This gives us the final ultimate formula:

$$\frac{\partial}{\partial t}\rho(\hat{U}+\tfrac{1}{2}v^2) = -\frac{\partial}{\partial x}\rho v_x(\hat{U}+\tfrac{1}{2}v^2) - \frac{\partial}{\partial y}v_y\rho(\hat{U}+\tfrac{1}{2}v^2)$$
$$-\frac{\partial}{\partial z}\rho v_z(\hat{U}+\tfrac{1}{2}v^2)$$
$$-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)$$
$$+\rho(v_x g_x + v_y g_y + v_z g_z)$$
$$-\left(\frac{\partial}{\partial x}P v_x + \frac{\partial}{\partial y}P v_y + \frac{\partial}{\partial z}P v_z\right)$$
$$-\left[\frac{\partial}{\partial x}(\tau_{xx}v_x + \tau_{xy}v_y + \tau_{xz}v_z) + \frac{\partial}{\partial y}(\tau_{yx}v_x + \tau_{yy}v_y + \tau_{yz}v_z) + \frac{\partial}{\partial z}(\tau_{zx}v_x + \tau_{zy}v_y + \tau_{zz}v_z)\right]$$
$$+S_c$$

In vector-tensor form this simplifies to:

$$\boxed{\frac{\partial}{\partial t}\rho(\hat{U}+\tfrac{1}{2}v^2) = -\underline{\nabla}\cdot\rho\underline{v}(\hat{U}+\tfrac{1}{2}v^2) - \underline{\nabla}\cdot\underline{q} + \rho\underline{v}\cdot\underline{g} - \underline{\nabla}\cdot P\underline{v} - \underline{\nabla}\cdot(\underline{\tau}\cdot\underline{v}) + S_c}$$

EQUATION OF ENERGY

LECTURE-20

81) Applying the identity: $\underline{\nabla}\cdot s\underline{w} = s\,\underline{\nabla}\cdot\underline{w} + \underline{w}\cdot\underline{\nabla}s$

$\downarrow$

we can transform:

$$\underline{\nabla}\cdot\underbrace{\rho\underline{v}}_{\underline{w}}\underbrace{(\hat{U}+\tfrac{1}{2}v^2)}_{s}$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}\rho\left(\hat{U}+\frac{1}{2}v^2\right)}_{\left(\hat{U}+\frac{1}{2}v^2\right)\frac{\partial \rho}{\partial t}+\rho\frac{\partial\left(\hat{U}+\frac{1}{2}v^2\right)}{\partial t}} + \left(\hat{U}+\frac{1}{2}v^2\right)\underline{\nabla}\cdot\rho\underline{v} + \rho\underline{v}\cdot\underline{\nabla}\left(\hat{U}+\frac{1}{2}v^2\right) = -\underline{\nabla}\cdot\underline{q} + \dots$$

$$\Rightarrow \left(\hat{U}+\frac{1}{2}v^2\right)\underbrace{\left[\frac{\partial\rho}{\partial t}+\underline{\nabla}\cdot\rho\underline{v}\right]}_{\text{eq.}^n\text{ of continuity}} + \rho\left(\frac{\partial\left(\hat{U}+\frac{1}{2}v^2\right)}{\partial t}+\underline{v}\cdot\underline{\nabla}\left(\hat{U}+\frac{1}{2}v^2\right)\right) = -\underline{\nabla}\cdot q + \dots$$

∴ this is Zero

$$\Rightarrow \boxed{\rho\frac{D\left(\hat{U}+\frac{1}{2}v^2\right)}{Dt} = -\underline{\nabla}\cdot q - \underline{\nabla}\cdot\left(\underset{\sim}{\tau}\cdot\underline{v}\right) + \rho\underline{v}\cdot\underline{g} - \underline{\nabla}\cdot\left(P\underline{v}\right) + S_c}$$

