = \frac{3}{2}\left[L^{-1}\left\{\frac{1}{s}\right\} - 4L^{-1}\left\{\frac{1}{s^3}\right\} + 4L^{-1}\left\{\frac{1}{s^5}\right\}\right]$$

$$= \frac{3}{2}\left[1 - 4\cdot\frac{t^2}{2!} + 4\cdot\frac{t^4}{4!}\right] = \frac{3}{2}\left[1 - 2t^2 + \frac{t^4}{6}\right]$$

$$= \frac{3}{2} - 3t^2 + \frac{t^4}{4} \quad \text{Ans}$$

**Que (6)** Evaluate : $L^{-1}\left\{\dfrac{s^2-3s+4}{s^3}\right\}$

**Sol$^n$** $\bar{f}(s) = \dfrac{s^2-3s+4}{s^3} = \dfrac{1}{s} - \dfrac{3}{s^2} + \dfrac{4}{s^3}$

Now, taking ILT on both sides,

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 1 - 3\frac{t^1}{\lfloor 1} + 4\frac{t^2}{\lfloor 2} = 1 - 3t + 2t^2 \quad \text{Ans}$$

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**Que (7)** Evaluate : $L^{-1}\left\{\dfrac{s+2}{s^2-4s+13}\right\}$ $13 < {4 \atop 9}$

$+4 \; +9$

**Sol$^n$** Let $L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\dfrac{s+2}{(s^2-4s+4)+9}\right\} = L^{-1}\left\{\dfrac{s+2}{(s-2)^2+3^2}\right\}$

$$= L^{-1}\left\{\frac{s-2+2+2}{(s-2)^2+3^2}\right\} = L^{-1}\left\{\frac{(s-2)+4}{(s-2)^2+3^2}\right\}$$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2+3^2}\right\}$$

Using shifting property,

$$L^{-1}\{\bar{f}(s)\} = e^{2t}L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + 4e^{2t}L^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$= e^{2t}\cdot\cos 3t + 4e^{2t}\cdot\frac{1}{3}\sin 3t$$

$$= e^{2t}\left(\cos 3t + \frac{4}{3}\sin 3t\right)$$

Ans

Que (8) Evaluate: $L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$

Sol$^n$ Here, $\bar{f}(s) = \frac{s+1}{s^2+s+1} = \frac{s+1}{(s^2+s+1/4)+(1-\frac{1}{4})}$

$(s^2+2\cdot s\cdot\frac{1}{2}+\frac{1}{4})$

$$= \frac{s+1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{(s+1/2)+(1/2)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Taking ILT on both sides,

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\}$$

Using shifting property,

$$L^{-1}\{\bar{f}(s)\} = e^{-t/2}L^{-1}\left\{\frac{s}{s^2+(\sqrt{3}/2)^2}\right\} + \frac{1}{2}e^{-t/2}L^{-1}\left\{\frac{1}{s^2+(\sqrt{3}/2)^2}\right\}$$

$$= e^{-t/2}\cdot\cos\frac{\sqrt{3}}{2}t + \frac{1}{\cancel{2}}e^{-t/2}\cdot\frac{1}{\frac{\sqrt{3}}{\cancel{2}}}\cdot\sin\frac{\sqrt{3}}{2}t$$

$$= e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right) \quad \text{Ans}$$

Que 10 Solve: $L^{-1}\left\{\log\left(\frac{s+3}{s+2}\right)\right\}$

Sol.n Let $\bar{f}(s) = \log\left(\frac{s+3}{s+2}\right) = \log(s+3) - \log(s+2)$

Differentiating w.r.t. s, we get

$$\frac{d}{ds}\left[\bar{f}(s)\right] = \frac{1}{s+3} - \frac{1}{s+2}$$

Taking ILT on both sides,

$$L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = L^{-1}\left\{\frac{1}{s+3}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\Rightarrow \quad -t\, f(t) = e^{-3t} - e^{-2t}$$

$$\Rightarrow \quad f(t) = \frac{e^{-2t} - e^{-3t}}{t}$$ Ans

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Que 11 Solve: $L^{-1}\left\{\log\frac{s^2+1}{s(s+1)}\right\}$

Sol.n Let $\bar{f}(s) = \log\frac{s^2+1}{s(s+1)} = \log(s^2+1) - \log s - \log(s+1)$

Differentiating w.r.t. s, we get

$$\frac{d}{ds}\bar{f}(s) = \frac{1}{s^2+1}.(2s) - \frac{1}{s} - \frac{1}{s+1} = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

