# LECTURE-9

14/08/2023

**NOTE:** We know

$$[\eta^{-1}][\eta] = \mathbb{1}_{n\times n}$$

identity matrix

we here define $\eta^{-1}$ as:

$$\eta^{cb}\eta_{ab} = \delta^{c}_{a}$$

**NOTE:** Only some $T^{k}_{j}$ can have an inverse (since it is actually a matrix)

(if we want it for $T_{ij}$ $\Rightarrow$ we must first raise one index)

**NOTE:** We saw previously:

$$\mathcal{L}(\phi, \partial_k\phi) = -(\partial_k\phi)(\partial^k\phi) - \underbrace{\tfrac{1}{2}m^2\phi^2}_{V(\phi)}$$

**NOTE:**

| Classical particle | Classical Field Theory |
|---|---|
| $t$, $\underset{\sim}{x}(t)$ | $\phi(t, \underset{\sim}{x})$ |
| $\downarrow$ the d.o.f. are given by function of time | $\Downarrow$ |
| $\downarrow$ thus, here we have 3 degrees of freedom | $\{\phi_{\underset{\sim}{k}}(t)\}$ $\rightsquigarrow$ here there are (for a field) infinite no. of degrees of freedom |

$$\left(\phi(t, \underset{\sim}{x}) = \int_0^\infty \frac{d^3\underset{\sim}{k}}{(2\pi)^3}\, \phi_{\underset{\sim}{k}}(t)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\right)$$

$\Downarrow$ Inverse Fourier Transform

$\hookrightarrow$ this is why there are infinite d.o.f

50] We saw that the eq.$^n$ of motion is obtained from:

$$\partial_k \Pi^k = \frac{\partial \mathcal{L}}{\partial \phi}$$

Taking: $\mathcal{L} = -(\partial_k\phi)(\partial^k\phi) - \frac{1}{2}m^2\phi^2 \Rightarrow$ we get:

Eq.$^n$ of Motion: $-\partial_k\partial^k\phi + m^2\phi = 0$

Now, since: $\partial_k \partial^k = -\partial_0^2 + \nabla^2$

$\Rightarrow$ we have: $\ddot{\phi} - \nabla^2 \phi + m^2 \phi = 0$

Using the fact that: $\phi(t, \underset{\sim}{x}) = \int_0^\infty \frac{d^3 \underset{\sim}{k}}{(2\pi)^3} \phi_{\underset{\sim}{k}}(t) e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}}$ (Inverse Fourier Transform)

$\Downarrow$

this is applied to that eq$^n$: and then the common integral can be removed:

$$\therefore \quad \ddot{\phi}_{\underset{\sim}{k}} + |\underset{\sim}{k}|^2 \phi_{\underset{\sim}{k}} + m^2 \phi_k(t) = 0$$

$$\Rightarrow \boxed{\ddot{\phi}_{\underset{\sim}{k}} + \omega_{\underset{\sim}{k}}^2 \phi_{\underset{\sim}{k}} = 0}$$

thus the $\phi_{\underset{\sim}{k}}$ behaves like a Harmonic Oscillator

(from $E_{\underset{\sim}{k}}^2 = |p_{\underset{\sim}{k}}|^2 + m^2$

$\Rightarrow \hbar^2 \omega_{\underset{\sim}{k}}^2 = \hbar^2 |\underset{\sim}{k}|^2 + m^2$

and by taking units where $\hbar = 1$ we get the eq$^n$.)

{ NOTE: If we quantized this theory $\downarrow$ then we would get particles with energy $E_{\underset{\sim}{k}}$ and momentum $p_{\underset{\sim}{k}}$ }

51] We have discussed the scalar field Lagrangian $\downarrow$ Now we want to build a Lagrangian for the Electromagnetic field;

$$\mathcal{L}_{EM}(A_k, \partial_i A_k)$$

* here, $A_\ell(t, \underset{\sim}{x}) = \int \frac{d^3 \underset{\sim}{k}}{(2\pi)^3} \tilde{A}_\ell(t, \underset{\sim}{k}) e^{i \underset{\sim}{k} \cdot \underset{\sim}{x}}$ (Inverse Fourier Transform for $A_\ell(t, \underset{\sim}{x})$)

this is written as $\tilde{A}$ just to show that it is a different function from A

52] Electromagnetism obeys the <u>Principle of Superposition</u>

↓

thus the Lagrangian must be of linear form

$$\mathcal{L}(\partial_i A_k)$$

Some Invariants we know:

$$F_{ab}F^{ab} = 2(|\underset{\sim}{B}|^2 - |\underset{\sim}{E}|^2)$$

$$\epsilon_{abcd}F^{ab}F^{cd} = -8\,\underset{\sim}{E}\cdot\underset{\sim}{B}$$

these are the 2 potential terms which might contribute to $\mathcal{L}$

(where, $F_{ab} = \partial_a A_b - \partial_b A_a$)

Also, we require Gauge invariance:

↓

and since $F^{ab}$ is invariant as well

↓

∴ We can build the Lagrangian as: $\mathcal{L}(F^{ab})$

{ <u>NOTE</u>: 
$$\epsilon^{abcd}F_{ab}F_{cd} = 2\epsilon^{abcd}F_{ab}\partial_c A_d$$
$$= 2\partial_c(\epsilon^{abcd}F_{ab}A_d)$$

So, $$A = \int_V d^4x\, \partial_c(\epsilon^{abcd}F_{ab}A_d) = \int_{\partial V} d^3x\, n_c\,\epsilon^{abcd}F_{ab}A_d$$

*( this is already a surface term & thus, will not contribute to the action ) }

Thus, out of $F_{ab}F^{ab}$ and $\epsilon_{abcd}F^{ab}F^{cd}$, we only need to consider $F_{ab}F^{ab}$

$$\Rightarrow \therefore \mathcal{L}(F^{ab}) \propto \int d^4x\, F_{ab}F^{ab} \Rightarrow$$ In CGS units for Maxwell's eq^n's we must use:

$$\mathcal{L}_{EM}(F^{ab}) = -\frac{1}{16\pi}F_{ab}F^{ab}$$

53] We have:

$$A = \underbrace{\int d^4x\, J^k A_k}_{\text{Source}} - \frac{1}{16\pi}\underbrace{\int d^4x\, F_{ab}F^{ab}}_{\text{Field}}$$

