MECHANICAL BEHAVIOUR OF MATERIALS [MLL251]

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RACHIT DHAR

Quizzes — 3 (or 2), 15% weightage. No cheat sheet

Attendance is essential for scoring marks.

Tutorials & Assignments : 4 - 5 assignments (maybe more)
15% weightage ⇒ can use computers to solve

Minor — 20%
Major — 30%
} ~~cheat sheet~~ may be allowed (generally)
Or maybe not [paper will be made easier without cheatsheet]

Term Paper — 20% ⇒ Theme: ~~Materials Science~~ & Society (focusing on Mechanical Behaviour or some other material theme)
⇓ make use of any of art, music, poetry, memes, ad, startup pitch, drama, etc.

⇓ (Not like conventional term paper)

Groups of around 5 people
prepare group by ~~around~~ mid-August

Name the team as well (marks are probably there for the name)

Create a report as well

Textbooks:

1. Mechanical Metallurgy, G.E. Dieter ← known as Bible
2. Mechanical Behavior of Materials by William H Hosford
3. Mechanical behavior of materials : Meyers and Chawla
} these 2 are good
4. "Mechanical Behaviour of Engineering Materials — Metals, Ceramics, Polymers, and Composites" by Joachim Rösler, Martin Bäker, Harald Harders

Hosford: short & concise ⇓ but may require line-to-line understanding & searching on internet

Meyers and Chawla: quite extensive

Rösler: good for polymers & composites

LECTURE-1

1] **Why we study Mechanical properties of materials?**

I) Why does stuff fail?
II) How does it fail
III) What type of stuff fails?
IV) Can we prevent it?

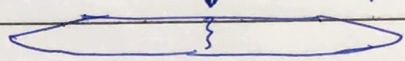
2] Atomic structure $\xrightarrow[\text{it to}]{\text{connect}}$ Microstructure

For BCC materials
eg: Steel :→ At v. low temperatures ($< -100°C$)
loses the ability to plastically deform ⟹ ductile to brittle transformation occurs *
↓
can crack easily (eg: Titanic)

LECTURE-2

3] **Types of Failures**

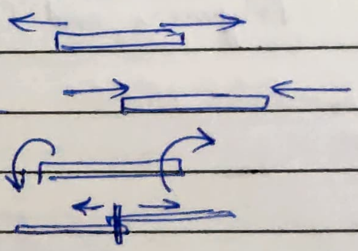
↓ cracked from top to bottom



Stress corrosion cracking
↓
due to weld failure
(Now welding is not done in ships)

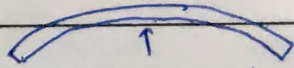
4] **Stress & Strain**

→ Tension
→ Compression
→ Torsion
→ Shear
→ Bending (combination stress)

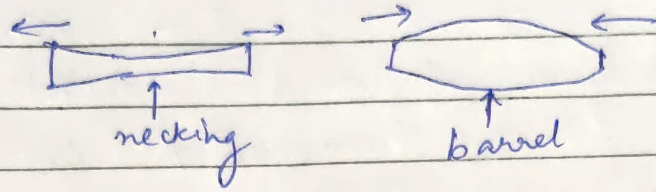


* All these are generally uniaxial in nature

tension here
↓



↑
Compression here

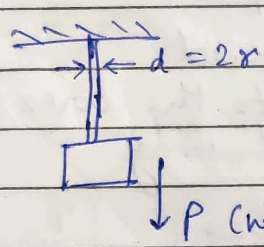


} These are NOT ideal cases

(In ideal, uniform change in thickness should occur everywhere, not just middle)

NOTE: In this course we'll most NOT deal with gradients
↓
mostly homogeneous behavior will be discussed

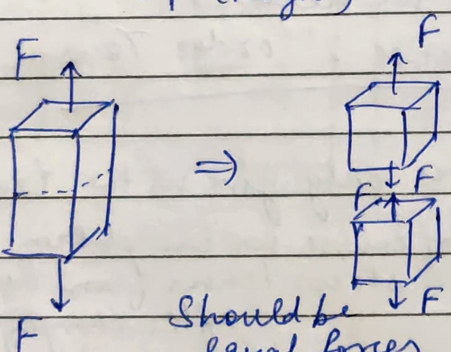
5] Stress



$A = \pi r^2$

Stress, $\sigma = \frac{P}{A} = \frac{P}{\pi r^2}$

Body is in Equilibrium



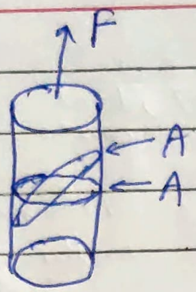
$\sigma = \frac{F}{A}$

Should be equal forces on both ends

($\because$ we are not accelerating, only internally forces should act)

6] Stress, $\sigma \rightarrow$ 2nd order Tensor
⇓
i.e. it is a specific form of Coord. transformation relating 2 vector

* When talking of Stress as a vector ⇒ referring to Stress Vector or Traction



$\sigma = \frac{F}{A}$

When talking of stress vector, it is $\frac{F}{A}$ and takes direction of Force

(i.e. Traction for A is $\frac{F}{A}$ along $\vec{F}$
and for A' is $\frac{F}{A'}$ along $\vec{F}$)

So stress vector need NOT be $\perp^r$ to plane.