Taking ILT on both sides,

$$L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = 2L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\Rightarrow \quad -t.\, f(t) = 2\cos t - 1 - e^{-t}$$

$$\Rightarrow \quad t\, f(t) = 1 + e^{-t} - 2\cos t$$

$$\Rightarrow \quad f(t) = \frac{1 + e^{-t} - 2\cos t}{t}$$ Ans

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Que (9) Prove that:

$$L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = \frac{2}{\sqrt{3}} \sinh\frac{t}{2} \cdot \sin\frac{\sqrt{3}}{2}t$$

Sol$^n$ let $\bar{f}(s) = \frac{s}{\underset{+s^2\;-s^2}{s^4+s^2+1}} = \frac{s}{(s^4+2s^2+1)-s^2} = \frac{s}{(s^2+1)^2-s^2}$

$$= \frac{s}{(s^2+s+1)(s^2-s+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2-s+1}$$

Using partial fraction technique,

$$s = (As+B)(s^2-s+1) + (Cs+D)(s^2+s+1)$$

$$s = As^3 - As^2 + As + Bs^2 - Bs + B + Cs^3 + Cs^2 + Cs + Ds^2 + Ds + D.$$

On equating the co-efficients of :

(i) $s^3$, $\quad 0 = A + C$

(ii) $s^2$, $\quad 0 = -A + \underbrace{B + C + D}$

(iii) $s$, $\quad 1 = \underbrace{A - B} + C + D \Rightarrow -B + D = 1$

(iv) constants, $\quad 0 = B + D \Rightarrow -A + C = 0$

$$\begin{array}{r} A + C = 0 \\ -A + C = 0 \\ \hline C = 0 \end{array}$$

$\Rightarrow A = 0$

Also, 
$$\begin{array}{r} B + D = 0 \\ -B + D = 1 \\ \hline D = 1/2 \end{array}$$

$\Rightarrow B = -1/2$

$$\therefore \frac{s}{s^4+s^2+1} = \frac{-1/2}{s^2+s+1} + \frac{1/2}{s^2-s+1}$$

Taking ILT on both sides,

$$L^{-1}\left\{\bar{f}(s)\right\} = -\frac{1}{2} L^{-1}\left\{\frac{1}{s^2+s+1}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{s^2+s+1}\right\}$$

$$= -\frac{1}{2} L^{-1}\left\{\frac{1}{(s^2+s+\frac{1}{4}) + (1-\frac{1}{4})}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{(s^2-s+\frac{1}{4}) + (1-\frac{1}{4})}\right\}$$

$$= -\frac{1}{2} L^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{(s-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\}$$

$$= -\frac{1}{2} L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

Using shifting property,

$$L^{-1}\{\bar{f}(s)\} = -\frac{1}{2} e^{-t/2}\cdot L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} + \frac{1}{2} e^{t/2} L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= -\frac{1}{\cancel{2}} e^{-t/2}\cdot \frac{1}{\frac{\sqrt{3}}{\cancel{2}}}\cdot \sin\frac{\sqrt{3}}{2}t + \frac{1}{\cancel{2}}\cdot e^{t/2}\cdot\frac{1}{\frac{\sqrt{3}}{\cancel{2}}}\cdot \sin\frac{\sqrt{3}}{2}t$$

$$= -\frac{1}{\sqrt{3}} e^{-t/2}\cdot \sin\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}} e^{t/2}\cdot \sin\frac{\sqrt{3}}{2}t$$

$$= \frac{1}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t \left(e^{t/2} - e^{-t/2}\right) \times \frac{2}{2}$$

$$\left\{ \frac{e^{t}-e^{-t}}{2} = \sinh t \right.$$

$$= \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t \left(\frac{e^{t/2} - e^{-t/2}}{2}\right)$$

$$= \frac{2}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}t \cdot \sinh\frac{t}{2} \Rightarrow LHS = RHS$$

Proved

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Que(12) Find the inverse laplace transform of $\cot^{-1}\left(\frac{s}{2}\right)$.

