MATHEMATICAL METHODS IN MATERIALS ENGINEERING

[MLL212]

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LECTURE 1 — **LINEAR ALGEBRA** — 09/01/2023

1] Some topics we'll cover first

1. Matrices
2. Operations on matrices
3. Linear Transformations

2] Linear Algebra

'm' simultaneous "linear" eq^{n}s with 'n' unknowns :

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\quad \ddots \qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}
$$

NOTE: eg:
$$
\left.\begin{aligned}
a_{11}x_1^2 + a_{12}x_2^2 &= b_1 \\
a_{21}x_1^2 + a_{22}x_2^2 &= b_2
\end{aligned}\right\}
$$
this is a non-linear system

3] Matrix

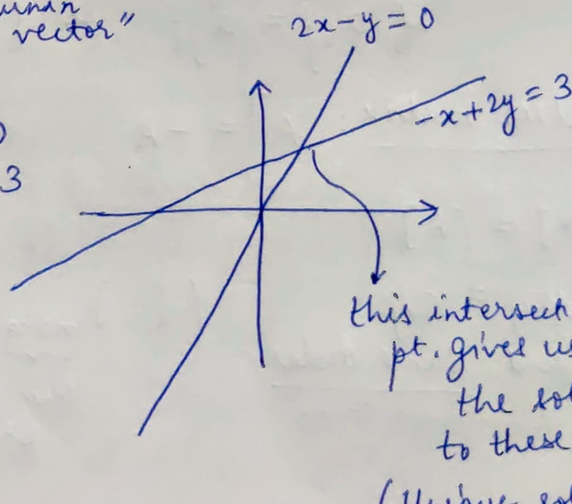
→ Row picture
→ Column picture

The row picture is as follows :

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$\Rightarrow \quad A\underset{\substack{\downarrow \\ \text{this is} \\ \text{a "column} \\ \text{vector"}}}{x} = b$ { This is called a Linear Transformation }

eg: $2x - y = 0$
$-x + 2y = 3$

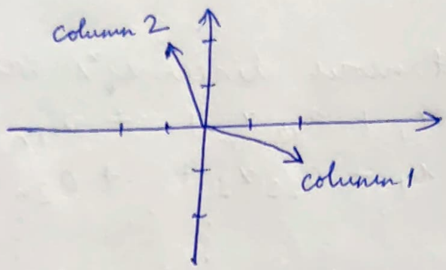


this intersection pt. gives us the sol^n to these eq^{n}s
(Unique sol^n)

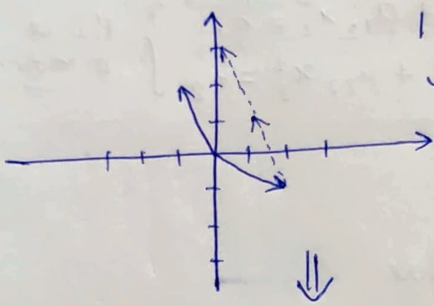
The Column picture: eg: $2x - y = 0$
$-x + 2y = 3$

$$\Rightarrow x\begin{bmatrix}2\\-1\end{bmatrix} + y\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

$\begin{bmatrix}2\\-1\end{bmatrix}$ ↓ column 1 $\qquad$ $\begin{bmatrix}-1\\2\end{bmatrix}$ ↓ column 2

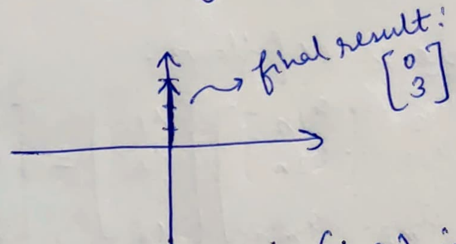


Trying for $x=1$, $y=2$:



$$\underbrace{1\begin{bmatrix}2\\-1\end{bmatrix}}_{\text{start at } \begin{bmatrix}2\\-1\end{bmatrix}} + \underbrace{2\begin{bmatrix}-1\\2\end{bmatrix}}_{\text{adding } \begin{bmatrix}-1\\2\end{bmatrix} \text{ twice}} = \begin{bmatrix}0\\3\end{bmatrix}$$

$\Downarrow$



$\therefore$ (1,2) is the solⁿ of the eqⁿs.

