LECTURE - 5

01/08/2023

NOTE: We have already seen:

$$A = -m\int d\tau = -m\int\sqrt{1-|\underset{\sim}{v}|^2}\,dt$$
$$= -m\int\sqrt{-dx_a\,dx^a}$$

This gives us:

$$\frac{d}{dt}\left(\frac{m\underset{\sim}{v}}{\sqrt{1-|\underset{\sim}{v}|^2}}\right) = 0$$

and, $\frac{du_a}{d\tau} = 0 \Rightarrow \frac{du_0}{d\tau} = 0 \;;\; \frac{d\underset{\sim}{u}}{d\tau} = 0$

(where, $u^k = (\gamma, \gamma\underset{\sim}{v})$
$u_k = (-\gamma, \gamma\underset{\sim}{v})$)

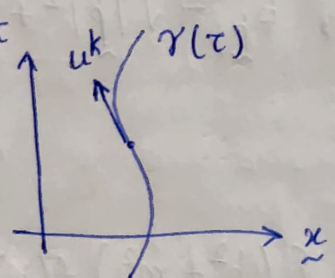
28] We have:

* $u^i = \frac{dx^i}{d\tau} \Rightarrow a^k = \frac{du^k}{d\tau}$

this is 4-acceleration

* $u_a u^a = -1 \Rightarrow \underline{a^k u_k = 0} \Rightarrow$ thus, $a^k = (0, \underset{\sim}{a})$

∴ Acceleration has to be purely spatial

29] 

Consider a potential: $\phi(t, \underset{\sim}{x})$

Thus, in general we can try writing:

(i.e. As per newtonian mechanics, we could imagine :) $m\frac{du^a}{d\tau} = -\eta^{ab}\partial_b\phi$

{we are trying to check whether directly using the eqn. for "Force in terms of potential" of classical mechanics could work for relativity?}

But since we know that $u_k a^k = 0 \Rightarrow u_a\eta^{ab}\partial_b\phi = 0$

$\Rightarrow u^b\partial_b\phi = 0$

$\Rightarrow \frac{\partial x^b}{\partial \tau} \frac{\partial \phi}{\partial x^b} = 0 \Rightarrow \frac{d\phi}{d\tau}\Big|_{\text{trajectory}} = 0 \Rightarrow \phi$ is constant on the trajectory

but since this is not true, this means we CANNOT use that form of the eq.^{n} directly in relativity

30] One thing that works similarly in special relativity is Statistical Mechanics.

This can be shown as follows : We consider a distribution function $f(\underset{\sim}{x}, \underset{\sim}{p}, t)$ on some phase space

$\downarrow$

Thus, the volume can be written (even in relativity) as:

Classical version : $f(\underset{\sim}{x}, \underset{\sim}{p}, t)\, d^3\underset{\sim}{p}\, d^3\underset{\sim}{x} = dN$

In relativity $\rightsquigarrow$ instead we use $d^4 p = dp^0\, d^3\underset{\sim}{p}$

Here p^0 is something like energy and thus, we would want it to be positive (i.e. $p^0 > 0$)

NOTE: { Earlier, we saw:

$$p_k p^k = -m^2$$

$$\Rightarrow -E_{\underset{\sim}{p}}^2 + |\underset{\sim}{p}|^2 = -m^2$$

$$\left(\text{or, } E_{\underset{\sim}{p}} = \sqrt{|\underset{\sim}{p}|^2 + m^2}\right) \}$$

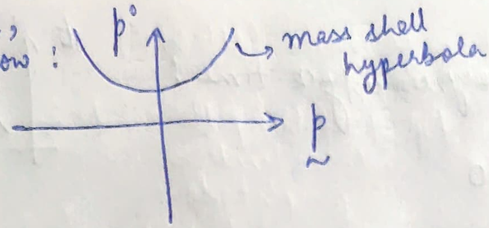
Thus, here in order to have this ($p^0 > 0$) property we use:

$$\int d^4 p\, f(p) = \int dp^0\, d^3\underset{\sim}{p}\; \theta(p^0)\, \delta(p^2 + m^2)\, f(p)$$

where, θ is the Heaviside step function: $\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$$= \int dp^0\, d^3\underset{\sim}{p}\; \underbrace{\theta(p^0)\, \delta(p^2 + m^2)}_{\text{this can be simplified}}\, f(p)$$

{ Since, We know :



and, $p^2 + m^2 = -p^{0\,2} + |\underset{\sim}{p}|^2 + m^2$

$$= -p^{0\,2} + E_{\underset{\sim}{p}}^2$$

Also, $\delta(f(x)) = \sum_i \dfrac{\delta(x - x_r^{(i)})}{|f'(x_r^{(i)})|}$ }

$\Rightarrow$ Hence, we have : $\displaystyle\int f(p)\, d^4p = \int dp^0\, d^3\underset{\sim}{p}\, \frac{\delta(p^0 - E_p)}{2E_p} f(p^0, \underset{\sim}{p})$

since $(p^0 + E_p)$ can be ignored

$$= \underbrace{\int \frac{d^3\underset{\sim}{p}}{2E_p} f(E_{\underset{\sim}{p}}, \underset{\sim}{p})}_{\text{this is Lorentz Invariant (L.I.)}}$$

