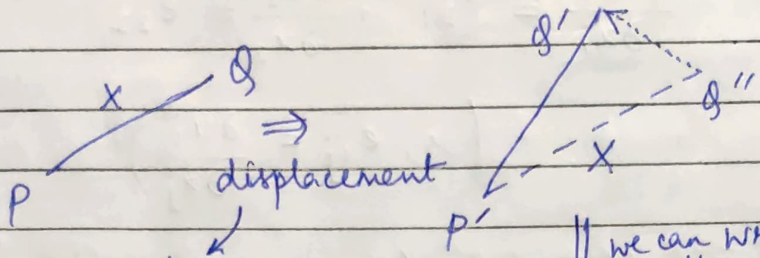
37] <u>2D-Strain</u>

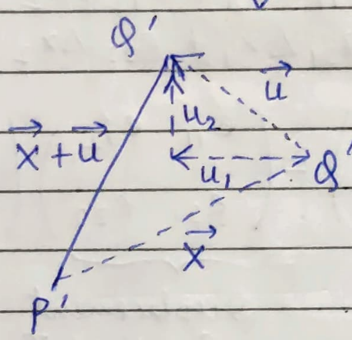
We can break a general displacement into: <u>rotation</u> + <u>deformation</u>



$\Rightarrow$ displacement

Translation
+
Rotation
+
Deformation

$\Downarrow$ we can write the displacement vector ($\vec{u}$) in terms of its components



$(\vec{u} = u_1\hat{i} + u_2\hat{j})$

38] We have, $\vec{u} = u_1\hat{i} + u_2\hat{j}$ $\Rightarrow$ u_1 and u_2 $\downarrow$ * continuous functions of x_1 & x_2

Using Taylor's Series:

For any general function, $L = f(x,y)$

We can say $\Rightarrow$ $$L = f(x,y) = f(x_0,y_0) + \frac{\left(\frac{\partial L}{\partial x}\right)_0 dx}{1!} + \frac{\left(\frac{\partial^2 L}{\partial x^2}\right)_0 dx^2}{2!} + \dots$$

$$+ \frac{\left(\frac{\partial L}{\partial y}\right)_0 dy}{1!} + \frac{\left(\frac{\partial^2 L}{\partial y^2}\right)_0 dy^2}{2!} + \dots$$

Thus, applying this to Δu, and using small displacement assumption:

$$\Delta u_1 = \frac{\partial u_1}{\partial x_1}\Delta x_1 + \frac{\partial u_1}{\partial x_2}\Delta x_2, \quad \Delta u_2 = \frac{\partial u_2}{\partial x_1}\Delta x_1 + \frac{\partial u_2}{\partial x_2}\Delta x_2$$

Thus, in general if we define:

$$\frac{\partial u_1}{\partial x_1} \equiv e_{11} \quad , \quad \frac{\partial u_1}{\partial x_2} \equiv e_{12}$$

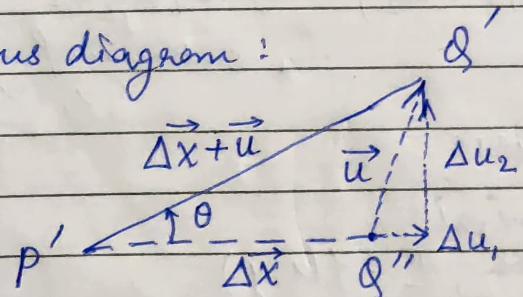
$$\frac{\partial u_2}{\partial x_1} \equiv e_{21} \quad , \quad \frac{\partial u_2}{\partial x_2} \equiv e_{22}$$

$$\left(\text{or, } e_{ij} = \frac{\partial u_i}{\partial x_j}\right)$$

Then, $\Delta u_1 = e_{11}\Delta x_1 + e_{12}\Delta x_2$
$\Delta u_2 = e_{21}\Delta x_1 + e_{22}\Delta x_2$

$$\Rightarrow \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} = \underbrace{\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}}_{\text{Displacement Gradient Tensor}} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$$