This equation by itself however has almost no use for us

82] Derivation of equation of Mechanical Energy

★ NOTE: This is NOT obtained from conservation of energy → it is obtained from Newton's IInd Law

We know:

$$\underline{F} = m\underline{a} = m\frac{d\underline{v}}{dt}$$

$$\Rightarrow \underline{v}\cdot\underline{F} = m\underline{v}\cdot\frac{d\underline{v}}{dt}$$

We also know that:

$$\frac{d}{dt}(\underline{v}\cdot\underline{v}) = \underline{v}\cdot\frac{d\underline{v}}{dt} + \frac{d\underline{v}}{dt}\cdot\underline{v}$$

$$= 2\underline{v}\cdot\frac{d\underline{v}}{dt}$$

$$= \frac{1}{2}\frac{d}{dt}(v^2)$$

$$= \frac{d}{dt}\left(\frac{1}{2}v^2\right)$$

Considering the situation:

$\downarrow g_z$, i.e. F_x and F_y are zero

(i.e. $\underline{g} = -g_z \hat{z}$) $\Rightarrow$ $\underline{F} = -m g_z \hat{z}$

Thus, we have:

$$-m v_z g_z = m \frac{d}{dt}\left(\frac{1}{2}v^2\right)$$

We know that the eqn. of motion for fluid is:

$$\rho \frac{D\underline{v}}{Dt} = -\underline{\nabla} P + \rho \underline{g} - \underline{\nabla} \cdot \underset{\sim}{\tau}$$

Taking $\underline{v}\cdot$ $\Rightarrow$ $$\rho \underline{v} \cdot \frac{D\underline{v}}{Dt} = -\underline{v} \cdot (\underline{\nabla} P) + \underline{v} \cdot \rho \underline{g} - \underline{v} \cdot \underline{\nabla} \underset{\sim}{\tau}$$

$$\Rightarrow \rho \frac{D\left(\frac{1}{2}v^2\right)}{Dt} = -\underline{v} \cdot \underline{\nabla} P + \underline{v} \cdot \rho \underline{g} - \underline{v} \cdot \underline{\nabla} \underset{\sim}{\tau}$$

For the case of $\rho = \text{const}$, No shear forces:

$$\rho \frac{D\left(\frac{1}{2}v^2\right)}{Dt} = -\underline{v} \cdot \underline{\nabla} P + \underline{v} \cdot \rho \underline{g}$$

We can define gravity using a potential: $\underline{g} = -\underline{\nabla} \phi$

$$\Rightarrow \rho \frac{D\left(\frac{1}{2}v^2\right)}{Dt} = -\underline{v} \cdot \underline{\nabla} P + \underline{v} \cdot \rho (-\underline{\nabla} \phi)$$

$$= -\underline{v} \cdot \underline{\nabla} P - \rho \underline{v} \cdot \underline{\nabla} \phi$$

Since both P and ϕ are not functions of time, so we can add those terms to create a substantial derivative:

$$\rho \frac{D\left(\frac{1}{2}v^2\right)}{Dt} = -\underline{v} \cdot \underline{\nabla} P - \rho \underline{v} \cdot \underline{\nabla} \phi - \frac{\partial P}{\partial t} - \rho \frac{\partial \phi}{\partial t}$$

$$\Rightarrow \rho \frac{D\left(\frac{1}{2}v^2\right)}{Dt} = -\frac{DP}{Dt} - \rho \frac{D\phi}{Dt}$$

$$\Rightarrow \frac{D}{Dt}\left(\frac{1}{2}\rho v^2 + P + \phi\right) = 0$$

$$\Rightarrow \boxed{\frac{1}{2}\rho v^2 + P + \phi = \text{const.}}$$

which is
BERNOULLI'S
EQUATION

(applicable only for const. ρ and no shear forces and no temp. change)

In general, the eq^n of mechanical energy (is for No shear and const. ρ):

$$\boxed{\rho \frac{D(\frac{1}{2}v^2)}{Dt} = -\underline{v}\cdot\underline{\nabla}P - \rho\underline{v}\cdot\underline{\nabla}\phi}$$

* this is actually in general! $+\rho\underline{v}\cdot\underline{g}$

EQUATION OF MECHANICAL ENERGY

Now, the general eq^n of mechanical energy (that we got by taking $\underline{v}\cdot$) will also have the shear term.