Sol$^n$ Let $\bar{f}(s) = \cot^{-1}\left(\frac{s}{2}\right) = \tan^{-1}\left(\frac{1}{s/2}\right) = \tan^{-1}\left(\frac{2}{s}\right)$ $\left\{ \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right) \right.$

Differentiating both sides w.r.t. s,

$$\frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\tan^{-1}\left(\frac{2}{s}\right) = \frac{1}{1+(2/s)^2}\cdot 2\left(-\frac{1}{s^2}\right) = \frac{-2}{\cancel{s^2}\left(\frac{s^2+2^2}{\cancel{s^2}}\right)}$$

$$= \frac{-2}{s^2+2^2}$$

Taking ILT on both sides,

$$L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = -2\, L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$$\Rightarrow -t\cdot f(t) = -\cancel{2}\cdot\frac{1}{\cancel{2}}\sin 2t$$

$$\Rightarrow f(t) = \frac{\sin 2t}{t}$$ Ans

# CONVOLUTION THEOREM

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$, then

$$L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\cdot g(t-u)\,du.$$

Que(1) Applying convolution theorem, solve $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Sol.$^n$ $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)}\right\}$

Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{1}{s^2+a^2}$

$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t)$

Also, $L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\cdot\sin at = g(t)$

Using convolution theorem,

$$L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = \int_0^t \cos au\cdot\frac{1}{a}\sin a(t-u)\,du.$$

Use formula : $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$= \frac{1}{2a}\int_0^t 2\cos au\cdot\sin a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t \{\sin a(u+t-u) - \sin a(u-t+u)\}\,du$$

$$= \frac{1}{2a}\int_0^t \{\sin at - \sin a(2u-t)\}\,du$$

$$= \frac{1}{2a}\left[\sin at\,|u|_0^t + \left|\frac{\cos a(2u-t)}{2a}\right|_0^t\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right] = \frac{t\sin at}{2a} \quad \text{Ans}$$

Que (2) Using convolution theorem, evaluate $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

Sol$^n$ $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\}$

Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{s}{s^2+b^2}$

$\therefore L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t)$

Also, $L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt = g(t)$

Using convolution theorem,

$$L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\} = \int_0^t \cos au \cdot \cos b(t-u)\,du.$$

$\boxed{\text{Use formula: } 2\cos A\cos B = \cos(A+B)+\cos(A-B)}$

$$= \frac{1}{2}\int_0^t 2\cos au\cdot\cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \left\{\cos(au+bt-bu)+\cos(au-bt+bu)\right\}du$$

$$= \frac{1}{2}\int_0^t \left[\cos\{(a-b)u+bt\}+\cos\{(a+b)u-bt\}\right]du$$

$$= \frac{1}{2}\left|\frac{\sin\{(a-b)u+bt\}}{a-b}+\frac{\sin\{(a+b)u-bt\}}{a+b}\right|_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b}-\frac{\sin bt}{a-b}+\frac{\sin(at+bt-bt)}{a+b}-\frac{\sin(-bt)}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at-\sin bt}{a-b}+\frac{\sin at+\sin bt}{a+b}\right]$$

$$= \frac{1}{2(a^2-b^2)}\left[a\sin at+b\sin at-a\sin bt-b\sin bt+a\sin at+a\sin bt-b\sin at-b\sin bt\right]$$

$$= \frac{2(a\sin at-b\sin bt)}{2(a^2-b^2)} = \frac{a\sin at-b\sin bt}{a^2-b^2}$$

Ans

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# Application of Laplace Transform

Que ① Solve:

$$(D^3 - 3D^2 + 3D - 1)y = t^2 e^t \, ;$$

given that: $y(0) = 1, \; y'(0) = 0, \; y''(0) = -2$.

Solⁿ Given: $(D^3 - 3D^2 + 3D - 1)y = t^2 e^t$

$$\Rightarrow \quad y''' - 3y'' + 3y' - y = t^2 e^t$$

Taking LT on both sides,

$$L\{y'''\} - 3L\{y''\} + 3L\{y'\} - L\{y\} = L\{t^2 e^t\}$$

$$\Rightarrow \left[s^3 L\{y\} - s^2 y(0) - s\,y'(0) - y''(0)\right] - 3\left[s^2 L\{y\} - s\,y(0) - y'(0)\right] + 3\left[s\,L\{y\} - y(0)\right] - L\{y\} = \frac{2}{(s-1)^3}$$

$$\Rightarrow \quad L\{y\}(s^3 - 3s^2 + 3s - 1) - s^2(1) - (-2) + 3s - 3 = \frac{2}{(s-1)^3}$$

$$\Rightarrow \quad L\{y\}\,(s-1)^3 = s^2 - 3s + 1 + \frac{2}{(s-1)^3}$$

$$\Rightarrow \quad L\{y\} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{(s^2 - 2s + 1) - s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{(s-1)^2 - s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$L\{y\} = \frac{1}{s-1} - \frac{s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$L\{y\} = \frac{1}{s-1} - \frac{s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{(s-1)+1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$L\{y\} = \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$y = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - L^{-1}\left\{\frac{1}{(s-1)^3}\right\} + 2L^{-1}\left\{\frac{1}{(s-1)^6}\right\}$$