39] Using previous diagram:



($\vec{u}$ is given by $\Delta u_1 \hat{i} + \Delta u_2 \hat{j}$)

For small displacements (du) $\Rightarrow$ Angle will be small

$\therefore$ Since $\theta \to 0 \Rightarrow \tan\theta \approx \theta$

So we can write, $\theta \approx \dfrac{\Delta u_2}{\Delta x_1 + \Delta u_1} \approx \dfrac{\Delta u_2}{\Delta x_1} = e_{21}$

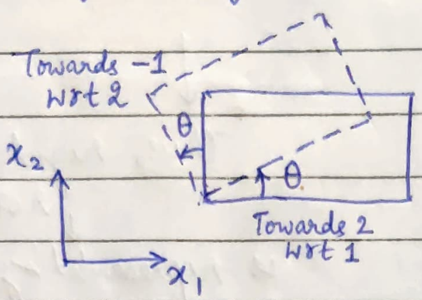
(since Δu_1 is very small)

Hence we have: $\boxed{e_{21} = \theta}$

angle of rotation TOWARDS 2-dir^n w.r.t. 1-dir^n

40] If $i=j \Rightarrow e_{ij} \Rightarrow$ Only linear stretching (zero rotation)
$\downarrow$
i.e. strain

* <u>Rigid Body Rotation</u> $\Rightarrow$ No stretching, only rotation



e_{21} = Toward 2 wrt 1 $= \theta$

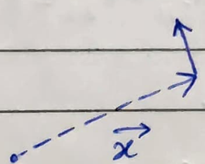
e_{12} = Towards 1 wrt 2

$= -$(Towards −1 wrt. 2) $= -\theta$.

$\Rightarrow e_{21} = \theta, \; e_{12} = -\theta$

Thus, we get: $\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$ ← anti-symmetric tensor (matrix)

41] <u>3D Rotation</u>



↳ For rigid-body rot?
$\downarrow$
$\vec{x}$ and $\vec{u} \Rightarrow$ orthogonal
(i.e. $\vec{x} \cdot \vec{u} = 0$)

[* NOTE: We can write $\Delta\vec{u}$.
(We have used u just for simplicity)]

Thus, $\vec{x} \cdot \vec{u} = 0$

$\Rightarrow x_1 u_1 + x_2 u_2 + x_3 u_3 = 0 \Rightarrow$ Here, we know:
$u_i = \sum_j e_{ij} x_j$
(or, $e_{ij} x_j$ in Einstein notation)

$\Rightarrow x_i e_{ij} x_j = 0$ (Einstein notation)

$\Rightarrow x_1 x_1 e_{11} + x_1 x_2 e_{12} + x_1 x_3 e_{13}$
$+ x_2 x_1 e_{21} + x_2 x_2 e_{22} + x_2 x_3 e_{23}$
$+ x_3 x_1 e_{31} + x_3 x_2 e_{32} + x_3 x_3 e_{33} = 0$

Possible solutions : 1) Trivial solution : all $e_{ij} = 0$

2) Or:

$e_{12} = -e_{21}$, $e_{13} = -e_{31}$, $e_{23} = -e_{32}$

and $e_{11} = e_{22} = e_{33} = 0$

$\downarrow$

Thus, rigid body rotation $\downarrow$ is represented by

* anti-symmetric matrix/tensor

{ Which make sense since : $\Leftarrow$

1) Rotation towards i wrt j $= -$(Rotation towards j wrt i)

and, 2) $e_{ii} = 0$ makes sense as we want pure rotation, and no deformation }

42] Rotation Tensor

Using: $e_{12} = -e_{21}$

$e_{13} = -e_{31}$ and $e_{11} = e_{22} = e_{33} = 0$

$e_{23} = -e_{32}$

$\Downarrow$

$$\underset{\sim}{e} = \begin{pmatrix} 0 & e_{12} & e_{13} \\ -e_{12} & 0 & e_{23} \\ -e_{13} & -e_{23} & 0 \end{pmatrix}$$