Here, we know, $\underline{v}\cdot(\underline{\nabla}\cdot\underset{\sim}{\tau}) = \underline{\nabla}\cdot(\underset{\sim}{\tau}\cdot\underline{v}) - \underset{\sim}{\tau}:\underline{\nabla}\underline{v}$

Thus: $$\rho\frac{D(\frac{1}{2}v^2)}{Dt} = -\underline{v}\cdot\underline{\nabla}P \underbrace{-\rho\underline{v}\cdot\underline{\nabla}\phi}_{\text{or just } +\rho\underline{v}\cdot\underline{g}} - \underline{\nabla}\cdot(\underset{\sim}{\tau}\cdot\underline{v}) + \underset{\sim}{\tau}:\underline{\nabla}\underline{v}$$

Again we know:
$$\underline{\nabla}\cdot(P\underline{v}) = P\,\underline{\nabla}\cdot\underline{v} + \underline{v}\cdot\underline{\nabla}P$$

$$\Rightarrow \rho\frac{D(\frac{1}{2}v^2)}{Dt} = P\,\underline{\nabla}\cdot\underline{v} - \underline{\nabla}\cdot(P\underline{v}) + \rho\underline{v}\cdot\underline{g} - \underline{\nabla}\cdot(\underset{\sim}{\tau}\cdot\underline{v}) + \underset{\sim}{\tau}:\underline{\nabla}\underline{v}$$

Subtracting eq^n of mechanical energy from eq^n of total energy:

$$\Rightarrow \boxed{\rho\frac{D\hat{U}}{Dt} = -\underline{\nabla}\cdot\underline{q} - P\,\underline{\nabla}\cdot\underline{v} - \underset{\sim}{\tau}:\underline{\nabla}\underline{v} + S_e}$$

EQUATION OF THERMAL ENERGY

Now, the eq^n of mechanical energy is:

$$\rho \frac{D(\frac{1}{2}v^2)}{Dt} = P \underline{\nabla} \cdot \underline{v} - \underline{\nabla} \cdot (P\underline{v}) + \rho \underline{v} \cdot \underline{g} - \underline{\nabla} \cdot (\underset{\sim}{\tau} \cdot \underline{v})$$
$$+ \underbrace{\underset{\sim}{\tau} : \underline{\nabla}\, \underline{v}}_{\text{this can be written as}}$$

$$-(-\underset{\sim}{\tau} : \underline{\nabla}\, \underline{v})$$

The term is called

* <u>Viscous dissipation term</u>

$\Rightarrow$ i.e. $-\underset{\sim}{\tau} : \underline{\nabla}\, \underline{v}$

$\downarrow$

this quantity is ALWAYS positive *

$\downarrow$

for Newtonian fluid it takes the form:

$$\boxed{-\underset{\sim}{\tau} : \underline{\nabla}\, \underline{v} = \rho \phi_v}$$

called the dissipation function for Newtonian fluids

Hence this means that some part of the mechanical energy is always going down (due to $-(-\underset{\sim}{\tau} : \underline{\nabla}\, \underline{v})$)

$\downarrow$

and gets converting to thermal energy

83) <u>Temperature explicit form of eq^n of thermal energy</u>

The eq^n of thermal energy is:

$$\rho \frac{D\hat{U}}{Dt} = -\underline{\nabla} \cdot \underline{q} - P \underline{\nabla} \cdot \underline{v} + (-\underset{\sim}{\tau} : \underline{\nabla}\, \underline{v}) + S_c$$

