RACHIT DHAR

TRANSPORT PHENOMENA [CLL110]

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Concepts Involved

Fluid Mechanics
Heat Transfer
Mass Transfer
} often have similar equations

Was quite good but not in print now
↓
(First edition in 1969)

* **Textbook**: Transport Phenomena, 2nd edition
— Bird, Stewart and Lightfoot

⇓

Can get first edition if possible (2nd edition is slightly more mathematically complex)

NOTE: Should be able to solve Level I & II problems in this course (& perhaps a few from level III or IV too)

Minor — 20%
Major — 40%
Tutorial — 40%

75% attendance minimum required (or else one grade lowered)

Parts of Course

NOTE: vector and Tensor form of eq^{n}s remain SAME in all coord. systems. (But we shall derive the eq^{n}s in cartesian coord.s)

1. Vector and Tensor analysis ⇒
2. Axiom 1 — Mass is conserved ⇒ Eq^n of Continuity
3. Axiom 2 — Momentum is conserved ⇒ Equation of motion for fluids
4. Axiom 3 — Moment of Momentum is conserved ⇒ Shear stress is symmetric (we will NOT be deriving this in this course)
5. Axiom 4 — Energy is conserved ⇒ Equation of energy
   - Eq^n of Mechanical energy (Bernoulli eq^n)
   - Eq^n of Thermal energy
6. Axiom 5 — Mass of each species in a multi-component mixture is also conserved
   ⇓
   Equation of continuity for multi-component system

VECTOR AND TENSOR ANALYSIS

LECTURE-1

1] **Notation** : Small letters : $a, b, c, \ldots$ ← Scalars

We will use : $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ ← as vectors

(alternatively: $\vec{u}$ or $\underset{\rightarrow}{u}$)

Greek letters : $\underset{\sim}{\tau}, \underset{\sim}{\sigma}, \underset{\sim}{\Omega}$ ← Second-order Tensors

(alternatively : $\vec{\vec{\tau}}$ or $\underset{\rightrightarrows}{\tau}$)

2] **n^{th} order tensor** : Has 3^n components

↳ (3 bcz of 3 dimensions)

Thus, Scalar : Zero-order tensor → $3^0 = 1$ component

Vector : First-order tensor → $3^1 = 3$ components

Second-order Tensor → $3^2 = 9$ components

NOTE : However, just any 3×3 matrix doesn't mean that it will be called a 2^{nd} order Tensor.
Another condition is required, which we will discuss later.
(Just as any 3×1 matrix isn't a vector; a vector must also have magnitude and direction)

3] **Kronecker Delta** (δ_{ij})

↓ where, $i = 1, 2,$ or 3
$j = 1, 2,$ or 3

Can be represented as:

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix}$$

$$\delta_{ij} = \begin{cases} +1 & , \text{if } i = j \\ 0 & , \text{if } i \neq j \end{cases}$$

⇓

But actually it is NOT necessary to write it like this
(δ_{ij} is not a matrix.
It is a second-order tensor)

4] <u>Alternating unit tensor</u> (ϵ_{ijk}) $\Rightarrow$ Has total $3 \times 3 \times 3 = 27$ components

$\downarrow$

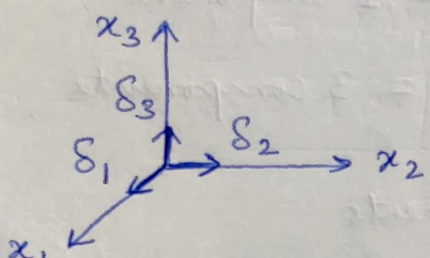
This is NOT actually a tensor

(the use of the word "tensor" here is somewhat of a misnomer)

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } i,j,k \text{ are in cyclic or clockwise rotation} \quad (\text{eg: } \epsilon_{123}, \epsilon_{231}, \epsilon_{312}) \\ -1, & \text{if } i,j,k \text{ are in anti-cyclic order} \quad (\text{eg: } \epsilon_{132}, \epsilon_{213}) \\ 0, & \text{if any two values are the same} \quad (\text{eg: } \epsilon_{223}, \epsilon_{131}) \end{cases}$$

LECTURE-2

5]



$\underbrace{\underline{\delta_1}, \underline{\delta_2}, \underline{\delta_3}}$

$\Downarrow$

these 3 are <u>unit vectors</u> (and NOT the kronecker delta)

6] An <u>important/useful property</u>:

$$\boxed{\sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} \epsilon_{hjk} = 2 \delta_{ih}}$$

δ_{ih} $\rightarrow$ kronecker delta

Here: $i, h \rightarrow$ occurs just once $\Rightarrow$ <u>Free indices</u>

$j, k \rightarrow$ they are repeated twice $\Rightarrow$ <u>Dummy indices</u> (i.e. thus you can replace them with any variable)

eg: For $i=1, h=1$:

$$\sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{1jk} \epsilon_{1jk} = 2 \delta_{11} = 2$$

For $i=1, h=2$:

$$\sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{1jk} \epsilon_{2jk} = 2 \delta_{12} = 0$$

, and so on ...