We can now break-up the displacement gradient into Strain tensor & Rigid body Rotation tensor

$$\Rightarrow \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = \underbrace{\begin{bmatrix} \text{Strain Tensor} \end{bmatrix}}_{\substack{\text{Stretching/shearing is involved} \\ \text{(and no rotation)}}} + \underbrace{\begin{bmatrix} \text{Rigid Body Rotation Tensor} \end{bmatrix}}_{\substack{\text{No stretching/shearing} \\ \downarrow \\ \text{only rotation}}}$$

Extracting strain using the following method:

Breaking displacement gradient components into

Anti-symmetric → Rotation: $\omega_{ij} \equiv \frac{1}{2}(e_{ij} - e_{ji}) = -\omega_{ji}$

Symmetric → Strain: $\varepsilon_{ij} \equiv \frac{1}{2}(e_{ij} + e_{ji}) = \varepsilon_{ji}$

Thus, $\Rightarrow$

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = \underbrace{\begin{pmatrix} e_{11} & \frac{e_{12}+e_{21}}{2} & \frac{e_{13}+e_{31}}{2} \\ \frac{e_{12}+e_{21}}{2} & e_{22} & \frac{e_{23}+e_{32}}{2} \\ \frac{e_{13}+e_{31}}{2} & \frac{e_{23}+e_{32}}{2} & e_{33} \end{pmatrix}}_{\text{Strain Tensor } \underset{\sim}{\varepsilon}}$$

$$+ \underbrace{\begin{pmatrix} 0 & \frac{e_{12}-e_{21}}{2} & \frac{e_{13}-e_{31}}{2} \\ -\left(\frac{e_{12}-e_{21}}{2}\right) & 0 & \frac{e_{23}-e_{32}}{2} \\ -\left(\frac{e_{13}-e_{31}}{2}\right) & -\left(\frac{e_{23}-e_{32}}{2}\right) & 0 \end{pmatrix}}_{\text{Rigid body rotation tensor } \underset{\sim}{\omega}}$$

$\Downarrow$

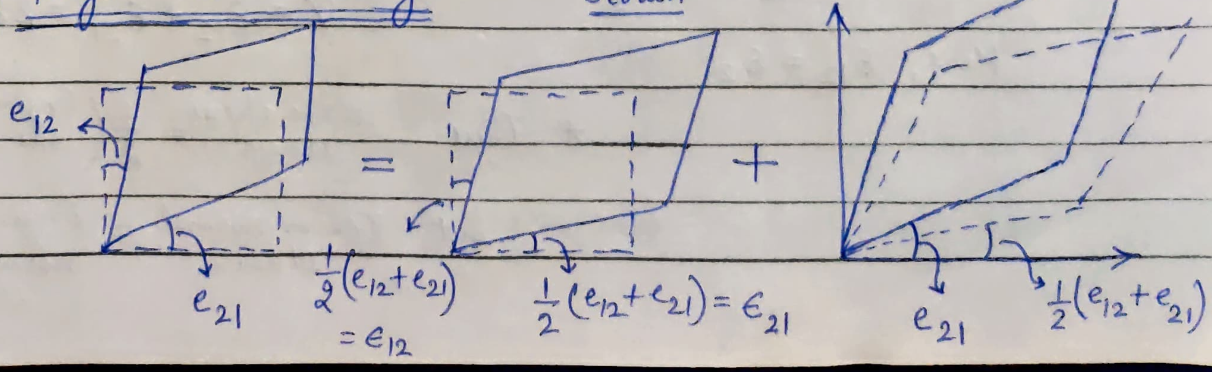
* **NOTE**: This (displ. gradient matrix) need NOT be symmetric

We can now, if we want, write $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\omega}$ by replacing e_{ij} with $\frac{\partial u_i}{\partial x_j}$, $\forall i, j$