Temp. explicit form of eq^n can be obtained as follows:

We can write from thermodynamics:

$$\hat{U} = \hat{U}(T, \underbrace{\hat{V}}_{\text{volume}})$$

$$d\hat{U} = \left(\frac{\partial \hat{U}}{\partial T}\right)_{\hat{V}} dT + \left(\frac{\partial \hat{U}}{\partial \hat{V}}\right)_T d\hat{V}$$

$$= C_v\, dT + \underbrace{\left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right]}_{\text{this term is Zero ONLY for Ideal gases}} d\hat{V}$$

*

{ This is because:

$$P\hat{V} = RT$$

$$\Rightarrow \left(\frac{\partial P}{\partial T}\right)_{\hat{V}} = \frac{R}{\hat{V}}$$

$$\text{Thus} \Rightarrow \left[-P + \frac{RT}{\hat{V}}\right] = -P + P = 0 \;\}$$

Hence, we can write:

$$\frac{D\hat{U}}{Dt} = C_v \frac{DT}{Dt} + \left[-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right]\frac{D\hat{V}}{Dt}$$

We know that also: $\hat{V} = \frac{1}{\rho}$ and $\frac{D\rho}{Dt} + \rho\,\underline{\nabla}\cdot\underline{v} = 0$

$$\Downarrow$$

$$\text{thus, } \frac{D\hat{V}}{Dt} = -\frac{1}{\rho^2}\frac{D\rho}{Dt} = -\frac{1}{\rho^2}\left(-\rho\,\underline{\nabla}\cdot\underline{v}\right) = \frac{\underline{\nabla}\cdot\underline{v}}{\rho}$$

So finally the eq.^{n} of thermal energy is explicit form is obtained using:

$$\underbrace{\rho\left(C_v\frac{DT}{Dt} + \left(-P + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right)\frac{\underline{\nabla}\cdot\underline{v}}{\rho}\right)}_{\rho C_v \frac{DT}{Dt} - P\,\underline{\nabla}\cdot\underline{v} + T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\underline{\nabla}\cdot\underline{v}} = -\underline{\nabla}\cdot\underline{q} - P\,\underline{\nabla}\cdot\underline{v} + \left(-\underset{\sim}{\tau} : \underline{\nabla}\underline{v}\right) + S_c$$

Hence, we get:

$$\boxed{\rho C_v \frac{DT}{Dt} = -\underline{\nabla}\cdot\underline{q} - T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\underline{\nabla}\cdot\underline{v} + \left(-\underset{\sim}{\tau} : \underline{\nabla}\underline{v}\right) + S_c}$$

EQUATION OF THERMAL ENERGY IN EXPLICIT FORM

* { can be seen from Appendix }

84] Special cases :

(i) For ρ constant :

$$\rho C_p \frac{DT}{Dt} = -\underline{\nabla} \cdot \underline{q} + \underbrace{(-\underset{\sim}{\tau} : \underline{\nabla}\, \underline{v})}_{} + S_c$$

in general can be written as

$\mu \phi_v$

↓

but if nothing mentioned in question

↓

it can be taken as Zero

($\because$ it is practically very small)

{ and similarly as we did in fluid mechanics, here also we can take $-\underline{\nabla} \cdot q$ as $k \nabla^2 T$ }

↓

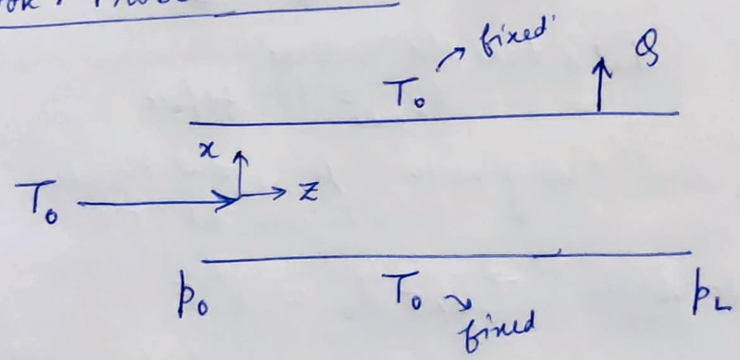
if k is also a constant

LECTURE-21

(ii) Thus, if both ρ and k are constant :

$$\rho C_p \frac{DT}{Dt} = k \nabla^2 T + (-\underset{\sim}{\tau} : \underline{\nabla}\, \underline{v}) + S_c$$

