AXIOM 2

LECTURE-8 [Continued]

* **NOTE**: For momentum flux → choose a coord. system ↓ and point an arrow along dir.^{n} of momentum flow

41] **Axiom 2** : Momentum is conserved

(**NOTE**: Here, shear forces act as well, so we need to include their contribution.)

To prove any axiom, we must test it in a given control volume itself.

Now, in control volume $\xrightarrow{\text{we know}}$ Rate of accumulation of momentum in CV
= Rate of inflow of momen. in CV
− Rate of outflow of mome. by convective transports.
$+ \sum F$

{ Is obtained by Newton's IInd law: $\sum F = ma$ ↳ applied for fluids }

(**NOTE**: For solids: generally, inflow rate = outflow rate, as all particles move with similar velocity)

Here, $\sum F$ = sum of all forces $\xrightarrow{\text{i.e.}}$ $\sum F$

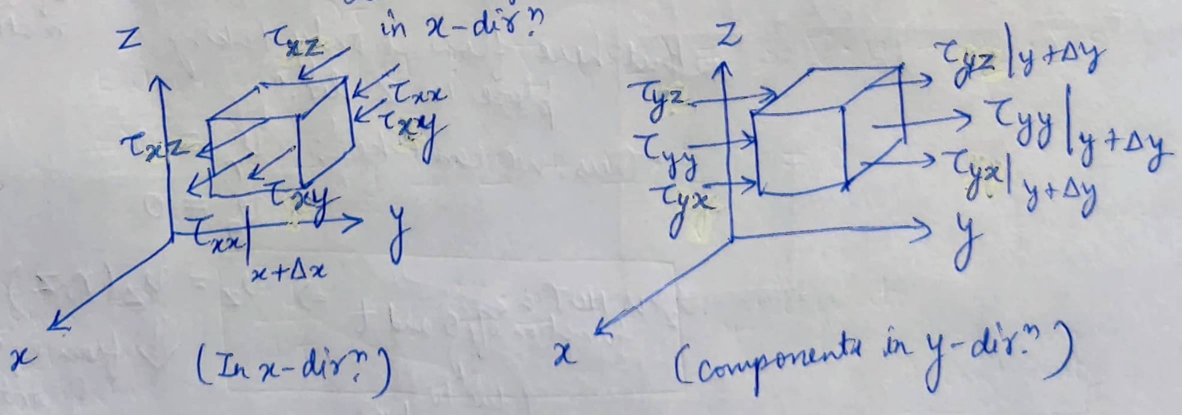
- Surface Forces (depends on surface area; Surf. area ↑ ⇒ F ↑ (eg: Pressure))
  - Shear Forces (due to velocity gradient ↳ which comes from boundary cond.^{n})
  - Pressure Forces
- Body Forces (Some external force, like gravity, etc. + depends on mass)

42] Here we will show $\underset{\sim}{\tau}$ as momentum flux (**NOTE**: In reality it's not But we will use it for purpose of mass heat transfer)

$$\underset{\sim}{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Notice that these are all mom. flux. in x-dir.^{n}

*(eg: τ_{xy} → y-directed momentum BUT flowing in x-dir.^{n})



(In x-dir.^{n}) (components in y-dir.^{n})

(and similarly for flowing in z-dir.^{n}) ...

NOTE: $\underset{\sim}{\tau}$ will be a symmetric matrix
(Proof of this comes from 3rd axiom)

43] Shell Momentum Balancing

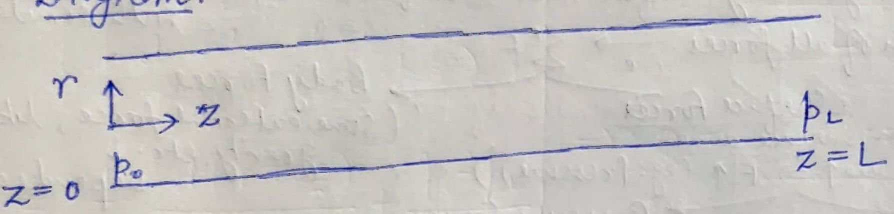
Flow through horizontal pipe/tube

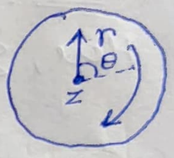
Most important part (of solving) → writing the assumptions ⟶ (MUST write these when answering a problem in exams)

⇓

Assumptions: Steady, fully developed, laminar flow.
Fluid has constant density (ρ) and viscosity (μ).
Horizontal pipe/tube.