LECTURE-8

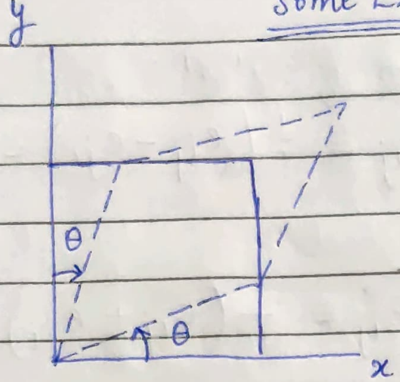
43] Physical Understanding

Strain | Rotation

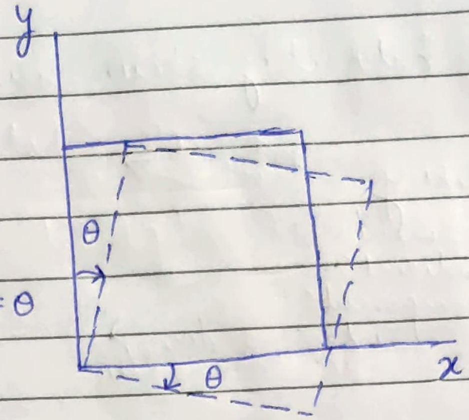


Some Examples

44]



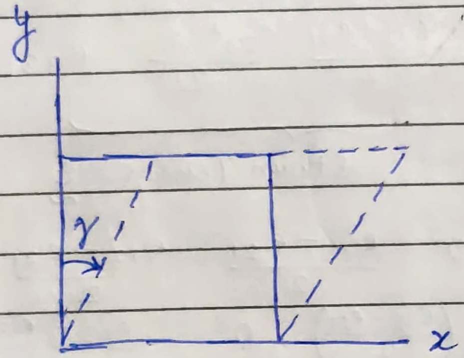
Here, $e_{xy} = e_{yx}$



$e_{xy} = \theta$

$e_{yx} = -\theta$

Thus $\Rightarrow$ $e_{xy} = -e_{yx}$



Here, $e_{xy} = \gamma$, $e_{yx} = 0$

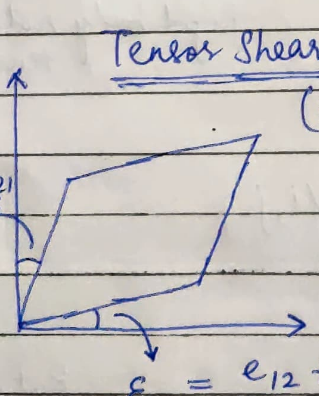
[* Always remember
↓
e_{xy} means the angle moved "towards x, wrt. y"
(and similarly for e_{yx})]

45]

Tensor Shear Strain

(ε_{ij}, where $i \neq j$)

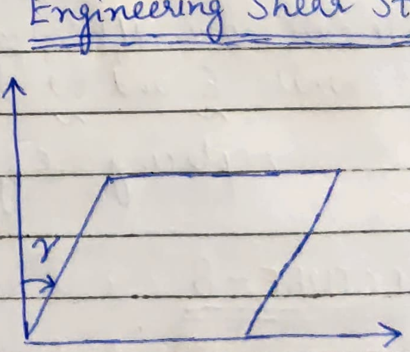
$\varepsilon_{21} = \frac{e_{12}+e_{21}}{2}$



$\varepsilon_{12} = \frac{e_{12}+e_{21}}{2}$

Here, $\varepsilon_{12} = \varepsilon_{21}$

Engineering Shear Strain



$\gamma = \varepsilon_{12} + \varepsilon_{21} = 2\varepsilon_{12}$

* Thus, Tensor shear strain $= \frac{1}{2}$ (Engineering shear strain)

but, Tensor normal strain $=$ Engineering normal strain

46] Principal Strains

Definition: Three mutually $\perp^r$ directions in the body

which remain mutually $\perp^r$ during deformation

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \xrightarrow[\text{(Basis)}]{\text{Change of axis}} \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

Principal Strains ⇒ (i.e. $\varepsilon_1, \varepsilon_2, \varepsilon_3$)