(in CGS units)

{ NOTE: We saw that the first 2 Maxwell's eq$^n$s. are simply obtained as:

$$\partial_c \, {}^{(*)}F^{cd} = 0$$

$$\hookrightarrow \quad \nabla \cdot \underset{\sim}{B} = 0 \qquad \nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$

} these are source free Maxwell's eq$^n$s

(i.e. they are NOT eq$^n$s of motion since we don't need to vary the action to obtain them
↓
they are simply identities ) }

Thus, we have:

$$\delta \mathcal{A} = \int d^4x \, J^k \delta A_k - \frac{1}{4\pi} \underbrace{\int d^4x \, F^{ab} \delta(\partial_a A_b)}$$

{ since, $\delta(F_{ab}F^{ab}) = 2F^{ab}\delta F_{ab}$
$= 4F^{ab}\delta(\partial_a A_b)$ }

$$\Downarrow$$

$$\int d^4x \, F^{ab} \partial_a(\delta A_b)$$

$$\Rightarrow \quad \delta \mathcal{A} = \int d^4x \, J^b \delta A_b - \frac{1}{4\pi}\left[\int d^4x \, \partial_a (F^{ab}\delta A_b) - (\partial_a F^{ab})\delta A_b\right]$$

$$\therefore \Rightarrow \quad \delta A = \int d^4x \left(J^b + \frac{1}{4\pi}\partial_a F^{ba}\right)\delta A_b - \frac{1}{4\pi}\underbrace{\int_V d^4x \, \partial_a(F^{ab}\delta A_b)}_{-\frac{1}{4\pi}\int_{\partial V} d^3x \, n_a F^{ab} \delta A_b}$$

* {NOTE: Check the exact calculation from Padmanabhan's textbook}

* Thus, from here we get:

$$\boxed{\partial_a F^{ab} = -4\pi J^b} \leftarrow$$

Source gives 2 maxwell's eq$^n$s

From here we can see: $\partial_b \partial_a F^{ab} = -4\pi \partial_b J^b = 0 \iff \boxed{\partial_a J^a = 0}$

The Continuity eq$^n$

{ Hence, the maxwell's eq$^n$s imply charge conservation, & charge conservation implies the Maxwell eq$^n$s }

54] For scalar field we saw:

$$\Pi^k = \frac{\partial \mathcal{L}}{\partial(\partial_k \phi)}$$

For the vector field $A_\ell$ $\quad (A_0 = -\phi, \underset{\sim}{A})$

we have:

$$\frac{\partial \mathcal{L}}{\partial(\partial_k A_\ell)} = p^{(k)\ell}$$

↳ i.e. for every $k$ we have a 4-vector

$$* \quad p^{(i)k} = \frac{\partial \mathcal{A}}{\partial A_k} = \frac{F^{ik}}{4\pi}$$

{NOTE: This is NOT really a 2-index tensor, since it doesn't transform that way}

$$\text{Thus, } p^{(i)0} = \frac{F^{i0}}{4\pi} = E^\alpha$$

* Exercise: (for Minor) For $\mathcal{L} = \frac{-1}{16\pi} \underbrace{\frac{2(|\underset{\sim}{B}| - |\underset{\sim}{E}|^2)}{F_{ab}F^{ab}}}$ $\Rightarrow$ write this in terms of $(\phi, \underset{\sim}{A})$

Ques: If we take $\mathcal{L} = \dot{x}$
$\Rightarrow$ find Hamiltonian $\mathcal{H}(p, x)$

(NOTE: This Lagrangian happens to be of non-invertible type)

# LECTURE - 10

18/08/2023

**NOTE:** when building a Lagrangian
$\downarrow$
we need to identify the dynamical variables
$\downarrow$
& then build Lagrangian & Eq$^{n}$s of motion for those dynamical variables

$$\mathcal{L}\left([\underset{\sim}{x}, \underset{\sim}{v}], [\phi, \partial\phi], [A, \partial A], \ldots\right)$$

OR OR

(you vary against the dynamical variable you want)

eg: For fields we use variation where $\phi$ is taken as dynamical variable

$$\mathcal{A}[\phi, \ldots] = \int d^4x \; \underbrace{\mathcal{L}(\phi, \partial\phi)}_{\frac{1}{2}\partial_a\phi\,\partial^a\phi - V(\phi)}$$

$x$ here is NOT a dynamical variable

* **NOTE:** Reference for Functional differentiation:
The First Appendix of: "Quantum effects in Gravity" by Wihitzki & Mukhanov

{this is important for understanding functional derivatives}

eg: $\dfrac{df(x)}{dx}$ here $x$ is varied ; $\dfrac{\delta \mathcal{A}}{\delta x}$ functional derivative (i.e. the function itself is being varied)

**NOTE:** We previously saw:

$$\mathcal{L}_{EM}(\cancel{A}, \partial A) = -\frac{1}{16\pi} F_{ab} F^{ab}$$

depends only on this

$$\Rightarrow \partial_k F^{kl} = 4\pi J^l$$

(which gave the 2 source Maxwell's equations)

55] We know the Maxwell's equations are! (constants are ignored)

$$\nabla \cdot \underset{\sim}{E} = \rho \quad -①$$

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} \quad -②$$

$$\nabla \cdot \underset{\sim}{B} = 0 \quad -③$$

$$\nabla \times \underset{\sim}{B} = \underset{\sim}{J} + \frac{\partial \underset{\sim}{E}}{\partial t} \quad -④$$

implies $\underset{\sim}{B} = \nabla \times \underset{\sim}{A} \quad -③^*$

$$\Rightarrow \nabla \times \underset{\sim}{E} = -\nabla \times \dot{\underset{\sim}{A}}$$

$$\Rightarrow \nabla \times (\underset{\sim}{E} + \dot{\underset{\sim}{A}}) = 0$$

$$\Rightarrow \underset{\sim}{E} + \dot{\underset{\sim}{A}} = -\nabla \phi$$

$$\Rightarrow \underset{\sim}{E} = -\nabla\phi - \dot{\underset{\sim}{A}} \quad -②^*$$

