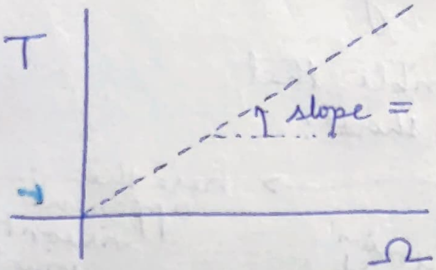
Now, at $r = R$, $\frac{v_\theta}{r} = \Omega$

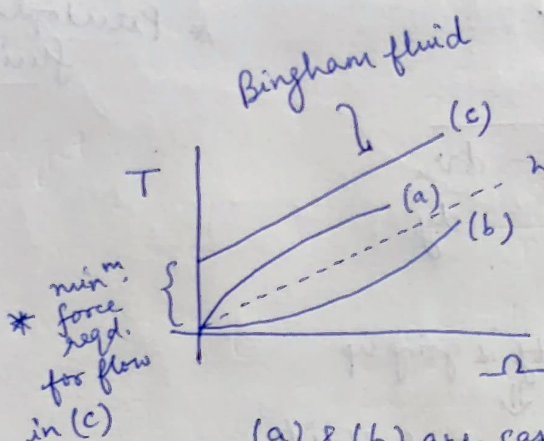
$$\therefore \Omega = \frac{T}{4\pi\mu L}\left(\frac{1}{k^2R^2} - \frac{1}{R^2}\right)$$

$$\Rightarrow \boxed{T = \frac{4\pi\mu L R^2 \Omega k^2}{(1-k^2)}} \leftarrow \text{Relation b/w } T \text{ and } \Omega$$



$$\text{slope} = \frac{4\pi\mu L R^2 k^2}{(1-k^2)}$$

If we did NOT have a Newtonian fluid
↓
on performing this same experiment using the viscometer
↓
we would see that the plot of T vs Ω would NOT give a straight line through origin



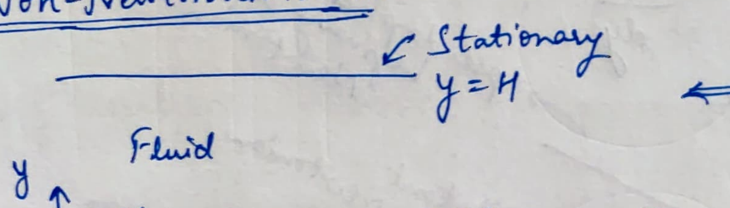
(a) & (b) are case for Non-Newtonian fluids