* But when we require force to be measured along dir^n $\perp^r$ or parallel only

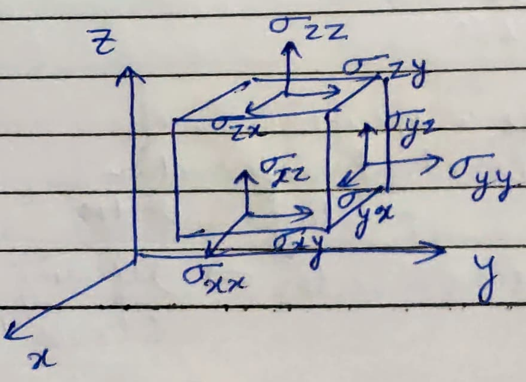
then we need
Stress 2nd order Tensor

($\because$ $\sigma = \frac{F}{A}$ and $\sigma = \frac{F}{A'}$ by themselves only give us the traction)
[of course we can obtain stress tensor from this too after changing it]

NOTE! To avoid confusion, for tensor $\Rightarrow$ we'll use "Stress"

for vector $\Rightarrow$ we'll use "Traction"
(sometimes symbol '$\vec{t}$' used)

7] Stress



$6 \times 3 = 18$ components
(9 others are on the backside of the cube)

But for back planes, this can be simplified

$\Rightarrow$ For body to be in static eqbm. ; $\sigma_{ij} = \sigma_{(-i)(-j)}$

So our 18 components actually becomes 9 components

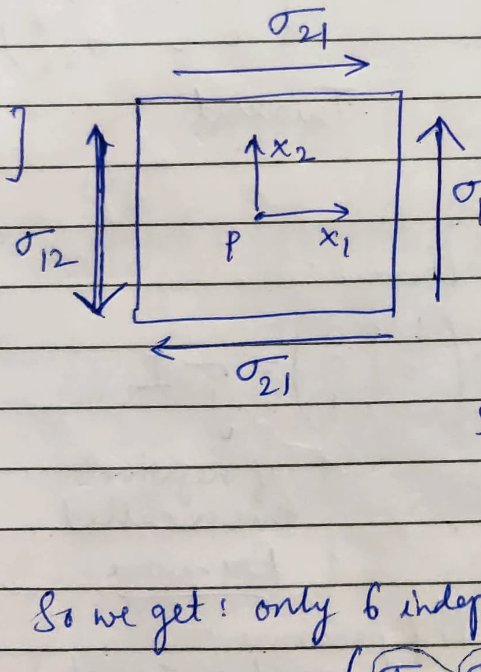
So stress tensor is :

Cauchy stress Tensor, $\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$

or

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

8]



$\Rightarrow$ Cannot be $\downarrow \sigma_{12}$ in this dirn (otherwise body will ROTATE)

So this implies that :

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

So we get : only 6 independent components

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$$

Shear stresses

Normal stresses

$\leftarrow$ So stress tensor is symmetric

Similarly, Strain tensor is also symmetric.

* NOTE :- Both stress & strain tensors are field tensors (i.e. at each point in space they have some value)

9] Examples

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Uniaxial stress

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

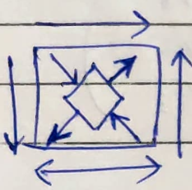
Plane stress

$$\underset{\sim}{\sigma} = \begin{pmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Pure shear

$\Downarrow$

this can be equivalently written as:



$$\begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & -\sigma_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

Hydrostatic or Spherical Stress State

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

Triaxial

Can become:

$$\sigma \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \sigma I$$

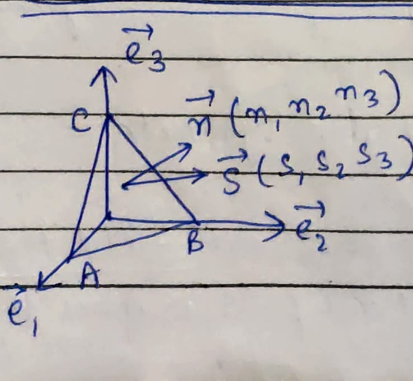
if negative this is called Pressure

i.e. (pressure must be equal)

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix}$$

Cylindrical stress

10] Relation b/w Traction & Stress

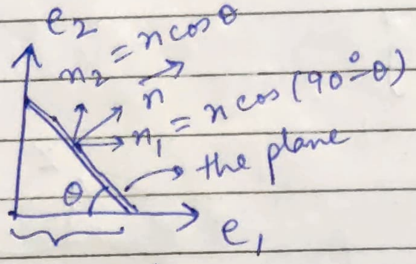


← normal to plane ABC

(Traction vector)

Doing force balance along dir^n 1:

$$F_1 = S_1 \times Ar(ABC) = \sigma_{11}\, Ar(OBC) + \sigma_{21}\, Ar(OAC) + \sigma_{31}\, Ar(OAB)$$



$A\cos\theta = A n_2$

So the same expression becomes

$$\Rightarrow F_1 = \sigma_{11}\, Ar(ABC)\, n_1 + \sigma_{21}\, Ar(ABC)\, n_2 + \sigma_{31}\, Ar(ABC)\, n_3$$

$$\Rightarrow S_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

When performed for all 3 dir^{n}s we get S_2 and S_3 similarly

In Einstein notation, this can be written as:

$$\boxed{S_i = \sigma_{ij}\, n_j}$$ (implicitly a summation is present) just not drawn