Diagram:





(Viewing through hole of pipe)

Next, we assume velocity profile intuitively:

We have to write down:
r momentum balance
θ mome. balance
and, z mome. balance

In general we have
⇓
$$\left.\begin{aligned} v_r &= v_r(r,\theta,z,t) \\ v_\theta &= v_\theta(r,\theta,z,t) \\ v_z &= v_z(r,\theta,z,t) \end{aligned}\right]$$
These are for general case
⇓
Need to solve general eq^{n}s of motion for this

* But here flow is only taking place in z-dir.^{n}

Thus, we can take: $v_r = 0$
$v_\theta = 0$

v_z is NOT a function of θ and t (due to symmetry) $\Rightarrow v_z = v_z(r,z)$

⇓ from eq^{n}s of continuity we obtain now that

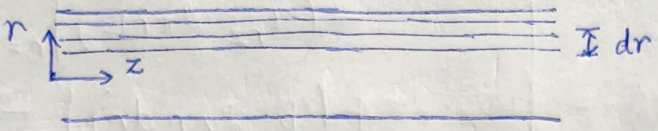
$$\frac{\partial v_z}{\partial z} = 0$$

Thus it also does NOT depend on z

* Required velocity profile:

$$\boxed{v_r = 0,\ v_\theta = 0,\ \therefore v_z = v_z(r)} \Longleftarrow$$

Thus, the only thing that we need to change for control volume is dividing into layers of thickness 'dr' ("shells")



Since flow of momentum is only in the z-direction:

for θ balance ↓ · for r balance ↓ · for z balance ↓

$$\underset{\sim}{\tau} = \begin{pmatrix} \vdots & \vdots & \tau_{rz} \\ \vdots & \vdots & \tau_{\theta z} \\ \vdots & \vdots & \tau_{zz} \end{pmatrix}$$

* we only need to be concerned with these 3 components (i.e. $\tau_{rz}, \tau_{\theta z}, \tau_{zz}$)

But even among these we see that $\tau_{\theta z} = 0$ and $\tau_{zz} = 0$

{ Because $\tau_{zz} = -\mu \dfrac{dv_z}{dz}$

and $\tau_{\theta z} = -\mu \dfrac{dv_z}{d\theta}$

But here since $v_z = v_z(r)$

↓

thus $\dfrac{dv_z}{dz} = 0$

and $\dfrac{dv_z}{d\theta} = 0$ }

⇓

Thus, we have reduced the 9 components of $\underset{\sim}{\tau}$ to only one component needed to work with this case:

i.e. $\boxed{\tau_{rz}}$ *

LECTURE-9

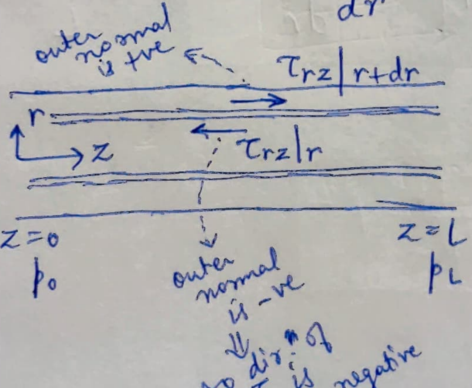
44] { * NOTE: Previously we talked about accumulation of momentum in CV. for case of shear forces

For case of momentum flux ⇒ will be Rate of inflow of momentum by viscous transport − Rate of outflow of momentum by viscous transport }

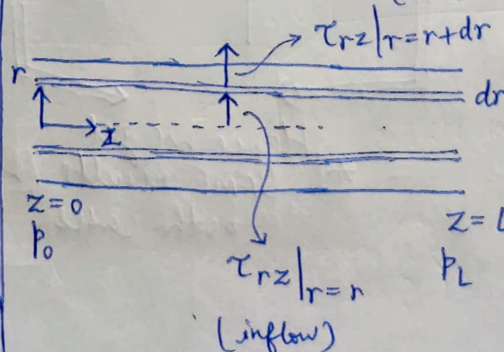
$+\sum F$ ↳ (and this can only be pressure forces and gravity [NOT surface force])

Taking both cases:

$\tau_{rz} = \mu \dfrac{dv_z}{dr}$



$\tau_{rz} = -\mu \dfrac{dv_z}{dr}$



(momentum in z-dirⁿ but flowing in r dirⁿ) convective mome-transports cancel out

$$0 = (\cdots) - (\cdots)$$
$$+ (\tau_{rz} \cdot 2\pi r L)\Big|_{r+dr}$$
$$- (\tau_{rz} \cdot 2\pi r L)\Big|_{r}$$
$$+ (p \cdot 2\pi r\, dr)\Big|_{z=0}$$
$$- (p \cdot 2\pi r\, dr)\Big|_{z=L}$$

$\Downarrow$ after performing division by vol^m

Finally we get same formula (but with +ve sign of derivative)

$$\Rightarrow 0 = \frac{1}{r}\frac{d(r\tau_{rz})}{dr} + \frac{p_0 - p_L}{L}$$

$\Downarrow$ Here using:

$$\tau_{rz} = \mu \frac{dv_z}{dr}$$

$\downarrow$ we get:

$$0 = \frac{\mu}{r}\frac{d\left(r\frac{dv_z}{dr}\right)}{dr} + \frac{(p_0 - p_L)}{L}$$

v_z is same for both → so inflow & outflow of mome - cancels out

$$0 = v_z(\rho v_z 2\pi r\, dr)\Big|_{z=0} - v_z(\rho v_z 2\pi r\, dr)\Big|_{z=L}$$
$$+ (\tau_{rz} \cdot 2\pi r L)\Big|_{r=r}$$
$$- (\tau_{rz} \cdot 2\pi r L)\Big|_{r=r+dr}$$
$$+ (p_0 \cdot 2\pi r\, dr)\Big|_{z=0} - (p_L \cdot 2\pi r\, dr)\Big|_{z=L}$$