85] Book : Problem 10H-2



(it appears Temp. is same everywhere so it can't change ↓ but it will still change, due to heating up by * viscous dissipation)

Assumptions :

Steady state

ρ, μ, k are constant

$v_x = v_y = 0$, $v_z = v_z(x)$

From fluid solution : $v_z = \dfrac{p_0 - p_L}{4 \mu L} \left(1 - \left(\dfrac{x}{\beta}\right)^2\right) = v_{max}\left(1 - \left(\dfrac{x}{\beta}\right)^2\right)$

Also, $T(x,z) \Rightarrow T(x)$ only

Now, $(-\underset{\sim}{\tau} : \underline{\nabla}\, \underline{v}) = \underset{\substack{\downarrow \\ \text{for Newtonian} \\ \text{case}}}{\mu \phi_v}$

Hence, we can write:

$$\rho C_p \frac{DT}{Dt} = k \nabla^2 T + \mu \phi_v + \overset{0 \text{ (in this problem)}}{S_c}$$

Thus, we have!

$$\rho C_p \left(\frac{\partial T}{\partial t} + \underbrace{\underline{v} \cdot \underline{\nabla} T}_{0} \right) = k \nabla^2 T + \overbrace{\mu \left(\frac{\partial v_z}{\partial x} \right)^2}^{\phi_v \text{ obtained from Appendix}}$$

Using boundary conditions:

$x = B,\ T = T_0$

and $x = -B,\ T = T_0$

We get the solution: $T = T_0 + \frac{\mu}{k} \frac{v_{z_{max}}^2}{3} \left(1 - \left(\frac{x}{B} \right)^4 \right)$

Thus, $T_{max} - T_0 = \frac{\mu}{k} \frac{v_{z_{max}}^2}{3}$

Let's take the values:

$v_{z_{max}} = 100$ ft/sec

using water, μ_{water} and k_{water} taken at 25°C

We obtain, $T_{max} - T_0 \approx 1°F$ ↳ very small

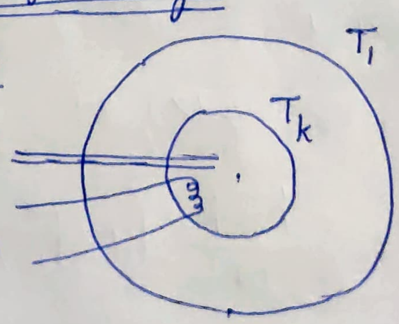
86] Transpiration cooling / heating

Problem 10.5-3

W_r air $\xrightarrow{T_k}$

(mass flow rate)

Q_0



Assumptions

Steady state

k, ρ, C_p constant

$v_\theta = v_\phi = 0$

$v_r = v_r(r)$

From eq^n of continuity: $\frac{1}{r^2}\frac{d}{dr}(r^2 v_r) = 0$

$$\Rightarrow r^2 v_r = \text{const.}$$

$$\Rightarrow \underbrace{4\pi r^2 \rho v_r}_{w_r} = \text{const}$$

$$\Rightarrow v_r = \frac{w_r}{4\pi\rho r^2}$$

Now, viscous dissipation is negligible

and, $T(r,\theta,\phi) \Rightarrow T(r)$

Thus, $$\rho C_p\left(\cancelto{0}{\frac{\partial T}{\partial t}} + \underline{v}\cdot\nabla T\right) = k\nabla^2 T + \mu\phi_v$$

$$\rho C_p\left(v_r\frac{dT}{dr}\right) = k\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)\right]$$