$*\left(\because \dfrac{d^3\underset{\sim}{p}}{2E_p} \text{ is L.I. measure}\right)$

31] $E_{\underset{\sim}{p}}\, d^3\underset{\sim}{x} = mu^0\, dt \dfrac{d^3\underset{\sim}{x}}{dt}$

or

(in fact we can just directly write) :

$$= mu^0\, d^3\underset{\sim}{x}$$

$$= m \frac{dx^0}{d\tau} d^3\underset{\sim}{x}$$

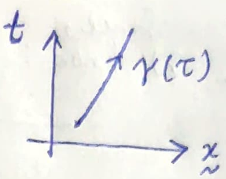
Since dx^0 is the same as dt $\Rightarrow$ $\therefore E_{\underset{\sim}{p}}\, d^3\underset{\sim}{x} = \dfrac{m}{d\tau} dt\, d^3\underset{\sim}{x} = \dfrac{m}{d\tau} d^4x$

$\Downarrow$

Hence, $E_{\underset{\sim}{p}}\, d^3\underset{\sim}{x}$ is also Lorentz Invariant

* Thus, from these results we see that the 'classical' statistical mechanics is Lorentz Invariant, and thus we can still use its results directly in relativity (i.e. they are already 'relativistic')

32] For straight line trajectory in space-time:



$$\frac{du^a}{d\tau} = 0$$

$$\Rightarrow \boxed{u^k \partial_k u^a = 0} \rightsquigarrow \text{eq}^n \text{ of straight line trajectory}$$

$$\Rightarrow \frac{dx^k}{d\tau}\frac{\partial u^a}{\partial x^k} = 0$$

In General Relativity
the eq^n of straight line is given by:

$$\boxed{u^k \nabla_k u^a = 0}$$

(NOTE: The meaning of ∇_k will be explained later)

33] In order to take into account interactions
↓
we consider a potential field $\phi(t, \underset{\sim}{x})$ → this is a scalar field

Some Lorentz Invariant theories we can imagine:

$$A = \underbrace{-m\int d\tau}_{\text{for free case}} + \lambda \int \phi \, d\tau$$

λ ↓ Coupling constant → this is also Lorentz Invariant

We could also take into account a vector field $A_k(t, \underset{\sim}{x})$:

$$A = -m\int d\tau + \lambda\int\phi\, d\tau + q\int A_k \frac{dx^k}{d\tau} d\tau$$

Similarly, we could also use a tensor field $B_{ij}(t, \underset{\sim}{x})$

$$A = -m\int d\tau + \lambda\int\phi\, d\tau + \underbrace{q\int A_k \frac{dx^k}{d\tau} d\tau}_{\text{this explains Electromagnetism}} + \underbrace{k\int\sqrt{B_{ij}dx^i dx^j}}_{\text{this explains gravity}} + \dots \text{(so on)}$$

$\lambda\int\phi\, d\tau$: this term won't be reqd.

NOTE: Beyond B_{ij}, it turns out that we don't really require any other terms for any problem

34] Considering the case for vector field interaction $A_k(t, \underset{\sim}{x})$:

↓

We can write the action $\mathcal{A}$ and Lagrangian $\mathcal{L}$ as:

$$\mathcal{A} = -m \int \sqrt{1-v^2}\, dt + q \int A_k \frac{dx^k}{d\tau} \sqrt{1-|\underset{\sim}{v}|^2}\, dt$$

$$\mathcal{L} = -m\sqrt{1-|\underset{\sim}{v}|^2} + q A_k u^k \sqrt{1-|\underset{\sim}{v}|^2}$$

NOTE: $A^k = (A^0, \underset{\sim}{A})$
$= (\phi, \underset{\sim}{A})$

and, $A_k = (-\phi, \underset{\sim}{A})$

which gives us:

$$\boxed{\mathcal{L} = -m\sqrt{1-|\underset{\sim}{v}|^2} - q\phi + q\,\underset{\sim}{A}\cdot\underset{\sim}{v}}$$

This gives us the Lagrangian describing a charged particle in an E.M. field

* NOTE: These results are valid in general relativity as well, except that the $\sqrt{1-|\underset{\sim}{v}|^2}$ would be modified in G.R.

Interesting Points

The above concept of writing an action & lagrangian also applies to Quantum Mechanics

↓

and it helps explain how Quantum Field Theory is obtained by relativistic quantum mechanics

We know in Quantum Mechanics:

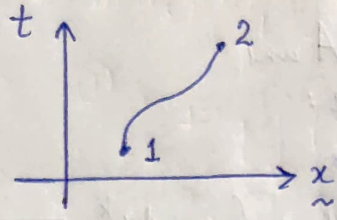
↓

instead of a path we have an amplitude for a path

also there is the use of Hamiltonian:

$$\mathcal{H}\psi = i\dot{\psi}$$

In Non-Relativistic Quantum Mechanics (NRQM):



For propagation from 1 to 2
we can write an amplitude:

$$\langle x_2\, t_2 | x_1\, t_1 \rangle$$ (or simply: $\langle 2|1 \rangle$)