4] **Symmetric matrix**: [A] is symmetric if $a_{ij} = a_{ji}$, or $[A]^T = [A]$

Skew-symmetric matrix: $A^T = -A$

NOTE: If $[A] = [B]$
$\Rightarrow a_{ij} = b_{ij}$
* and, same order $\left([\ ]_{\underbrace{m\times n}_{\text{this is the order of the matrix}}} \right)$

Upper triangle : $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$

Lower triangle : $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$

5] Addition

① $([A]+[B])[C] = [A][C] + [B][C]$

② $[A]+[B] = [B]+[A]$ {Commutative}

③ Associative : $([A]+[B])+[C] = [A] + ([B]+[C])$

④ Scalar multiplication :

$$c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

6] Product

$$[A]_{m\times n}[B]_{n\times l} = [C]_{m\times l}$$

also, $[A][B] \neq [B][A]$, in general

NOTE! LAPACK library in FORTRAN and MATLAB have useful functions for matrix multiplication.

In general, let's say we have:

$$a_{11}x_1 + a_{12}x_2 = y_1$$
$$a_{21}x_1 + a_{22}x_2 = y_2$$

This can be written as a linear transformation :

$$Ax = y$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \; A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Now, we can write 'x' itself as a linear transformation !

$$x = Bw$$

$$b_{11}w_1 + b_{12}w_2 = x_1$$
$$b_{21}w_1 + b_{22}w_2 = x_2$$

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Thus, we can say:

$$ABw = y$$

Here,

$$y_1 = a_{11}(b_{11}w_1 + b_{12}w_2) + a_{12}(b_{11}w_1 + b_{12}w_2)$$
$$y_2 = a_{21}(b_{21}w_1 + b_{22}w_2) + a_{22}(b_{21}w_1 + b_{22}w_2)$$

{ NOTE: There may be errors in the exact indices used above }

Taking this as, $y = Cw$

$$\Rightarrow \quad c_{11} = a_{11}b_{11} + a_{12}b_{21}$$
$$c_{12} = a_{11}b_{12} + a_{12}b_{22}$$

⋮ and so on

Which is the motivation for the way we do matrix multiplication.

H.W.: (a) $\begin{bmatrix} 3 & 5 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 1 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} -2 & 4 \\ -4 & 6 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

(c) If $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

find: (i) $A^4 - 4A^2 + 2I$

(ii) $A^2 - 3A + I$

(iii) $(3A - 2B)$, $3(2A - B)$

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7] Solutions to system of linear equations

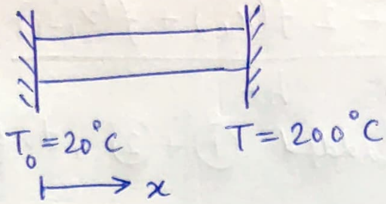
We will discuss:

→ Cramer's Rule

→ Gauss Elimination

→ LL Decomposition

8] eg:

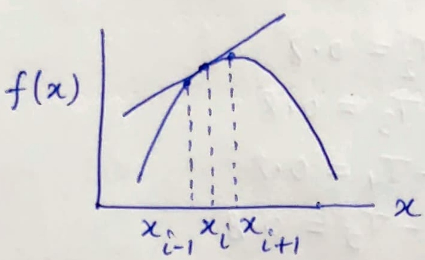


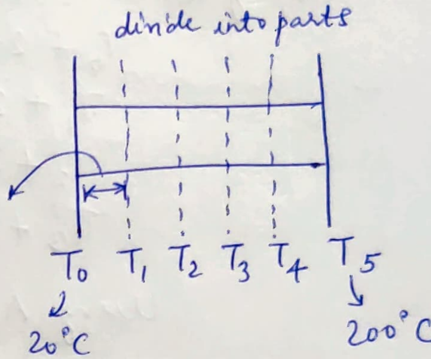
Here, T is a function of x
i.e. $T(x)$
* (here, it is not changing with time)

We can say: $\dfrac{d^2T}{dx^2} = h(T - T_a)$ (we can take full derivative here, since $T = T(x)$ only)

h → heat transfer coefficient

To solve this we can use Finite difference approximation





say, this has a length: $\Delta x = 2$

⇓

$$f'(x) = \frac{df}{dx}$$

$$= \lim_{\Delta x \to 0} \frac{f(x_{i+1}) - f(x_i)}{\Delta x} \quad \{\text{Forward Difference}\}$$

$$= \lim_{\Delta x \to 0} \frac{f(x_i) - f(x_{i-1})}{\Delta x} \quad \{\text{Backward Difference}\}$$