7] Summation Convention

$$\boxed{\text{No. of summations} = \text{No. of dummy indices}}$$

i.e. whenever we see a dummy index
→ we know that there is a summation going on

↓

Thus, we don't need to write the summation sign again and again

An example by a property ⇒

$$\boxed{\epsilon_{ijk}\,\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}$$

↓

hidden summation over k taking place here

(invisible summation sign/symbol)

Here:

$i, j, m, n \rightarrow$ Free indices

$k \rightarrow$ dummy index

Also, another important fact related to the summation convention is:

$$\boxed{\text{No. of equations} = 3^n \quad (\text{where } n = \text{No. of Free indices})}$$

8] Vectors

↓

- 3 components: $(v_1, v_2, v_3) \Rightarrow$ Has magnitude
- Also has direction

We write, $\underline{v} = v_1\,\underline{\delta_1} + v_2\,\underline{\delta_2} + v_3\,\underline{\delta_3}$

$$= \sum_{i=1}^{3} v_i\,\underline{\delta_i}$$

So we can write,

$$\left.\begin{aligned} \underline{v} &= v_i\,\underline{\delta_i} \\ &= v_j\,\underline{\delta_j} \\ &= v_k\,\underline{\delta_k} \end{aligned}\right\}$$

NOTE!

We can change/replace the variable
↓
since they are dummy variables

$$\Rightarrow \boxed{\underline{v} = \sum_{i=1}^{3} v_i\,\underline{\delta_i} = v_i\,\underline{\delta_i} = v_j\,\underline{\delta_j} = v_k\,\underline{\delta_k}}$$

So we can also write:

$$\underline{v} \cdot \underline{w} = v_i \underline{\delta_i} \cdot w_j \underline{\delta_j}$$

or

$$= v_i \underline{\delta_i} \cdot w_k \underline{\delta_k}$$

9] Addition / Subtraction of vectors

$$\underline{v} + \underline{w} = v_i \underline{\delta_i} + w_i \underline{\delta_i}$$

$$= (v_i + w_i) \underline{\delta_i}$$

10] Multiplication : Scalar and Vector

$$\underline{w} = a \underline{v}$$

$$\Rightarrow w_i \underline{\delta_i} = a v_i \underline{\delta_i}$$

$$\therefore \Rightarrow \boxed{w_i = a v_i} \Rightarrow \begin{aligned} w_1 &= a v_1 \\ w_2 &= a v_2 \\ w_3 &= a v_3 \end{aligned}$$

11] Multiplication of Vector and Vector

I] $\underline{v}\,\underline{w} \Rightarrow 2^{nd}$ order tensor (when no sign in b/w 2 vectors $\downarrow$ order gets added $\downarrow$ 2^{nd} order tensor is formed)

II] $\underline{v} \cdot \underline{w} \Rightarrow 0^{th}$ order tensor (when $\cdot$ sign (i.e dot product) $\downarrow$ combined order is reduced by 2)

III] $\underline{v} \times \underline{w} \Rightarrow 1^{st}$ order tensor (when $\times$ sign (i.e. cross product) $\downarrow$ Combined order is reduced by 1)

12] Dot Product

$$\boxed{\delta_i \cdot \delta_j = \delta_{ij}}$$

* Important Note : Use different indices for 2 vectors, even if they are same vectors

OR

If same indices, then use summation

$$\underline{v} \cdot \underline{w} = v_i \underline{\delta_i} \cdot w_j \underline{\delta_j}$$
$$= v_i w_j \, \underline{\delta_i} \cdot \underline{\delta_j}$$
$$= v_i w_j \, \delta_{ij}$$
$$= \sum_{i=1}^{3} \sum_{j=1}^{3} v_i w_j \delta_{ij}$$
$$= v_1 w_1 \delta_{11} + v_2 w_2 \delta_{22} + v_3 w_3 \delta_{33} + \left(\text{other terms}\right) \nearrow 0$$

$[\because \delta_{ij} = 0 \text{ when } i \neq j]$

$$= v_1 w_1 + v_2 w_2 + v_3 w_3$$
$$= \sum_{i=1}^{3} v_i w_i$$

compaction operation

$$\text{or} = \sum_{j=1}^{3} v_j w_j$$

i is dummy variable (thus, replaceable)

13] Vector Product / Cross Product

$$\underline{v} \times \underline{w} = v_i \underline{\delta_i} \times w_j \underline{\delta_j}$$
$$= v_i w_j \left(\underline{\delta_i} \times \underline{\delta_j} \right)$$

* NOTE: Another property: $\boxed{\underline{\delta_i} \times \underline{\delta_j} = \epsilon_{ijk} \underline{\delta_k}}$ $\Rightarrow$ (Introduce a third variable)

For eg:

$\underline{\delta_1} \times \underline{\delta_2} = \underline{\delta_3}$

$\underline{\delta_2} \times \underline{\delta_3} = \underline{\delta_1}$

$\underline{\delta_3} \times \underline{\delta_1} = \underline{\delta_2}$

$\underline{\delta_2} \times \underline{\delta_1} = -\underline{\delta_3}$

$\underline{\delta_3} \times \underline{\delta_2} = -\underline{\delta_1}$

$\underline{\delta_1} \times \underline{\delta_3} = -\underline{\delta_2}$

$\underline{\delta_1} \times \underline{\delta_1} = \underline{0}$

$\underline{\delta_2} \times \underline{\delta_2} = \underline{0}$

$\underline{\delta_3} \times \underline{\delta_3} = \underline{0}$

For instance, another property is:

$$* \quad \boxed{\underline{u} \times (\underline{v} \times \underline{w}) = \underline{v}\,(\underline{u} \cdot \underline{w}) - \underline{w}\,(\underline{u} \cdot \underline{v})}$$

Proof: $\quad LHS = \underline{u} \times (\underline{v} \times \underline{w})$

$$= u_i v_j w_k \left[\underline{\delta_i} \times (\underline{\delta_j} \times \underline{\delta_k}) \right]$$

$$= u_i v_j w_k \, \epsilon_{jkl} \, (\underline{\delta_i} \times \underline{\delta_l})$$

(say) $\boxed{\underline{Z_m} = u_i v_j w_k \, \epsilon_{jkl} \, \epsilon_{ilm} \, \underline{\delta_m}}$

$\Rightarrow \underline{Z_m} = Z_m \underline{\delta_m} \Rightarrow Z_m = u_i v_j w_k \, \epsilon_{jkl} \, \epsilon_{ilm}$ (NOTE: $\epsilon_{ilm} = \epsilon_{mil}$)

LECTURE-3

14] Vector differential Operator (Del operator)

$$\underline{\nabla} = \underline{\delta_1} \frac{\partial}{\partial x_1}(\,) + \underline{\delta_2} \frac{\partial}{\partial x_2}(\,) + \underline{\delta_3} \frac{\partial}{\partial x_3}(\,)$$

acts "on" something

OR

$$\underline{\nabla} = \underline{\delta_i} \frac{\partial}{\partial x_i}$$

15] Gradient of a Scalar Field

$$S(x_1, x_2, x_3, t)$$

Scalar Field

may be a function of time also

Thus, its gradient is: $\quad \underline{\nabla} S = \underline{\delta_i} \frac{\partial}{\partial x_i} S$

(say) $\underline{w} = \boxed{\underline{\nabla} S = \frac{\partial S}{\partial x_i} \underline{\delta_i}}$

$\Downarrow$

$$\underline{w} = w_i \underline{\delta_i}$$

Hence, $\boxed{w_i = \frac{\partial S}{\partial x_i}}$ $\leftarrow$ Here, i is free index

Thus, $w_1 = \frac{\partial s}{\partial x_1}$, $w_2 = \frac{\partial s}{\partial x_2}$, $w_3 = \frac{\partial s}{\partial x_3}$

* Gradient of S → indicates the direction ↓ of maximum rate of change of S ↓ at a certain (x_1, x_2, x_3, t)

16] Taking: $\underline{\nabla}\, \underline{v} \Rightarrow$ second order tensor

$\underline{\nabla} \cdot \underline{v} \Rightarrow$ Divergence of a vector field (OR dot product of del operator & vector)

$\underline{\nabla} \times \underline{v} \Rightarrow$ Curl of a vector field (OR, cross product of del operator & vector)

Divergence of Vector Field

17] $\underline{\nabla} \cdot \underline{v} = \underline{\delta}_i \frac{\partial}{\partial x_i} \cdot v_j \underline{\delta}_j$

$= \frac{\partial v_j}{\partial x_i} \underline{\delta}_i \cdot \underline{\delta}_j$ ⟸ DO NOT change order of δ_i and δ_j (Here it doesn't matter, but in cross product it WILL make a difference)

$= \frac{\partial v_j}{\partial x_i} \delta_{ij}$

By expanding this (compaction operation) ⇒ $\underline{\nabla} \cdot \underline{v} = \frac{\partial v_j}{\partial x_j}$

$$\boxed{\underline{\nabla} \cdot \underline{v} = \frac{\partial v_i}{\partial x_i}}$$

18] Curl of Vector Field

$\underline{\nabla} \times \underline{v} = \underline{\delta}_i \frac{\partial}{\partial x_i} \times v_j \underline{\delta}_j$

$\underline{w}$ (say)

$= \frac{\partial v_j}{\partial x_i} \underline{\delta}_i \times \underline{\delta}_j$

$$\boxed{\underline{\nabla} \times \underline{v} = \frac{\partial v_j}{\partial x_i} \epsilon_{ijk} \underline{\delta}_k}$$

Hence, this implies:

$$\underline{w} = w_k \underline{\delta_k} = \frac{\partial v_j}{\partial x_i} \epsilon_{ijk} \underline{\delta_k}$$

$$\Rightarrow \quad w_k = \frac{\partial v_j}{\partial x_i} \epsilon_{ijk}$$

eg: $w_1 = \frac{\partial v_j}{\partial x_i} \epsilon_{ij1}$, $w_2 = \frac{\partial v_j}{\partial x_i} \epsilon_{ij2}$, ...