↓

in fact this is simply an integral formulation of the propagator eq^n: $\mathcal{H}\psi = i\dot{\psi}$

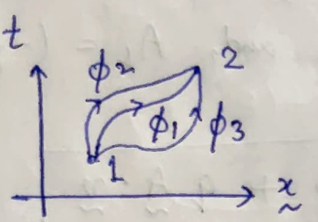
⇐

i.e. $\psi(2) = \int d^3\underset{\sim}{x} \, \langle 2|1 \rangle \, \psi(1)$

↳ this is the integral form

↓ this is differential form

Now, in NRQM, for any path from 1 to 2
we can consider the number obtained
by computing the action for that path
↓
∴ for each path, we can obtain such a number (say, ϕ)

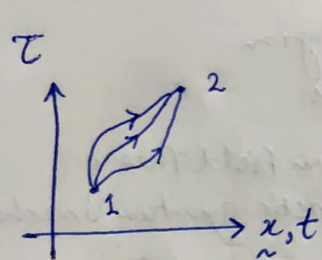


It turns out that the probability amplitude is obtained by summing over these no.s for all possible paths

$$\langle 2|1 \rangle = \sum_{\text{paths}} \phi[\underset{\sim}{x}(t)]$$ ⇒ This is Feynman's Path Integral Formulation *

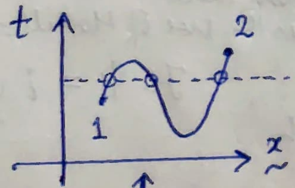
Here, an important thing to note is that although all possible paths from 1 to 2 are taken, these <u>must ONLY be going forward in time</u>.

The difference in <u>Relativistic Quantum Mechanics (RQM)</u> is that here we must consider space and time on equal footing
↓
but since something needs to be chosen which only goes forward, we use proper time (τ) for that:



⇓

But since it only needs to go forward in proper time, it could take 'any' path in time t:
(i.e. even backwards)



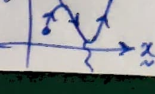
which we can see by plotting $\underset{\sim}{x}-t$

(one of these infinite paths is shown)

thus, at this time the particle exists simultaneously at 3 positions!

this could be interpreted as the particle (say, e^-) going forward in time, then backward & then forward again

↓ OR



e^- emits photon (γ) to become positron, which again emits γ to become e^- ...

LECTURE – 6

04/08/2023

NOTE: In previous lectures we discussed

↓

i) How free particle behave in special relativity

↓

ii) How Lorentz invariance confines the kinds of interactions that are possible in nature.

NOTE: We saw:

$$\mathcal{A}[x^i(\tau), A_k(x^i)] = -m\int d\tau + q\int A_k\, dx^k$$

this is used for describing photons (i.e. spin-0 type particles)

In a more generalized form we could also write:

$$\mathcal{A} = -m\int d\tau + q\int B_M A^M_k\, dx^k$$

(used in Yang-Mills Theory)

↓

thus, Lorentz invariance also allows this

35] $\mathcal{A} = -m\int d\tau + q\int A_k\, dx^k$

$$= -m\int \sqrt{1-v^2}\, dt + q\int A_k \frac{dx^k}{dt}\, dt$$

$\therefore\ \mathcal{L} = -m\sqrt{1-v^2} - q\phi + q\,\underset{\sim}{v}\cdot\underset{\sim}{A}$

$$\Rightarrow \mathcal{A} = -m\int \sqrt{-dx^a dx_a} + q\int A_k\, dx^k$$

(since: $A^k = (\phi, \underset{\sim}{A})$, $A_k = (-\phi, \underset{\sim}{A})$)

Hence we have:

$$\delta\mathcal{A} = -m\int \delta\left(\sqrt{-dx_a dx^a}\right) + q\int \delta A_k\, dx^k + q\int A_k\, \delta(dx^k)$$

$$= m u_a\, \delta x^a\Big|_1^2 - \int m\frac{du_a}{d\tau}\,\delta x^a\, d\tau + q\int \partial_i A_k\, \delta x^i\, dx^k + q\int A_k\, d(\delta x^k)$$

$*\left\{\text{where, } \partial_i A_k = \dfrac{\partial A_k}{\partial x^i}\right\}$

$$= \underbrace{m u_a \delta x^a \Big|_1^2}_{①} - \underbrace{\int m \frac{du_i}{d\tau} \delta x^i d\tau}_{③}$$

$$+ \underbrace{q \int (\partial_i A_k) u^k \delta x^i d\tau}_{④} + q \int_1^2 \frac{d}{d\tau} (A_k \delta x^k) d\tau$$

$$- q \int dA_k \, \delta x^k$$

$$+ \underbrace{q A_i \delta x^i \Big|_1^2}_{②} - q \int \partial_i A_k \, dx^i \, \delta x^k$$

can be written as

$$- q \int \partial_i A_k \, u^i \delta x^k d\tau$$

OR

$$\underbrace{- q \int \partial_k A_i \, u^k \delta x^i d\tau}_{⑤}$$

Combining ① & ②, and ③, ④ & ⑤ :

$$\delta A = (m u_i + q A_i) \delta x^i \Big|_1^2 + \int \left[- m \frac{du_i}{d\tau} + q (\partial_i A_k - \partial_k A_i) u^k \right] \delta x^i d\tau$$