$\therefore$ We get $\Rightarrow$ $\boxed{\vec{S} = \underset{\sim}{\sigma}\, \vec{n}}$

LECTURE-3

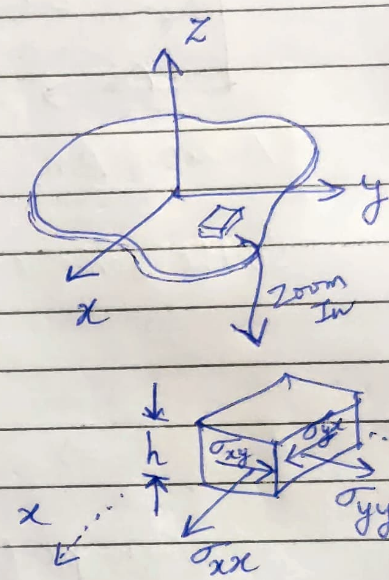
11] Previously we have seen: $\vec{S} = \underset{\sim}{\sigma}\, \vec{n}$ ($\underset{\sim}{\sigma}$ tensor helps relate $\vec{S}$ to $\vec{n}$ * [But NOT every matrix that relates two vectors is necessarily a tensor])

Now, taking case of Zero forces (No traction) on a plane [eg: x-y plane]:

$$\vec{S} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \underset{\sim}{\sigma}\, \vec{n} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{Bmatrix} 0 \\ 0 \\ n \end{Bmatrix}$$

$$= \begin{Bmatrix} \sigma_{13}\, n \\ \sigma_{23}\, n \\ \sigma_{33}\, n \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$



Thus, this is condition of Plane stress

plane stress $\Rightarrow$ when $\sigma_3 = 0$ (in this case)

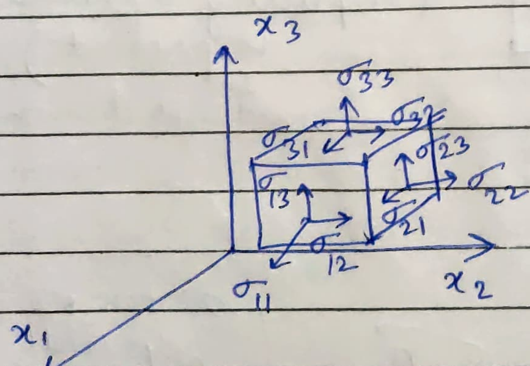
$\downarrow$

$\therefore$ we need to only be concerned with

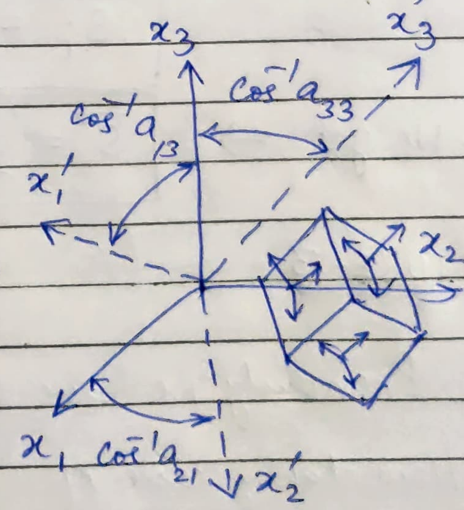
σ_x, σ_y and τ_{xy}

$(\sigma_{xy} = \sigma_{yx})$

12] Stress Transformation



$\Longrightarrow$ transform to another basis



$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

$\downarrow$

is ~~an~~ an orthonormal (cartesian) basis $\vec{e}\,(\vec{x}_1\, \vec{x}_2\, \vec{x}_3)$

$$\underset{\sim}{\sigma}' = \begin{pmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{12}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{13}' & \sigma_{23}' & \sigma_{33}' \end{pmatrix}$$

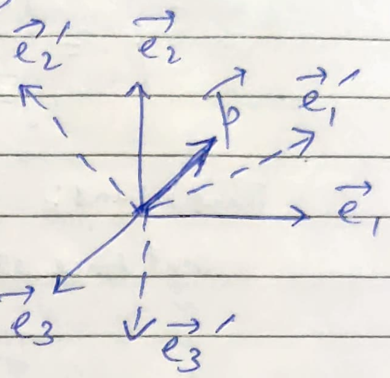
is in another orthonormal (cartesian) basis!

$\vec{e}\,'(\vec{x}_1', \vec{x}_2', \vec{x}_3')$

Now we need to find σ' from a given σ (i.e. relation b/w σ and σ')

13] **Starting with vectors**

Taking the case of vector $\rightarrow$ & changing the basis from $\vec{e}$ to $\vec{e}'$



We can write:

$$\vec{p} = p_1\vec{e}_1 + p_2\vec{e}_2 + p_3\vec{e}_3 = \sum_{i=1}^{3} p_i\vec{e}_i = p_i\vec{e}_i \quad \text{(in Einstein summation convention)}$$

Also, $\vec{p} = \sum p_k' \vec{e}_k' = p_k' \vec{e}_k'$

Dot product of $\vec{p}$ with $\vec{e}_j'$:

$$\vec{p}\cdot\vec{e}_j' = p_1\vec{e}_1\cdot\vec{e}_j' + p_2\vec{e}_2\cdot\vec{e}_j' + p_3\vec{e}_3\cdot\vec{e}_j'$$

$$= p_i\vec{e}_i\cdot\vec{e}_j'$$

$$\Rightarrow \boxed{p_j' = \sum_{i=1}^{3} p_i e_i\cdot e_j' = p_i e_i\cdot e_j'}$$