56] $\underbrace{\text{Flat space}}_{\text{fourier domain}}$ and source free ($\rho = 0$, $\underset{\sim}{J} = 0$):

We define: $$\underset{\sim}{A}(t, \underset{\sim}{x}) = \int \frac{d^3\underset{\sim}{k}}{(2\pi)^3} \underset{\sim}{A}_{\underset{\sim}{k}}(t)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}$$

$$\phi(t, \underset{\sim}{x}) = \int \frac{d^3\underset{\sim}{k}}{(2\pi)^3} \phi_{\underset{\sim}{k}}(t)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}$$

* these kinds of bases can only be used for flat space

Thus this means:

②* : $\underset{\sim}{E} = -\nabla\phi - \dot{\underset{\sim}{A}} \xrightarrow[\text{becomes}]{} \underset{\sim}{E}_{\underset{\sim}{k}} = -i\underset{\sim}{k}\phi_{\underset{\sim}{k}} - \dot{\underset{\sim}{A}}_{\underset{\sim}{k}}$

③* : $\underset{\sim}{B} = \nabla \times \underset{\sim}{A} \xrightarrow[\text{becomes}]{} \underset{\sim}{B}_{\underset{\sim}{k}} = i\underset{\sim}{k} \times \underset{\sim}{A}_{\underset{\sim}{k}}$

Also, ④ : $\xrightarrow{\text{becomes}}$ $i\underset{\sim}{k} \times \underset{\sim}{B}_{\underset{\sim}{k}} = \dot{\underset{\sim}{E}}_{\underset{\sim}{k}}$

and ① $\xrightarrow[\text{becomes}]{}$ $\underset{\sim}{k} \cdot \underset{\sim}{E}_{\underset{\sim}{k}} = 0$

* We will now be working in <u>Coulomb Gauge</u> ($\phi = 0$, $\underset{\sim}{A} \neq 0$) → actually we take a gauge choice such that also: $\nabla \cdot \underset{\sim}{A} = 0$

i.e. $\left\{ \begin{array}{l} \underset{\sim}{A} \to \underset{\sim}{A} + \nabla f = A' \\ \phi \to \phi - \frac{\partial f}{\partial t} = \phi' \end{array} \right.$ ⇝ we choose such an $f$ so that $\phi' = 0$ }

[NOTE: Refer to:
"Quantum field theory: How What Why"
by Paddy
[Topic: Gauge Transformations in Electrodynamics)]

NOTE: ( the fact that
$\underset{\sim}{k} \cdot \underset{\sim}{A}_{\underset{\sim}{k}} = 0$
is a constraint on $\underset{\sim}{A}_k$ )

In Coulomb Gauge we have:

$$i\underset{\sim}{k} \times (i\underset{\sim}{k} \times \underset{\sim}{A}_{k}) = -\ddot{\underset{\sim}{A}}_{k}$$

(since here:
$\underset{\sim}{E}_{\underset{\sim}{k}} = -\dot{\underset{\sim}{A}}_{k}$
$\underset{\sim}{B}_{k} = i\underset{\sim}{k} \times \underset{\sim}{A}_{k}$
$i\underset{\sim}{k} \times \underset{\sim}{B}_{k} = \dot{\underset{\sim}{E}}_{k}$)

$$\Rightarrow |\underset{\sim}{k}|^2 \underset{\sim}{A}_{k} = -\ddot{\underset{\sim}{A}}_{k}$$

$$\Rightarrow \boxed{\ddot{\underset{\sim}{A}}_{k} + \omega_{\underset{\sim}{k}}^2 \underset{\sim}{A}_{k} = 0}$$

(where, $\omega_{\underset{\sim}{k}}^2 = c^2 |\underset{\sim}{k}|^2$ )

Thus, Electromagnetic field (and thus, "light") is simply a collection of <u>harmonic oscillators</u>.

* Hence, due to Flat space + Source free and Gauge Invariance, light (photons) despite being spin 1 particles have only 2 polarizations.

[NOTE: Hence also, when we write the Lagrangian:

$$\frac{p^2}{2} - \frac{\omega^2 x^2}{2}$$

ignoring $m$

$\underset{\sim}{E}_k$ plays the role of $\underset{\sim}{p}$

$\underset{\sim}{A}_k$ or $\underset{\sim}{B}_k$ plays the role of $\underset{\sim}{x}$

$\Rightarrow$ so just as $\{q, p\} = 1$
$\downarrow$
$[q, p] = i\hbar$
Similarly is for E and B

i.e. $\underset{\sim}{E}$ and $\underset{\sim}{B}$ also CANNOT be measured simultaneously in quantum mechanics:

$$[\underset{\sim}{E}, \underset{\sim}{B}] = i\hbar\,\delta(\underset{\sim}{x}-\underset{\sim}{y})$$ ]

* NOTE: Electrodynamics survives relativity, since it is already Lorentz Invariant.

57] We know: Gravity

$$\left.\begin{aligned} m\frac{d^2\underset{\sim}{x}}{dt^2} &= -m\nabla\phi \\ \nabla^2\phi &= 4\pi G\rho \end{aligned}\right\}$$ this is Newtonian Gravity *

(i.e. $\underbrace{\rho(\underset{\sim}{x})}_{\text{mass density}} \xrightarrow[\text{helps us obtain}]{} \phi(\underset{\sim}{x})$ )

(1) One major problem: Change is instantaneous

(i.e. since $\rho$ and $\phi$ are not dependent on $t$, thus change in $\rho$ immediately changes $\phi(\underset{\sim}{x})$ )

this is NOT compatible with relativity

(2) In electrodynamics:

$$m\frac{du_i}{d\tau} = qF_{ik}u^k$$

(particles are acted upon differently)

But in gravitation:
$$\cancel{m}\frac{d^2\underset{\sim}{x}}{dt^2} = -\cancel{m}\nabla\phi$$

i.e. all particles are acted upon in the same way

58] One thing we could try is (for solving (1)):

Take: $\phi(t, \underset{\sim}{x})$, $\rho(t, \underset{\sim}{x})$

and now: $\nabla^2 \xrightarrow[\text{becomes}]{} \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)$

