POST-MINOR-2

LECTURE 16 (27/03/2023)

69] Fourier Transforms :

Fourier cosine transform

For even function $f(x) \rightarrow$ Fourier transform is:

$$f(x) = \int_0^\infty A(\omega)\cos(\omega x)\,d\omega \quad -①$$

$$\left\{ A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos(\omega v)\,dv \right\} \quad -②$$

Let $A(\omega) = \sqrt{\frac{2}{\pi}}\,\underbrace{\hat{f_c}(\omega)}_{\text{"Fourier transform"}}$ → cosine transform

$v = x$

Substitute $A(\omega)$ in ①:

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f_c}(\omega)\cos(\omega x)\,d\omega}$$

INVERSE FOURIER TRANSFORM

$A(\omega)$ in ②:

$$\Rightarrow \boxed{\hat{f_c}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\omega x)\,dx}$$

FOURIER TRANSFORM OF $f(x)$

* Note the diff. b/w ω & x

Notation : We write:

$F(f) \rightarrow$ Fourier transform

$F^{-1}(f) \rightarrow$ Inverse fourier transform

If $f(x)$ is odd : $F(f)$ or $\hat{f_s}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\omega x)\,dx$

$F^{-1}(f)$ or $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f_s}(\omega)\sin(\omega x)\,d\omega$

70] Linearity

If f & g have Fourier transforms, so does $\underbrace{af+bg}$

i.e. this will also have Fourier transforms eg: $F_c(af+bg) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} (af+bg) \cos(wx)\, dx$

$$\boxed{F_c(af+bg) = aF_c(f) + bF_c(g)}$$

71] Derivatives

$$F_c(f'(x)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f'(x) \cos(wx)\, dx$$

Using Integration by Parts →

$$= \sqrt{\frac{2}{\pi}} \left[f(x)\cos(wx)\Big|_0^{\infty} + w\int_0^{\infty} f(x)\sin(wx)\, dx \right]$$

* [Here we are assuming: $f(x) \to 0$ when $x \to \infty$]

$$= 0 + w F_s(f(x))$$

$$\therefore \boxed{F_c(f'(x)) = w\, F_s(f(x))}$$

and, $\boxed{F_s(f'(x)) = -w F_c(f(x))}$

72] Complex Form of Fourier Integral

$$f(x) = \int_{-\infty}^{\infty} (A(w)\cos wx + B(w)\sin wx)\, dw$$

here, $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(wv)\, dv$

$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(wv)\, dv$

Thus, we get: $f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \underbrace{[\cos(wv)\cos(wx) + \sin(wv)\sin(wx)]}_{\text{Cos}(A-B) \text{ form}} dv \right] dw$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v)\cos(w(v-x))\, dv \right] dw$$

$$= \frac{2}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v) \cos\underbrace{(wv - wx)}_{\text{even func.}} dv\right] dw$$

Euler formula: $e^{ix} = \cos x + i \sin x$ $\quad x \to (wv - wx)$

multiply by $f(v)$

$$f(v)\cos(wx - wv) + i f(v) \sin(wx - wv)$$

$$= f(v)\, e^{i(wx - wv)}$$

Complex Fourier int.: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\, e^{iw(x-v)}\, dv\, dw$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\, e^{-iwv}\, dv\right] e^{iwx}\, dw$$

$$\boxed{\hat{f} = F(f) \Rightarrow \hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx}$$

$$\boxed{f = F^{-1}(f) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw}$$

eg: Find F.T. $f(x) = e^{-ax}$ if $x > 0$
$\quad = 0$ if $x < 0$

$$F(f) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} 0 \cdot e^{-iwx}\, dx + \int_0^\infty e^{-ax} e^{-iwx}\, dx\right]$$

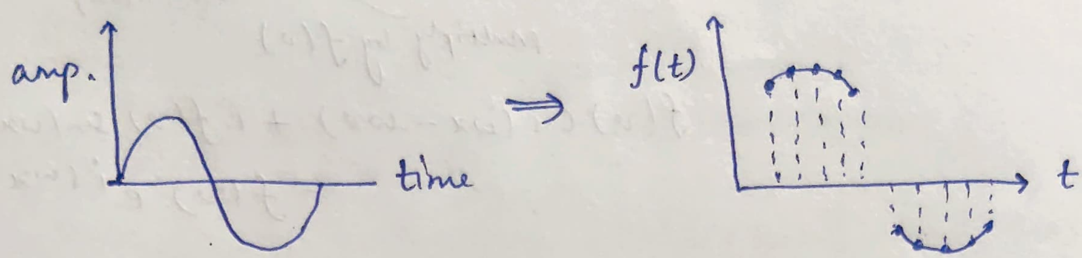
$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x(a+iw)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-e^{-x(a+iw)}}{a+iw}\right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}\,(a+iw)}$$

73] <u>Discreet Fourier Transforms</u>. (or Fast Fourier Transforms)

* { NOTE: Intensity of spectra $\propto \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$ (ω : frequency) }



We must now go back to summations.

$f(x) \rightarrow$ equally spaced nodes x_k

Measure k^{th} frequency for n^{th} samples ⇒ $n \rightarrow N$

$$x_k = \frac{2\pi k}{N}$$

n → samples
k → freq. is the k^{th} one

$$q(x) = \sum_{n=0}^{N-1} c_n e^{inx_k}$$

$$q(x_k) = f(x_k) \qquad (k = 0, 1, \ldots, N-1$$
$$n = 0, 1, \ldots, N-1)$$

* (Refer to the complete derivation from the textbooks)

We get : $c_n = \frac{1}{N} \sum_{k=0}^{n-1} f_k e^{-inx_k}$, $f_k = f(x_k)$

$$\therefore \boxed{\hat{f}_n = N c_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}}$$

→ F

→ $\vec{f}$

$F_N \rightarrow N \times N$ matrix

each sample will have a fourier transform ⇒ we can thus get a matrix of fourier transforms

$$\hat{f} = F_N \vec{f}$$

$$\hookrightarrow \hat{f} = [\hat{f}_0 \; \hat{f}_1 \cdots \hat{f}_{N-1}]$$

$$\vec{f} = [f_1 \; f_2 \cdots f_{N-1}]^T \qquad x_k = \frac{2\pi k}{N}$$

Elements of $F_N \Rightarrow [e_{nk}] = e^{-inx_k} = e^{-i2\pi nk/N}$

$F_N [e_{nk}] = w^{nk}$ (where, $w = e^{-i2\pi/N}$)

eg: $N=4$, $\vec{f} = [0\ 1\ 4\ 9]^T$

$$\hat{f} = F_N \vec{f} = F_4 \vec{f}$$

$$w = e^{-2\pi/4} = e^{-i\pi/2} = -i$$

$$\therefore F_4 = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix}$$

$$\hat{f} = F_4 \vec{f}$$

$$\Downarrow$$

$$\hat{f} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}$$

<u>NOTE:</u> $N^2 \rightarrow$ DFT

$$\underbrace{N \log_2 N \rightarrow \text{FFT}}$$

$\Downarrow$

Thus this helps makes computations much faster

M block for N samples

$M \times N$

<u>LECTURE 17</u> (03/04/2023)

{ Given in Lecture Slides }

↳ Discussion on Symmetries

* [Successive <u>Lectures are also in the form</u> of Slides]