(this acts as the "Electromagnetic Tensor")

* Defining : $F_{ij} = \partial_i A_j - \partial_j A_i$

We get :

$$\delta A = (m u_i + q A_i) \delta x^i \Big|_1^2 + \int \left[- m \frac{du_i}{d\tau} + q F_{ik} u^k \right] \delta x^i d\tau$$

$$\Downarrow$$

E.O.M. : $\boxed{m \frac{du_i}{d\tau} = q F_{ik} u^k}$

Also, $P_i = \frac{\partial A}{\partial x^i} = m u_i + q A_i$

$$= p_i + q A_i$$

36] We know: $p = \frac{\partial \mathcal{L}}{\partial \underset{\sim}{v}}$

Also, $F_{ij} = \partial_i A_j - \partial_j A_i$ $\rightsquigarrow$ * Hence F_{ij} is antisymmetric

$$\Rightarrow F_{0\alpha} = \partial_0 A_\alpha - \partial_\alpha A_0$$
$$= \partial_0 A_\alpha + \partial_\alpha \phi = -E_1$$
$$= \dot{\vec{A}} + \vec{\nabla}\phi$$

(since: $\vec{E} = -\nabla\phi - \frac{\partial \underset{\sim}{A}}{\partial t}$)

Also, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$

$$\Rightarrow F_{12} = \partial_1 A_2 - \partial_2 A_1 = B_3$$

Hence, it is clear that under Lorentz invariance
↓
electric & magnetic fields are not separate, but actually parts of a single Electromagnetic field tensor F_{ij}

* NOTE: $F_{i'j'} = L_{i'}^{a} L_{j'}^{b} F_{ab}$

37] $$m \frac{du_\alpha}{d\tau} = q F_{\alpha k} u^k$$

$$\Rightarrow m \frac{du_1}{d\tau} = q F_{1k} u^k = q F_{10} u^0 + q F_{12} u^2 + q F_{13} u^3 \quad (F_{11} = 0)$$

$$\Rightarrow m \frac{dt}{d\tau} \frac{du_1}{dt} = q E_1 u^0 + q (\underset{\sim}{v} \times \underset{\sim}{B})_1$$

which is simply the Lorentz Force Law
↓
Thus Lorentz invariance directly gives rise to electromagnetic forces

38] Also, since $\frac{du_i}{d\tau} = a_i$

$$\Rightarrow m a_i = q F_{ik} u^k$$

$$\Rightarrow m u^i a_i = q \underbrace{F_{ik} u^k u^i}_{\substack{\text{due to symmetry} \\ \text{this will become} \\ \text{zero}}} = 0 \Rightarrow$$

which satisfies the fact that

$* \; u^i a_i = 0$

39] Also, $P_i P^i = -m^2$

$$\Rightarrow \eta^{ab} (m u_a + q A_a)(m u_b + q A_b) = -m^2$$

$$\Rightarrow \eta^{ab} \frac{\partial A}{\partial x^a} \frac{\partial A}{\partial x^b} = -m^2 \longrightarrow$$

Hamilton-Jacobi equation

40] We know: $F_{ab} = \partial_a A_b - \partial_b A_a$

If we change the 4-potential by some arbitrary manner!

$$A_k \longrightarrow A_k + \partial_k f$$

then! $F_{ab} \rightarrow \partial_a A_b + \cancel{\partial_a \partial_b f} - \partial_b A_a - \cancel{\partial_b \partial_a f}$

$$= F_{ab}$$

i.e. F_{ab} remains invariant under such transformation of the potential

$\Downarrow$

* This is called

<u>Gauge Invariance</u>

{ Similarly we can see this is ordinary electromagnetism where we would write!

$$\phi \rightarrow \phi - \partial_t f$$

$$\vec{A} \rightarrow \vec{A} + \nabla f$$

}

{ **NOTE:** Thus, despite the fact that A^k should have 4 degrees of freedom,

$\downarrow$

Gauge invariance leads to only 2 degrees of freedom

$\downarrow$

* this also leads to the fact that photons have only 2 polarizations }

* **NOTE:** Gauge invariance is also responsible for explaining charge conservation

41] We know the Stokes theorem in 3-dimensions:

$$\int_V \nabla \cdot \underset{\sim}{A} \, d^3x = \int_S d\vec{S} \, \underset{\sim}{A} \cdot \hat{n}$$

Similarly, the Stokes theorem generalizes to 4-dimensions (and in fact it could be done for arbitrary dimensions):

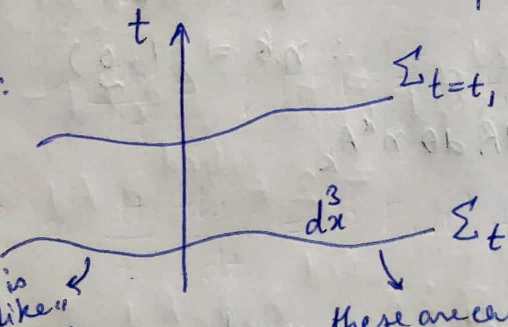
$$* \Rightarrow \int_V (\partial_k A^k) \, \underbrace{d^4x}_{dt\, d^3x} = \int_{\partial V} d^3x \, n^k A_k$$