$\left\{ \text{i.e. } \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) \phi(t, \underset{\sim}{x}) = 4\pi G \rho(t, \underset{\sim}{x}) \right\}$

(just as in E.M. $\rho$ was a component of $J^k = (\rho, \underset{\sim}{J})$

what is $\rho$ (mass density) a component of? )

59] Mass density, $\rho = \frac{M}{V} = \frac{\overset{\text{mass}}{\sum m_i}}{V}$ or $\frac{\sum_i \text{mass-energy}}{V}$

Transforming to moving frame:

$V' = V/\gamma$, $m \longrightarrow \gamma m$

This would imply that $\rho' \sim \gamma^2 \rho$

* (but this should not happen unless $\rho$ is a 2-index object)

$\{ \text{eg: } T_{a'b'} = L^i_{a'} L^j_{b'} T_{ij} \}$

this is going to be the source of gravity

$\gamma \quad \gamma$

Thus, if $\rho$ should be a 2-index object

$\therefore$ * $\phi$ should ALSO be a 2-index object

It was observed that
if we change the
metric for flat space
to instead a field (that changes from
place to place):

$$ds^2 = \eta_{ab}\, dx^a dx^b \longrightarrow g_{ab}(t, \underset{\sim}{x})\, dx^a dx^b$$

is Also symmetric (has 10 components)

* it turns out that $\phi$ is one of the components of this metric in curved spacetime

{ there are 2 derivatives of the metric tensor $g$:

$\partial g$ , $\partial^2 g$

this relates
* to Curvature }

# LECTURE-11

19/08/2023

NOTE: For relativistic particles we have:

$$\mathcal{A} = -m\int d\tau = -m\int \sqrt{-\frac{dx_a}{d\tau}\frac{dx^a}{d\tau}}\, d\tau$$

alternatively, a diff. lagrangian could be used as well (this would give the same eq?s of motion):

$$\mathcal{A} = \int \frac{m}{2}\dot{x}_a \dot{x}^a \, d\tau$$

* One problem with these is for massless particles (since $m = 0$ & $d\tau = 0$ for them)

There we change this to:

$$\mathcal{A} = \int \underbrace{\left(\frac{1}{2}\frac{\dot{x}_a\dot{x}^a}{e} - \frac{m^2 e}{2}\right)}_{\mathcal{L}(\dot{x}^a,\, e)} d\lambda \quad , \text{where } e = e(\lambda)$$

non-dynamical ⇝ "means that its NOT changing"

This is called the:
* BRST
Einbein Action for relativistic point particle

This Lagrangian can be varied as follows:

$$\frac{\partial \mathcal{L}}{\partial e} = -\frac{1}{2e^2}\dot{x}_a\dot{x}^a - \frac{m^2}{2} = 0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{e}}\right)$$

$$\Rightarrow \therefore e = \frac{1}{m}\sqrt{-\dot{x}_a\dot{x}^a}$$

So also, $\dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}^a}\right) = \dfrac{\partial \mathcal{L}}{\partial x^a} = 0$

$$\therefore \frac{d}{dt}\left(\frac{\dot{x}_a}{e}\right) = 0 \Rightarrow \frac{1}{e}\ddot{x}_a = 0$$

NOTE: A dynamic variable is that whose equation of motion is second-order in time.

60] We saw that, in Newtonian gravity
↓
the source of gravity is :

$$\nabla^2 \phi = 4\pi G \rho \rightsquigarrow \rho \text{ was the source of gravity here}$$

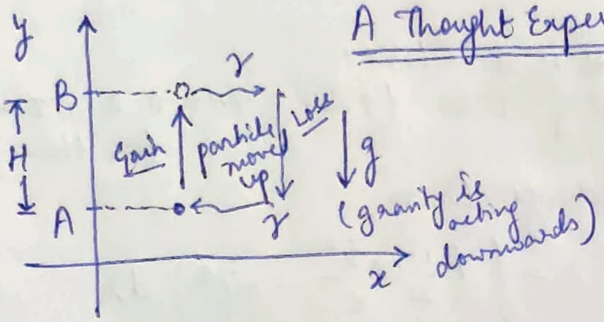
$$m \frac{d\underset{\sim}{v}}{dt} = -m \nabla \phi$$

⇒ we would now want this to be $\rho(t, \underset{\sim}{x})$
↓
but that implies we would require a 2-index object

Also:

$$ds^2 = \eta_{ab}\, dx^a dx^b \longrightarrow g_{ab}(x^k)\, dx^a dx^b$$

61] A Thought Experiment




At A : $E_A = mc^2 = \hbar\omega_A$

At B : $E_B = mc^2 + mgH$
$= \hbar\omega_B$

↓ so we can write :

when particle goes up, it gains energy
↓
and when it goes down it loses energy

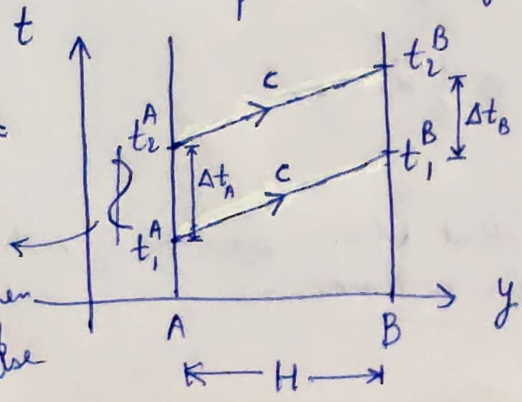
$$\hbar\omega_B = mc^2 + mgH$$
$$= mc^2\left(1 + \frac{gH}{c^2}\right)$$

$$\Rightarrow \boxed{\omega_B = \omega_A\left(1 + \frac{gH}{c^2}\right)}$$

the relation of how frequency of photon changes

Let's consider the following scenario, where a wave pulse is sent from A to B

* geometrically we would expect $\Delta t_A = \Delta t_B$




A receives front of pulse first, & then back of pulse

Now, no. of pulses recieved by both must be same!