So overall we will write:

$$\boxed{w_k = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_i} \epsilon_{ijk}}$$

For instance, $w_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}$

$w_2 = \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}$ or $-\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}$, and so on ...

19] Laplacian of a Scalar Field

Let's say we have $\underline{\nabla} \cdot \underline{v}$

But this $\underline{v}$ itself is:

$$\underline{v} = \underline{\nabla} s$$

Thus $\Rightarrow$ $\underline{\nabla} \cdot \underline{\nabla} s = \underline{\delta_i} \frac{\partial}{\partial x_i} \cdot \underline{\delta_j} \frac{\partial}{\partial x_j} s$

$$= \frac{\partial}{\partial x_i} \frac{\partial s}{\partial x_j} \underline{\delta_i} \cdot \underline{\delta_j}$$

$$= \frac{\partial}{\partial x_i} \frac{\partial s}{\partial x_j} \delta_{ij}$$

$$= \frac{\partial}{\partial x_i} \frac{\partial s}{\partial x_i}$$

Compaction operation (i.e. can be seen by expanding)

$$= \frac{\partial^2 S}{\partial x_i^2} \quad \text{or} \quad \frac{\partial^2 S}{\partial x_1^2} + \frac{\partial^2 S}{\partial x_2^2} + \frac{\partial^2 S}{\partial x_3^2}$$

⇒ This can be separated as an operator:

$$\underline{\nabla} \cdot \underline{\nabla} S = \underbrace{\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right)}_{\text{defined as Laplacian operator}} S$$

Thus, Laplacian operator: $\boxed{\nabla^2 = \frac{\partial^2}{\partial x_i^2}}$

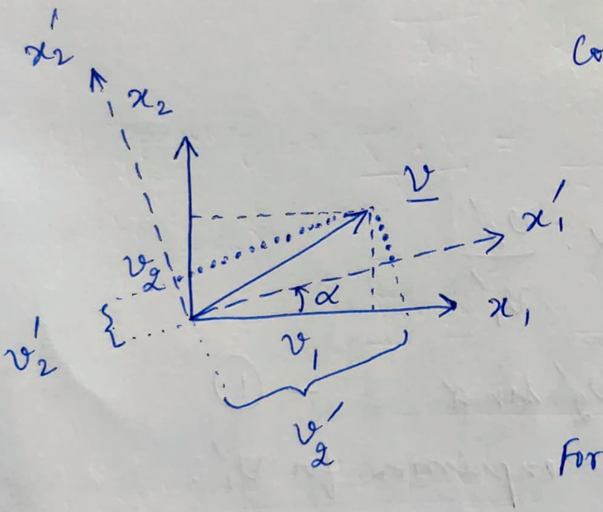
20] Rigorous Definition of a Vector

We represent vector by: $\underline{v} = v_i \underline{\delta_i}$

Now the rigorous definition is:

1] It has 3 components

2] The vector must follow the transformation rule when the coordinate system is rotated



The transformation rule is:

$$\boxed{v_p' = l_{ip} \, v_i}$$

For eg: $v_1' = l_{i1} v_i$

$$= \sum_{i=1}^{3} l_{i1} v_i$$

$$= l_{11} v_1 + l_{21} v_2 + l_{31} v_3$$

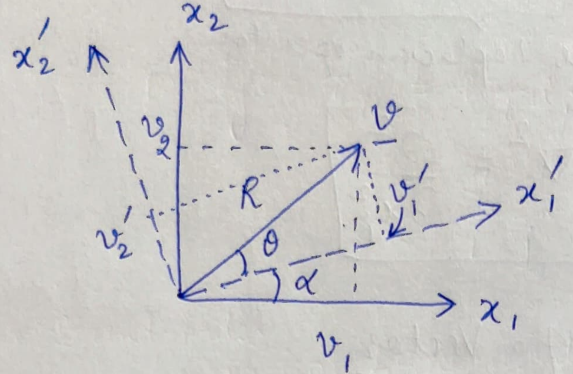
Similarly for $i = 2$ and 3.

Now, here, l_{ip} = cosine of angle b/w (i^{th} old axis & p^{th} new axis) for instance : $l_{11} = \cos\alpha$

$l_{12} = \cos(90^\circ + \alpha)$

$\vdots$ and so on

Now let's show why this is the case:



$$v_1 = R\cos(\alpha+\theta)$$

$$v_1' = R\cos\theta$$

$$\Rightarrow \frac{v_1}{v_1'} = \frac{\cos(\alpha+\theta)}{\cos\theta} = \cos\alpha - \sin\alpha\tan\theta$$

Since, $\dfrac{v_2'}{v_1'} = \tan\theta$

$$\Rightarrow \frac{v_1}{v_1'} = \cos\alpha - \sin\alpha \cdot \frac{v_2'}{v_1'}$$

$$\Rightarrow v_1 = v_1'\cos\alpha - v_2'\sin\alpha \qquad - (1)$$

But we now want such an expression for v_1':

Now, $v_2 = R\sin(\alpha+\theta)$

$v_2' = R\sin\theta$

$$\frac{v_2}{v_2'} = \frac{\sin(\alpha+\theta)}{\sin\theta} = \cos\alpha + \sin\alpha\cot\theta$$