Dividing by volume of CV (i.e. $2\pi r\, dr \cdot L$)

$$\Rightarrow 0 = \frac{(\tau_{rz} \cdot 2\pi r L)\Big|_{r=r} - (\tau_{rz} \cdot 2\pi r L)\Big|_{r=r+dr}}{2\pi r L\, dr} + \frac{p_0 - p_L}{L}$$

$$\Rightarrow 0 = \frac{(r\tau_{rz})\Big|_{r} - (r\tau_{rz})\Big|_{r+dr}}{r\, dr} + \frac{p_0 - p_L}{L}$$

$\Rightarrow$ Using definition of derivative:

$$0 = -\frac{d(r\tau_{rz})}{r\, dr} + \frac{p_0 - p_L}{L}$$

Using, $\tau_{rz} = -\mu \dfrac{dv_z}{dr}$

$$\Rightarrow 0 = \frac{\mu}{r}\frac{d\left(r\frac{dv_z}{dr}\right)}{dr} + \frac{(p_0 - p_L)}{L}$$

* Hence we get SAME results for both cases

After rearranging this becomes: { for our case of momentum flux [i.e. $\tau_{rz} = -\mu \frac{dv_z}{dr}$] }

$$\frac{1}{r}\frac{d(r\tau_{rz})}{dr} = \frac{p_0 - p_L}{L}$$

$$\Rightarrow \frac{d(r\tau_{rz})}{dr} = \frac{p_0 - p_L}{L} \cdot r$$

$$\Rightarrow r\tau_{rz} = \frac{p_0 - p_L}{L} \cdot \frac{r^2}{2} + C_1$$

$$\Rightarrow \tau_{rz} = \left(\frac{p_0 - p_L}{2L}\right) r + \frac{C_1}{r}$$

Here we know that τ_{rz} is finite, even at $r = 0$ so this means that $C_1 = 0$

$$\Rightarrow \boxed{\underbrace{\tau_{rz}}_{-\mu \frac{dv_z}{dr}} = \left(\frac{p_0 - p_L}{2L}\right) r}$$

$$\Rightarrow v_z = -\left(\frac{p_0 - p_L}{2\mu L}\right)\frac{r^2}{2} + C_2 = -\left(\frac{p_0 - p_L}{4\mu L}\right) r^2 + C_2$$

Taking boundary condition,
at $r = R$, $v_z = 0$

We get $\Rightarrow 0 = -\left(\frac{p_0 - p_L}{4\mu L}\right) R^2 + C_2$

$$\Rightarrow C_2 = \left(\frac{p_0 - p_L}{4\mu L}\right) R^2$$

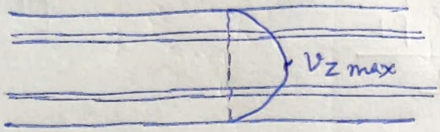
Thus; $v_z = -\left(\frac{p_0 - p_L}{4\mu L}\right) r^2 + \left(\frac{p_0 - p_L}{4\mu L}\right) R^2$

$$= \boxed{\frac{p_0 - p_L}{4\mu L}(R^2 - r^2)}$$

$$= \frac{p_0 - p_L}{4\mu L} \cdot R^2\left(1 - \frac{r^2}{R^2}\right) = \boxed{v_{z_{max}}\left(1 - \frac{r^2}{R^2}\right)}$$

Hence, $\boxed{v_{z_{max}} = \frac{p_0 - p_L}{4\mu L} R^2}$

Hence, we have obtained Parabolic velocity profile.



To get flow rate:

$$\int dQ = \int v_z \cdot 2\pi r \, dr$$

$$\Rightarrow \quad v_{z_{avg}} \pi R^2 = Q = \int_0^R v_z \cdot 2\pi r \, dr$$

$$\Rightarrow \quad v_{z_{avg}} = \frac{\int_0^R v_z \cdot 2\pi r \, dr}{\pi R^2}$$

$$\Rightarrow \quad v_{z_{avg}} = \frac{\int_0^R v_{z_{max}} \left(1 - \frac{r^2}{R^2}\right) \cdot 2r \, dr}{R^2}$$

$$= \frac{2 v_{z_{max}}}{R^2} \left[\int_0^R \left(1 - \frac{r^2}{R^2}\right) r \, dr \right]$$

$$= \frac{2 v_{z_{max}}}{R^2} \left[\frac{r^2}{2} - \frac{r^4}{4R^2} \right]_0^R$$

$$= \frac{2 v_{z_{max}}}{R^2} \left[\frac{R^2}{2} - \frac{R^4}{4R^2} \right]$$

$$= \frac{2 v_{z_{max}}}{R^2} \cdot \frac{R^2}{4} = \boxed{\frac{v_{z_{max}}}{2}}$$