$$\therefore N = \frac{\Delta t_A}{\omega_A} = \frac{\Delta t_B}{\omega_B}$$

For light pulse, geometrically we expect $\Delta t_A = \Delta t_B$

* But if $\omega_A \neq \omega_B$ ⇒ then $\boxed{\Delta t_A \neq \Delta t_B}$

So it turns out that in reality $\Delta t_A \neq \Delta t_B$ * { despite the fact that speed of light 'c' MUST be same in all frames }

↓

the actual reason for this is that we are * treating the 2 gaps ($\Delta t_A$ & $\Delta t_B$) on a flat plane (i.e. the entire spacetime plot is being assumed wrongly to be flat)

↓

but in reality this is not the case

62] Now, $\Delta t_A = \Delta t_B \dfrac{w_A}{w_B} = \Delta t_B \left(1 - \dfrac{gH}{c^2}\right)$

writing, $gH = \phi_B - \phi_A$

$$= \Delta t_B \left(1 - \frac{\phi_B}{c^2} + \frac{\phi_A}{c^2}\right)$$

this can be approximately given by:

$$\Delta t_A = \underbrace{\Delta t_B \left(1 - \frac{\phi_B}{c^2}\right)}_{\Delta t_\infty}\left(1 + \frac{\phi_A}{c^2}\right)$$

$$\Rightarrow \boxed{\Delta t_A = \Delta t_\infty \left(1 + \frac{\phi_A}{c^2}\right)}$$

By squaring this eq$^n$, and using the approximation that $\dfrac{\phi}{c^2} \ll 1$ : called * Weak-field limit

$$(\Delta t)^2 = (\Delta t_\infty)^2 \left(1 + \frac{2\phi}{c^2}\right)$$

Thus, in the weak-field limit we see:

$$ds^2 = \eta_{ab}\, dx^a dx^b$$

$$= -c^2 dt^2 + |d\underset{\sim}{x}|^2 \xrightarrow[\text{changes to}]{} \underbrace{g_{00}\, dt^2 + g_{01}\, dt\, dx^1 + \ldots}$$

⇓ is given by

$$ds^2 = -c^2\underbrace{\left(1 + \frac{2\phi}{c^2}\right)}\, dt^2 + |d\underset{\sim}{x}|^2$$

* this shows that gravity/acceleration also causes time-dilation

we see that gravity/acceleration causes the rate of ticking of the clock to change

NOTE: The principle of equivalence is applicable
only locally (in small spatial & temporal interval)

$\downarrow$

where acceleration
doesn't really vary

63] Thus we see that
we must now use:

$$ds^2 = g_{ab}\, dx^a\, dx^b$$

$\downarrow$

where, $g_{ab}(x^k)$

We want to know:

(1) How particles move in this spacetime?

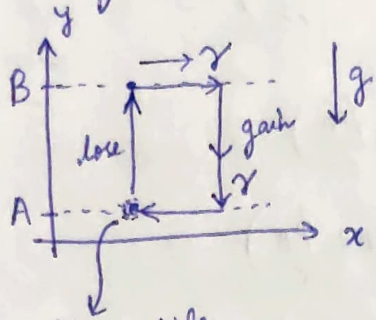
(2) What is the source of this $g_{ab}$?

(i.e. what the hell is causing this curvature?)

# LECTURE-12

21/08/2023

**NOTE:** Previously we saw: * [Correction to previous discussion]




the particles are moved up
↓
* they get converted to a photon
↓
and the photon comes down and gains energy
* (converted back to particles)

* {"This concept is responsible for "Gravitational redshift"}

$$E_A = mc^2 = \hbar\omega_A$$
$$E_B = mc^2 - mgh = \hbar\omega_B$$

$$\hbar\omega_B = mc^2 - mgh$$
$$= mc^2\left(1 - \frac{gh}{c^2}\right)$$
$$= \hbar\omega_A\left(1 - \frac{gh}{c^2}\right)$$

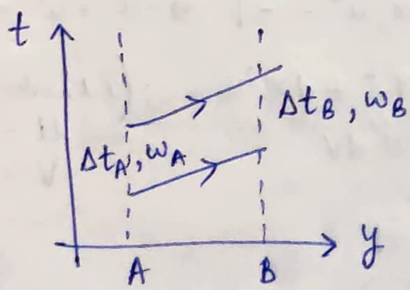
$$\Rightarrow \omega_B = \omega_A\left(1 - \frac{gh}{c^2}\right)$$

and

$$\omega_A = \omega_B\left(1 + \frac{gh}{c^2}\right)$$

$\left\{\text{for } \frac{gh}{c^2} \ll 1\right\}$

**NOTE:** * {Another correction}




$$N = \omega_A \Delta t = \omega_B \Delta t$$

Since $\omega_B \neq \omega_A$

$$\Rightarrow \Delta t_A \neq \Delta t_B$$

* Thus:

$$\Delta t_A = \frac{\omega_B \Delta t_B}{\omega_A} = \Delta t_B\left(1 - \frac{gh}{c^2}\right)$$
$$= \Delta t_B\left(1 - \frac{\phi_B}{c^2} + \frac{\phi_A}{c^2}\right)$$

{where, $gh = \phi_B - \phi_A$}

$$\therefore \Delta t_A \simeq \underbrace{\Delta t_B\left(1 - \frac{\phi_B}{c^2}\right)}_{\Delta t_f}\left(1 + \frac{\phi_A}{c^2}\right)$$

(approximation)

* Hence:

$$\Delta t^2 = \Delta t_f^2\left(1 + \frac{\phi}{c^2}\right)^2 \Leftarrow$$
$$\simeq \left(1 + \frac{2\phi}{c^2}\right)(\Delta t_f)^2$$

For the relation $\omega_B = \omega_A\left(1 - \frac{gh}{c^2}\right)$

$\Downarrow$

we can define :

$$* \quad \omega_\infty = \omega_A\left(1 + \frac{\phi}{c^2}\right)$$

{this can be with plus sign since $\phi$ has a $-$ve value }

( We have :

$$\omega_B = \omega_A\left(1 - \frac{\phi_B}{c^2} + \frac{\phi_A}{c^2}\right)$$

For $\omega_\infty$, $\phi \to 0$

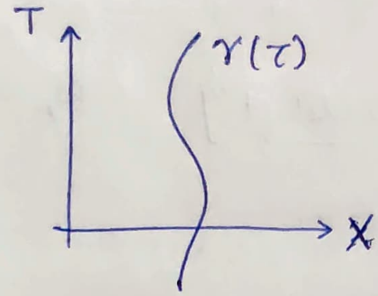
and thus,

$$\phi_A = -gh$$

$$\therefore \Rightarrow \omega_\infty = \omega_A\left(1 + \frac{\phi}{c^2}\right)$$