Since, $\dfrac{v_1'}{v_2'} = \cot\theta$

Thus, $\dfrac{v_2}{v_2'} = \cos\alpha + \sin\alpha \dfrac{v_1'}{v_2'}$

$$\Rightarrow v_2 = v_2' \cos\alpha + v_1' \sin\alpha \quad - (2)$$
$$= v_1' \sin\alpha + v_2' \cos\alpha$$

Do (1) $\times \cos\alpha$ and (2) $\times \sin\alpha$:

$$v_1 \cos\alpha = v_1' \cos^2\alpha - v_2' \sin^2\alpha$$

and, $v_2 \sin\alpha = v_1' \sin^2\alpha + v_2' \sin^2\alpha$

Adding: $v_1 \cos\alpha + v_2 \sin\alpha = v_1'$

Hence, $v_1' = v_1 \cos\alpha + v_2 \sin\alpha$
$$= v_1 l_{11} + v_2 l_{21}$$

In general, we'll get: $v_p' = l_{ip} v_i \Rightarrow$ eg: $v_1' = v_1 l_{11} + v_2 l_{21}$

[Similarly for v_2', we will see:

$$v_2' = v_1 l_{12} + v_2 l_{22}$$

(where $l_{12} = \sin\alpha$
$l_{22} = \cos\alpha$)]

21] Thus we have seen, a vector:

1] Has 3 components

2] These 3 components follow the following transformation rules:

$$v_p' = l_{ip} v_i$$

And we write a vector as: $\underline{v} = v_i \underline{S}_i$

22] Rigorous Definition for Second-order Tensor

2nd order Tensor $\rightarrow$ 9 components

$$\underset{\sim}{\tau} \Rightarrow \begin{matrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{matrix}$$

* NOTE: This is NOT a matrix (written this way just for beauty. We can also write them in a line)

these are called diagonal components

We represent it by $\underset{\sim}{\tau}$
and components are τ_{ij}

* The Transformation rule that MUST be followed, is : $\boxed{\tau'_{mn} = l_{im} l_{jn} \tau_{ij}}$

For eg: $\tau'_{11} = l_{i1} l_{j1} \tau_{ij}$

$$= \sum_i \sum_j l_{i1} l_{j1} \tau_{ij}$$

The Second-order tensor will be represented as :

$$\boxed{\underset{\sim}{\tau} = \tau_{ij} \underline{\delta_i} \underline{\delta_j}}$$

[We will show why this is true, below]

LECTURE-4

23] Taking vectors $\underline{v}$ and $\underline{w}$:

$\underline{v} = v_i \underline{\delta_i}$

$v'_m = l_{im} v_i$ — ①

Also, $\underline{w} = w_j \underline{\delta_j}$

$w'_n = l_{jn} w_j$ — ②

Multiplying ① and ② together:

$v'_m w'_n = l_{im} l_{jn} v_i w_j$ $\Rightarrow$ Thus, this shows that $\underline{v}\,\underline{w}$ is a second-order tensor

Now, $\underline{v}\,\underline{w} = (v_1 \underline{\delta_1} + v_2 \underline{\delta_2} + v_3 \underline{\delta_3})(w_1 \underline{\delta_1} + w_2 \underline{\delta_2} + w_3 \underline{\delta_3})$

$$= v_1 w_1 \underline{\delta_1}\,\underline{\delta_1} + v_1 w_2 \underline{\delta_1}\,\underline{\delta_2} + v_1 w_3 \underline{\delta_1}\,\underline{\delta_3}$$
$$+ v_2 w_1 \underline{\delta_2}\,\underline{\delta_1} + v_2 w_2 \underline{\delta_2}\,\underline{\delta_2} + v_2 w_3 \underline{\delta_2}\,\underline{\delta_3}$$
$$+ v_3 w_1 \underline{\delta_3}\,\underline{\delta_1} + v_3 w_2 \underline{\delta_3}\,\underline{\delta_2} + v_3 w_3 \underline{\delta_3}\,\underline{\delta_3}$$

This shows that we have 9 components:

$$\left.\begin{matrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{matrix}\right\}$$ and all these components follow the transformation rule as shown before

So:
We know, $\underline{v} = v_i \underline{\delta_i}$

$\underline{w} = w_j \underline{\delta_j}$

$$\Rightarrow \underline{v}\,\underline{w} = v_i w_j \underline{\delta_i}\,\underline{\delta_j}$$

Thus, $\boxed{\underset{\sim}{\tau} = \tau_{ij} \underline{\delta_i}\,\underline{\delta_j}}$ ← representation of second-order tensor

NOTE: $\underline{v}\,\underline{w}$ ⇒ this is called a dyadic product of two vectors

24] We know, $\underline{\delta_i} \cdot \underline{\delta_j} = \delta_{ij}$

$\underline{\delta_i} \times \underline{\delta_j} = \epsilon_{ijk} \underline{\delta_k}$

