Spatial: $u(x,y,t) = U(x,y)\,T(t)$

$$\Rightarrow T\frac{\partial^2 U}{\partial x^2} + T\frac{\partial^2 U}{\partial y^2} = \frac{1}{c^2}U\frac{\partial^2 T}{\partial t^2}$$

Dividing by UT on both sides:

$$\frac{1}{U}\left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right] = \frac{1}{Tc^2}\frac{\partial^2 T}{\partial t^2} = -k^2$$

Taking ; $U(x,y) = X(x)\,Y(y)$

$$\Rightarrow \frac{1}{U}\left[YX'' + Y''X\right] = -k^2 \quad ; \quad T'' = -k^2c^2T$$

$$\boxed{T(t) = d_0 \cos kct + d_1 \sin kct}$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = -k^2$$

$$\Rightarrow -k_1^2 - k_2^2 = -k^2$$

Thus: $\dfrac{X''}{X} = -k_1^2 \quad , \quad \dfrac{Y''}{Y} = -k_2^2$

$$\therefore \boxed{\begin{aligned} X(x) &= a_0 \cos k_1 x + a_1 \sin k_1 x \\ Y(y) &= b_0 \cos k_2 y + b_1 \sin k_2 y \end{aligned}}$$

Boundary conditions: $X(x) = a_0 \cos k_1 x + a_1 \sin k_1 x$

$$X\left(-\frac{L_x}{2}\right) = X\left(\frac{L_x}{2}\right) = 0 \Rightarrow a_0 \cos\frac{k_1 L_x}{2} - a_1 \sin\frac{k_1 L_x}{2} = 0$$

and, $a_0 \cos\dfrac{k_1 L_x}{2} + a_1 \sin\dfrac{k_1 L_x}{2} = 0$

Adding: $\quad 2a_0 \cos\dfrac{k_1 L_x}{2} = 0$

$$\Rightarrow a_0 = 0 \text{ or } \cos\frac{k_1 L_x}{2} = 0$$

If $a_0 = 0$: $X(x) = a_1 \sin\dfrac{k_1 L_x}{2} = 0 \Rightarrow \sin\dfrac{k_1 L_x}{2} = 0 \Rightarrow \dfrac{k_1 L_x}{2} = n\pi$

$$\Rightarrow k_1 = \frac{2n\pi}{L_x}$$

$$\therefore \boxed{X(x) = a_1 \sin\frac{2n\pi x}{L_x}}$$

If $\cos\dfrac{k_1 L_x}{2} = 0 \Rightarrow \dfrac{k_1 L_x}{2} = \dfrac{(2n+1)\pi}{2} \Rightarrow k_1 = \dfrac{(2n+1)\pi}{L_x}$

$$\therefore X(x) = a_0 \cos\frac{(2n+1)\pi x}{L_x}$$

$$\therefore \quad X(x) = a_0 \cos \frac{(2n+1)\pi x}{L_x} + a_1 \sin \frac{2n\pi x}{L_x}$$

$$Y(y) = b_0 \cos \frac{(2n+1)\pi y}{L_y} + b_1 \sin \frac{2n\pi y}{L_y}$$

Thus, we obtain:

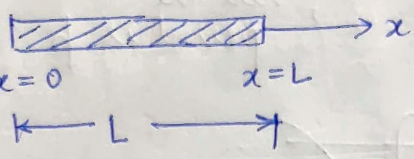
$$\boxed{\begin{aligned} U(x,y) &= \sum_{n=0}^{\infty} \left(a_0 \cos \frac{(2n+1)\pi x}{L_x} + a_1 \sin \frac{2n\pi x}{L_x}\right)\left(b_0 \cos \frac{(2n+1)\pi y}{L_y} + b_1 \sin \frac{2n\pi y}{L_y}\right) \\ T(t) &= d_0 \cos kct + d_1 \sin kct \end{aligned}}$$

{ LECTURE 14 onwards
continued ahead
(in notebook) }

Lecture 14 (14/03/2023)

60] 1-D Heat Equation

$$\frac{1}{a^2}\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$



Initial condition, $t = 0$:

$$u(x,0) = g(x) \quad (0 < x < L)$$

Boundary condn; $u(0,t) = u(L,t) = 0$
$(t > 0)$

$$u(x,t) = X(x)\,T(t)$$

$\Rightarrow$ Wave eqn: $\frac{1}{a^2} XT' = TX''$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{a^2 T} = -k^2$$