)

64] We have particles moving in an inertial frame:




For timelike curves, $u_k u^k = -1 \Rightarrow -u^{0^2} + u^{1^2} = -1$ — (1)

In 1-D: $ds^2 = -dT^2 + dX^2$ ; (introduce: $U = T - X$, $V = T + X$)

$= -dU\,dV$

$\underbrace{\hphantom{U = T+X}}$

$\Downarrow$

$\dot{U} = \dot{T} - \dot{X}$

$\dot{V} = \dot{T} + \dot{X}$

Now, $-\dot{T}^2 + \dot{X}^2 = -1$

(from (1) )

{ NOTE: $u^k = \frac{dx^k}{d\tau} = \left(\frac{dT}{d\tau}, \frac{dX}{d\tau}\right) = (\dot{T}, \dot{X})$ }

Thus, this can be written as :

$$\dot{U}\dot{V} = +1 \qquad -(2)$$

also, we define: $\ddot{U}\ddot{V} = -K^2(\tau)$ — (3)

Differentiating ②:

$$\ddot{u}\dot{V} + \dot{u}\ddot{V} = 0$$

$$\Rightarrow \frac{\ddot{u}}{\dot{u}} = \frac{-\ddot{V}}{\dot{V}}$$

Now from ②, since $\dot{V} = \frac{1}{\dot{u}}$

$$\Rightarrow \frac{\ddot{u}^2}{\dot{u}^2} = -\ddot{V}\ddot{u} = K^2(\tau) \quad \text{(from ③)}$$

$$\Rightarrow \frac{\ddot{u}}{\dot{u}} = \pm K(\tau)$$

choosing +ve sign for $\frac{\ddot{u}}{\dot{u}}$ (and thus, −ve for $\frac{\ddot{V}}{\dot{V}}$):

$$\frac{\ddot{u}}{\dot{u}} = +K(\tau) \quad ; \quad \frac{\ddot{V}}{\dot{V}} = -K(\tau)$$

$$\Rightarrow \dot{u} = e^{+\int K(\tau)\,d\tau} \quad \text{and} \quad \dot{V} = e^{-\int K(\tau)\,d\tau}$$

{NOTE: All these integrals are on the trajectory, since they are being performed for the observer}

* For uniform acceleration: $K(\tau) = K$

$$\Rightarrow \dot{u} = e^{K\tau} \quad ; \quad \dot{V} = e^{-K\tau}$$

Thus $\Rightarrow$ $u = \int e^{K\tau}\,d\tau \quad ; \quad V = \int e^{-K\tau}\,d\tau$

$$= \frac{1}{K}e^{K\tau} + \underbrace{c_1} \qquad = -\frac{1}{K}e^{-K\tau} + \underbrace{c_2}$$

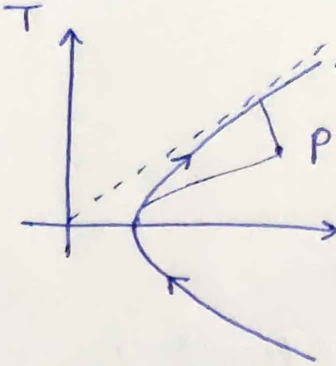
these constants of integration can be set (here in the discussion they won't be relevant)

Hence, $T - X = \frac{1}{K}e^{K\tau} \quad ; \quad T + X = -\frac{1}{K}e^{-K\tau}$

$$\Rightarrow \boxed{X = \underbrace{\frac{1}{K}\cosh K\tau}_{*\ g(\tau)} \quad ; \quad T = \overbrace{\frac{1}{K}\sinh K\tau}^{f(\tau)\ *}}$$

(for uniform acceleration)

65] This has been performed in the frame of global observer!




P ~> global observer gives coord.s $(T_P, X_P)$ ~> these coords. will instead be some $(t_p, x_p)$

But for an observer along trajectory (who is at rest in his own frame)

{ for this frame we can use lowercase letters: $x, t, u, v$ }

It turns out that

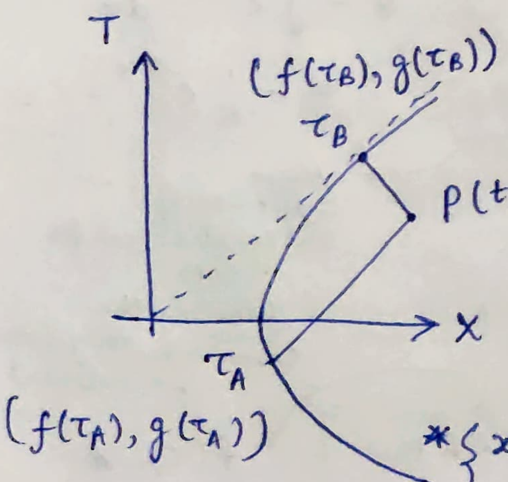
$$\underbrace{ds^2 = -dU\,dV = f(u,v)\,du\,dv}_{\text{i.e. any transformation will only require a change in this manner}}$$

We know:

* $ds^2 = f(u,v)\,du\,dv$ ⇒ (this can be shown ahead)
* $\dot{U} = e^{K\tau}$ ; $\dot{V} = e^{-K\tau}$ } ⇒ we have found this
* $X_{traj.} = \frac{1}{K}\cosh K\tau$ ; $T_{traj.} = \frac{1}{K}\sinh K\tau$

i.e. $g(\tau)$ ; i.e. $f(\tau)$

66] Now we want to go away from the trajectory (in the global frame itself)




$$T - f(\tau_A) = X - g(\tau_A)$$

{ NOTE: $t, x$ are the accelerated observer's coordinates }

$$f(\tau_B) - T = -(g(\tau_B) - X)$$

* $\left\{ x = \frac{(\tau_B - \tau_A)}{2} \quad t = \frac{(\tau_B + \tau_A)}{2} \right\}$