(c) is also a non-newtonian fluid of a special type → called * Bingham fluids

{ * Syllabus for Minor Examination will be till here }

LECTURE-14

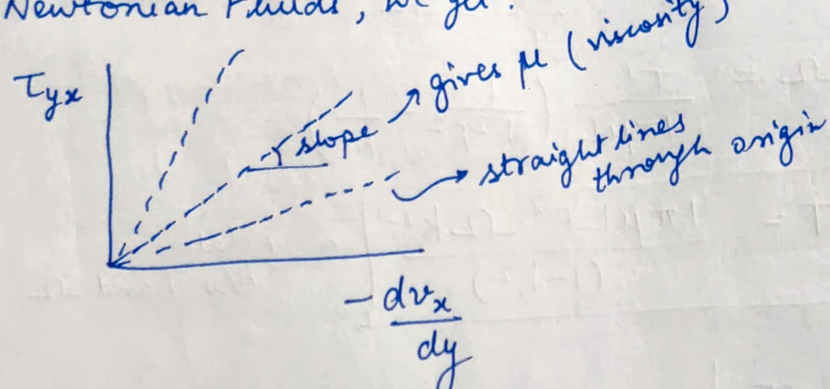
57] Non-Newtonian Fluids



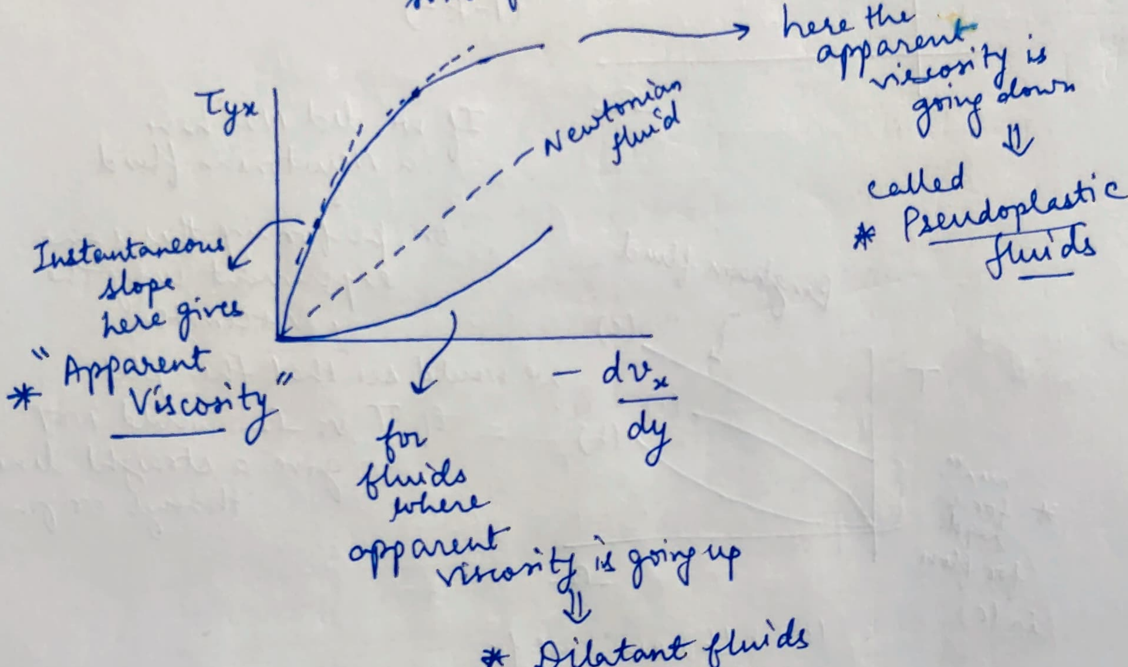
⟸ This is what we wrote
↓
performing an experiment

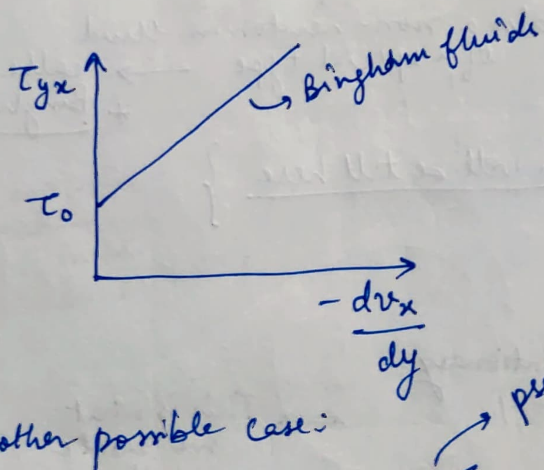
$$\frac{F}{A} = \tau_{yx} \quad , \quad -\frac{dv_x}{dy} \approx \frac{v}{H}$$

For Newtonian Fluids, we get:

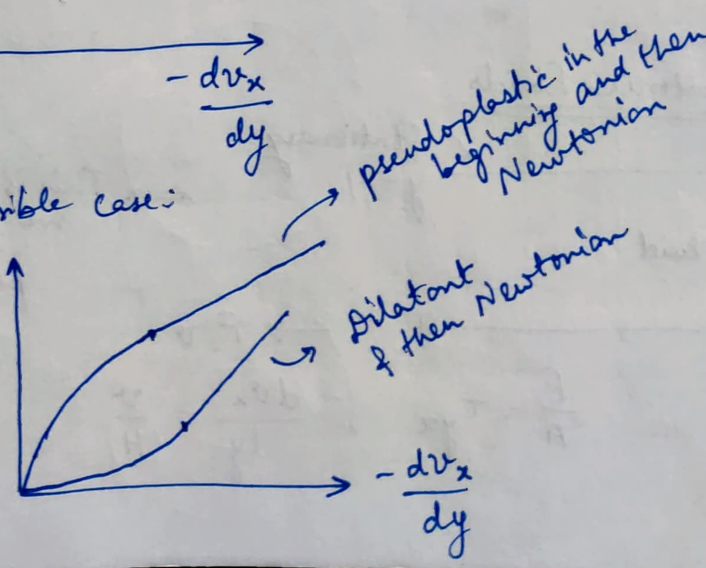


But in real-life it's possible that some fluids shows:





Another possible case:



NOTE: We may have to deal with such fluids (eg: Polymers, etc.) is real-life, and thus fluid mechanics has to be studied for them.