By shifting property.

$$y = e^t L^{-1}\left\{\frac{1}{s}\right\} - e^t L^{-1}\left\{\frac{1}{s^2}\right\} - e^t L^{-1}\left\{\frac{1}{s^3}\right\} + 2e^t L^{-1}\left\{\frac{1}{s^6}\right\}$$

$$= e^t(1) - e^t \cdot \frac{t^1}{\lfloor 1} - e^t \frac{t^2}{\lfloor 2} + 2e^t \frac{t^5}{\lfloor 5}$$

$$y = e^t\left(1 - t - \frac{t^2}{2} + \frac{t^5}{60}\right)$$

which is the reqd. solution. Ans.

Que② Solve: $ty'' + 2y' + ty = \cos t$ ; given $y(0) = 1$

Sol$^n$ Taking LT on both sides,

$$L\{ty''\} + 2L\{y'\} + L\{ty\} = L\{\cos t\}$$

$$\Rightarrow -\frac{d}{ds}L\{y''\} + 2L\{y'\} - \frac{d}{ds}L\{y\} = \frac{s}{s^2+1^2}$$

$$\Rightarrow -\frac{d}{ds}\left[s^2L\{y\} - sy(0) - y'(0)\right] + 2\left[sL\{y\} - y(0)\right] - \frac{d}{ds}L\{y\} = \frac{s}{s^2+1}$$

$$\Rightarrow -\left[s^2\frac{d}{ds}L\{y\} + L\{y\}\cdot(2s) - (1)(1) - 0\right] + 2sL\{y\} - 2 - \frac{d}{ds}L\{y\} = \frac{s}{s^2+1}$$

$$\Rightarrow -s^2\frac{d}{ds}L\{y\} - 2sL\{y\} + 1 + 2sL\{y\} - 2 - \frac{d}{ds}L\{y\} = \frac{s}{s^2+1}$$

$$\Rightarrow (-s^2-1)\frac{d}{ds}L\{y\} - 1 = \frac{s}{s^2+1}$$

$$\Rightarrow -(s^2+1)\frac{d}{ds}L\{y\} = 1 + \frac{s}{s^2+1}$$

$$\Rightarrow -\frac{d}{ds}L\{y\} = \frac{1}{s^2+1} + \frac{s}{(s^2+1)^2}$$

Taking ILT on both sides,

$$L^{-1}\left\{-\frac{d}{ds}\bar{f}(s)\right\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$\Rightarrow t\cdot y = \frac{1}{1}\cdot\sin t + L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$\Rightarrow ty = \sin t + L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} \quad \text{— ①}$$

Solving by convolution theorem,

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = L^{-1}\left\{\frac{s}{s^2+1}\cdot\frac{1}{s^2+1}\right\}$$

Let $\bar{f}(s) = \frac{s}{s^2+1}$ and $\bar{g}(s) = \frac{1}{s^2+1}$

then, $L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t = f(t)$

Also, $L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{1}{1}\cdot\sin t = \sin t = g(t)$

Using convolution theorem,

$$L^{-1}\left\{\frac{s}{s^2+1}\cdot\frac{1}{s^2+1}\right\} = \int_0^t \cos u\cdot\sin(t-u)\,du.$$

Use formula: $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$= \frac{1}{2}\int_0^t 2\cos u\cdot\sin(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\sin(u+t-u) - \sin(u-t+u)]\,du$$

$$= \frac{1}{2}\int_0^t [\sin t - \sin(2u-t)]\,du.$$

$$= \frac{1}{2}\left[\sin t\,|u|_0^t + \left|\frac{\cos(2u-t)}{2}\right|_0^t\right]$$

$$= \frac{1}{2}\left[\sin t\,(t-0) + \frac{1}{2}\{\cos t - \cos(-t)\}\right]$$

$$= \frac{t\sin t}{2}$$

from (1), $ty = \sin t + L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

$$ty = \sin t + \frac{t\sin t}{2}$$

$$y = \left(1+\frac{t}{2}\right)\frac{\sin t}{t}$$

Ans..

which is the reqd. solution.