From boundary conditions:

$$\left.\begin{aligned} r &= \lambda R, \quad T = T_k \\ r &= R, \quad T = T_1 \end{aligned}\right\} \text{will be used later}$$

$$\Rightarrow r^2\frac{\rho C_p}{k} v_r \frac{dT}{dr} = \frac{d}{dr}\left(r^2\frac{dT}{dr}\right)$$

replace with; $\frac{w_r}{4\pi\rho r^2}$

$$\Rightarrow \underbrace{\frac{w_r C_p}{4\pi k}}_{\text{let's call this } R_0}\frac{dT}{dr} = \frac{d}{dr}\left(r^2\frac{dT}{dr}\right)$$

$$\rightarrow R_0\frac{dT}{dr} = r^2\frac{d^2T}{dr^2} + 2r\frac{dT}{dr}$$

Taking $\frac{dT}{dr} = Y$

$$\Rightarrow R_o Y = r^2 \frac{dY}{dr} + 2rY$$

$$\Rightarrow r^2 \frac{dY}{dr} = (R_o - 2r) Y$$

$$\Rightarrow \frac{dY}{Y} = \frac{(R_o - 2r)}{r^2} dr$$

$$\Rightarrow \log Y = -\frac{R_o}{r} - 2 \log r + C_2$$

$$\Rightarrow \log(r^2 Y) = \frac{-R_o}{r} + C_2$$

$$\Rightarrow r^2 Y = C_3 e^{-R_o/r}$$

$$\Rightarrow r^2 \frac{dT}{dr} = C_3 e^{-R_o/r}$$

$$\Rightarrow \frac{dT}{dr} = C_3 \frac{e^{-R_o/r}}{r^2}$$

We can integrate this to find T:

$$T = (C_3 \qquad) + C_4$$

From boundary cond^{n}s:

$$\frac{T - T_1}{T_k - T_1} = \frac{e^{-R_o/r} - e^{+R_o/R}}{e^{-R_o/\lambda R} - e^{R_o/R}}$$ $\Big\}\rightarrow$ from here $\frac{dT}{dr}$ can be computed

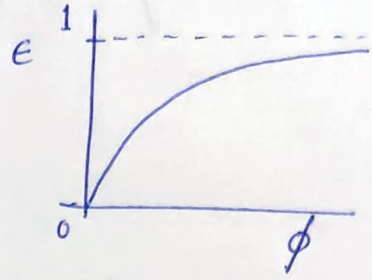
So we get, $Q = 4\pi \lambda^2 R^2 \left(-k \frac{dT}{dr}\right)\Big|_{r=\lambda R}$

$$= \frac{4\pi k R_o (T_1 - T_k)}{e^{(R_o/\lambda R (1-\lambda))} - 1}$$

Now, $R_o = \frac{W_r C_p}{4\pi k}$, so for the case $W_r = 0 \Rightarrow Q_o = \frac{4\pi k \lambda R (T_1 - T_k)}{1-\lambda}$

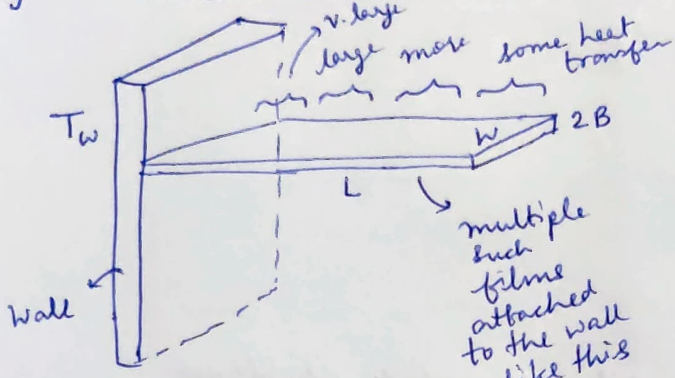
∴ The efficiency can be obtained as:

$$\epsilon = \frac{Q_0 - Q}{Q_0}$$



$\left(\text{where, } \phi = \frac{R_0(1-\lambda)}{\lambda R}\right)$

87) <u>Heat conduction in a thin film</u>



T_a

$\Rightarrow$ ∴ heat transfer $\propto T - T_a$

↓

thus, longer length is preferable

(but length must be limited by practicality)