Multiplication by a Scalar (of Second-order Tensor)

$$\underset{\sim}{\tau}\, s = \tau_{ij} \underline{\delta_i}\,\underline{\delta_j}\, s$$

$$= \underbrace{\tau_{ij} s}_{\text{also looks like a tensor}}\, \underline{\delta_i}\,\underline{\delta_j}$$

So let's say, $\underset{\sim}{\sigma} = \underset{\sim}{\tau}\, s$

$$\Rightarrow \sigma_{ij} \underline{\delta_i}\,\underline{\delta_j} = \tau_{ij}\, s\, \underline{\delta_i}\,\underline{\delta_j}$$

$$\Rightarrow \boxed{\sigma_{ij} = \tau_{ij}\, s}$$

25] Tensor Product of Two Second-order Tensors

$$\underset{\sim}{\sigma} \cdot \underset{\sim}{\tau} \rightarrow (2+2-2=2)$$

Thus, it is also a second-order tensor

Now, $\underset{\sim}{\sigma} \cdot \underset{\sim}{\tau} = \sigma_{ij} \, \underline{\delta_i} \, \underline{\delta_j} \cdot \tau_{mn} \, \underline{\delta_m} \, \underline{\delta_n}$

$$= \sigma_{ij} \tau_{mn} \, \underline{\delta_i} \, \underline{\delta_j} \cdot \underline{\delta_m} \, \underline{\delta_n}$$

$$= \sigma_{ij} \tau_{mn} \, \delta_{jm} \, \underline{\delta_i} \, \underline{\delta_n}$$

a scalar

$\Rightarrow$ Replacing j by m to remove δ_{jm} $\Rightarrow$ $\underset{\sim}{\sigma} \cdot \underset{\sim}{\tau} = \sigma_{im} \tau_{mn} \, \underline{\delta_i} \, \underline{\delta_n}$

$\downarrow$

let's call this $\underset{\sim}{\Omega}$

$$\Rightarrow \underset{\sim}{\Omega} = \Omega_{in} \, \underline{\delta_i} \, \underline{\delta_n} = \sigma_{im} \tau_{mn} \, \underline{\delta_i} \, \underline{\delta_n}$$

$$\Rightarrow \boxed{\Omega_{in} = \sigma_{im} \tau_{mn}}$$

for eg: $\Omega_{11} = \sigma_{1m} \tau_{m1}$

$$= \sigma_{11} \tau_{11} + \sigma_{12} \tau_{21} + \sigma_{13} \tau_{31}$$

26] Scalar Product of Two Tensors (or Double-Dot Product)

$$\underset{\sim}{\sigma} : \underset{\sim}{\tau} \Rightarrow (2+2-2-2=0)$$

$\downarrow$

thus, scalar quantity

Now $\Rightarrow$ $\underset{\sim}{\sigma} : \underset{\sim}{\tau} = \sigma_{ij} \, \underline{\delta_i} \, \underline{\delta_j} : \tau_{mn} \, \underline{\delta_m} \, \underline{\delta_n}$

$$= \sigma_{ij} \tau_{mn} \, \underline{\delta_i} \, \underline{\delta_j} : \underline{\delta_m} \, \underline{\delta_n}$$

$= \sigma_{ij} \tau_{mn} \delta_{jm} (\underline{\delta_i} \cdot \underline{\delta_n})$ → one dot remaining

$= \sigma_{ij} \tau_{mn} \delta_{jm} \delta_{in}$ → change $n = i$ (Compaction operation)

change $m = j$

$$= \boxed{\sigma_{ij} \tau_{ji}} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} \tau_{ji}$$

27] Vector Product of Tensor and Vector

$\underset{\sim}{\tau} \cdot \underline{v} \Rightarrow (2+1-2 = 1 \rightarrow$ vector$)$

$\underset{\sim}{\tau} \times \underline{v} \Rightarrow (2+1-1 = 2 \rightarrow$ thus, this is tensor product of tensor and vector$)$

Now, $\underset{\sim}{\tau} \cdot \underline{v} = \tau_{ij} \underline{\delta_i} \underline{\delta_j} \cdot v_k \underline{\delta_k}$

$= \tau_{ij} v_k \underline{\delta_i} \underline{\delta_j} \cdot \underline{\delta_k}$

$= \tau_{ij} v_k \delta_{jk} \underline{\delta_i}$

$k = j$ (compaction)

$= \tau_{ij} v_j \underline{\delta_i}$

Let's say $\underline{w} = \underset{\sim}{\tau} \cdot \underline{v}$

$\underline{w} = w_i \underline{\delta_i} = \tau_{ij} v_j \underline{\delta_i}$

Hence, $\boxed{w_i = \tau_{ij} v_j} = \sum_{j=1}^{3} \tau_{ij} v_j$

28] Tensor Product of Tensor and Vector (Not so commonly used)

$\underset{\sim}{\tau} \times \underline{v} = \tau_{ij} \underline{\delta_i} \underline{\delta_j} \times v_k \underline{\delta_k}$