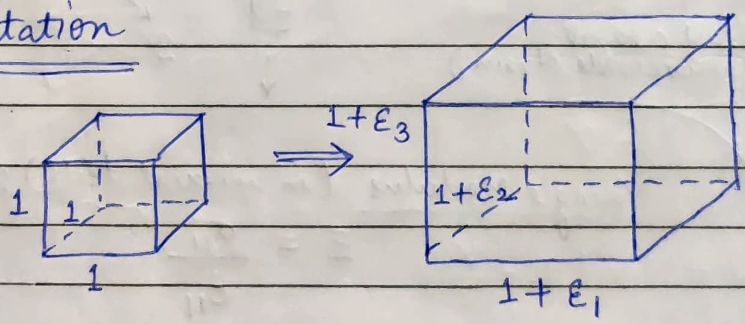
⇓

* Principal axes can ALWAYS be determined

* NOTE: For isotropic solids

principal strain axes coincide with principal stress axes

47] Dilatation



$$\Delta V = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1$$

$$= 1 + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + \left(\text{higher order terms in } \varepsilon_i\varepsilon_j,\ \varepsilon_i\varepsilon_j\varepsilon_k\right) - 1$$

$$= \underbrace{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}_{\text{Dilation}}$$

$$= \text{Trace of strain tensor} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{ii}$$

48] Components of Strain Tensor

We can break the strain tensor into

- Hydrostatic components
- Deviatoric components

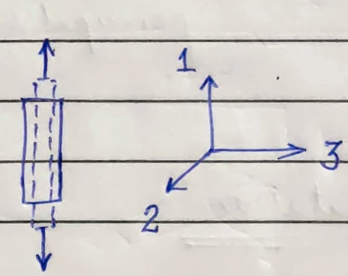
Hydrostatic

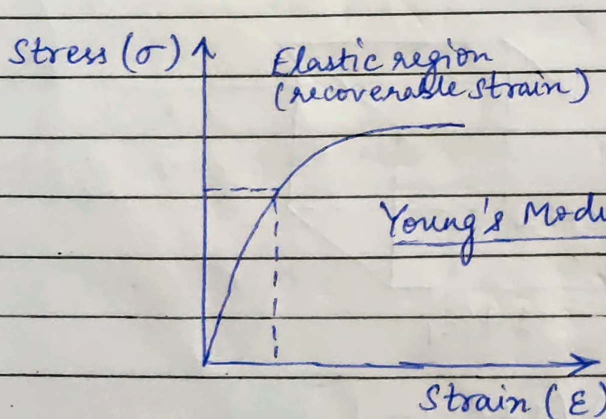
$$\varepsilon_m = \frac{\varepsilon_{ii}}{3} = \frac{1}{3}\,tr(\varepsilon)$$

The strain deviator, ε'_{ij} or deviatoric component of strain tensor will be:

$$\begin{pmatrix} \varepsilon_{11}-\varepsilon_m & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22}-\varepsilon_m & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}-\varepsilon_m \end{pmatrix}$$

49] Hooke's Law





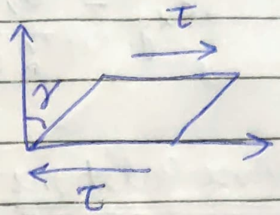
Young's Modulus (in uniaxial tests):

$$E = \frac{\sigma_{11}}{\varepsilon_{11}}$$

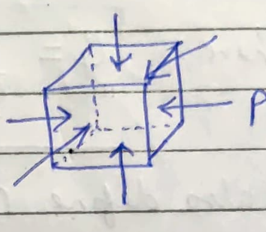
LECTURE-9

50] Other Moduli

Shear Modulus : $G = \dfrac{\tau}{\gamma}$



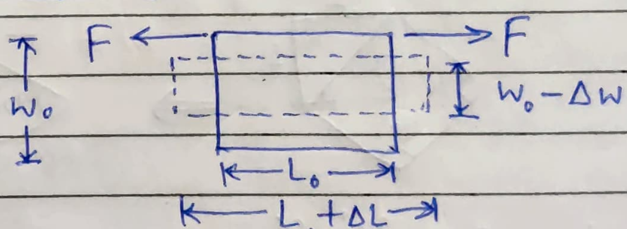
Bulk Modulus : $K = \dfrac{P}{-\left(\dfrac{\Delta V}{V}\right)}$