By Taylor Series expansion:

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots \quad -①$$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2!} f''(x) - \dots \quad -②$$

$$f(x+2h) = f(x) + 2h f'(x) + \frac{(2h)^2}{2!} f''(x) + \dots \quad -③$$

Doing ③ − 2 × ① :

$$f(x+2h) - 2f(x+h) = -f(x) + h^2 f''(x)$$

$$\Rightarrow \frac{f(x+2h) + f(x) - 2f(x+h)}{h^2} = f''(x)$$

Taking: $x + 2h \to x_{i+1}$
$x + h \to x_i$
$x \to x_{i-1}$

thus, $f''(x) = \dfrac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$

(NOTE: We could also just have added ① + ②, to find $f''(x)$)

Using this, we can apply this to our original eq^n :

$$\frac{d^2T}{dx^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{(2)^2} = 0.01\,(T_i - 20)$$

This gives us :

$$T_0 + 2.04\,T_1 - T_2 = 0.8$$
$$T_1 + 2.04\,T_2 - T_3 = 0.8$$
$$T_2 + 2.04\,T_3 - T_4 = 0.8$$
$$T_3 + 2.04\,T_4 - T_5 = 0.8$$

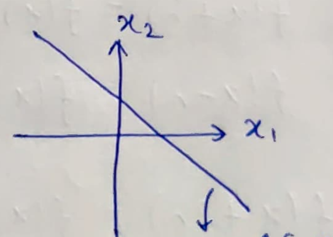
Using $Ax = b$:

$$\Rightarrow \begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{bmatrix}$$

9] 3 types of solutions for system of linear eq^ns

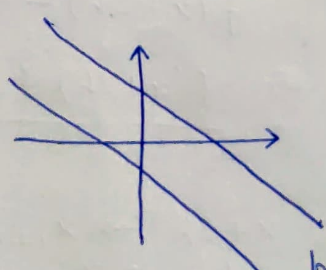
1. Consistent, unique
2. Consistent, infinite
3. Inconsistent, no sol^n

eg: $x_1 + x_2 = 2$
$2x_1 + 2x_2 = 4$



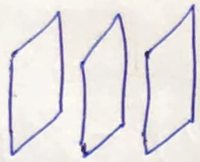
both lines overlap
(infinite sol^ns)

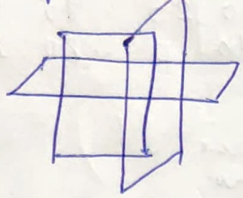
eg: $x_1 + x_2 = 2$
$2x_1 + 2x_2 = -1$
No solution



parallel lines

Similarly, in planes also
we have multiple cases :-





. . . . and so on

parallel
(No sol^n)

Unique sol^n

10] Cramer's Rule

$$2x_1 + x_2 + 2x_3 = -1$$
$$x_1 + x_3 = -1$$
$$-x_1 + 3x_2 - 2x_3 = 7$$

Augmented matrix:

$$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 1 & 2 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 3 & -2 & 7 \end{array}\right]$$

Cramer's rule : $x_i = \dfrac{\det(A_i)}{\det(A)}$ A_i is obtained by replacing i^{th} column of A with constant column

$$\therefore |A_1| = \begin{vmatrix} -1 & 1 & 2 \\ -1 & 0 & 1 \\ 7 & 3 & -2 \end{vmatrix} = -1(-3) - 1(-5) + 2(-3)$$
$$= 2$$

⋮ and so on for A_2 and A_3

11] Gauss Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

We want to make the lower triangular part zero

NOTE: If $b_1, b_2, \dots, b_n$ are all zero (0)
↓
called Homogeneous set of linear eq^{n}s
(they have a trivial sol^n, i.e. all x_i being zero)

G.E. method:

(I) Unique sol^n:

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$
$$10x_2 + 25x_3 = 90$$
$$20x_1 + 10x_2 = 80$$

pivot point

$$\Rightarrow \quad \tilde{A} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

we want to make these zero

3 Transformations are allowed in G.E. :

1. Interchange rows
2. Multiply the row by a scalar
3. Add/subtract a row multiplied by some scalar

Taking the above $\tilde{A}$:

$$R_2 \to R_2 + R_1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

Moving the zero row to the bottom :

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 20 & 10 & 0 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \to R_2 - 20 R_1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Similarly after further steps: (interchanging R_2 & R_3, & then $R_3 \to R_3 - 3R_1$)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now we can from here perform Back-substitution, and thus obtain all values of x_i