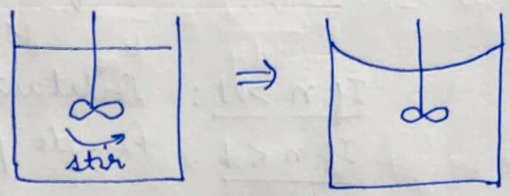
58] We write in general:

$$\tau_{yx} = -\eta\left(\frac{dv_x}{dy}\right) \cdot \frac{dv_x}{dy}$$

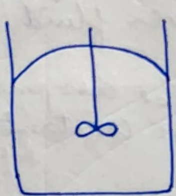
i.e. (where $\mu = \eta\left(\frac{dv_x}{dy}\right)$)

apparent viscosity

For ordinary fluids:

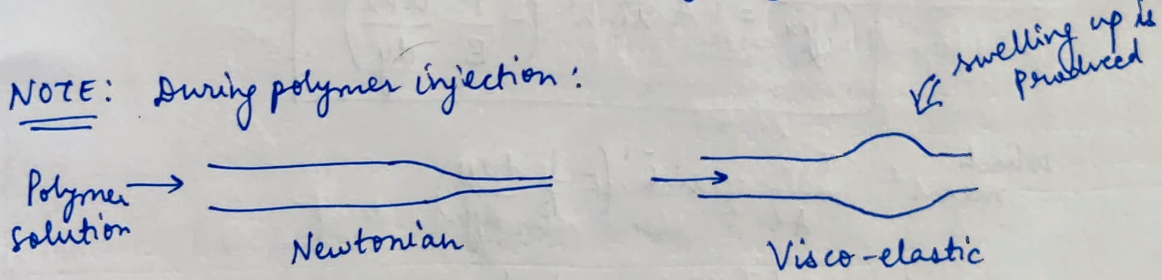


But for visco-elastic fluids:



{ *Study of fluids of these kinds is called Rheology }

NOTE: During polymer injection:



NOTE: Here, we will discuss two of the most popular models for Non-newtonian fluids

59] In general we can write:

$$\tau_{yx} = -\eta \frac{dv_x}{dy}$$

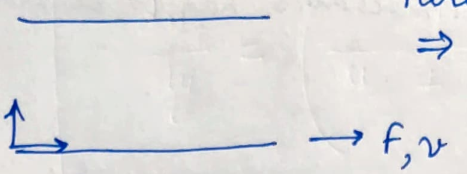
this is a general viscosity that may vary

Here $\Rightarrow \eta = \eta\left(\frac{dv_x}{dy}\right)$

some function of $\frac{dv_x}{dy}$

$$\left(\text{i.e. } \tau_{yx} = -\eta\left(\frac{dv_x}{dy}\right)\frac{dv_x}{dy}\right)$$

First Model : The Power Law model says that: * **Power Law or Ostwald-de-Waele Model**



Here:

$$\Rightarrow \boxed{\tau_{yx} = -m\left|\frac{dv_x}{dy}\right|^{n-1}\frac{dv_x}{dy}}$$

(i.e. there are 2 constants)

If $n=1$:
This reduces to Newtonian case:

$$\tau_{yx} = -m\frac{dv_x}{dy}$$

If $n>1$: Dilatant case
If $n<1$: Pseudo-plastic

{ Generally we observe that m and n change only slightly for any given fluid
↓
so they can be assumed to be constant }

60] **Eyring Model:**

$$\boxed{\tau_{yx} = A\sinh^{-1}\left(-\frac{1}{B}\frac{dv_x}{dy}\right)}$$

which means: $\dfrac{\tau_{yx}}{A} = \sinh^{-1}\left(-\dfrac{1}{B}\dfrac{dv_x}{dy}\right)$

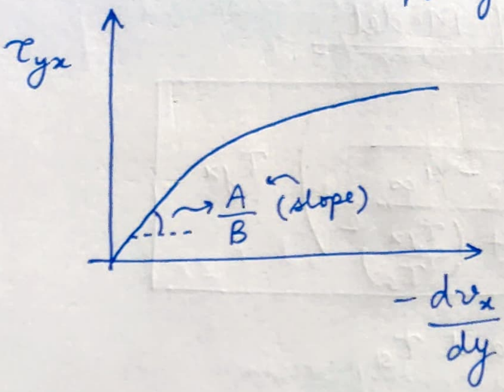
$$\Rightarrow \sinh\left(\frac{\tau_{yx}}{A}\right) = -\frac{1}{B}\frac{dv_x}{dy}$$

For $\tau_{yx}\to 0$ (small values of τ_{yx}) : $\sinh\left(\dfrac{\tau_{yx}}{A}\right) \approx \dfrac{\tau_{yx}}{A}$

$\Rightarrow$ gives $\dfrac{\tau_{yx}}{A} = -\dfrac{1}{B}\dfrac{dv_x}{dy}$

↓

$\tau_{yx} = -\dfrac{A}{B}\dfrac{dv_x}{dy}$ $\Rightarrow$ i.e. Shows Newtonian behavior for small τ_{yx}



For general model of Eyring

we see this plot (looks like Pseudo-plastic curve) *

61] **Ellis Model:**

$$\boxed{-\frac{dv_x}{dy} = \left\{ \phi_0 + \phi_1 |\tau_{yx}|^{\alpha-1} \right\} \tau_{yx}}$$