This implies that we can write:

$$\begin{bmatrix} p_1' \\ p_2' \\ p_3' \end{bmatrix} = \begin{pmatrix} e_1'\cdot e_1 & e_1'\cdot e_2 & e_1'\cdot e_3 \\ e_2'\cdot e_1 & e_2'\cdot e_2 & e_2'\cdot e_3 \\ e_3'\cdot e_1 & e_3'\cdot e_2 & e_3'\cdot e_3 \end{pmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

In Einstein notation:

$$p_i' = \sum_{j=1}^{3} (e_i'\cdot e_j) p_j = \sum_{j=1}^{3} a_{ij} p_j = a_{ij} p_j$$

where a_{ij} (is the dot product of new-i & old-j dir^s) OR cosine between ...

Taking this matrix as A (i.e. $A = [a_{ij}]$) $\Rightarrow$ We get: $\boxed{\vec{p}' = A\vec{p}}$ $\rightarrow$ A is called Transformation matrix

Now, in order to do the reverse
i.e. go from new to old system:

$$\Rightarrow p_i = \sum_{j=1}^{3} (e_i \cdot e_j') p_j' = \sum_{j=1}^{3} a_{ij}^T p_j' = a_{ij}^T p_j'$$

(or $a_{ji} p_j'$)

This means: $\vec{p} = A^T \vec{p}\,'$

But since also: $\vec{p}\,' = A\vec{p} \Rightarrow A^{-1}\vec{p}\,' = \vec{p}$

Thus we get: $\boxed{A^T = A^{-1}}$ ← for Transformation matrix
↓
Inverse is given by the Transpose

14] Extending to 2nd Rank Tensor

Old system: $\vec{p} = \underset{\sim}{B}\vec{v}$ or $p_i = B_{ij} v_j$ (such as $\vec{T} = \underset{\sim}{\sigma}\vec{n}$)

New system: $p_m' = B_{mn}' v_n'$ (such as $\vec{T}' = \underset{\sim}{\sigma}' \vec{n}'$)

Now using ⇒ $p_i' = a_{ij} p_j$, we get that:

$$p_m' = a_{mn} p_n = B_{mn}' v_n'$$
$$= B_{mn}' a_{np} v_p$$

Thus implying ⇙

$$a_{mn} p_n = B_{mn}' a_{np} v_p$$

(such as: $A\vec{T} = \underset{\sim}{\sigma}' A\vec{n}$)

From the begining $(p_i = B_{ij} v_j) \Rightarrow$

(Taking $p_n = B_{np} v_p$)

$$\therefore a_{mn} B_{np} v_p = B_{mn}' a_{np} v_p$$

(such as: $A\underset{\sim}{\sigma}\vec{n} = \underset{\sim}{\sigma}' A\vec{n}$)

∴ In Matrix form

$$[A][B]\vec{v} = [B'][A]\vec{v}$$

$$\Rightarrow [B]\vec{v} = [A^{-1}][B'][A]\vec{v} = [A^T][B'][A]\vec{v}$$

$$\Rightarrow \boxed{[B] = [A^T][B'][A]}$$

(such as: $\underset{\sim}{\sigma} = A^T \underset{\sim}{\sigma}' A$)

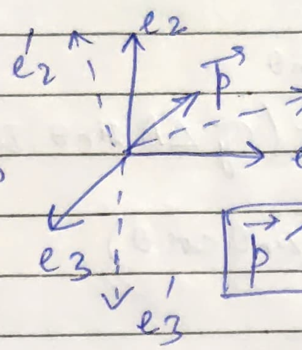
and also, $\boxed{[B'] = [A][B][A^T]}$

and $\underset{\sim}{\sigma}' = A\underset{\sim}{\sigma}A^T$

LECTURE-4

Starting with vectors:

In previous lecture we discussed:



$$\boxed{\vec{p}' = A\vec{p}}$$

$\Longrightarrow$ upto

Transformation for Rank-2 Tensors:

$$\underset{\sim}{\sigma} = A^T \underset{\sim}{\sigma}' A$$

and $\boxed{\underset{\sim}{\sigma}' = A \underset{\sim}{\sigma} A^T}$

15] We can write: $\underset{\sim}{\sigma}' = A \underset{\sim}{\sigma} A^T$

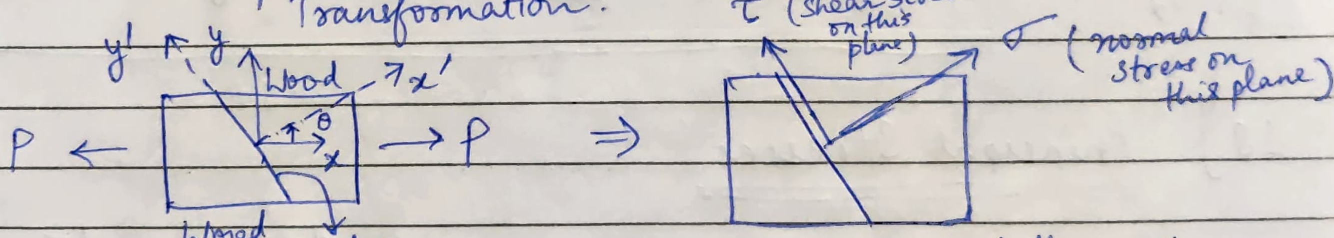
$$\Downarrow$$

$$\sigma_{ij}' = \sum_{k,l}^{3} a_{ik} a_{jl} \sigma_{kl} = a_{ik} a_{jl} \sigma_{kl}$$

for eg: $\sigma_{11}' = a_{1k} a_{1l} \sigma_{kl} = \sum_{k=1}^{3} a_{1k} \left(\sum_{l=1}^{3} a_{1l} \sigma_{kl} \right)$