(i.e. using R_3 to get x_3
then using R_2 to get x_2
then finally using R_1 to get x_1)

LECTURE 3

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12] Gauss Elimination method

(I) Unique solⁿ → we have looked at this case

(II) Infinite solⁿs

(III) No solution

eg: If during solving we get:

$$\left[\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 12 \end{array}\right] \leftarrow \text{not possible [Inconsistent]}$$

$\tilde{}$ → tilda it is bcz it is changed from $b_1, b_2 \dots b_m$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ & c_{22} & \cdots & c_{2n} & \tilde{b}_2 \\ & & \ddots & \vdots & \vdots \\ & & k_{rs} \cdots & k_{rn} & \tilde{b}_r \\ & & & & \vdots \\ & & & & \tilde{b}_m \end{array}\right]$$

← This type of matrix is called Row Echelon matrix

If first element is 1 called Reduced Row Echelon form

3 cases:

$r \leq m$ and $a_n \neq 0$, $c_{22} \neq 0, \dots, k_m \neq 0$

and all the entries in Δ and $\square$ are 0.

(1) One unique solⁿ : $r = n$ and $\tilde{b}_{r+1}, \tilde{b}_m$ if present $= 0$

(2) Infinite solⁿs : $r < n$ and $\tilde{b}_{r+1} \cdots \tilde{b}_m = 0$

(3) No solⁿ : $r < m$ and one of the entries of $\tilde{b}_{r+1} \dots \tilde{b}_m \neq 0$

13] LU Decomposition

$$[A]\{x\} = \{b\}$$

$$\Rightarrow [A]\{x\} - \{b\} = \vec{0} \quad -(1)$$

We take: $[U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \{x\} = \{y\}$

Thus, $[U]\{x\} - \{y\} = \vec{0} \quad -(2)$

Also; $[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

$$[L]\left([U]\{x\} - \{y\}\right) = [A]\{x\} - \{b\}$$

We use this as follows:

① $[L][U] = [A]$ (Convert A into an L and U product)

② $[L]\{y\} = \{b\}$ (solve for y)

③ $[U]\{x\} = \{y\}$ (solve for x)

eg: $A = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$a_{11} = u_{11} = 3$

$a_{12} = u_{12} = 5$

$a_{13} = u_{13} = 2$

Next, $a_{21} = u_{11} l_{12} = 0 \Rightarrow l_{12} = 0$

$a_{22} = u_{12} l_{12} + u_{22} = 8 \Rightarrow u_{22} = 8$

$a_{23} = l_{12} u_{13} + u_{23} = 2 \Rightarrow u_{23} = 2$

Next, $a_{31} = u_{11} l_{31} = 6 \Rightarrow l_{21} = 2$

$a_{32} = l_{21} u_{12} + l_{22} u_{22} = 10 + 8 l_{22} = 2$

$\Rightarrow l_{22} = -1$

$a_{33} = l_{21} u_{13} + l_{22} u_{23} + u_{33} = 8$

$\Rightarrow u_{33} = 6$

$\therefore [L] \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 8 \\ -7 \\ 26 \end{Bmatrix} \Rightarrow \begin{matrix} y_1 = 8 \\ y_2 = -7 \\ y_3 = 3 \end{matrix}$

$[U] \Rightarrow \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} x_3 = 1/2 \\ x_2 = -1 \\ x_1 = 4 \end{matrix}}$ which is the required solution

NOTE:

If $[A]$ is symmetric

$(\text{i.e. } A^T = A)$

and A is Positive definite, i.e. each element of A is +ve

Then, $[A] = [L][L]^T$

$(\text{i.e. } [U] = [L]^T)$

This is called Cholesky Decomposition.