For $\phi_0 = 0$, $\alpha = 1$ $\Rightarrow$ we get $-\frac{dv_x}{dy} = \phi_1 \tau_{yx}$

i.e. Newtonian Behavior

For $\phi_0 = 0$ $\Rightarrow$ Power Law Model

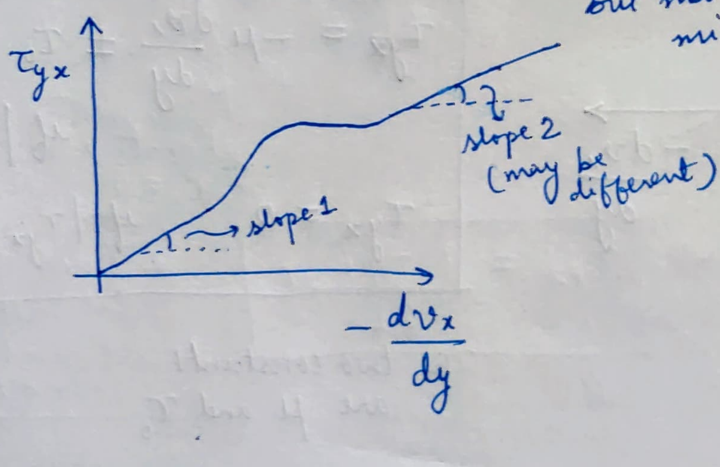
For $\phi_1 = 0$ $\Rightarrow$ Newtonian Model

For $\alpha > 1$ $\Rightarrow$ Model approaches Newton's law of viscosity for $\tau_{yx} \to 0$

For $\alpha < 1$ $\Rightarrow$ Model approaches Newton's law of viscosity for $\tau_{yx} \to \infty$

62]

NOTE: There are some fluids which show Newtonian behaviour in beginning & end as well but not in the middle



Explained by: <u>Reiner-Philippoff Model</u>

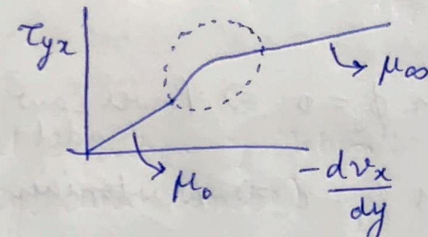
$$\boxed{-\frac{dv_x}{dy} = \left(\frac{1}{\mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + \left(\frac{\tau_{yx}}{\tau_s}\right)^2}} \right) \tau_{yx}}$$

Constants are → μ_0, μ_∞ and τ_s

For $\tau_{yx} \to 0 \Rightarrow -\frac{dv_x}{dy} = \frac{1}{\mu_0} \tau_{yx} \Rightarrow \tau_{yx} = -\mu_0 \frac{dv_x}{dy}$

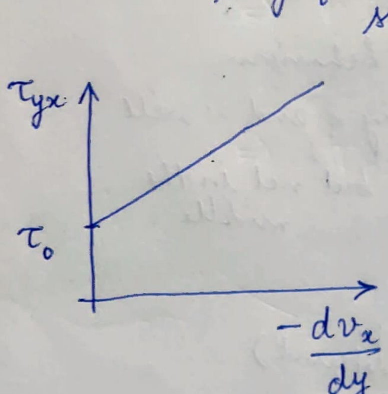
For $\tau_{yx} \to \infty \Rightarrow -\frac{dv_x}{dy} = \frac{1}{\mu_\infty} \tau_{yx} \Rightarrow \tau_{yx} = -\mu_\infty \frac{dv_x}{dy}$

Hence we get the curve required:



63] <u>Bingham Model</u>

We will require some minimum critical stress
↓
only after which it will show velocity gradient



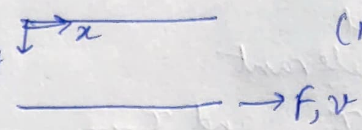
Model is given by:

$$\boxed{\begin{aligned} \tau_{yx} &= -\mu \frac{dv_x}{dy} \pm \tau_0 \quad , \text{if } |\tau_{yx}| \geq \tau_0 \\ \tau_{yx} &= 0 \quad , \text{if } |\tau_{yx}| < \tau_0 \end{aligned}}$$

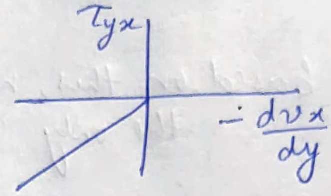
The two constants are μ and τ_0

* { **NOTE:** By reversing coordinate system, you can change the graph:

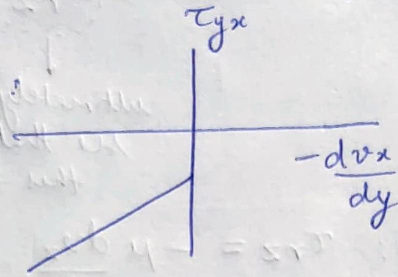
eg:



(Newtonian fluid) ⇒



For Bingham fluid:



(reason: because here $\frac{dv_x}{dy}$ is +ve)
↓
so τ_{yx} becomes −ve

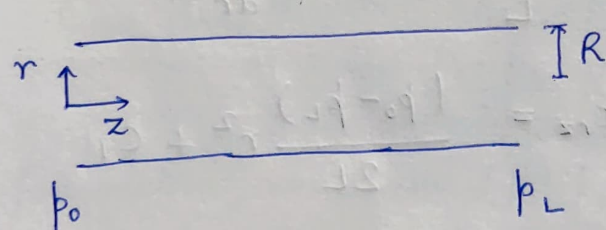
So that's why we have:

* { when τ_{yx} is +ve ⇒ we use $+\tau_0$ in that Bingham model

and, when τ_{yx} is −ve ⇒ we use $-\tau_0$

* **NOTE:** The two most important models are Power Law model & Bingham model.

64] Power Law or Bingham Fluid Flowing through a horizontal pipe



Assumptions:

Steady flow
Fully developed flow
Laminar flow
ρ is constant

and { 1. μ, τ_0 are constant (if Bingham model)
or
2. m and n are constant (Power Law model)

Intuitive velocity profile:

$v_r = 0$

$v_\theta = 0$

$v_z = v_z(r, z)$

⇓

from Equation of Continuity:

$$\frac{\partial v_z}{\partial z} = 0 \Rightarrow v_z = v_z(r)$$

[* NOTE: Here we cannot use Navier-Stokes equations since this is not a Newtonian fluid.]

So based on this, we can see that the only components relevant are: τ_{rz} and τ_{zr}

ultimately we can see that only this is relevant

Now, for Newtonian fluid: $\tau_{rz} = -\mu \dfrac{dv_z}{dr}$

For Non-Newtonian: $\tau_{rz} = -\eta \dfrac{dv_z}{dr}$ $\Rightarrow$ Hence, same τ components will be present whether it is Newtonian or Non-Newtonian

(only difference is that $\eta = \eta\left(\dfrac{dv_z}{dr}\right)$)

Now, from General Equation of Motion: (using Cylindrical coord-s)

z-comp: $0 = -\dfrac{dp}{dz} - \dfrac{1}{r}\dfrac{d(r\tau_{rz})}{dr}$

(just like we did earlier) $= \dfrac{p_0 - p_L}{L} - \dfrac{1}{r}\dfrac{d(r\tau_{rz})}{dr}$

$\Rightarrow r\tau_{rz} = \dfrac{(p_0 - p_L)}{2L} r^2 + C_1$

$\Rightarrow \tau_{rz} = \dfrac{p_0 - p_L}{2L} r + \dfrac{C_1}{r}$

and here we get that $C_1 = 0$

$\therefore \Rightarrow \boxed{\tau_{rz} = \dfrac{p_0 - p_L}{2L} r}$

$\therefore$ This relation is valid for both Newtonian & Non-Newtonian fluids

* Thus, till here we will get same answer ALWAYS

LECTURE-15

{NOTE: In the power law model for expression of $\left|\frac{dv_z}{dr}\right|$:

if $\frac{dv_z}{dr} > 0 \Rightarrow \left|\frac{dv_z}{dr}\right|$ can be written as $\left(\frac{dv_z}{dr}\right)$

if $\frac{dv_z}{dr} < 0 \Rightarrow \left|\frac{dv_z}{dr}\right| = -\left(\frac{dv_z}{dr}\right)$ }

Thus in our case, using power law model:

$$\tau_{rz} = -m\left(-\frac{dv_z}{dr}\right)^{n-1}\frac{dv_z}{dr}$$

$$= \frac{p_0 - p_L}{2L} r$$

Using this we can simplifty for $\frac{dv_z}{dr}$, and then integrate to obtain v_z.

<u>Similarly in Bingham model:</u>

(i) $\tau_{rz} = -\mu \frac{dv_z}{dr} \pm \tau_0$. if $|\tau_{rz}| \geq \tau_0$

(ii) else, $\frac{dv_z}{dr} = 0$ if $|\tau_{rz}| < \tau_0$

From the expression, $\tau_{rz} = \frac{p_0 - p_L}{2L} r$, we know that τ_{rz} is <u>positive</u>.