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Que(3) Solve: $\frac{d^2x}{dt^2} + 9x = \cos 2t$ ; Given: $x(0)=1$ and $x(\pi/2) = -1$.

Soln ∵ $\frac{d^2x}{dt^2} + 9x = \cos 2t$

$\Rightarrow (D^2+9)x = \cos 2t$

$\Rightarrow x'' + 9x = \cos 2t$

Taking LT on both sides,

$L\{x''\} + 9L\{x\} = L\{\cos 2t\}$

$\Rightarrow [s^2 L\{x\} - s\,x(0) - x'(0)] + 9L\{x\} = \frac{s}{s^2+2^2}$

Assume $x'(0) = A$

$\Rightarrow [s^2 L\{x\} - s(1) - A] + 9L\{x\} = \frac{s}{s^2+4}$.

$\Rightarrow L\{x\}(s^2+9) = s + A + \frac{s}{s^2+4}$

$\Rightarrow L\{x\} = \frac{s}{s^2+9} + \frac{A}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$

$= \frac{s}{s^2+9} + \frac{A}{s^2+9} + \frac{1}{5}\left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right)$

$L\{x\} = \frac{4}{5}\left(\frac{s}{s^2+9}\right) + \frac{A}{s^2+9} + \frac{1}{5}\left(\frac{s}{s^2+4}\right)$

$\Rightarrow x = \frac{4}{5} L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + A\,L^{-1}\left\{\frac{1}{s^2+3^2}\right\} + \frac{1}{5} L^{-1}\left\{\frac{s}{s^2+2^2}\right\}$

$\Rightarrow x = \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t + \frac{1}{5}\cos 2t$

$$x = \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t + \frac{1}{5}\cos 2t$$

Put $x=-1$ when $t=\pi/2$,

$$-1 = \frac{4}{5}\cos\frac{3\pi}{2} + \frac{A}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\cos 2\frac{\pi}{2}$$

$$\Rightarrow -1 = 0 - \frac{A}{3} - \frac{1}{5} \Rightarrow A = \frac{12}{5}$$

$$\therefore\ x = \frac{4}{5}\cos 3t + \frac{12}{15}\sin 3t + \frac{1}{5}\cos 2t$$

$$x = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t$$

which is the reqd. solution. Ans..

# Laplace Transform of Periodic Functions

A function $f(t)$ is said to be periodic if there exists a constant $T$ $(T>0)$ such that $f(t+T) = f(t)$, for all values of $t$.

$$f(t+2T) = f(t+T+T) = f(t+T) = f(t)$$

In general, $f(t+nT) = f(t)$ for all $t$, where $n$ is an integer and $T$ is the period of the function.

If $f(t)$ is a periodic function with period $T$ i.e., $f(t+T) = f(t)$, then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

Que ① Find the laplace transform of

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Sol$^n$ Since $f(t)$ is a periodic function with period $\frac{2\pi}{\omega}$

$\Rightarrow T = \frac{2\pi}{\omega}$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} \cdot f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[ \int_0^{\pi/\omega} e^{-st} \cdot \sin \omega t\, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0\, dt \right]$$

Use formula : $\int e^{at} \cdot \sin bt\, dt = \frac{e^{at}}{a^2+b^2} (a \sin bt - b \cos bt)$

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}} \left[ \frac{e^{-st}}{s^2+\omega^2} (-s \cdot \sin \omega t - \omega \cos \omega t) \right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[ \frac{e^{-s\pi/\omega}}{s^2+\omega^2} (-s \cdot \sin \pi - \omega \cos \pi) - \frac{e^0}{s^2+\omega^2} (-s \cdot \sin 0 - \omega \cdot \cos 0) \right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[ -\frac{e^{-s\pi/\omega}}{s^2+\omega^2} (\omega \cdot \cos \pi) + \frac{1}{s^2+\omega^2} (\omega \cdot \cos 0) \right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[ \frac{e^{-s\pi/\omega} \cdot \omega + \omega}{s^2+\omega^2} \right]$$

$$\boxed{L\{f(t)\} = \frac{\omega \left( e^{-s\pi/\omega} + 1 \right)}{\left(1-e^{-2\pi s/\omega}\right)(s^2+\omega^2)}}$$

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**Que ②** Find the LT of $f(t) = \begin{cases} 1, & 0 \le t < a \\ -1, & a < t < 2a \end{cases}$ and

$f(t)$ is periodic with period $2a$.