Putting value for $v_{z_{max}}$:

$$v_{z_{avg}} = \frac{p_0 - p_L}{4\mu L} \cdot \frac{R^2}{2} = \boxed{\frac{p_0 - p_L}{8\mu L} R^2}$$

Thus, $Q = v_{z_{avg}} \cdot \pi R^2 = \boxed{\dfrac{\pi (p_0 - p_L) R^4}{8 \mu L}}$

$$= \frac{\pi (p_0 - p_L) D^4}{128 \mu L}$$

HAGEN-POISEUILLE EQUATION

{ NOTE: Remember that this is valid only under the assumption that we previously took
i.e. steady flow, constant density, constant μ, etc. }

45] From our previous result, we have:

$$\tau_{rz} = \frac{p_0 - p_L}{2L} r$$ } → this was momentum flux

Thus, for shear force:

$$(\tau_{rz})_f = -\frac{p_0 - p_L}{2L} r$$ } shear force

NOTE: Friction factor is defined as: $\boxed{f = \dfrac{\overbrace{(\tau_w)}^{\text{shear force on wall}}}{\frac{1}{2} \rho v_{z_{avg}}^2}}$

Now, * $(\tau_w) = -(\tau_{rz})_f \Big|_{r=R} = -(-\tau_{rz})\Big|_{r=R}$

$$= \tau_{rz}\Big|_{r=R}$$

$$= \frac{p_0 - p_L}{2L} \cdot R$$

v. important
(there is -ve since the normal on the wall will point inwards
i.e. -ve dirn)

Hence, $$f = \frac{\dfrac{p_0 - p_L}{2L} R}{\frac{1}{2} \rho v_{z_{avg}} \cdot \underbrace{\dfrac{p_0 - p_L}{8 \mu L}}_{} R^2}$$

→ opened up one $v_{z_{avg}}$ using the previous formula

$$= \frac{8 R \mu}{\rho v_{z_{avg}} R^2} = \frac{16 \mu}{\rho v_{z_{avg}} D} \rightarrow (\text{replaced, } D = 2R)$$

$\frac{\mu}{\rho v_{z_{avg}} D}$ → inverse of * reynold's number

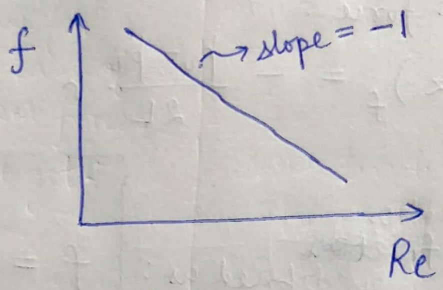
Hence, ⇒ $$\boxed{f = \frac{16}{Re}}$$ ⇒ Found For laminar flow

{ But also valid for Turbulent flow }

thus friction factor is also dimensionless

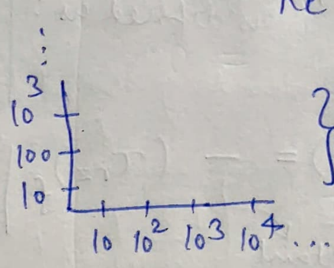
$$\Rightarrow \log f = \log 16 - \log Re$$

This can be used to draw a log-log graph:



Known as:
* Friction Factor Charts

{ NOTE: The scalings on log-log graphs look like:



}

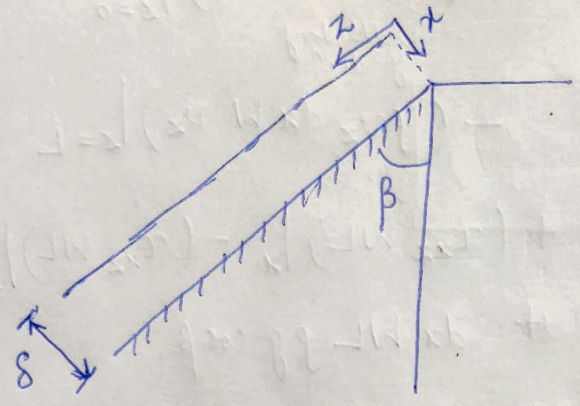
NOTE: These friction factor charts are needed for designing pipes → so that they can handle not only laminar, but also turbulent flow.

LECTURE-10

46] It turns out that, $f = \frac{16}{Re}$ ⇒ valid for Turbulent flow as well *

$$\left(\because \tau_w = \frac{p_0 - p_L}{2L} R \right.$$ is valid for turbulent flow too $\left.\right)$

47] **Falling Liquid Film on an Inclined Surface**



Assumptions:

Fully developed flow, steady flow, laminar flow.

Fluid has constant density and viscosity.