$= \tau_{ij} v_k \underline{\delta_i} \underline{\delta_j} \times \underline{\delta_k}$

$$\underset{\sim}{\Omega} = \underset{\sim}{\tau} \times \underline{v} = \tau_{ij} v_k \epsilon_{jkm} \underline{\delta_i} \, \underline{\delta_m}$$

$$\Rightarrow \underset{\sim}{\Omega} = \Omega_{im} \underline{\delta_i} \, \underline{\delta_m}$$

Thus, $\boxed{\Omega_{im} = \tau_{ij} v_k \epsilon_{jkm}}$

For eg: $\Omega_{11} = \tau_{1j} v_k \epsilon_{jk1} = \sum_j \sum_k \tau_{1j} v_k \epsilon_{jk1}$

29] $\underline{\nabla}$ operator involving Tensors and dyads

$$\underbrace{\underline{\nabla}}_{1} \cdot \underbrace{\underset{\sim}{\tau}}_{2} \Rightarrow (1+2-2 = 1 \rightarrow \text{vector}).$$

Now, $\underline{\nabla} \cdot \underset{\sim}{\tau} = \underline{\delta_i} \dfrac{\partial}{\partial x_i} \cdot \tau_{jk} \underline{\delta_j} \, \underline{\delta_k}$

$$= \frac{\partial}{\partial x_i} \tau_{jk} \, \underline{\delta_i} \cdot \underline{\delta_j} \, \underline{\delta_k}$$

$$= \frac{\partial \tau_{jk}}{\partial x_i} \delta_{ij} \, \underline{\delta_k}$$

$i = j$ (compaction)

$$= \frac{\partial \tau_{jk}}{\partial x_j} \underline{\delta_k}$$

Taking, $\underline{w} = w_k \underline{\delta_k} = \underline{\nabla} \cdot \underset{\sim}{\tau}$

$$\Rightarrow \boxed{w_k = \frac{\partial \tau_{jk}}{\partial x_j}}$$

For eg: $w_1 = \dfrac{\partial \tau_{j1}}{\partial x_j}$

$$= \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3}$$

30] $\underline{\nabla} \cdot (\underline{v}\, \underline{w}) = \underline{\delta}_i \frac{\partial}{\partial x_i} \cdot v_j \underline{\delta}_j\, w_k \underline{\delta}_k$

{* **NOTE:** Even if this was

$\underline{\nabla} \cdot (\underline{v}\, \underline{v})$

you MUST use different indices for both $\underline{v}$'s }

$$\underline{\nabla} \cdot (\underline{v}\, \underline{w}) = \frac{\partial v_j w_k}{\partial x_i}\, \underline{\delta}_i \cdot \underline{\delta}_j\, \underline{\delta}_k$$

$$= \frac{\partial v_j w_k}{\partial x_i}\, \delta_{ij}\, \underline{\delta}_k \qquad j = i$$

$$= \frac{\partial v_j w_k}{\partial x_j}\, \underline{\delta}_k$$

Let $\underline{u} = \underline{\nabla} \cdot (\underline{v}\, \underline{w})$

$\underline{u} = u_k \underline{\delta}_k \Rightarrow \boxed{u_k = \frac{\partial v_j w_k}{\partial x_j}}$

NOTE: $\underline{\nabla} \cdot (\underset{\sim}{\tau} \cdot \underline{v}) = \underline{v} \cdot (\underline{\nabla} \cdot \underset{\sim}{\tau}) + \underset{\sim}{\tau} \cdot \underline{\nabla}\, \underline{v}$

Q [Try to prove this] (H.W.)

* ⇒ True iff $\underset{\sim}{\tau}$ is Symmetric

(i.e. $\tau_{12} = \tau_{21}$, $\tau_{13} = \tau_{31}$ and $\tau_{23} = \tau_{32}$)

31] Time Derivatives in Transport Phenomena

NOTE: $x_1 \Rightarrow x$ $\quad \underline{\delta}_1 = \underline{i}$

$x_2 \Rightarrow y$ $\quad \underline{\delta}_2 = \underline{j}$

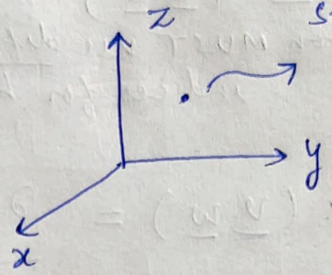
$x_3 \Rightarrow z$ $\quad \underline{\delta}_3 = \underline{k}$

We will be working on, say a function $c(x, y, z, t)$:

Then:

(1) Partial time derivative

$$\boxed{\frac{\partial c}{\partial t}}$$



stay at same point in space
⇓
and measure c at time t_1 and then at time t_2

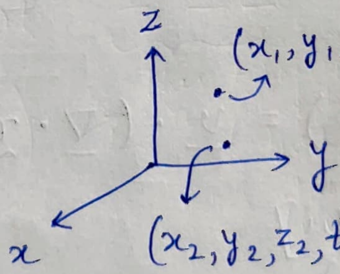
Means: $$\frac{(c)_{t_2} - (c)_{t_1}}{t_2 - t_1}$$