Here, $n_a = -\partial_a t = (-1, \underset{\sim}{0})$ for spacelike surfaces

and $n^a = \eta^{ab} n_b$

This can be thought of as:

eg:



Here, this is a "spacelike" surface

these are called Hypersurfaces

(Σ is the standard symbol for surface

$\downarrow$

Σ_t means t is kept constant over that surface)

We know: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

Similarly, we can have timelike surfaces, and also null surfaces

eg: surface of a light cone is a null surface

eg: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$= -dt^2 + \underbrace{dr^2}_{-du\,dv} + \underbrace{r^2 d\Omega^2}_{r^2(u,v)\,d\Omega^2}$ ← spherical coord's

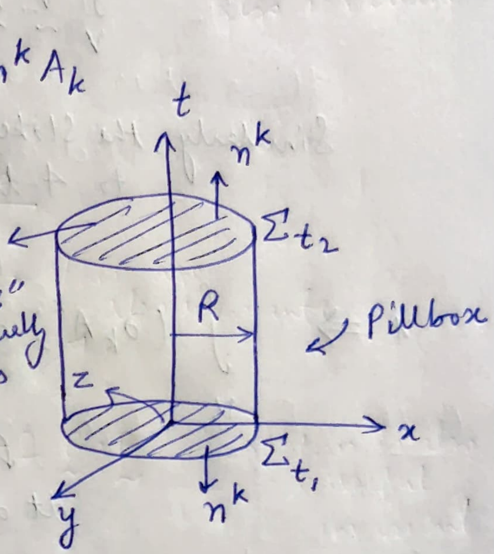
Taking:

$\begin{cases} u = t - r \\ v = t + r \end{cases}$ (where, $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$)

these are null coordinates

42] We have:

$$\int_V (\partial_k A^k)\, d^4x = \int_{\partial V} d^3x\, n^k A_k$$



* these "surfaces" are actually volumes

$$\Rightarrow \int_{\Sigma_{t_1}} d^3x\, n^k A_k + \int_{\Sigma_{t_2}} d^3x\, n^k A_k + \int_{\partial V_3} d^3x\, n^k A_k$$

n^k is upwards for Σ_{t_2}

$$= -\int_{\Sigma_{t_1}} d^3x\, A_0 + \int_{\Sigma_{t_2}} d^3x\, A_0 + \int_{\partial V_3} dt\, dR\, d\theta\, n^k A_k$$

$\because n^k$ unit vector is downwards for Σ_{t_1}

$\therefore n_k = -\partial_k t = (-1, \underset{\sim}{0})$

$\therefore n^k = (1, \underset{\sim}{0})$

Thus, If $\underbrace{\partial_k A^k = 0}_{\text{some conservation law}} \Rightarrow \int d^3x\, A^0$ is conserved

Since we know: $\partial_k J^k = 0 \Rightarrow \int d^3x\, J^0$ is conserved $\{ J^k = (\rho, \underset{\sim}{j}) \}$

$$\therefore \frac{\partial \rho}{\partial t} + \nabla \cdot \underset{\sim}{J} = 0 \Rightarrow \int_{\Sigma_t} d^3x\, \rho = \underline{Q} \text{ is conserved}$$

* (i.e. Charge is conserved)

LECTURE-7

08/08/2023

NOTE: Previously we saw interaction of relativistic charged particle with vector field

$\downarrow$

gave rise to Lorentz force

We saw: $F_{ij} = \partial_i A_j - \partial_j A_i$ $\rightsquigarrow$ Electromagnetic field Tensor

where, $A^k = (\phi, \underset{\sim}{A})$

Also $\Rightarrow$ $m \dfrac{du_k}{d\tau} = q F_{kj} u^j$

NOTE: The action was:

$$\mathcal{A} = -m\int d\tau + q\int A_k \, dx^k$$

there is a reason why we use "lower" index for A_k

$A_k(t, \underset{\sim}{x}) \rightsquigarrow$ called "One form field" *

Here, A_k are called <u>Connections</u> *

{ Remember, Electrodynamics has property of Gauge invariance

$\downarrow$

i.e. A_k can be changed without changing the field F_{ij} }

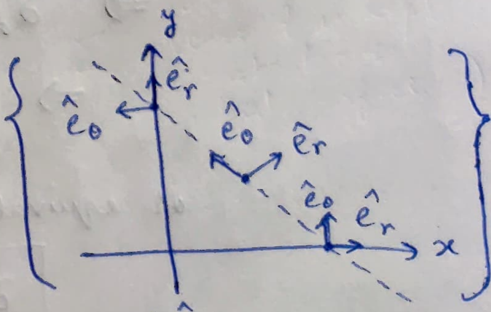
* A_k is a connection

$\downarrow$

since it also us to move from point to point in space $\rightsquigarrow$ without changing the field

(we will see in more detail about connections later in G.R.)

eg:



But $\hat{e}_r$ & $\hat{e}_\theta$ are changing

(at each point on the line $\hat{e}_x, \hat{e}_y$ is same)

43] One invariant we observe for F_{ij} is

*

$\Downarrow$

$$F_{ij} F^{ij}$$

In terms of electric and magnetic fields it can be written as:

$$\boxed{F_{ij} F^{ij} = 2(B^2 - E^2)}$$

this is an important invariant

* Another definition is:

$$^{(*)}F_{ab} = \epsilon_{abcd} F^{cd}$$

this is also invariant

called

"Hodge dual of F_{ab}"

* Similarly, $^{(*)}F_{ab} F^{ab}$

these are also invariants

AND

$$^{(*)}F_{ab}\, ^{(*)}F^{cd} = -2\, \underset{\sim}{E} \cdot \underset{\sim}{B}$$

$\Rightarrow$ this is also a useful invariant

44] We can consider:

$$^{(*)}F^{ab} = \epsilon^{abcd} F_{cd} = \epsilon^{abcd} \overbrace{(\partial_c A_d - \partial_d A_c)}^{\text{this is antisymmetric}}$$

$$= 2\epsilon^{abcd} \partial_c A_d$$

$$\Rightarrow \partial_a\, ^{(*)}F^{ab} = 2\epsilon^{abcd} \partial_a \partial_c A_d = 0$$

$$\Rightarrow \boxed{\partial_a\, ^{(*)}F^{ab} = 0}$$

this is a constraint on EM field tensor

or equivalently $\downarrow$

$$\boxed{\partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} = 0}$$

* this contains 2 maxwell's eq^n's

These eq^{n}s can be obtained just from looking at $\partial_a {}^{(*)}F^{ab} = 0$:

* Source free Maxwell eq^{n}s :

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} \quad ; \quad \nabla \cdot \underset{\sim}{B} = 0$$

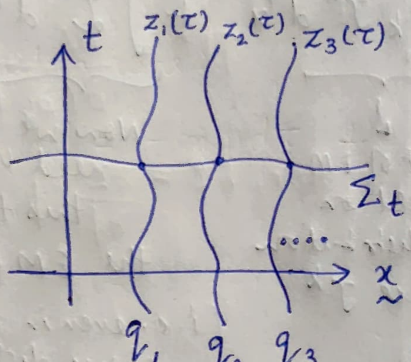
$$\Downarrow$$

from these we can write:

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$$

$$\underset{\sim}{E} = -\nabla \phi - \frac{\partial \underset{\sim}{A}}{\partial t}$$

45] Consider the situation:



(3 charged particles moving in spacetime)

$\Downarrow$

OR in general N such particles

$\Rightarrow$ we can write: $\rho = \sum q_s \, \delta(\underset{\sim}{x} - \underset{\sim}{z_s}(\tau))$ ← ONLY spatial

Thus, we have:

$$\mathcal{A} = \sum_{s=1}^{N} \underbrace{-m_s \int d\tau_s}_{\text{this term is not v. relevant to us right now}} + \underline{\int A_k \frac{dx^k}{d\tau} d\tau \, \rho \, \underbrace{dV}_{\text{spatial volume}}}$$

$$\Downarrow$$

$$+\int A_k u^k d\tau \, \rho \, d^3x$$

$$\Downarrow$$

$$\int A_k u^k \rho \frac{d\tau}{dt} dt \, d^3x$$

$$\Downarrow$$

$$\int A_k \frac{u^k}{u^0} \rho \, d^4x$$

$$\Downarrow$$

$$\int A_k \frac{dx^k}{dt} \rho \, d^4x$$

(since A_k and d^4x are Lorentz invariant)

this should also be L.I.

Here:

$$(J^0, \underset{\sim}{J}) = \left(\rho \frac{dx^0}{dt}, \rho \frac{dx^\alpha}{dt}\right) = (\rho, \rho \underset{\sim}{v})$$

As "4-current"

So we can define: $\Leftarrow$

$$\boxed{J^k \equiv \rho \frac{dx^k}{dt}}$$

NOTE: Here we are only taking a flow of multiple (or a continuum of) point charges

↓

we have NOT considered interaction of one charge with the other charges

* The A_k is due to an "External field" (i.e. created from some other charge distribution somewhere else)

46] Hence we have the Generalization of the action to a continuum (current-density):

$$\mathcal{A} = \dots + \underbrace{\int A_k J^k d^4x}_{\text{effect of EM field on charge distribution of source } J^k} + \underbrace{\int \mathcal{L}\, d^4x}_{\text{Now we just need to find this term for the EM field itself}}$$

(i.e. even in absence of charge distribution, the field itself will have a Lagrangian, & its own dynamics)

* NOTE: In general we should write:

$$\mathcal{A} = \underbrace{\int A_k J^k d^4x}_{\text{for source } J^k} + \underbrace{\int \mathcal{L}\, d^4x}_{\text{for EM field}} - \underbrace{\cancel{m\int d\tau}}_{\text{neglected}} - \cancel{q\int A_k\, dx^k}$$

neglected (it is for the particle itself)

this is the effect of the "test charge" itself, which is small enough and thus is neglected here