Here we can write:

$$\left.\begin{array}{l} v_z(x, y, z, t) \\ v_x(x, y, z, t) \\ v_y(x, y, z, t) \end{array}\right\} \xRightarrow[\text{steady flow}]{\text{due to}} \begin{array}{l} v_z(x, y, z) \\ v_x(x, y, z) \\ v_y(x, y, z) \end{array}$$

$(\text{i.e. } v_i \neq f(t))$ $\Downarrow$

In this case:

$\therefore v_x = 0, \; v_y = 0$

and $v_z(x, z)$

(Not of y due to symmetry)

$$\frac{\partial v_z}{\partial z} = 0 \xleftarrow[\text{of continuity}]{\text{from Eq}^n}$$

$\Downarrow$

$$v_z = v_z(x)$$

Now we can write $\underset{\sim}{\tau}$ as:

$$\underset{\sim}{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

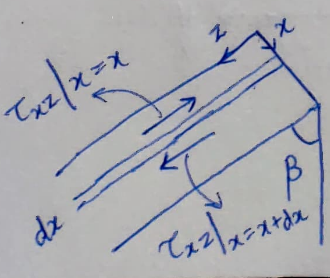
$\Downarrow$

momentum only along z $\Rightarrow$ but here also since $v_z = v_z(x)$ only

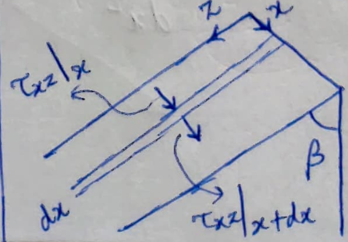
$\Downarrow$

So only τ_{xz} will be non-zero

$$\tau_{xz} = \mu \frac{dv_z}{dx}$$



$$\tau_{xz} = -\mu \frac{dv_z}{dx}$$



Everything here will be the same, except just signs will change next to τ_{xz}

Thus for the case of shear forces we get:-

$$0 = \frac{d\tau_{xz}}{dx} + \rho g \cos\beta$$

$$\left(\tau_{xz} = \mu \frac{dv_z}{dx} \right)$$

By z-shell momentum balance:

$$0 = (\rho v_z \, dx \, W \, v_z)\Big|_{z=0}$$

width along y-dir.^{n}

$$- (\rho v_z \, dx \, W \, v_z)\Big|_{z=L}$$

$$+ (\tau_{xz} WL)\Big|_x - (\tau_{xz} WL)\Big|_{x+dx}$$

$$+ \; dx \, WL \, \rho g \cos\beta$$

(* NOTE: Pressures at p_0 and p_L are equal for freely falling film
↓
since open to atmosphere, and free fall (not forced) taking place
↓
thus, here the flow is only due to gravity.)

Now, dividing by volume of Control volume: (i.e. by $dx\,WL$):

$$0 = \frac{(\tau_{xz})\big|_x - (\tau_{xz})\big|_{x+dx}}{dx} + \rho g \cos\beta$$

$$\Rightarrow \quad 0 = -\frac{d\tau_{xz}}{dx} + \rho g \cos\beta$$

$$\left(\tau_{xz} = -\mu \frac{dv_z}{dx} \right)$$

Give the same result:

$$0 = \mu \frac{d^2 v_z}{dx^2} + \rho g \cos\beta$$

Now, from the momentum flux equation:

$$\frac{d\tau_{xz}}{dx} = \rho g \cos\beta$$

$$\Rightarrow \tau_{xz} = \rho g x \cos\beta + C_1$$

$$\Rightarrow -\mu \frac{dv_z}{dx} = \rho g x \cos\beta + C_1$$

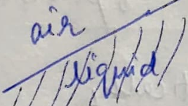
$$\Rightarrow \frac{dv_z}{dx} = -\frac{\rho g x \cos\beta}{\mu} - \frac{C_1}{\mu}$$

Thus, we get $\Rightarrow v_z = -\dfrac{\rho g \cos\beta}{2\mu} x^2 - \dfrac{C_1}{\mu} x + C_2$

We need two boundary conditions:

1) The first we know is: $v_z\big|_{x=\delta} = 0$

2) Here, there is liquid-air interface



$\Rightarrow$ thus $\tau_{xz}\big|_{air} = \tau_{xz}\big|_{liq}$ at interface

$$\Rightarrow -\mu_{air} \frac{dv_{z\,air}}{dx} = -\mu_{liq} \cdot \frac{dv_{z\,liq}}{dx}$$

(If this was some other fluid, other than air, we would use this equation, and solve for both fluids together

$\downarrow$

but for the case of air we can take the observation that air cannot sustain the shear forces

$\downarrow$

thus, $\left(\dfrac{dv_{z\,liq}}{dx}\right)_{\text{at interface}} \propto \underbrace{\dfrac{\mu_{air}}{\mu_{liq}}}_{\text{will be } \approx 0} (\ldots)$)

Hence, we see that:

In the eq^n: $\tau_{xz} = \rho g x \cos\beta + C_1$

Since $\tau_{xz} = 0$ at $x = 0 \Rightarrow C_1 = 0$

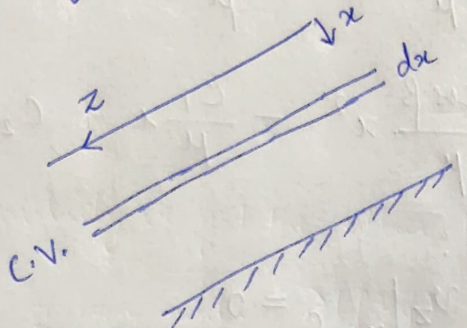
Thus, $\boxed{\tau_{xz} = -\mu \frac{dv_z}{dx} = \rho g x \cos\beta}$

Similarly, as at $x = \delta$, $v_z = 0$

$$\therefore \Rightarrow \boxed{v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu}\left(1 - \frac{x^2}{\delta^2}\right)}$$

$\left\{ \begin{array}{l} * \text{ NOTE: Mass flow rate} = \rho v A \\ \text{and, Volume flow rate} = vA \end{array} \right.$
(where A is area through which flow is taking place) }

Taking the case of mass flow rate:



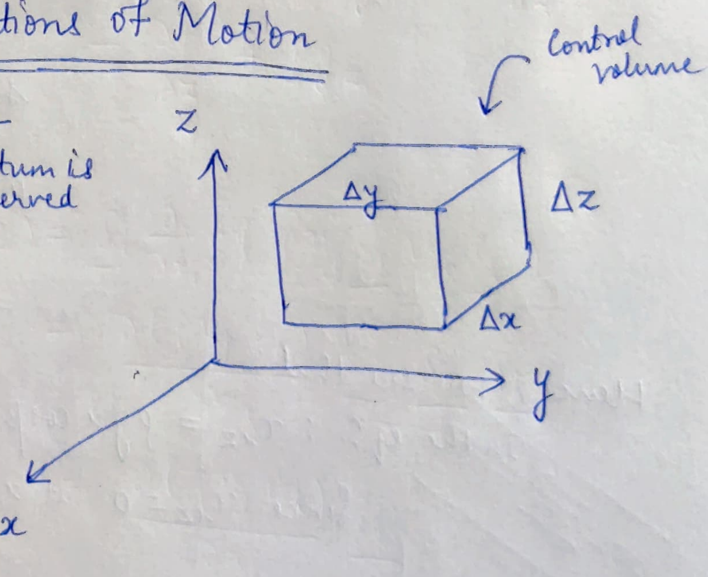
$$dQ = \rho v_z \, dx \, W$$

$$Q = \int_0^\delta \rho v_z \, dx \, W = \rho W \delta \cdot \frac{1}{\delta}\int_0^\delta v_z \, dx$$

$$\therefore Q = \rho v_{z_{avg}} W \delta$$

48] NOTE: There are very limited no. of problems that can be solved Analytically
{and most of them are covered in this course}
⇓
thus any other problem with an analytical solution
↓
is just a variation of these problems.

49] Derivation of Equations of Motion
↓
Based on Axiom 2
↓
i.e. momentum is Conserved



$$\underline{v} \left\{ \begin{array}{l} v_x(x,y,z,t) \\ v_y(x,y,z,t) \\ v_z(x,y,z,t) \end{array} \right.$$

and $\rho(x,y,z,t)$

{Taking most general case ↳ No assumptions}

So here, we can write:

* (Net = inflow − outflow)

Rate of accumulation of momentum in C.V. (I) = Net rate of inflow of momentum by convection (II)

+ Net rate of inflow of momentum by viscous transport (III)

+ Pressure forces on C.V. (IV)

+ Gravity forces on C.V. (V)

LECTURE-11

Writing the x-component:

$$\text{I}: \quad \frac{\partial(\rho\,\Delta x\,\Delta y\,\Delta z\,v_x)}{\partial t}$$

$$\begin{aligned}\text{II}: \quad & (\rho v_x\,\Delta y\,\Delta z\,v_x)\big|_x - (\rho v_x\,\Delta y\,\Delta z\,v_x)\big|_{x+\Delta x} \\ & + (\rho v_y\,\Delta x\,\Delta z\,v_x)\big|_y - (\rho v_y\,\Delta x\,\Delta z\,v_x)\big|_{y+\Delta y} \\ & + (\rho v_z\,\Delta x\,\Delta y\,v_x)\big|_z - (\rho v_z\,\Delta x\,\Delta y\,v_x)\big|_{z+\Delta z}\end{aligned}$$

III: In the momentum flux:

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

for x-dir^n momentum

So we have:

flowing in x

$$\begin{aligned} & (\tau_{xx}\,\Delta y\,\Delta z)\big|_x - (\tau_{xx}\,\Delta y\,\Delta z)\big|_{x+\Delta x} \\ \text{flowing in y} \quad & + (\tau_{yx}\,\Delta x\,\Delta z)\big|_y - (\tau_{yx}\,\Delta x\,\Delta z)\big|_{y+\Delta y} \\ \text{flowing in z} \quad & + (\tau_{zx}\,\Delta x\,\Delta y)\big|_z - (\tau_{zx}\,\Delta x\,\Delta y)\big|_{z+\Delta z}\end{aligned}$$

$$\text{IV}: \quad (p\,\Delta y\,\Delta z)\big|_x - (p\,\Delta y\,\Delta z)\big|_{x+\Delta x}$$

$$\text{V}: \quad \rho g_x\,\Delta x\,\Delta y\,\Delta z$$

Now dividing by $\Delta x \Delta y \Delta z$

and taking the limit : $\Delta x \to 0$

$\Delta y \to 0$

$\Delta z \to 0$

we get:

x-component: $$\frac{\partial(\rho v_x)}{\partial t} = -\frac{\partial(\rho v_x v_x)}{\partial x} - \frac{\partial(\rho v_y v_x)}{\partial y} - \frac{\partial(\rho v_z v_x)}{\partial z} - \frac{\partial(\tau_{xx})}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \tau_{zx}}{\partial z} - \frac{\partial p}{\partial x} + \rho g_x$$

Similarly we get the other two components as:

y-component: $$\frac{\partial(\rho v_y)}{\partial t} = -\frac{\partial(\rho v_x v_y)}{\partial x} - \frac{\partial(\rho v_y v_y)}{\partial y} - \frac{\partial(\rho v_z v_y)}{\partial z} - \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \tau_{yy}}{\partial y} - \frac{\partial \tau_{zy}}{\partial z} - \frac{\partial p}{\partial y} + \rho g_y$$

z-component: $$\frac{\partial(\rho v_z)}{\partial t} = -\frac{\partial(\rho v_x v_z)}{\partial x} - \frac{\partial(\rho v_y v_z)}{\partial y} - \frac{\partial(\rho v_z v_z)}{\partial z} - \frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} - \frac{\partial \tau_{zz}}{\partial z} - \frac{\partial p}{\partial z} + \rho g_z$$

Combining these 3 terms :

[We know, $$\underline{\nabla} \cdot (\rho \underline{v}\,\underline{v}) = \underline{\delta_i} \frac{\partial}{\partial x_i} \cdot (\rho v_j \underline{\delta_j}\, v_k \underline{\delta_k})$$

$$= \frac{\partial(\rho v_j v_k)}{\partial x_i} \underbrace{\underline{\delta_i} \cdot \underline{\delta_j}}_{\delta_{ij}} \underline{\delta_k}$$

$$= \frac{\partial(\rho v_j v_k)}{\partial x_j} \underline{\delta_k}$$

For instance, $k=1$:

$$\frac{\partial(\rho v_j v_x)}{\partial x_j} = \sum_{j=1}^{3} \frac{\partial(\rho v_j v_x)}{\partial x_j}$$

$\downarrow$
which is exactly the IInd term]

Thus we have obtained:

$$\boxed{\frac{\partial(\rho \underline{v})}{\partial t} = -\underline{\nabla}\cdot(\rho \underline{v}\,\underline{v}) - \underline{\nabla}\cdot\underset{\sim}{\tau} - \underline{\nabla}p + \rho \underline{g}}$$

GENERAL EQUATION
OF MOTION

{NOTE: Navier-Stokes equation is a special case of the general equation of motion}

We can now substitute: $\underline{u} = \rho\underline{v}$
and $\underline{w} = \underline{v}$

So we get the similar answer as we studied in tutorials:

$$\underline{\nabla}\cdot(\underline{u}\,\underline{v}) = \underline{u}\cdot\underline{\nabla}\underline{w} + \underline{w}\,\underline{\nabla}\cdot\underline{u}$$

Thus $\Rightarrow$ $\underline{\nabla}\cdot(\rho\underline{v}\,\underline{v}) = \rho\underline{v}\cdot\underline{\nabla}\underline{v} + \underline{v}\,\underline{\nabla}\cdot(\rho\underline{v})$

Hence the expanded equation becomes:

$$\underline{v}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\underline{v}}{\partial t} + \rho\underline{v}\cdot\underline{\nabla}\underline{v} + \underline{v}\,\underline{\nabla}\cdot(\rho\underline{v})$$
$$= -\underline{\nabla}\cdot\underset{\sim}{\tau} - \underline{\nabla}p + \rho\underline{g}$$

which gives us:

$$\underline{v}\underbrace{\left(\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{v})\right)}_{\text{this is the Equation of continuity} \downarrow \text{Hence it will be zero}} + \rho\underbrace{\left(\frac{\partial\underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\underline{v}\right)}_{\text{Substantial derivative}} = -\underline{\nabla}\cdot\underset{\sim}{\tau} - \underline{\nabla}p + \rho\underline{g}$$

Thus $\Rightarrow$ we finally obtain the form that we use for solving problems

$$\boxed{\rho\frac{D\underline{v}}{Dt} = -\underline{\nabla}\cdot\underset{\sim}{\tau} - \underline{\nabla}p + \rho\underline{g}}$$

50] <u>Some special cases:</u>

i) If viscosity is zero (i.e. $\mu = 0$) $\Rightarrow$ τ term becomes zero in the equation

ii) If stationary fluid (i.e. no motion) $\downarrow$ (i.e. $\underline{v} = 0$) $\Rightarrow$ $\rho \frac{D\underline{v}}{Dt}$ and $-\underline{\nabla} \cdot \underset{\sim}{\tau}$ are both zero

(* this is what Civil Engineers work with $\downarrow$ eg: in dams, etc.)