$\therefore \quad T - X = f(\tau_A) - g(\tau_A) = \frac{1}{K}\sinh K\tau_A - \frac{1}{K}\cosh K\tau_A$

$T + X = f(\tau_B) + g(\tau_B) = \frac{1}{K}\sinh K\tau_B + \frac{1}{K}\cosh K\tau_B$

$$\Rightarrow \quad T - X = -\frac{1}{K} e^{-K(t-x)}$$

$$T + X = \frac{1}{K} e^{K(t+x)}$$

* Now, $ds^2 = -dT^2 + dX^2$

$$= e^{2Kx}(-dt^2 + dx^2)$$

$$= -e^{2Kx}dt^2 + \underbrace{e^{2Kx}dx^2}_{d\bar{x}^2}$$

(we can write, $\bar{x} = \int e^{Kx}dx$ )

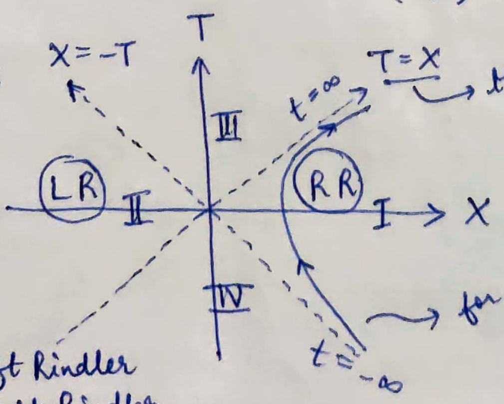
Thus, $ds^2 = -f(\bar{x})\, dt^2 + d\bar{x}^2$

* Conceptually, in the global frame (inertial frame)
↓
it is observing something undergoing acceleration
↓
However, within that accelerated (i.e. non-inertial) frame, the observer would find himself to be at rest
↓
This means that the effect

* of acceleration in coord. transformation in the interval gets instead interpretted as a change in geometry (i.e. change in metric) of the spacetime itself

i.e. $(ds^2)$

NOTE:




$T=X$ → this is the horizon (like a one way membrane)

→ for an "eternally accelerated observer" (i.e. it has always been accelerating) {from $t=-\infty$ to $+\infty$}

↳ only this region is accessible (rest of spacetime is NOT accessible to it)

LR: Left Rindler
RR: Right Rindler

# LECTURE 13

22/08/2023

**NOTE:** Principle of Equivalence

↓

it is the idea that acceleration is equivalent to a change in metric (or "gravity")

↓

Also, it tells that we can't tell to what extent does acceleration have an effect versus gravity (due to some source)

**NOTE:** Earlier (with $\eta$), there was always a coordinate transformation

↓

that would allow us to come to an $x, y, z$ frame

↓

Thus the spacetime was flat

**NOTE:** We saw previously:

67] $$ds^2 = -dT^2 + dX^2 = -dU\,dV$$

$$\left.\begin{aligned} \dot{U}\dot{V} &= 1 \\ \ddot{U}\ddot{V} &= -K^2(\tau) \end{aligned}\right\}$$

So, $\dfrac{\ddot{U}}{\dot{U}} = -K \Rightarrow \dot{U} = c\exp\left(-\int^{\tau} K(\tau')\,d\tau'\right) = F(\tau)$

and, $\dot{V} = \dfrac{1}{F(\tau)}$

i.e. $K(\tau) = K$ ← In the (uniformly) accelerated frame: $(t, x)$ or equiv. $(u, v)$

⇓

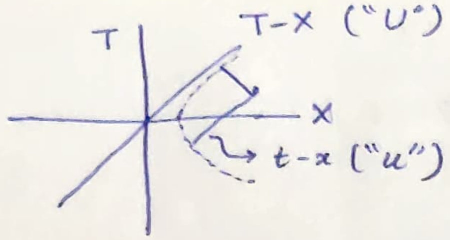
$U = U(u)$ ; $V = V(v)$

{* U and V are light rays ($T-X$ and $T+X$) and $u$ and $v$ are light rays in the accelerated frame}

In this frame: $x = 0$, $u = v = t = \tau$

$$\therefore ds^2 = -\frac{dU}{du}\frac{dV}{dv}\,du\,dv$$

* { NOTE:




Thus $U$ must map to $u$ only & $V$ must map to $v$ only

$\Downarrow$

$$\therefore U = U(u), \quad V = V(v) \; \}$$

$$ds^2 = -\frac{dU}{du}\frac{dV}{dv}\,du\,dv$$

$$= -\frac{F(u)}{F(v)}\,du\,dv$$

$$= -\exp\left(-\int_v^u K\,d\tau\right)du\,dv$$

$$= -\exp(-K(u-v))\,du\,dv$$

$$= \exp(2Kx)(-dt^2+dx^2)$$

{ $\frac{dU}{du} = \frac{dU}{d\tau}\frac{d\tau}{du}\Big|_{traj=1}$

on traj.: $= F(\tau)$ → on the trajectory it is a function of $\tau$

off traj: $= F(u)$ → off the traj., it will be a function of $u$

Similarly, $\frac{dV}{dv} = \frac{1}{F(v)}$ }

$\Downarrow$

Thus: $$ds^2 = e^{2Kx}(-dt^2+dx^2)$$
$$= e^{2Kx}\eta_{ab}\,dx^a dx^b$$

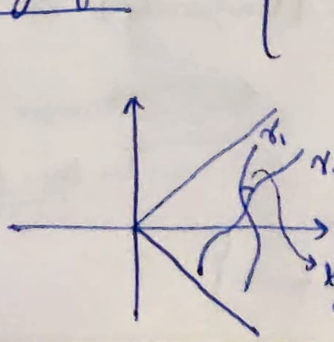
* { This is still actually a flat metric $\Rightarrow$ i.e. $ds^2 = g_{ab}\,dx^a dx^b$ }
where this is for flat spacetime here

$$\therefore ds^2 = \underbrace{e^{2Kx}\eta_{ab}}_{g_{ab}}\,dx^a dx^b$$

* The relation of the form:

$$g_{ab} = \Omega^2(x^i)\,\eta_{ab}$$

is called Conformally flat $\Rightarrow$ { This means that distances may NOT be preserved + But angles WILL be preserved in transformation b/w frames