**Sol$^n$** $\because L\{f(t)\} = \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T e^{-st} \cdot f(t)\,dt$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} \cdot f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[ \int_0^a e^{-st}(1)\,dt + \int_a^{2a} e^{-st}(-1)\,dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[ \left| \frac{e^{-st}}{-s} \right|_0^a - \left| \frac{e^{-st}}{-s} \right|_a^{2a} \right]$$

$$= \frac{-1}{s(1-e^{-2as})} \left[ (e^{-as} - e^{0}) - (e^{-2as} - e^{-as}) \right]$$

$$= \frac{-1}{s(1-e^{-2as})} \left[ -1 + 2e^{-as} - e^{-2as} \right]$$

$$= \frac{1}{s(1-e^{-2as})} \left[ 1^2 - 2(1)(e^{-as}) + (e^{-as})^2 \right]$$

$$= \frac{1}{s(1-e^{-2as})} (1-e^{-as})^2 = \frac{(1-e^{-as})^2}{s[1^2 - (e^{-as})^2]}$$

$$= \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})}$$

Ans

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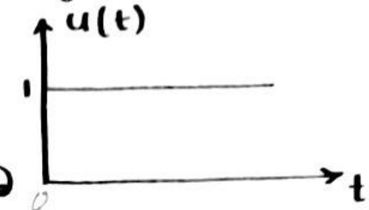
## Unit Step Function

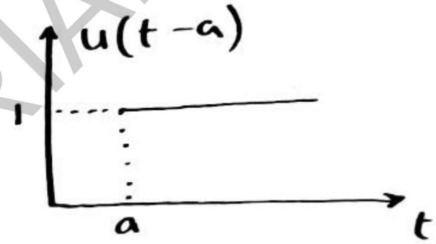
Unit step function is defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$




The delayed (or) displaced unit step function $u(t-a)$ represents the function $u(t)$ which is displaced by a distance 'a' to the right.

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

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## Laplace Transform of Unit Step Functions

1. LT of unit step fn. $u(t)$:

$\because$ We know, $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

$$\therefore L\{u(t)\} = \int_0^\infty e^{-st} u(t)\, dt = \int_0^\infty e^{-st}\, dt = \left| \frac{e^{-st}}{-s} \right|_0^\infty = \frac{1}{s}$$

2. LT of displaced unit step function $u(t-a)$:

$\because$ We know, $u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$

$$\therefore L\{u(t-a)\} = \int_0^\infty e^{-st} \cdot u(t-a)\, dt$$

$$= \int_0^a e^{-st}\, 0\, dt + \int_a^\infty e^{-st} (1)\, dt = \left| \frac{e^{-st}}{-s} \right|_a^\infty = \frac{e^{-as}}{s}$$

3. LT of function $f(t)\cdot u(t-a)$

$$f(t)\,u(t-a) = \begin{cases} 0 & , \ t < a \\ f(t) & , \ t \geq a \end{cases}$$

then $L\{f(t)\,u(t-a)\} = e^{-as}\,L\{f(t+a)\}$

4. LT of function $f(t-a)\,u(t-a)$

$\because$ We Know,

$$f(t-a)\,u(t-a) = \begin{cases} 0 & , \ t < a \\ f(t-a) & , \ t \geq a \end{cases}$$

then, $L\{f(t-a)\,u(t-a)\} = e^{-as}\cdot \bar{f}(s)$

41 Que (1) Find the LT of $\sin t\; u\left(t-\frac{\pi}{2}\right) - u\left(t-\frac{3\pi}{2}\right)$

Sol.$^n$ $\because$ We Know, $L\{u(t-a)\} = \dfrac{e^{-as}}{s}$

and $L\{f(t)\cdot u(t-a)\} = e^{-as}\cdot L\{f(t+a)\}$

$$\therefore L\left\{\sin t\cdot u\left(t-\frac{\pi}{2}\right) - u\left(t-\frac{3\pi}{2}\right)\right\}$$

$$= L\left\{\sin t\cdot u\left(t-\frac{\pi}{2}\right)\right\} - L\left\{u\left(t-\frac{3\pi}{2}\right)\right\}$$

$$= e^{-\pi s/2}\cdot L\left\{\sin\left(t+\frac{\pi}{2}\right)\right\} - \frac{e^{-3\pi s/2}}{s}$$

$$= e^{-\pi s/2}\cdot L\{\cos t\} - \frac{e^{-3\pi s/2}}{s}$$

$$= e^{-\pi s/2}\cdot \frac{s}{s^2+1} - \frac{e^{-3\pi s/2}}{s}$$ Ans

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Que(2) Find the LT of

$$f(t) = \begin{cases} \sin 2t, & 2\pi < t < 4\pi \\ 0, & \text{otherwise} \end{cases}$$