51] Poisson's Ratio

$$\nu = \frac{-\varepsilon_{33}}{\varepsilon_{11}} = \frac{-\varepsilon_{22}}{\varepsilon_{11}}$$

$$\boxed{\nu = -\left(\frac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}}\right)} = -\left(\frac{\frac{-\Delta W}{W_0}}{\frac{\Delta L}{L_0}}\right)$$



Value For: metals → ~0.33

ceramics → ~0.25

polymers → ~0.40

52] Application of Poisson's Ratio : Using Principle of Superposition

Normal strain in any one dir^n

↳ will be obtained by contributions of all the 3 normal stresses

Thus ⇒ $$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz})$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{zz})$$

and, $$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

For the shear strains, we have just the simple relations:

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{2G}, \quad \varepsilon_{yz} = \frac{\sigma_{yz}}{2G}, \quad \varepsilon_{zx} = \frac{\sigma_{zx}}{2G}$$

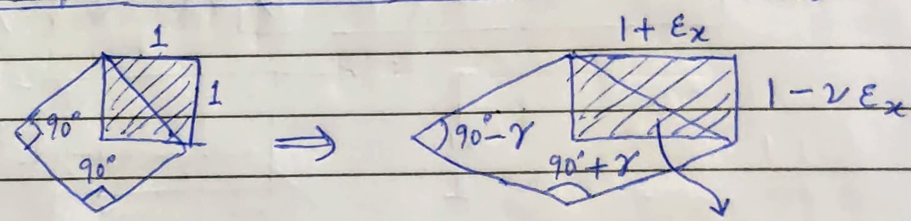
53] Some More Relations

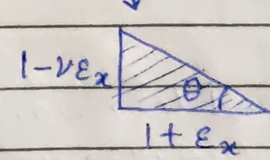
We also have: $\boxed{G = \frac{E}{2(1+\nu)}}$ and $\boxed{K = \frac{E}{3(1-2\nu)}}$

and we also define Compressibility (β) as, $\boxed{\beta = \frac{1}{K}}$

LECTURE-10

54] Relationship b/w Shear Modulus and Poisson's Ratio





Here, $\theta = \dfrac{90° - \gamma}{2}$

Thus, $\tan\theta = \tan\left(45° - \frac{\gamma}{2}\right)$

$$= \frac{\tan 45° - \tan\left(\frac{\gamma}{2}\right)}{1 + \tan 45° \tan\left(\frac{\gamma}{2}\right)}$$

For small γ: $\tan\theta \approx \dfrac{1 - \frac{\gamma}{2}}{1 + \frac{\gamma}{2}}$ — ①

But also, we have: $\tan\theta = \dfrac{1 - \nu\varepsilon_x}{1 + \varepsilon_x}$ — ②

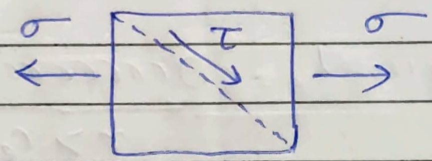
Thus, from 1 and 2 we get:

$$\gamma = \frac{2\varepsilon_x(1+\nu)}{2 - \underbrace{\varepsilon_x(\nu-1)}_{\text{this can be ignored (as strain is small)}}}$$

$$\Rightarrow \gamma = \varepsilon_x(1+\nu)$$

$$\Rightarrow \frac{\tau}{G} = \frac{\sigma}{E}(1+\nu) \quad — (3)$$

Now, since in this case we know:



$$\tau = \frac{\sigma}{2} \quad (4)$$

(can also be seen using the formula: $\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2}$