Temporal part: $T' = -k^2 a^2 T$

$$\Rightarrow \frac{dT}{dt} = -k^2 a^2 T$$

$$\Rightarrow \ln T = -k^2 a^2 t + c$$

$$\Rightarrow \boxed{T(t) = c\,\exp(-k^2 a^2 t)}$$

Spatial part: $X'' = -k^2 X$

$$X(x) = A\cos kx + B\sin kx$$

B.C.: $x = 0 \Rightarrow A = 0$
$x = L \Rightarrow B \sin kx = 0 \quad (B \neq 0)$

$$\therefore \sin kL = \sin n\pi$$

$$\Rightarrow k = \frac{n\pi}{L}$$

$$\therefore \boxed{u(x,t) = \sum_{n=1}^{\infty} G_n \exp\left(-\frac{a^2 n^2 \pi^2 t}{L^2}\right) \sin\left(\frac{n\pi x}{L}\right)}$$

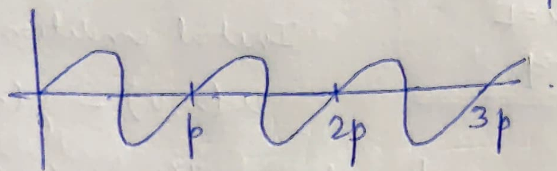
$(G_n = C_n B_n)$

61] Fourier Series

For a periodic function f :

$$f(x) = f(x+p)$$, here the period is p

eg:



If there is an Infinite dimensional space of a function $f(x)$: which is periodic

then it can be expressed as: $f(x) = \sum_{n=1}^{\infty} c_n b_n(x)$, where $b_n(x)$: basis function

(either continuous or discontinuous both can be expressed this way)

We know: $f(x+2p) = f(x+p+p) = f(x+p) = f(x)$

and since, $\cos(x+2\pi) = \cos(x)$

$\sin(x+2\pi) = \sin(x)$

⋮ so on

Thus, function of $f(x)$ having period 2π are:

1 , $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, ..., $\cos nx$, $\sin nx$, ...

a constant function is always periodic

∴ Trigonometric "series" can be written as:

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$\Rightarrow \boxed{f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

Fourier Series of $f(x)$

eg: vector: $\vec{v} = \sum_n c_n \vec{b}_n$, $\vec{b}_n$: basisvector

$$\vec{v} \cdot \vec{b}_j = \sum_n c_n \vec{b}_n \cdot \vec{b}_j \quad \begin{pmatrix} n \neq j \to 0 \\ n = j \to |\vec{b}_j|^2 \end{pmatrix}$$

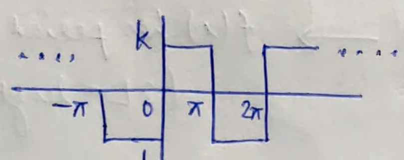
$$\Rightarrow \vec{v} \cdot \vec{b}_j = c_j$$

Similarly, here:

Fourier coefficients
$$\left\{ \begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \end{aligned} \right.$$

62] Example: Find the fourier coeff. of $f(x)$,

$$f(x) = \begin{cases} -k & , \text{if } -\pi < x < 0 \\ k & , \text{if } 0 < x < \pi \end{cases} \quad \text{and } f(x+2\pi) = f(x)$$



$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$= \frac{1}{2\pi} \left(\int_{-\pi}^{0} f(x) \, dx + \int_{0}^{\pi} f(x) \, dx \right)$$

$$= \frac{1}{2\pi} (-k + k)\pi = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \left(-k \frac{\sin nx}{n} \Big|_{-\pi}^{0} - k \frac{\sin nx}{n} \Big|_{0}^{\pi} \right)$$

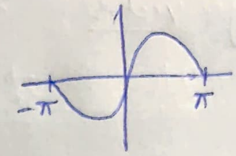
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \left(k \frac{\cos nx}{n} \Big|_{-\pi}^{0} + k \frac{\cos nx}{n} \Big|_{0}^{\pi} \right)$$

$$\therefore b_n = \frac{2k}{n\pi} (-\cos n\pi + 1)$$

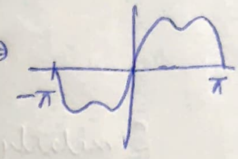
$$\therefore f(x) = \frac{4k}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right)$$

If we look at the partial sums:

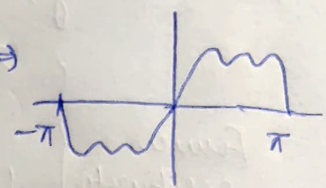
$$S_1 = \frac{4k}{\pi}\sin x \Rightarrow$$



$$S_2 = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3}\right) \Rightarrow$$

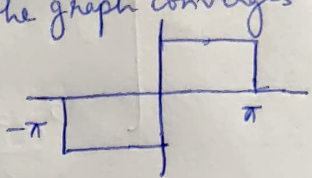


$$S_3 = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\right) \Rightarrow$$



⋮ and so on

{ the graph converges towards:



}

63] <u>Change of scale</u>

$$g(v) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi + b_n \sin n\pi$$

$g(v)$ has period $2\pi \longrightarrow f(x)$ has period $2L$ ($2L$ = length)