Solⁿ Expressing $f(t)$ in terms of unit step function.

$$f(t) = \sin 2t \cdot u(t-2\pi) - \sin 2t \cdot u(t-4\pi)$$

Taking LT on both sides,

$$L\{f(t)\} = L\{\sin 2t \cdot u(t-2\pi) - \sin 2t \cdot u(t-4\pi)\}$$

$$= L\{\sin 2t \cdot u(t-2\pi)\} - L\{\sin 2t \cdot u(t-4\pi)\}$$

Use formula:

$$L\{f(t) \cdot u(t-a)\} = e^{-as} L\{f(t+a)\}$$

$$L\{f(t)\} = e^{-2\pi s} L\{\sin 2(t+2\pi)\} - e^{-4\pi s} L\{\sin 2(t+4\pi)\}$$

$$= e^{-2\pi s} L\{\sin(4\pi + 2t)\} - e^{-4\pi s} L\{\sin(8\pi + 2t)\}$$

$$= e^{-2\pi s} L\{\sin 2t\} - e^{-4\pi s} L\{\sin 2t\}$$

$$= e^{-2\pi s}\left(\frac{2}{s^2+4}\right) - e^{-4\pi s}\left(\frac{2}{s^2+4}\right)$$

$$= \left(\frac{2}{s^2+4}\right)\left(e^{-2\pi s} - e^{-4\pi s}\right)$$

Ans

Que (3) Find the LT of $f(t) = \begin{cases} \cos t, & 0 < t \le \pi \\ \cos 2t, & \pi < t \le 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$

Sol.$^n$ Expressing $f(t)$ in terms of unit step function.

$$f(t) = [\cos t \cdot u(t-0) - \cos t \cdot u(t-\pi)] + [\cos 2t \, u(t-\pi) - \cos 2t \, u(t-2\pi)] + \cos 3t \cdot u(t-2\pi)$$

$$= \cos t \cdot u(t-0) + (\cos 2t - \cos t) u(t-\pi) + (\cos 3t - \cos 2t) u(t-2\pi)$$

Taking LT on both sides.

$$L\{f(t)\} = L\{\cos t \cdot u(t-0)\} + L\{(\cos 2t - \cos t) \cdot u(t-\pi)\} + L\{(\cos 3t - \cos 2t) \cdot u(t-2\pi)\}$$

Use formula: $L\{f(t) \cdot u(t-a)\} = e^{-as} L\{f(t+a)\}$

$$L\{f(t)\} = e^{0} L\{\cos(t+0)\} + e^{-\pi s} L\{\cos 2(t+\pi) - \cos(t+\pi)\} + e^{-2\pi s} L\{\cos 3(t+2\pi) - \cos 2(t+2\pi)\}$$

$$= L\{\cos t\} + e^{-\pi s} L\{\cos(2\pi + 2t) - \cos(\pi + t)\} + e^{-2\pi s} L\{\cos(6\pi + 3t) - \cos(4\pi + 2t)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2t + \cos t\} + e^{-2\pi s} L\{\cos 3t - \cos 2t\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Ans.

Que(4) Find the LT of $(1+2t-3t^2+4t^3)\,u(t-2)$ and hence evaluate

$$\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)\,u(t-2)\,dt$$

Sol$^n$: $\because$ We know

$$L\{f(t)\cdot u(t-a)\} = e^{-as}\,L\{f(t+a)\}$$

$\therefore L\{(1+2t-3t^2+4t^3)\,u(t-2)\}$

$$= e^{-2s}\cdot L\{1+2(t+2)-3(t+2)^2+4(t+2)^3\}$$

$$= e^{-2s}\cdot L\{1+2(t+2)-3(t^2+4t+4) + 4(t^3+6t^2+12t+8)\}$$

$$= e^{-2s}\cdot L\{25+38t+21t^2+4t^3\}.$$

$$= e^{-2s}\left(\frac{25}{s}+38\cdot\frac{1}{s^2}+21\cdot\frac{\lfloor 2}{s^3}+4\cdot\frac{\lfloor 3}{s^4}\right)$$

$$= e^{-2s}\left(\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right)$$

$\because L\{f(t)\cdot u(t-a)\} = \int_0^\infty e^{-st}\cdot f(t)\,u(t-a)\,dt$

$$= \int_0^\infty e^{-st}(1+2t-3t^2+4t^3)\,u(t-2)\,dt$$

$$= e^{-2s}\left(\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right)$$

Put $s=1$.