$= \frac{\sigma - 0}{2}$)

along x

along y

Thus, finally from 3 and 4 we obtain:

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

55] Strain Energy

Defined by, $dU_0 = \underset{\sim}{\sigma} : d\underset{\sim}{\varepsilon}$

$= \sigma_{ij}\, d\varepsilon_{ij}$

(where ":" is the Frobenius inner product

↓

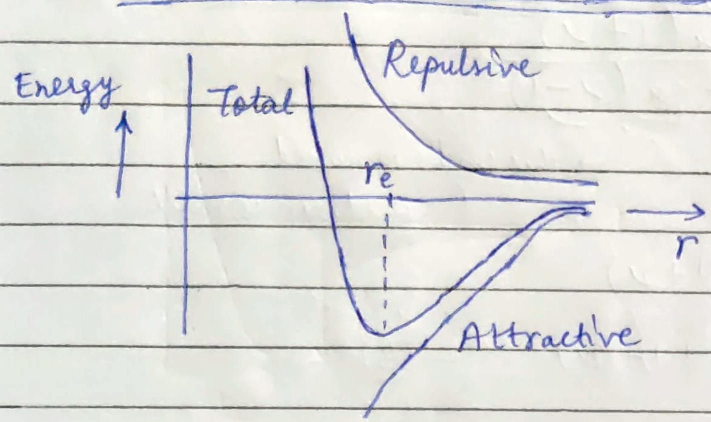
element by element product & summation

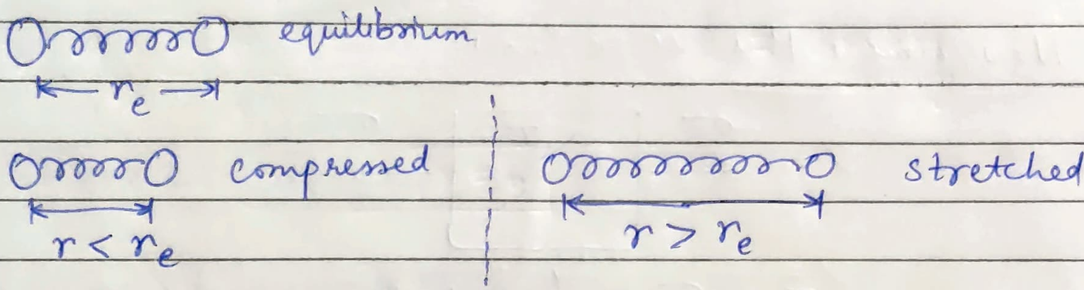
$\Rightarrow \underset{\sim}{A} : \underset{\sim}{B} = \sum_i \sum_j A_{ij} B_{ij}$)

This gives us, $U_0 = \frac{1}{2}\underset{\sim}{\sigma} : \underset{\sim}{\varepsilon}$
(for Linear elasticity)

$= \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$

56] Atomic model for Elastic Behaviour





We consider a model of potential energy as:

$$W = -\frac{A}{r^n} + \frac{B}{r^m}$$

Attractive energy   Repulsive energy

(A, B, n, m are constants, $m > n$)

So we get: Restoring force, $F = -\dfrac{dW}{dr}$

We know, restoring force for spring, $F = -kx$

Thus, stiffness of spring : $\dfrac{dF}{dx} = k$

Stiffness of atomic spring : $\dfrac{dF}{dr} = \dfrac{d^2W}{dr^2} = S$ (definition for S)

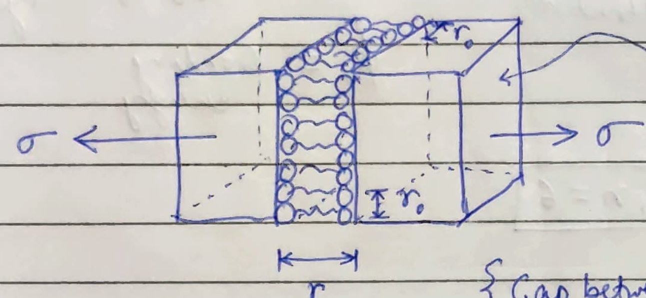
Thus, Stiffness $S = \dfrac{dF}{dr} = \dfrac{d^2U}{dr^2}$