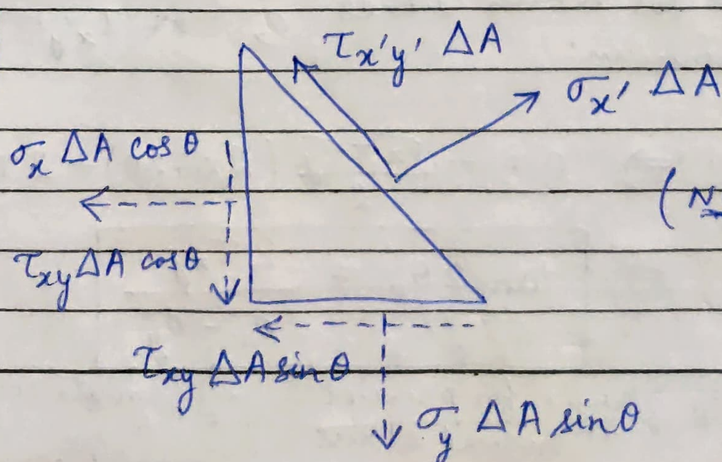
16] We can also write σ as: $\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix}$

One example of 2D Transformation:



We can ask the question "Can the glue sustain the forces?"

17]



(NOTE: These "dotted" vector forces are actually not in those directions. [They would be in the opposite dir^{n}s]

We have drawn them just to "cancel out" the forces from the new system)

So we will get; along x dirⁿ :

$$\sigma_{x'} \Delta A - (\tau_{xy} \Delta A \sin\theta)\cos\theta - (\tau_{xy} \Delta A \cos\theta)\sin\theta$$
$$- (\sigma_x \Delta A \cos\theta)\cos\theta - (\sigma_y \Delta A \sin\theta)\sin\theta = 0$$

$$\Rightarrow \quad \sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta)$$

And along y dirⁿ :

$$0 = \tau_{x'y'} \Delta A + (\tau_{xy} \Delta A \sin\theta)\sin\theta - (\tau_{xy} \Delta A \cos\theta)\cos\theta$$
$$- (\sigma_y \Delta A \sin\theta)\cos\theta + (\sigma_x \Delta A \cos\theta)\sin\theta$$

Using Trigonometric Identities
these can be written as :

$$\boxed{\begin{aligned} \sigma_{x'} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \sigma_{y'} &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\ \tau_{x'y'} &= -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{aligned}}$$

obtained by simply putting $\theta + 90°$ in the equation for $\sigma_{x'}$

18] Principal Stresses

From the above equations, we can do $\frac{d\sigma_{x'}}{d\theta}$ and $\frac{d\tau_{x'y'}}{d\theta} = 0$ to obtain the angles for which either $\sigma_{x'}$ or $\tau_{x'y'}$ are maximum or minimum:

$$\frac{d\sigma_{x'}}{d\theta} = 0 \Rightarrow 0 = -\frac{\sigma_x - \sigma_y}{2}(2\sin 2\theta) + 2\tau_{xy}\cos 2\theta$$

$$\Rightarrow \boxed{\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}}$$

θ_p : angle for principal normal stress (i.e. angle corresponding to max. & min. normal stress)

Similarly: $\frac{d\tau_{x'y'}}{d\theta} = 0 \Rightarrow \boxed{\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}}$

θ_s : angle at which element will be subjected to max^m & min^m shear

LECTURE-5 (Questions)

Find eigenvectors & eigenvalues for:

Q ~ $A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$

$$\left| \begin{pmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix} \right| = 0$$

$$(\lambda+5)(2+\lambda) - 4 = 0$$

$$2\lambda + \lambda^2 + 5\lambda + 10 - 4 = 0$$

$$\lambda^2 + 7\lambda + 6 = 0$$

$$\Rightarrow (\lambda+6)(\lambda+1) = 0$$

$$\Rightarrow \lambda = -1 \text{ or } -6$$

For $\lambda = -1 \Rightarrow \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For $\lambda = -6 \Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x_2} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Q ~ Diagonalize:

$A = \begin{pmatrix} -1 & 5 \\ 5 & 1 \end{pmatrix}$

$$\begin{vmatrix} -1-\lambda & 5 \\ 5 & 1-\lambda \end{vmatrix} = 0 \Rightarrow -(1-\lambda)(1+\lambda) - 25 = 0$$

$$\Rightarrow \lambda^2 - 26 = 0$$

$$\Rightarrow \cancel{\lambda^2 - 2\lambda - 24 = 0}$$

$$\Rightarrow \cancel{(\lambda-6)(\lambda+4) = 0}$$

$\cancel{\lambda = 6 \text{ or } -4}$ $\lambda = -5.099$ and 5.099

$\lambda = 6 \Rightarrow \begin{pmatrix} -7 & 5 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ and so on...