$$\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)\,u(t-2)\,dt$$

$$= e^{-2}\left(\frac{25}{1}+\frac{38}{1^2}+\frac{42}{1^3}+\frac{24}{1^4}\right)$$

$$= \frac{129}{e^2} \quad \text{Ans}$$

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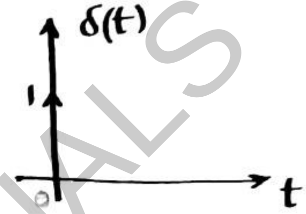
## Unit Impulse Function "DIRAC DELTA FUNCTION"

It has zero amplitude everywhere except at $t=0$.

At $t=0$, the amplitude of the function is infinitely large such that the area under its curve is equal to one unit

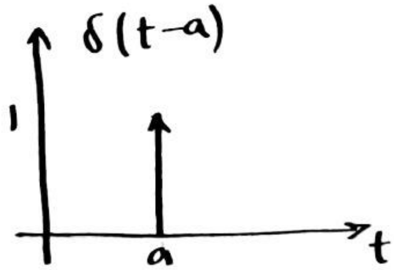
Hence, it is defined as




$$\delta(t)=0, \quad t\neq 0$$

and $$\int_{-\infty}^{\infty}\delta(t)\,dt=1, \quad t=0$$

Similarly, the delayed (or) displaced delta (or) unit impulse function $\delta(t-a)$ represents the function $\delta(t)$ which is displaced by a distance 'a' to the right.




$$\delta(t-a)=0, \quad t\neq a$$

and $$\int_{-\infty}^{\infty}\delta(t-a)\,dt=1, \quad t=a.$$

## Some properties of Unit Impulse Functions:

1. $\int_{-\infty}^{\infty} f(t)\ \delta(t)\ dt = f(0)$

2. $\int_{0}^{\infty} f(t)\ \delta(t)\ dt = f(0)$

3. $\int_{-\infty}^{\infty} f(t)\ \delta(t-a)\ dt = f(a)$

4. $\int_{0}^{\infty} f(t)\ \delta(t-a)\ dt = f(a)$

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## Laplace Transform of Unit Impulse Functions:

1. LT of $\delta(t)$

$\because$ We know, $\delta(t) = 0\ ,\ t \neq 0$

and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1\ ,\ t = 0$

then, $L\{\delta(t)\} = 1$

2. LT of $\delta(t-a)$

$\because$ We know, $\delta(t-a) = 0\ ,\ t \neq a$

and $\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1\ ,\ t = a$

then, $L\{\delta(t-a)\} = e^{-as}$.

3. LT of $f(t)\cdot\delta(t-a)$

$\because$ We know that, $f(t)\cdot\delta(t-a) = 0\ ,\ t \neq a$

and $\int_{0}^{\infty} f(t)\,\delta(t-a)\, dt = f(a)\ ,\ t = a$.

then, $L\{f(t)\cdot\delta(t-a)\} = e^{-as}\cdot f(a)$

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**Que (1)** Evaluate: $\int_0^\infty \cos 2t \cdot \delta\left(t-\frac{\pi}{4}\right) dt$

**Sol.ⁿ** $\because$ We know that

$$\int_0^\infty f(t) \cdot \delta(t-a)\, dt = f(a)$$

Here, $a = \pi/4$ and $f(t) = \cos 2t$

$$\therefore \int_0^\infty \cos 2t \cdot \delta\left(t-\frac{\pi}{4}\right) dt = \cos 2\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0$$

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**Que (2)** Evaluate: $\int_0^\infty t^m (\log t)^n \delta(t-3)\, dt$

**Sol.ⁿ** $\because$ We know that

$$\int_0^\infty f(t)\, \delta(t-a)\, dt = f(a)$$

Here, $a = 3$ and $f(t) = t^m (\log t)^n$

$$\Rightarrow \int_0^\infty t^m (\log t)^n \cdot \delta(t-3)\, dt = f(3)$$

$$= 3^m (\log 3)^n$$

Ans

# THANK YOU SO MUCH