$(L \;\&\; W \gg 2B)$

Due to large area

↓

there will be large heat transfer

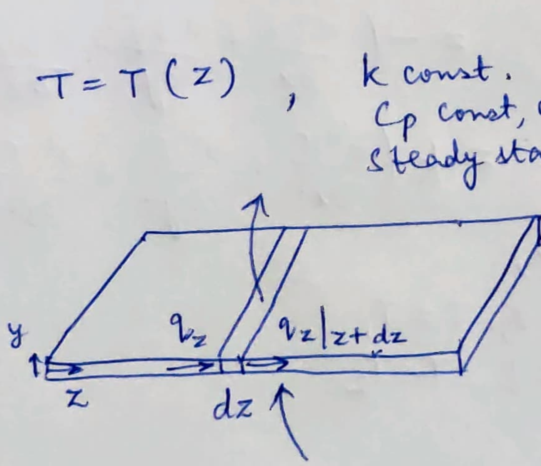
Efficiency, η = $\dfrac{\text{heat which actually dissipates by addition of film}}{\text{Heat dissipated if the whole film was at temp. } T_w}$

(effectiveness of film)

↓

$(\eta > 1)$

$T = T(z)$, k const.
Cp const, etc...
Steady state



$$0 = W(2B)\, q_z\big|_z - W(2B)\, q_z\big|_{z+dz} + h(T-T_a)\,dz\, W - (-h(T-T_a))\,dz\, W$$

$$\Rightarrow \frac{d}{dz}\left(-k\frac{dT}{dz}\right) = \frac{dq_z}{dz} = \frac{-h\,(T-T_a)}{B}$$

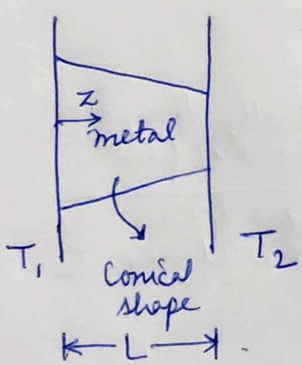
Boundary condⁿs : $z = 0,\ T = T_w$

$z = L,\ \dfrac{dT}{dz} = 0,\quad q_z\Big|_{z=L} = h\,(T - T_a)\Big|_{z=L}$

LECTURE-22

88] Reducing dimensions to solve complex problems in Transport Phenomena

eg:



$T(x, y, z, t) \Rightarrow T(x, y, z)$

Steady state

k is constant

∴ Here :

$$0 = -\underline{\nabla} \cdot \underline{q}$$

$$= -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)$$

$$= k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

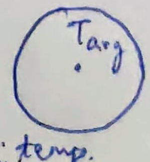
Boundary condⁿs :

$z = 0,\ T = T_1$

$z = L,\ T = T_2$

* But for approximation :

We look at cross-section and assign an avg. temp. for it



$\Rightarrow$ thus T reduces to $T_{avg}(z)$

So we have:

$$(A\, q_z)\big|_z - (A\, q_z)\big|_{z=z+dz} = 0 \qquad (A \text{ is also changing})$$

$$\Rightarrow \frac{d(A q_z)}{dz} = 0$$

Now, $A = \pi r^2$, and $r = r(z)$

{ From this method you get the answer for approximate/avg. temperature
↓
the real answer is extremely complex to solve, & will take a very long time }