* (Also, we have neglected interactions b/w charges)

LECTURE-8

11/08/2023

NOTE: $X^{(ab)}$ $\Rightarrow$ means symmetric in $a \leftrightarrow b$ $\Rightarrow$ i.e. $X^{(ab)} = X^{ba}$

$T^{(ab)c}$ $\Rightarrow$ means symmetric only in $a \leftrightarrow b$

or

$T^{\underline{a}b\underline{c}}$ $\Rightarrow$ symmetric only in $a \leftrightarrow c$

$X^{[ab]}$ $\Rightarrow$ antisymmetric in $a \leftrightarrow b$ $\Rightarrow$ i.e. $X^{[ab]} = -X^{ba}$

NOTE: We have seen:

$$\partial_c \, {}^{(*)}F^{cd} = 0 \underset{\text{gives 2 Maxwell Eq}^{\text{n}}\text{s}}{\Longrightarrow} : \nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} , \quad \nabla \cdot \underset{\sim}{B} = 0$$

NOTE: Also, we saw how to move from single particle to a continuum source:

$$q \int A_k \, dx^k \longrightarrow \int A_k J^k \, d^4x \quad , \quad \text{where: } J^k(t, \underset{\sim}{x})$$

47]

Point particle (Classical)	Classical Field
* t, $\underset{\sim}{x}(t)$, $\dot{x}(t)$ — t: parameter; $\underset{\sim}{x}(t)$: degrees of freedom; $\dot{x}(t)$: velocity	* We have a field $\phi(t, \underset{\sim}{x})$ — ϕ plays the role of: degrees of freedom; $t, \underset{\sim}{x}$: parameter (**NOTE:** this is just a symbol it could be scalar, vector, etc. in reality) and instead of $\dot{x}$ we have: $\partial_a \phi = (\partial_0 \phi, \vec{\nabla}\phi) = (\dot{\phi}, \nabla\phi)$ — "velocity"
* $\mathcal{L}(\underset{\sim}{x}, \dot{\underset{\sim}{x}})$	* Here we have: $\mathcal{L}(\phi, \partial_a \phi)$ — this is actually called the "Lagrange density" (i.e. per unit volume)

* $A[\underset{\sim}{x}(t)\,;1,2] = \int_1^2 \mathcal{L}(x,\dot{x})\,dt$

* $\delta A = \int_1^2 \Big(\underbrace{\frac{dp}{dt}}_{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\underset{\sim}{x}}}\right)} - \frac{\partial \mathcal{L}}{\partial \underset{\sim}{x}} \Big) \delta \underset{\sim}{x}(t)\,dt - (p\,\delta x)\Big|_1^2$

{ i.e. conjugate momentum is given by: $\underset{\sim}{p} = \frac{\partial \mathcal{L}}{\partial \dot{\underset{\sim}{x}}}$ }

{ Here, $\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q}\delta q + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}} \underbrace{\delta \dot{q}}_{\frac{d}{dt}(\delta q)}}_{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\delta q - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right)}$ }

* By fixing end pts.; & $\delta A = 0$:

$$\frac{dp}{dt} = \frac{\partial \mathcal{L}}{\partial \underset{\sim}{x}}$$

$(\delta \underset{\sim}{x}(t_1) = \delta \underset{\sim}{x}(t_2) = 0)$

48] -

* Here we get, the Hamiltonian as:

$$\mathcal{H}(\underset{\sim}{p},\underset{\sim}{x}) = \underset{\sim}{p}\,\dot{\underset{\sim}{x}} - \mathcal{L}$$

* $A = \int d^4x\, \mathcal{L}(\phi, \partial_a \phi) = \int dt \underbrace{\int d^3x\, \mathcal{L}(\phi,\partial_a\phi)}_{L} = \int dt\, L$

* $\delta A = \int_1^2 \left(\partial_k \Pi^k - \frac{\partial \mathcal{L}}{\partial \phi}\right)\delta\phi\, d^4x - \int d^4x\, \partial_a(\Pi^a \delta\phi)$

{ i.e. canonical momentum is given by: $\Pi^k = \frac{\partial \mathcal{L}}{\partial(\partial_k \phi)}$ }

{ Here, $\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_a\phi)}}_{\Downarrow}\delta(\partial_a\phi)$

and this part is further manipulated }

Here, using the same pillbox method:

$$\int_V d^4x\, \partial_a(\Pi^a \delta\phi) = \int_{\partial V} d^3x\, n_a \Pi^a \delta\phi$$

* Fix end pts. & $\delta A = 0$:

$$\boxed{\partial_k \Pi^k = \frac{\partial \mathcal{L}}{\partial \phi}}$$

$(\delta\phi(t_1,\underset{\sim}{x}_1) = \delta\phi(\underset{\sim}{x}_2,t_2) = 0)$

* Here we have a 2-index object:

$$\boxed{T^k_{\ l} = \Pi^k \partial_l \phi - \delta^k_l \mathcal{L}}$$

This object is called the Energy-Momentum Tensor

kronecker delta

{ NOTE: Here, time translation symmetry gives Energy conservation, Space translation symmetry gives Momentum conservation }

* Here we saw that:

$$\partial_a J^a = 0$$
$$\Downarrow$$
$$\int J^0 d^3x = Q$$
is conserved

In Newtonian mechanics we build:

49] $\mathcal{L}(\underset{\sim}{x}, \underset{\sim}{\dot{x}}) = \underbrace{\frac{1}{2} m \dot{x}^2}_{\text{Kinetic}} - \underbrace{V}_{\text{potential}}$

* How to build the Lagrangian?