To check whether (i) or (ii) will apply:

τ_{rz} is max^m at $r = R$

$\Rightarrow \therefore$ if $\frac{p_0 - p_L}{2L} R < \tau_0$, then $\frac{dv_z}{dr}$ will be zero everywhere

$\therefore$ For motion, minimum $(p_0 - p_L)$ required is:

$$\underbrace{(\tau_{rz})_{max}}_{\text{i.e. at } r=R} = \frac{(p_0 - p_L)_{min}}{2L} R \geq \tau_0$$

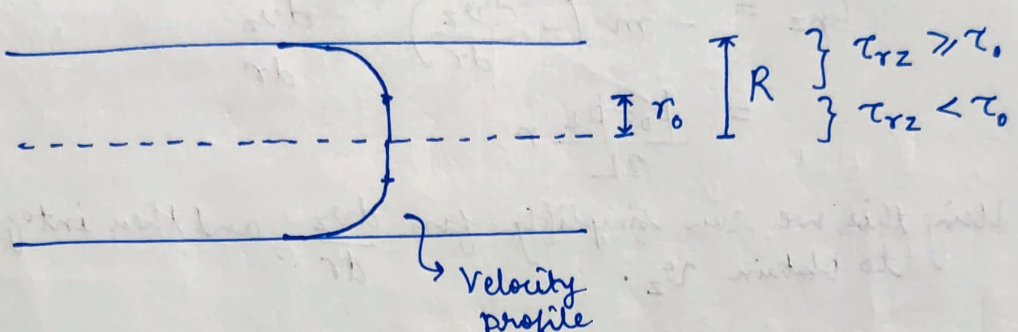
$$\Rightarrow \boxed{(p_0 - p_L)_{min} \geq \frac{2L\tau_0}{R}}$$

* Minimum reqd. pressure difference for fluid to move.

Now, at the center τ_{rz} has to be zero
& and it will increase towards $r \to R$

So there must be a
* region from $r=0$ to $r=r_0$
where $\frac{dv_z}{dr} = 0$

Thus: we get



Hence the solution will be in 2 parts:

$r \geq r_0$: $\quad \tau_{rz} = -\mu \frac{dv_z}{dr} + \tau_0$

$r < r_0$: $\quad -\frac{dv_z}{dr} = 0$

Solving: for $r \geq r_0$:

$$\tau_{rz} = \frac{p_0 - p_L}{2L} r = -\mu \frac{dv_z}{dr} + \tau_0$$

$$\Rightarrow \mu \frac{dv_z}{dr} = \tau_0 - \frac{p_0 - p_L}{2L} r$$

$$\Rightarrow \frac{dv_z}{dr} = \frac{\tau_0}{\mu} - \frac{p_0 - p_L}{2\mu L} r$$

Integrating $\Rightarrow$ $v_z = \frac{\tau_0}{\mu} r - \frac{p_0 - p_L}{2\mu L} \frac{r^2}{2} + C_2$

Now $v_z = 0$ at $r = R$

$$\Rightarrow 0 = \frac{\tau_0}{\mu} R - \frac{p_0 - p_L}{4\mu L} R^2 + C_2$$

∴ We obtain:

$$v_z = \frac{\tau_0}{\mu} r - \frac{p_0 - p_L}{4\mu L} r^2 + \frac{p_0 - p_L}{4\mu L} R^2 - \frac{\tau_0}{\mu} R$$

$$= \frac{p_0 - p_L}{4\mu L}(R^2 - r^2) + \frac{\tau_0}{\mu}(r - R)$$

For $r \leq r_0$: $\frac{dv_z}{dr} = 0$

$$\Rightarrow v_z = C_3$$

Thus, $C_3 = v_z \big|_{r = r_0}$ which we get from the above formula.

$$= \frac{p_0 - p_L}{4\mu L}(R^2 - r_0^2) + \frac{\tau_0}{\mu}(r_0 - R)$$

[* **NOTE**: Liquids are generally incompressible except for extreme conditions]

65] <u>3D - Non Newtonian Problem</u>

In 1D:

* In Newtonian case: $\tau_{yx} = -\mu \frac{dv_x}{dy}$

In 3D:

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

↓

these are all given by the 9 equations (as we saw in Newton's Law for Viscosity)

For simplicity, we can write:

$$\tau_{ij} = -\mu \Delta_{ij}$$

{ eg: $\tau_{11} = -\mu \Delta_{11}$
$\tau_{12} = -\mu \Delta_{12}$
⋮ etc... }

* **NOTE**: $\underset{\sim}{\Delta}$ is the Velocity-Gradient Tensor

{ Hence we have: $\tau_{rz} = -\mu \Delta_{rz}$
⋮
so on... }

Now, in Non-Newtonian case:

$$\tau_{ij} = -\eta\,\Delta_{ij} \quad \leftarrow \text{Model for Non-Newtonian fluid}^{*}$$

(where, $\eta = \eta(\Delta_{ij})$) (in general 3D case)