Make $t_2 - t_1 \to 0$ ($\Delta t = t_2 - t_1$) ⟹ Then, $$\lim_{\Delta t \to 0} \frac{(c)_{t_2} - (c)_{t_1}}{\Delta t} = \frac{\partial c}{\partial t}$$

↓ i.e. differentiated only w.r.t. time

* (i.e. observer remains stationary)

(2) Total time derivative



(x_1, y_1, z_1, t_1) ⇑ [At time t_1 at this location]

(x_2, y_2, z_2, t_2) ⇖ [At time t_2 at this location]

If you now calculate:

$$\frac{c_2 - c_1}{t_2 - t_1}$$

→ this is different from before
↓
* since location also has changed

when: $\Delta t = t_2 - t_1 \to 0$

⇓

$$\boxed{\frac{dc}{dt}}$$

⟹ this is given also by:

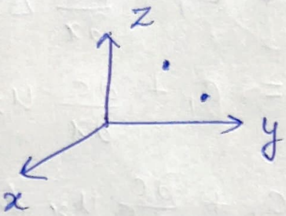
$$\boxed{\frac{dc}{dt} = \frac{\partial c}{\partial t} + u_x \frac{\partial c}{\partial x} + u_y \frac{\partial c}{\partial y} + u_z \frac{\partial c}{\partial z}}$$

(3) Substantial Time Derivative : Time derivative following the motion of the fluid

Taking $\underline{v}$ (NOT $\underline{u}$)

$\underline{u}$ is arbitrary

this is the velocity of the fluid (eg: river, sea, air, etc.)

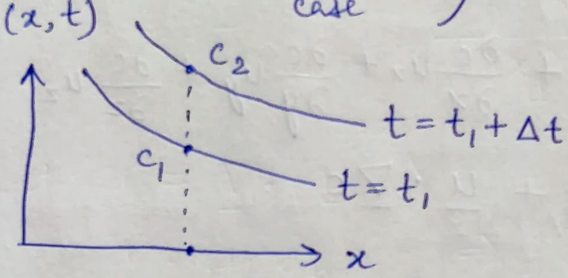


Written as : $\boxed{\dfrac{Dc}{Dt}}$ $\Rightarrow$ where $\dfrac{Dc}{Dt} = \dfrac{\partial c}{\partial t} + v_x \dfrac{\partial c}{\partial x} + v_y \dfrac{\partial c}{\partial y} + v_z \dfrac{\partial c}{\partial z}$

thus : $\downarrow$

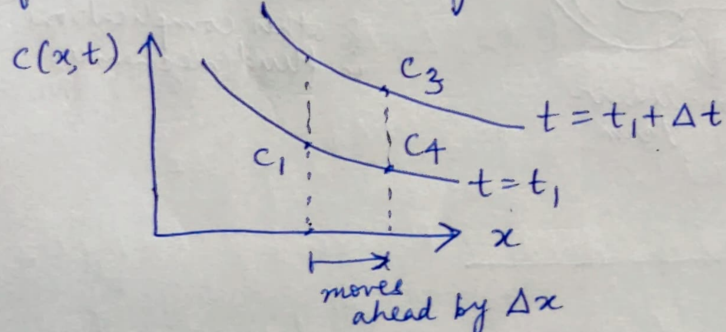
$$\boxed{\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + \underline{v} \cdot \underline{\nabla} c}$$

<u>For example</u> (Taking One-dimensional case)



$$\frac{\partial c}{\partial t} = \lim_{\Delta t \to 0} \frac{c_2 - c_1}{\Delta t}$$

If we move with velocity u_x:



Here,

$$c_4 = c_1 + \frac{\partial c}{\partial x} \Delta x$$

and

$$c_3 = c_4 + \frac{\partial c}{\partial t} \Delta t$$

$$\frac{dc}{dt} = \lim_{\Delta t \to 0} \frac{c_3 - c_1}{\Delta t}$$

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From the previous equations:

$$C_4 = c_1 + \frac{\partial c}{\partial x} \Delta x$$

$$= c_1 + \frac{\partial c}{\partial x} u_x \Delta t$$

$$\Rightarrow \quad c_3 = c_1 + \frac{\partial c}{\partial x} u_x \Delta t + \frac{\partial c}{\partial t} \Delta t$$

Now, $$\frac{dc}{dt} = \lim_{\Delta t \to 0} \frac{c_3 - c_1}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{\partial c}{\partial x} u_x \Delta t + \frac{\partial c}{\partial t} \Delta t}{\Delta t}$$

$$\frac{dc}{dt} = \frac{\partial c}{\partial x} u_x + \frac{\partial c}{\partial t}$$

Similarly in 3D we will get:

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} u_x + \frac{\partial c}{\partial y} u_y + \frac{\partial c}{\partial z} u_z$$

$$= \frac{\partial c}{\partial t} + \underline{u} \cdot \underline{\nabla} c$$

For instance: $$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \underbrace{\underline{u} \cdot \underline{\nabla}\, \underline{v}}$$

this term is responsible for turbulence & other complications in fluid mechanics.

(and, $$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + \underbrace{\underline{v} \cdot \underline{\nabla} \underline{v}}_{\Downarrow}$$)

* Non-linear term