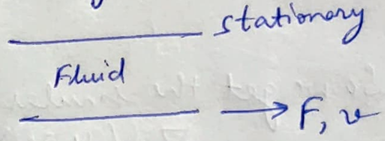
$$0 = -\underline{\nabla} p + \rho \underline{g}$$

iii) If ρ and μ are constant
$\Rightarrow$ we obtain the <u>Navier-Stokes Equations</u>

*{ <u>NOTE</u>: Incompressible $\downarrow$ ρ is constant
Newtonian fluid $\downarrow$ μ is constant }

$\underset{\sim}{\tau}$: Here we have <u>3D flow case</u>

(previously when we wrote τ_{yx} for the case:



this was 1-D flow $\Leftarrow$)

* Here, $\underset{\sim}{\tau}$ is <u>symmetric</u> $\longrightarrow$ is obtained as: Moment of momentum is conserved
$\Downarrow$
Gives us (not deriving it):

* { <u>NOTE</u>: All this is for case of Newtonian fluid $\Downarrow$ i.e. μ is Constant }

(<u>NOTE</u>: $\underline{\nabla} \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$)

Thus,
$$\tau_{xx} = -2\mu \frac{\partial v_x}{\partial x} + \frac{2}{3} \mu \underline{\nabla} \cdot \underline{v}$$
$$\tau_{yy} = -2\mu \frac{\partial v_y}{\partial y} + \frac{2}{3} \mu \underline{\nabla} \cdot \underline{v}$$
$$\tau_{zz} = -2\mu \frac{\partial v_z}{\partial z} + \frac{2}{3} \mu \underline{\nabla} \cdot \underline{v}$$

* { Remember: $\underline{\nabla} \cdot \underline{v} = 0$ if ρ is constant (from continuity eq^n) }

and for shear components:
$$\tau_{xy} = \tau_{yx} = -\mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$$
$$\tau_{yz} = \tau_{zy} = -\mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right)$$
$$\tau_{zx} = \tau_{xz} = -\mu \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right)$$

Thus, x-component of General Eq^n. of motion is :

$$\frac{\partial(\rho v_x)}{\partial t} = -\frac{\partial(\rho v_x v_x)}{\partial x} - \frac{\partial(\rho v_y v_x)}{\partial y} - \frac{\partial(\rho v_z v_x)}{\partial z}$$

$$- \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \tau_{zx}}{\partial z} + \underbrace{(\rho g)_x}_{\text{i.e. } \rho g_x} - \underbrace{(\nabla p)_x}_{\text{i.e. } \frac{\partial p}{\partial x}}$$

Let us examine :

The three terms with τ_{xx}, τ_{yx} and τ_{zx} can be simplified for the case of constant ρ and μ.

$$\Rightarrow \frac{\partial}{\partial x}\left(-2\mu \frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right)$$

$$+ \frac{\partial}{\partial z}\left(-\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right)$$

$$\Rightarrow -\mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_y}{\partial y \partial x} + \frac{\partial^2 v_z}{\partial z \partial x} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

Hence it becomes :

$$-\mu\left[\nabla^2 v_x + \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)}_{\text{becomes zero}}\right]$$

Thus, if ρ and μ are constant, the general equation of motion gets simplified to :

x-component :
$$\frac{\partial(\rho v_x)}{\partial t} = -\left(\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right) + \mu \nabla^2 v_x + \rho g_x - \frac{\partial p}{\partial x}$$

Thus, finally the vector form will be as follows :

$$\boxed{\frac{\partial(\rho \underline{v})}{\partial t} = -\underline{\nabla}\cdot(\rho \underline{v}\,\underline{v}) + \mu \nabla^2 \underline{v} + \rho \underline{g} - \underline{\nabla} p}$$

NAVIER-STOKES EQUATIONS

(NOTE : The non-linear term $-\underline{\nabla}\cdot(\rho \underline{v}\,\underline{v})$ is responsible for the problems of fluctuation which makes these equations incredibly hard to solve)

The problem with being non-linear is :

these can still almost cancel out

$$(v_{avg} + \underset{\text{a fluctuating part}}{\Delta v})(v_{avg} + \Delta v) = v_{avg}^2 + \overbrace{v_{avg}\Delta v + \Delta v \cdot v_{avg}} + (\Delta v)^2$$

but since this is squared → it is always positive ⇒ thus creates problems

{ **NOTE**: These Navier-Stokes equations are used in weather forecasting, etc. }