For small stretching, S is constant

$\therefore$ spring constant of bond $\Rightarrow$ $\boxed{S_0 = \left(\dfrac{d^2U}{dr^2}\right)_{r=r_0}}$

and, $\boxed{F = S_0 (r - r_0)}$

Now, we can see :



Thus, if this face is a unit area

$\Rightarrow$ i.e. it is 1×1

Let's say there are N_{side} atoms along any side

then :

$N_{side} r_0 = 1$

$N_{side} = \dfrac{1}{r_0}$

no. of atoms along a side

{Gap between two adjacent atoms on any one block is r_0

r_0 }

$\therefore$ Total no. of atoms in unit face $\Rightarrow N_{side}^2 = \dfrac{1}{r_0^2}$

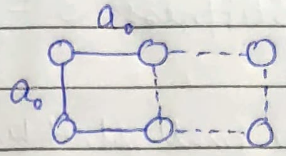
$\therefore$ $\boxed{\text{No. of bonds per unit area} = \dfrac{1}{r_0^2}}$ $= N$ (say)

Thus, we have : $\sigma = N S_0 (r - r_0)$

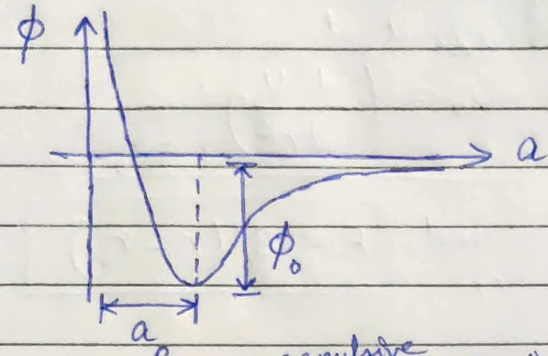
and we know: $\varepsilon_n = \dfrac{r - r_0}{r_0}$

$\Rightarrow$ Thus : $\boxed{E = \dfrac{\sigma}{\varepsilon_n} = \dfrac{S_0}{r_0}}$

57] <u>Model for Potential</u>



In a crystal



a_0 = inter-atomic spacing

ϕ = potential energy of two atom system

ϕ_0 = Bonding energy

$$\phi = \underbrace{\frac{A}{a^m}}_{\text{repulsive}} - \underbrace{\frac{B}{a^n}}_{\text{attractive}}$$

$\Rightarrow$ we take $\boxed{m = 12, n = 6}$

$$F = -\frac{\partial \phi}{\partial a} \rightarrow \text{inter-atomic force}$$

(i.e. restoring force)

58] Using Taylor Series expansion:

$$\phi\Big|_{\text{near } a_0} = \left(\frac{d\phi}{da}\right)_{a=a_0} \Delta a + \frac{1}{2}\left(\frac{d^2\phi}{da^2}\right)_{a=a_0} \Delta a^2 + \ldots$$

higher order terms $(O(\Delta a^3))$

$$\Rightarrow F = -\frac{d\phi}{d(\Delta a)}\Bigg|_{a=a_0} = -\left(\frac{d\phi}{da}\right)_{a=a_0} - \left(\frac{d^2\phi}{da^2}\right)_{a=a_0} \Delta a + \ldots$$

Now, at equilibrium distance $(a = a_0)$

$$\Rightarrow F_{a=a_0} = -\left(\frac{d\phi}{da}\right)_{a=a_0} = 0$$

~~which gives us~~ Hooke's law (of springs)

Thus, for small displacements ~~we get~~ : $\boxed{F = -\underbrace{\left(\frac{d^2\phi}{da^2}\right)_{a=a_0}}_{\text{curvature of curve } \phi} \Delta a = k\,\Delta a}$

Force $\propto$ displacement