(1) Power-Law Model

$$\eta = -m\,|\Delta_{ij}|^{n-1}$$

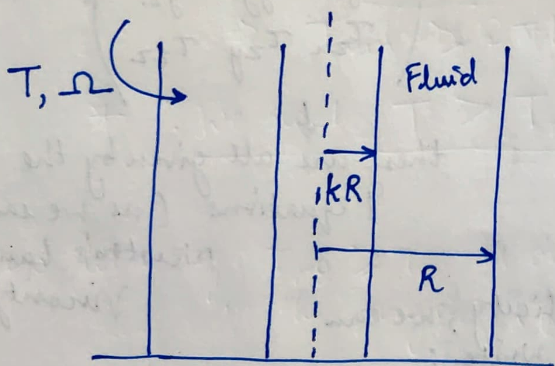
(2) Binghum Model

(i) $\tau_{ij} = -\mu\,\Delta_{ij} \pm \tau_0 \quad , \text{ if } |\tau_{ij}| \geq \tau_0$

(ii) $\Delta_{ij} = 0 \quad , \text{ if } |\tau_{ij}| \leq \tau_0$

{here, if τ_{ij} is +ve then $\tau_0 \rightarrow$ + sign
else if τ_{ij} is −ve then we use $-\tau_0$ }

66] Couette flow of Non-Newtonian fluids between two Concentric cylinders



Steady flow
Fully developed
Laminar
ρ constant

Here we obtain the following expression:

$$\tau_{r\theta} = -\frac{T}{2\pi L r^2}$$

LECTURE-16

{ We have taken: $v_r = v_z = 0$; $v_\theta = v_\theta(r)$
(Velocity profile)

From this, we can go to the Newtonian table and check all τ components

$\downarrow$

the only relevant component is $\tau_{r\theta}$

$$\Rightarrow \tau_{r\theta} = -\mu \Delta_{r\theta}$$

From the table we see:

$$\Delta_{r\theta} = r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)$$

Using eq^n. of motion in θ dir^n:

$$0 = -\frac{1}{r^2} \frac{d}{dr}\left(r^2 \tau_{r\theta}\right)$$

$$\Rightarrow r^2 \tau_{r\theta} = C_1$$

$$\Rightarrow \tau_{r\theta} = \frac{C_1}{r^2}$$

Now,
$$T = (\tau_{r\theta})_f \Big|_{r=R} 2\pi R L \times R$$

$$= (-\tau_{r\theta})\Big|_{r=R} 2\pi R L \times R$$

$$= -\frac{C_1}{R^2} 2\pi R^2 L$$

$$\Rightarrow \therefore C_1 = -\frac{T}{2\pi L}$$

Thus we obtain the formula:

$$\tau_{r\theta} = -\frac{T}{2\pi L r^2}$$

$\rightarrow$ since $\tau_{r\theta}$ is $-$ve thus $\Delta_{r\theta}$ is $+$ve ($\because \tau_{r\theta} = -\mu \Delta_{r\theta}$)

* With Power Law:

$$\tau_{r\theta} = -m \left|\Delta_{r\theta}\right|^{n-1} \Delta_{r\theta}$$

$$= -m \Delta_{r\theta}^{n-1} \Delta_{r\theta}$$

$$= -m \Delta_{r\theta}^{n}$$

$$\Rightarrow \left(\frac{T}{2\pi m L r^2}\right)^{1/n} = \Delta_{r\theta} = r \frac{d}{dr}\left(\frac{v_\theta}{r}\right)$$

This can be solved by integration.

(You can then put boundary conditions as well:

$$\text{at } r = kR, \; v_\theta = 0$$

$$\text{and } r = R, \; \frac{v_\theta}{R} = \Omega \;)$$

* With Binghom model:

$$\tau_{r\theta} = -\frac{T}{2\pi L r^2}$$

$$\Delta_{r\theta} = r \frac{d}{dr}\left(\frac{v_\theta}{r}\right)$$

We have: (i) $\tau_{r\theta} = -\mu \Delta_{r\theta} \pm \tau_0$, if $|\tau_{r\theta}| \geq \tau_0$

(ii) $\Delta_{r\theta} = 0$, if $|\tau_{r\theta}| \leq \tau_0$

Now, for $|\tau_{r\theta}|$ to be > than τ_0 it should be greater at least at $r = kR$ (bcz here $\tau_{r\theta}$ is maximum)

$$\therefore \frac{T_{min}}{2\pi L (kR)^2} \geq \tau_0$$

$$\Rightarrow T_{min} \geq 2\pi L (kR)^2 \tau_0$$

* ∴ We need $T > T_{min}$ for motion to occur

Now, let's say $\tau_{r\theta} \geq \tau_0$ is occuring in the range kR to r_0

$\Rightarrow$ and in r_0 to R, $\tau_{r\theta} < \tau_0 \Rightarrow$ i.e. $\Delta_{r\theta} = 0$