(after this, perform diagonalization)

LECTURE-5

19] Principal Stresses

Previously we have identified the planes → for which stress will be max or min

↓

Now, we have to find the magnitudes of these principal stresses

$$\begin{pmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

20] We want a relation for $\sigma_{x'}$ and $\tau_{x'y'}$ (which we can also denote by simply σ and τ)

↓

such that we have eliminated the presence of the Trig functions (and thus, also θ)

We know:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

and

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

So we can write:

$$\Big(\sigma - \underbrace{\frac{\sigma_x + \sigma_y}{2}}_{\sigma_{avg}}\Big)^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 \cos^2 2\theta + 2\left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta \cdot \tau_{xy}\sin 2\theta + \tau_{xy}^2 \sin^2 2\theta$$

and also, $\tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 \sin^2 2\theta + \tau_{xy}^2 \cos^2 2\theta$

Thus, by adding both, we get:

$$\boxed{(\sigma - \sigma_{avg})^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

21] From the previous equation, we get that:

I] Principal stress will occur when shear stress is zero

i.e. $$\boxed{\tau = 0 \quad \text{and} \quad \sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

II] Principal shear stress directly given by the above square root term

(Since it will occur when $\sigma = \sigma_{avg}$)

i.e. $$\boxed{\sigma = \sigma_{avg} \quad \text{and} \quad \tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

From these two expressions, we can obtain a relation b/w σ_1, σ_2 and τ_{max}:

$$\Rightarrow \quad \sigma_1 - \sigma_2 = 2\tau_{max}$$

or

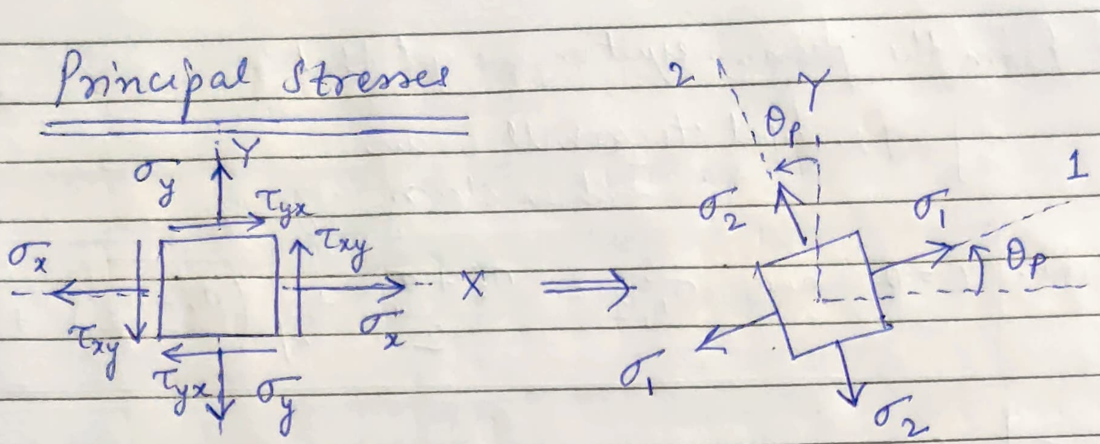
$$\boxed{\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}}$$

Also, another important property we obtain is:

$$\boxed{\sigma_1 + \sigma_2 = \sigma_x + \sigma_y}$$

$\Rightarrow$ (Later we shall see that the Trace of the stress tensor is an Invariant quantity)

22] Principal Stresses



Stresses in given Coord. system

Principal stresses

There will be a set of planes
↓
where only Normal stresses exist
↓
& shear stresses are zero

These planes are called → Principal planes
↓
and stresses are called → Principal stresses

* NOTE: Principal directions are orthogonal

23] Principal Stress in 3D

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Generally, $\sigma_1 > \sigma_2 > \sigma_3$

* NOTE: $\boxed{\text{Eigenvalues of Stress Tensor} = \text{Principal Stresses}}$

24] Eigenvalues and Eigenvectors of Matrices

of the form : $A\vec{x} = \lambda\vec{x}$ (eg: Schrödinger equation: $\hat{H}\psi = E\psi$)

for stresses, eg: $\underset{\sim}{\sigma}\vec{x} = \sigma_{principal}\vec{x}$

25] Eigenvalues

Starting with an equation of the form:

$$A\vec{x} = \lambda \vec{x}$$

* NOTE: The eigenvalue-matrix pair is unique in this way

$\vec{x} = \vec{0}$ is a solution (Trivial sol^n)

But for Non-Trivial solutions:

($\vec{x}$: eigenvectors
λ : eigenvalues)

$$A\vec{x} = \lambda \vec{x}$$

$$\Rightarrow \left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= \lambda x_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= \lambda x_n \end{aligned}\right\} \text{n-different equations}$$

$$\Rightarrow \begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \dots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}$$

$$\Rightarrow (A - \lambda I)\vec{x} = \vec{0}$$

(where I is the identity matrix)

or

$$\underbrace{\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix}}_{\text{characteristic matrix}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By Cramer's Rule, this homogeneous linear system of eq^{n}s has a non-trivial solution if and only if:

Characteristic Equation $\Leftarrow$ $\boxed{|A - \lambda I| = 0}$

26] Some important theorems :

i) A^T has same eigenvalues as A

ii) A and $A^n \longrightarrow$ same eigenvectors
But, if eigenvalue of A is λ, then for A^n it will be λ^n

iii) If $\det(A) = 0$, then one of the eigenvalues will be zero

iv) Symmetric matrix
↓
has ortho-normal basis of eigenvectors (i.e. all eigenvectors are $\perp^r$ to each other)

(* NOTE : Stress, strain
↓
both are symmetric matrices (technically 2^{nd} order tensors))

27] Diagonalization

Taking an $n \times n$ matrix A
Let's say its eigenvectors are given by:

$$X = \left(\{\vec{X}\} \ \{\vec{Y}\} \ \{\vec{Z}\} \right)$$

$$= \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

Then, $\boxed{D = X^{-1} A X}$ is called the diagonal matrix.
↓
the eigenvalues of A occur as entries on the main diagonal

$$\boxed{D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}} \Longleftarrow$$

28) We have the X matrix as:

$$X = \left(\{\vec{X}\}\ \{\vec{Y}\}\ \{\vec{Z}\}\right) = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

For this we know, $D = X^{-1} A X$

Now, for our case → we will be working with Stress & strain in place of A matrix

⟹ Since they are symmetric, this means $\vec{X}$, $\vec{Y}$ and $\vec{Z}$ are orthonormal (from previous theorems)

i.e. $\vec{X}\cdot\vec{Y} = 0$, $\vec{Y}\cdot\vec{Z} = 0$ and $\vec{X}\cdot\vec{Z} = 0$

From this it can be shown that ⟹ $\boxed{X^T = X^{-1}}$

and hence:

$\boxed{D = X^T A X}$, if X is orthogonal matrix

29] <u>Principal Stresses</u>

Instead of A matrix → using stress tensor, σ

We will get:

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

* Here, a row of X^T is actually the same as a row of the Transformation matrix A (the row represented the dir.^{n} cosines or components of a new axis with respect to the old axes)

⇓

* Thus, X^T will be equal to A

LECTURE-6

30] Stress Invariants

We know : $A\vec{x} = \lambda\vec{x}$

For stress : $\underset{\sim}{\sigma}\vec{x} = \sigma_{principal}\vec{x}$

$$A\vec{x} = \lambda\vec{x} \Rightarrow \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sigma_{principal}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Similarly, $(A - \lambda I)\vec{x} = \vec{0}$

$\Downarrow$ becomes

$$\begin{pmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, $\Rightarrow$ $$\det\begin{pmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma \end{pmatrix} = 0$$

$\Downarrow$ Upon expanding this we get :

$$\sigma^3 - \underbrace{(\sigma_{11} + \sigma_{22} + \sigma_{33})}_{I_1}\sigma^2 + \underbrace{(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2)}_{I_2}\sigma - \underbrace{(\sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{23}\sigma_{13}\sigma_{12} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2)}_{I_3} = 0$$

We can write this as :

$$\boxed{\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0}$$

$\rightsquigarrow$ this is the Characteristic Equation

Here, I_1, I_2 and I_3 are Invariants of the stress tensor

i.e. they do not change with coordinate transformation

I_1 = Trace → first invariant → given by:

$\sigma_{11} + \sigma_{22} + \sigma_{33}$

OR

σ_{ii} or $\sigma_{ij}\delta_{ij}$

I_2 = Sum of principal minors → 2nd invariant

This means:

$$I_2 = \underbrace{\begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix}}_{\text{by ignoring 3rd column \& 3rd row}} + \underbrace{\begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix}}_{\text{by ignoring 2nd column \& 2nd row}} + \underbrace{\begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix}}_{\text{by ignoring 1st column \& 1st row}}$$

$$= \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$$

I_3 = Determinant → 3rd invariant

$$= \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix} = \frac{1}{6}\left(\sigma_{ii}\sigma_{jj}\sigma_{kk} + 2\sigma_{ij}\sigma_{ik}\sigma_{ki} - 3\sigma_{ij}\sigma_{ij}\sigma_{kk}\right)$$

*NOTE: (1) For stress tensor (of any symmetric matrix)

eigen vectors CHANGE with coord. transformation

But eigen values remain same (since they represent principal stresses)

(2) Geometrically they are the 3 axes of the ellipsoid formed as $\sigma_{ij}x_i x_j$

(3) On the plane of max shear stresses → Normal stress may not be zero

But on plane of max normal stress (i.e. principal axes) → Shear stresses are zero.

31] Some important results:

1] Roots of $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$ $\Rightarrow$ are principal stresses

2] Trace of $\underset{\sim}{\sigma}$ is a scalar & invariant

(given by $\sigma_{11} + \sigma_{22} + \sigma_{33}$ i.e. $= \sigma_1 + \sigma_2 + \sigma_3$ } $\rightarrow$ principal stresses)