$v = kx$ such that $v = 2\pi \Rightarrow$ New period $x = 2L$

$$\Rightarrow 2\pi = 2Lk$$

$$\Rightarrow k = \frac{\pi}{L}$$

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

64] Half-range expansion

Odd function: $f(-x) = -f(x)$

Even function: $f(-x) = f(x)$

① Even $f(x)$: $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

② Odd function: $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Q→ Find the fourier series for function

(H.W.)

$$f(x) = \begin{cases} 0 & \text{if } -2<x<1 \\ k & \text{if } -1<x<1 \\ 0 & \text{if } 1<x<2 \end{cases}$$

$p = 2L = 4$ (i.e. $L=2$)

LECTURE 15 (17/03/2023)

65] Fourier Analysis

Fourier series was defined for a function (periodic) $f(x)$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos kx + b_n \sin kx$$

$$= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{k\pi x}{L} + b_n \sin \frac{k\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

66] Complete solution of 1-D Wave equation

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[H_n \cos\left(\frac{n\pi ct}{L}\right) + G_n \sin\left(\frac{n\pi ct}{L}\right)\right] \quad -①$$

Initial cond^n: $t=0$: $u(x,0) = f(x)$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)[H_n] = f(x)$$

$$H_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[G_n \cos\left(\cancelto{1}{\frac{n\pi ct}{L}}\right)\frac{n\pi c}{L}\right]$$

$$= \sum_{n=1}^{\infty}\left[G_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\cancelto{1}{\frac{n\pi ct}{L}}\right)\right]$$

Let $\lambda_n = \frac{n\pi c}{L}$

$$\therefore \Rightarrow \underbrace{\left.\frac{\partial u}{\partial t}\right|_{t=0}}_{g(x)|_{t=0}} = \sum_{n=1}^{\infty}\left[\underbrace{G_n\lambda_n}_{B_n}\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\Rightarrow G_n\lambda_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)g(x)\,dx$$

$$\Rightarrow G_n = \frac{2}{n\pi c}\int_0^L \sin\left(\frac{n\pi x}{L}\right)g(x)\,dx$$

In the case when $t=0$, $g(x)=0$

$$\therefore \text{Eq}^n\ ①:\ u(x,t) = \sum_{n=1}^{\infty} H_n \underbrace{\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)}_{\text{sin a cos b form}}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \frac{H_n}{2}\left(\sin\left(\frac{n\pi x}{L} - \frac{n\pi ct}{L}\right) + \sin\left(\frac{n\pi x}{L} - \frac{n\pi ct}{L}\right)\right)$$

$$\Rightarrow u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} H_n \sin\left(\frac{n\pi}{L}(x-ct)\right) + \frac{1}{2}\sum_{n=1}^{\infty} H_n \sin\left(\frac{n\pi}{L}(x+ct)\right)$$

$\therefore$ We get:

$$\boxed{u(x,t) = \frac{1}{2}\left[\underbrace{f^*(x-ct)}_{\text{forward propagation}} + \underbrace{f^*(x+ct)}_{\text{backward propagation}}\right]}$$

"Stationary wave" or "Standing wave"

67] Complete solution of 1D Heat Equation

$$u(x,t) = \sum_{n=1}^{\infty} G_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(\frac{-a^2 n^2 \pi^2 t}{L^2}\right) \quad - (1)$$

At $t=0$, $u(x,0) = g(x)$

Putting $t=0$ in (1): $u(x,0) = \sum_{n=1}^{\infty} G_n \sin\left(\frac{n\pi x}{L}\right) = g(x)$

This is Fourier Sine series of $g(x)$

$$\therefore G_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) g(x)\, dx$$

68] Fourier Integrals

Fourier series can be used for aperiodic $f(x)$ when $L \to \infty$.

Let $f_L(x)$ be the function with period $2L$

$$\therefore \Rightarrow f_L(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos w_n x + b_n \sin w_n x$$

$$\left(w_n = \frac{n\pi}{L}\right)$$

If $L \to \infty$, we can use variable of integration 'v'

Substitute a_0, a_n, b_n.

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\, dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v) \cos w_n v\, dv + \sin w_n x \int_{-L}^{L} f_L(v) \sin w_n v\, dv\right]$$

$$\Delta w = (n+1)\frac{\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

$$\therefore \Rightarrow \quad f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\,dv + \sin w_n x\,\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\,dv\right]$$

As $L \to \infty$:

$$f_L(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\,dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\,dv\right]$$

Let $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$

$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$

$$\therefore \boxed{f_L(x) = \int_0^{\infty}\left(A(w)\cos wx + B(w)\sin wx\right)dw}$$

Fourier Integral

If $f(x)$ is odd : Fourier sine integral $\rightarrow f(x) = \int_0^{\infty} B(w)\sin wx\,dw$

if even : Fourier cosine integral $\rightarrow f(x) = \int_0^{\infty} A(w)\cos wx\,dw$

* NOTE: Syllabus for Minor-2 : Everything after Minor-1