This angle is same in $(T,X)$ and $(t,x)$ frames }

## 68] Rindler Metric

* $ds^2 = e^{2Kx}(-dt^2 + dx^2) = \underbrace{e^{2Kx}\eta_{ab}}_{g_{ab}\, dx^a dx^b} dx^a dx^b$

* $dl = e^{Kx} dx$

$\therefore Kl = e^{Kx}$

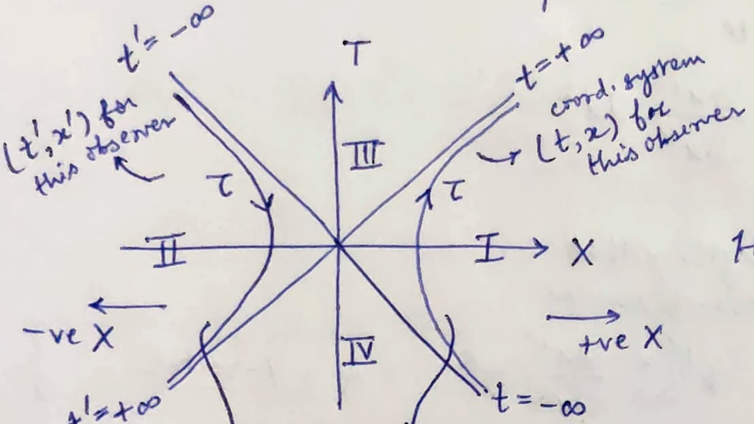
$\Rightarrow ds^2 = -K^2 l^2 dt^2 + dl^2$
$= -(1 + K\xi)^2 dt^2 + d\xi^2$ $\Big\} \Rightarrow$ i.e. there is no preferred coordinate system $\rightarrow$ i.e. frame

If we can transform the coord.s such that the structure remains preserved then $\rightarrow$ This is called

* General Covariance

$\Downarrow$ i.e.

$x^{i'} = f(x^i)$




The observers in I & II are two distinct observers
↓
they exist in two disjoint regions of spacetime

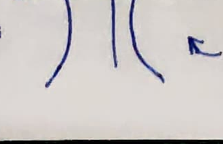
Here:

(I) can recieve signals from (IV) & send to (III)

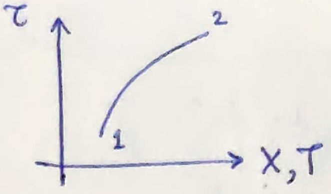
(II) can recieve signals from (IV) & send to (III)

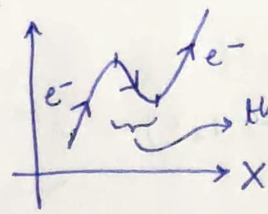
{ The $(t', x')$ frame observer appears to be going backwards in time in the frame of global observer $(T, X)$ $\rightarrow$ (eg: antiparticle) $\rightarrow$ i.e. for an antiparticle $\tau$ always moves opposite to T
↓
But in his own frame $(t', x')$ is going forward in time }

{ NOTE: We can draw these hyperbolas anywhere (not necessarily centered at origin in $(T, X)$)
↓
(due to translation symmetry) }

eg:  ← This is also allowed i.e.

NOTE:




→ This can be interpretted as a particle moving backwards in time

↓

OR

a positron (antielectron) moving forward in time

## 69] General Covariance

The coordinate transformation can be given by:

↓

$$x^{k'} = f(x^i)$$

Thus, we can write:

$$dx^{i'} = \frac{\partial x^{i'}}{\partial x^k} dx^k$$

* { In general we define: Anything that transforms as

$$q^{i'} = \underbrace{\frac{\partial x^{i'}}{\partial x^k}} q^k \quad : \text{is a 4-vector} \quad \}$$

↳ these can be non-linear (earlier, the Lorentz transformations were linear)

$$= L^{i'}_{k} q^k$$

↳ * This is NOT Lorentz transformation

(it is a general coordinate transformation)

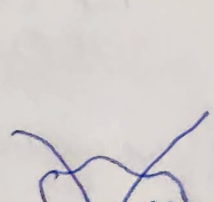
* $$T^{i'j'} = \frac{\partial x^{i'}}{\partial x^k} \frac{\partial x^{j'}}{\partial x^l} T^{kl} = L^{i'}_{k} L^{j'}_{l} T^{kl}$$

* Also, $g_{ab}(x^k)\, dx^a dx^b = ds^2$

* The neighbourhood of P is flat to the lowest order

↓

This is a condition reqd. for the spacetime to be "genuinely curved"



{ If we can transform such that neighbourhood is flat <u>to all orders</u> $\rightarrow$ then the spacetime is genuinely flat

$$eg: \quad dl^2 = \underbrace{x_2^2 dx_1^2 + dx_2^2}_{\text{can become}}$$

$dx^2 + dy^2 \leftarrow$ (flat)

But, $d\theta^2 + \sin^2\theta \, d\phi^2 \not\Rightarrow dx^2 + dy^2$ }

(sphere) can never become

If we perform a transformation:

$$x^i \longrightarrow x^{i'}$$

$$x^{i'} = f(x^i) = B^{i'}_j x^j + C^{i'}_{jk} x^j x^k + D^{i'}_{jkl} x^j x^k x^l + \ldots$$

$$\Rightarrow \therefore g_{ab}(x^k) = g_{ab}(P) + (\partial_c g_{ab})\Big|_P (x^{\prime c} - x^{\prime c}(P))$$

$$+ \frac{1}{2} \partial_c \partial_d g_{ab}\Big|_P (x^{\prime c} - x^{\prime c}(P))(x^{\prime d} - x^{\prime d}(P))$$

$$+ \ldots$$

If we can choose some $B^{i'}_j \Rightarrow$ such that $g_{ab}(P) = \eta_{ab}$

then the spacetime is <u>locally flat</u>

(and such a spacetime is a "<u>genuinely curved spacetime</u>")

{ <u>NOTE</u>: A "genuinely flat spacetime" is that where all higher order terms can be made zero

$\downarrow$

i.e. it is actually globally flat (not just "locally") }