$\downarrow$

* i.e. $\frac{v_\theta}{r}$ is constant for $r \leq r_0$

Now, at r_0:

$$\frac{T}{2\pi L r_0^2} = \tau_0$$

$$\Rightarrow r_0^2 = \left(\frac{T}{\tau_0}\right)\frac{1}{2\pi L}$$

If we go beyond the value for τ_o for all r values, i.e.:

$$\frac{T_{max}}{2\pi L R^2} \geq \tau_o$$

$$\Rightarrow \quad T_{max} \geq 2\pi L R^2 \tau_o$$

$\ast \quad \therefore$ for $T \geq T_{max}$, only eq^n (i) will apply everywhere

Thus, we have obtained:

(1) if $T \geq T_{max}$, then flow is governed by eq^n (i) of the model only and eq^n (ii) does not apply anywhere

i.e. viscous flow everywhere

(2) if $T \leq T_{min}$, then there will be No motion

(3) if $T_{min} < T < T_{max}$

$\Rightarrow$ if kR to r_o $\rightarrow$ eq^n (i) applies

if r_o to R $\rightarrow$ eq^n ( ) applies

this is the most difficult case (have to solve two differential eq^{n}s and have to equate both at $r = r_o$)

<u>Solution for case (1) : $T \geq T_{max}$</u>

Now, $\tau_{r\theta} = -\dfrac{T}{2\pi L r^2}$

$$\tau_{r\theta} = -\mu \Delta_{r\theta} - \tau_o$$

$$\tau_{r\theta} = -\mu\left(r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right) - \tau_o = -\frac{T}{2\pi L r^2}$$

$$\Rightarrow r\frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \frac{T}{2\pi\mu L r^2} - \frac{\tau_o}{\mu}$$

$$\Rightarrow \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \frac{T}{2\pi\mu L r^3} - \frac{\tau_o}{\mu r}$$

Thus, $\dfrac{v_\theta}{r} = -\dfrac{1}{2}\dfrac{T}{2\pi\mu L r^2} - \dfrac{\tau_o}{\mu}\ln r + C_2$

At $r = kR$, $\left(\frac{v_\theta}{r}\right)\Big|_{r=kR} = 0$

Thus, $$0 = -\frac{T}{4\pi\mu L (kR)^2} - \frac{\tau_0}{\mu}\ln(kR) + C_2$$

$$\Rightarrow C_2 = \frac{T}{4\pi\mu L k^2 R^2} + \frac{\tau_0}{\mu}\ln(kR)$$

Hence, we obtain:

$$\frac{v_\theta}{r} = -\frac{T}{4\pi\mu L r^2} - \frac{\tau_0}{\mu}\ln r + \frac{T}{4\pi\mu L k^2 R^2} + \frac{\tau_0}{\mu}\ln(kR)$$

$$= \frac{T}{4\pi\mu L}\left(\frac{1}{k^2R^2} - \frac{1}{r^2}\right) + \frac{\tau_0}{\mu}\ln\left(\frac{kR}{r}\right)$$

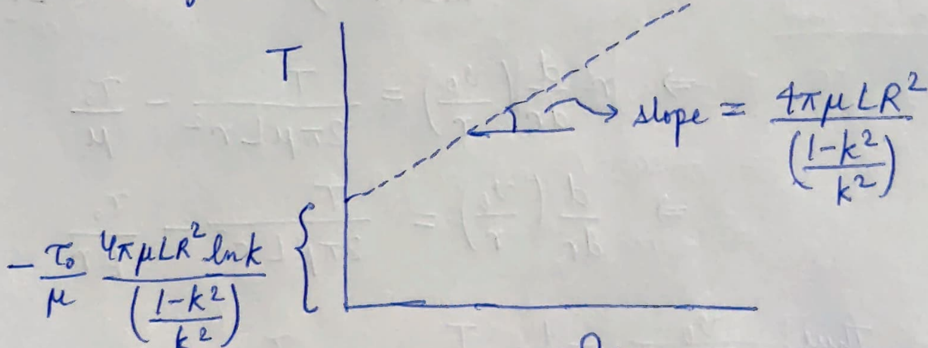
At $r = R$, $\left(\frac{v_\theta}{r}\right)\Big|_{r=R} = \Omega$

$$\Rightarrow \Omega = \frac{T}{4\pi\mu L}\left(\frac{1}{k^2R^2} - \frac{1}{R^2}\right) + \frac{\tau_0}{\mu}\ln k$$

$$\Rightarrow \frac{T}{4\pi\mu L R^2}\left(\frac{1}{k^2} - 1\right) + \frac{\tau_0}{\mu}\ln k = \Omega$$

$$\Rightarrow T = \Omega\,\frac{4\pi\mu L R^2}{\left(\frac{1-k^2}{k^2}\right)} + \left(-\frac{\tau_0}{\mu}\cdot\frac{4\pi\mu L R^2}{\left(\frac{1-k^2}{k^2}\right)}\ln k\right)$$

Using this, we can plot T vs Ω:



$\Rightarrow$ * Hence this value of slope can help us determine the viscosity μ