↓

Here, we firstly follow Relativistic Invariance

$$\therefore A = -m \int d\tau = -m \int \sqrt{-dx_a dx^a}$$

In particular:

$$T^0_0 = \Pi^0 \dot{\phi} - \mathcal{L} = -E$$

{ NOTE: Here, the symmetries of the field gives the conservation of Energy, Momentum, etc. }

The energy-momentum tensor can be written as:

$$\begin{pmatrix} E & p^\alpha \\ p^\alpha & \underbrace{\sigma^{\alpha\beta}}_{\text{stress tensor}} \end{pmatrix}$$

* Similarly here:

$$\partial_a T^a_b = 0$$

(since:

$$\partial_k T^k_l = (\partial_k \Pi^k) \partial_l \phi + \Pi^k \partial_k \partial_l \phi - \delta^k_l \frac{\partial \mathcal{L}}{\partial \phi} \partial_k \phi = 0 \;)$$

{ NOTE: In general:

$$\partial_k \Pi^k = \frac{\partial \mathcal{L}}{\partial \phi} \iff \partial_a T^a_b = 0 \; \}$$

(They imply each other)

Hence:

$$\partial_a T^a_b = 0 \underset{\text{implies}}{\Rightarrow} \int n_0 T^0_b d^3x \text{ is conserved}$$

$$\{ \text{as: } \int_V (\partial_a T^a_b) d^4x = \int_{\partial V} n_a T^a_b d^3x \}$$

and since $n_0 = 1$

$$\Rightarrow \boxed{\int T^0_b d^3x \text{ is conserved}}$$

* To build the Lagrangian:

$$\mathcal{L}(\phi, \partial_a \phi) = \underbrace{-\partial_a \phi \partial^a \phi}_{\text{Kinetic term}} - \underbrace{V(\phi)}_{\text{Potential term}}$$
$$= -\eta^{ab} \partial_a \phi \partial_b \phi - V(\phi)$$

which gives us:

$$-\eta^{ab}\partial_a\phi\,\partial_b\phi = \underbrace{\dot{\phi}^2}_{\text{here we have a plus sign}} - |\nabla\phi|^2$$

here we have a plus sign (and that is why we used a $-$ve sign for: $-\partial_a\phi\,\partial^a\phi$)

Hence the action will be:

$$\mathcal{A} = \int d^4x \left(-\partial_a\phi\,\partial^a\phi - V(\phi)\right)$$

(NOTE: The word "canonical" means standard

eg: canonical K.E. would be of the form $\frac{1}{2}m\dot{x}^2$

But if we use K.E. which is some other $f(\dot{x})$

this will be called non-canonical K.E.)

{NOTE: The lagrangian:

$$\mathcal{L} = \underbrace{-\partial_a\phi\,\partial^a\phi - V(\phi)}$$

this is called a Scalar field * (since ϕ is scalar)}

$\Downarrow$

Taking $V(\phi) = \frac{1}{2}k\phi^2$

(i.e., $\frac{\partial^2 V}{\partial\phi^2} = k = m^2$ (say))

we get:

$$\mathcal{L}(\phi;\partial_a\phi) = -\frac{1}{2}\partial_a\phi\,\partial^a\phi - \frac{1}{2}m^2\phi^2$$

this is relativistically invariant

(NOTE: Quantization of particles is done by:

$$\{q,p\} = 1 \rightarrow [q,p] = i$$)

(NOTE: Quantization of classical field is done by:

$$\{\phi,\pi\} \longrightarrow [\phi(t,\underset{\sim}{x}),\pi(t,\underset{\sim}{y})] = i\,\delta^3(\underset{\sim}{x}-\underset{\sim}{y})$$)

{NOTE: Here we have considered the simple case of $V(\phi) = \frac{1}{2}m^2\phi^2$

In general we could also have:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \underbrace{g\phi^3 + (\ldots)\phi^4 + \ldots}$$

these occur due to interaction of the field with itself

Here, the constants $m^2, g, \ldots$ etc. are called Coupling constants

(the reason why m^2 is used (m representing some 'mass') even though this is a field

is due to the fact that when this field is quantized, then this coupling const. can be shown to be representing the mass^2 of the quanta)

However, for EM fields, the other interaction terms turn out to be impossible since they violate gauge invariance

$\therefore V(\phi) = \frac{1}{2} m^2 \phi^2$ is taken }

NOTE: Complex scalar field Lagrangians are also possible:

$$\mathcal{L}(\phi^*, \partial_a \phi^*) = -\frac{1}{2} \partial_a \phi^* \partial^a \phi - \frac{1}{2} m^2 \phi^* \phi$$

(this is used for representing spin-0 particles and charged antiparticles)