* Thus, $I_1 = \sigma_1 + \sigma_2 + \sigma_3$

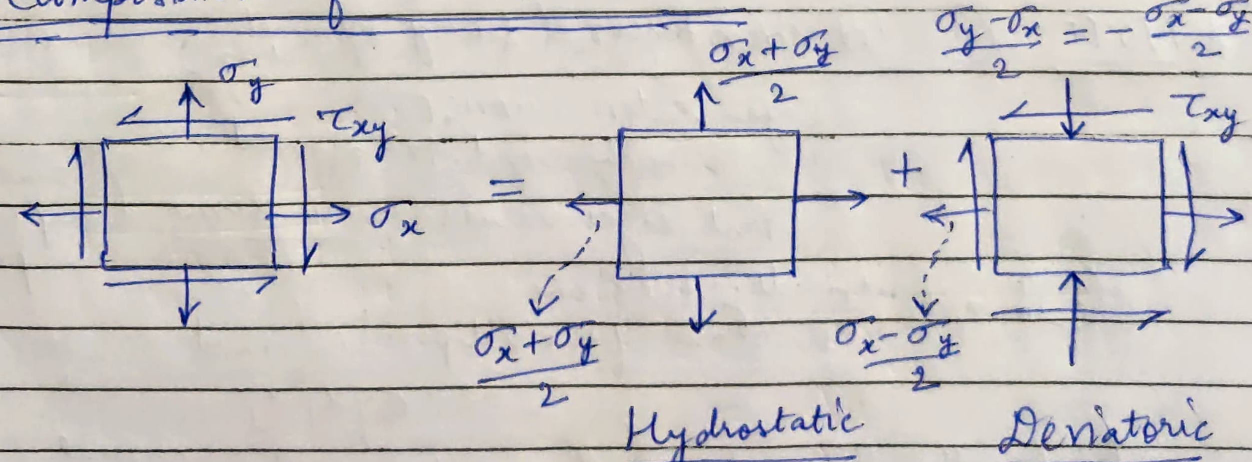
3] Maximum shear stress in 3D is:

$$\boxed{\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2}}$$

NOTE: Max.^{m} shear always occurs in coord. system orientation rotated 45° from principal coord. system

eg: If max & min principal stresses are σ_1 and σ_3 $\Rightarrow$ Max shear orientation is obtained by rotating the principal coord. system by 45° in the (1-3) plane.

32] Decomposition of Stress Tensor - 2D



Thus, we can decompose ("break") the stress tensor into:

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\sigma_x+\sigma_y}{2} & 0 \\ 0 & \frac{\sigma_x+\sigma_y}{2} \end{pmatrix}}_{\text{Hydrostatic}} + \underbrace{\begin{pmatrix} \frac{\sigma_x-\sigma_y}{2} & \tau_{xy} \\ \tau_{xy} & -\frac{\sigma_x-\sigma_y}{2} \end{pmatrix}}_{\text{Deviatoric}}$$

33] **Decomposition in 3D**

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} = \underbrace{\begin{pmatrix} \sigma_{11}-\sigma_n & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_n & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_n \end{pmatrix}}_{\text{Deviatoric}} + \underbrace{\begin{pmatrix} \sigma_n & 0 & 0 \\ 0 & \sigma_n & 0 \\ 0 & 0 & \sigma_n \end{pmatrix}}_{\text{Hydrostatic}}$$

where, $\sigma_n = \dfrac{\sigma_1+\sigma_2+\sigma_3}{3}$

34] **Decomposition Invariants** $\longrightarrow$ * NOTE: This discussion is important for the theory of plasticity

$$\underbrace{\begin{pmatrix} \sigma_{11}-\sigma_n & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_n & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_n \end{pmatrix}}_{\text{Deviatoric}} \Rightarrow I_1^{\text{Deviatoric}} = 0$$

$\Downarrow$

i.e. this is Pure Shear (bcz there must be a coord. transformation where all normal stresses will become zero)

$$\underbrace{\begin{pmatrix} \sigma_n & 0 & 0 \\ 0 & \sigma_n & 0 \\ 0 & 0 & \sigma_n \end{pmatrix}}_{\text{Hydrostatic}} \Rightarrow I_1^{\text{hydrostatic}} = I_1^{\sigma} = 3\sigma_h$$

$$= \sigma_{11}+\sigma_{22}+\sigma_{33}$$

$$= \sigma_1+\sigma_2+\sigma_3$$

$\left(\text{where, } \sigma_h = \dfrac{\sigma_1+\sigma_2+\sigma_3}{3}\right)$

35] Invariants of Deviatoric Stress Tensor

We know: $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$ (characteristic equation for stress tensor)

Let's say, characteristic eq^n for deviatoric stress tensor is:

$$s^3 - J_1 s^2 - J_2 s - J_3 = 0$$

(deliberately given $-$ve sign bcz it turns out to be that way when we calculate it)

From this we get:

1st Invariant: $J_1 = (\sigma_x - \sigma_m) + (\sigma_y - \sigma_m) + (\sigma_z - \sigma_m)$
$= 0$

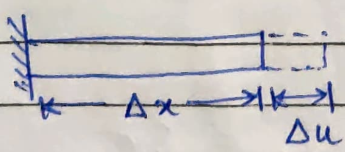
2nd Invariant: $J_2 = \frac{1}{6}\left[(\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + (\sigma_x - \sigma_y)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2\right)\right]$

3rd Invariant: J_3 = determinant of deviatoric stress matrix.

LECTURE-7

36] Strain

1D case:



becomes
$\Delta x \longrightarrow \Delta x + \Delta u$

$\therefore$ Strain (ε) $= \dfrac{\text{change in length}}{\text{original length}}$

$= \dfrac{\Delta u}{\Delta x}$

In the limit $(\Delta x \to 0)$ $\Rightarrow$ $\boxed{\varepsilon = \dfrac{\partial u}{\partial x}}$ (where, u = displacement)