Here, $l_{11} = \sqrt{a_{11}}$

$$l_{j1} = \frac{a_{j1}}{l_{11}} \quad , \; j = 2, 3, \ldots, n$$

$$l_{jj} = \sqrt{a_{jj} - \sum_{s=1}^{j-1} l_{js}^2} \quad , \; j = 2, 3, \ldots, n$$

$$l_{pj} = \frac{1}{l_{jj}} \left(a_{pj} - \sum_{s=1}^{j-1} l_{js} l_{ps} \right) \quad , \; p = j-1, \ldots, n$$

$(j \geq 2)$

Lecture 4   MLL212

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14] **Vector**

Two operations :

1] Vector addition

2] Scalar multiplication

In Linear Algebra :

1] Geometric vectors

2] Polynomials

$$f(x) = 2x^2 - 5x + 3$$

$$f(x) + g(x) = h(x)$$

$$cf(x) = f_1(x)$$

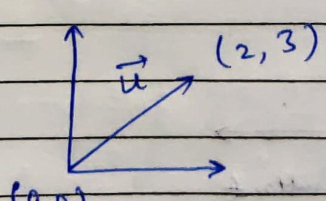
Similarly, $\vec{a} + \vec{b} = \vec{c}$

$$\underbrace{k}_{\text{scalar}}\vec{a} = \vec{b}$$

We will study :

- Vector Spaces
- Linear dependence
- Inner Product
- Transformations

15] **Vector Addition**



$\vec{u} = x\hat{i} + y\hat{j}$

$\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ or $(2 \ \ 3)$ ← * this is still column vecto

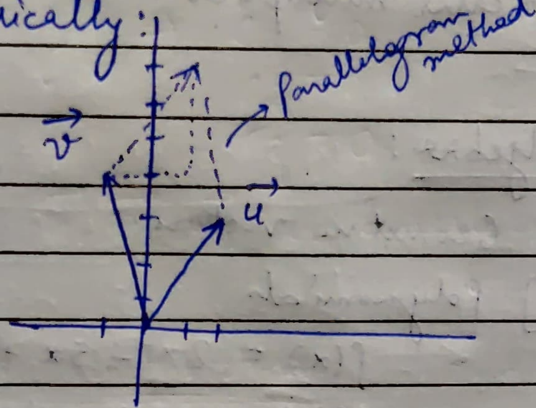
$\vec{u}^T = [2 \ \ 3]$ ← Row vector

Now, let's say we have:

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

and we have to add them.

Geometrically:



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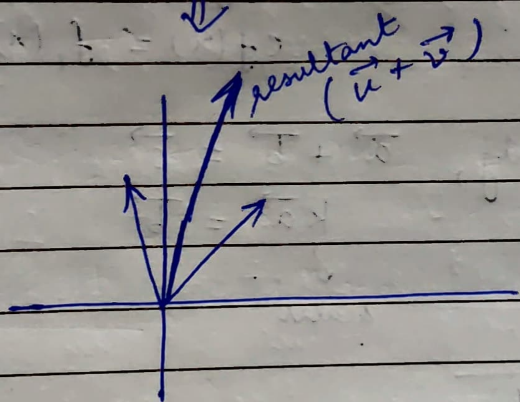
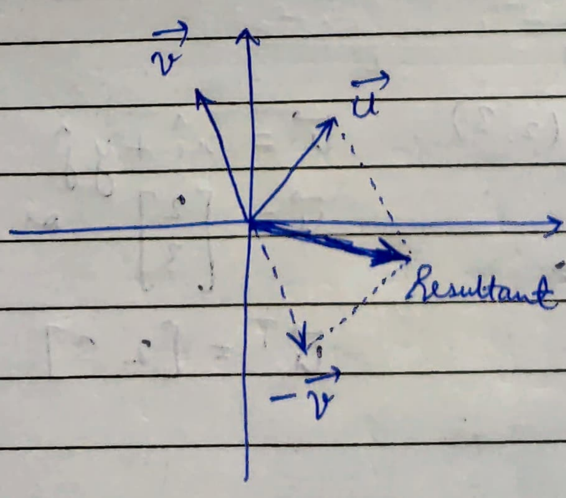


Similarly, for subtraction:

$$\vec{u} - \vec{v}$$
$$= \vec{u} + (-\vec{v})$$

↳ and now we can perform vector addition in this way



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NOTE! By combining vector add^n & scalar multiplication we get Linear combination:

$$a_1\vec{x_1} + a_2\vec{x_2} + \dots + a_n\vec{x_n}$$

eg: Line passing through $(0,0) \Rightarrow c\vec{u} + d\vec{v}$
Plane passing through $(0,0) \Rightarrow c\vec{u} + d\vec{v} + e\vec{w}$

* A vector space V consists of elements $\vec{a}, \vec{b}, \dots$ such that two algebraic operations are obeyed

16] **Linear Dependence**

$$\vec{u} = c\vec{v}$$

$\Rightarrow$ Here we say $\vec{u}$ is linearly dependent on $\vec{v}$

We have: $\vec{u} - c\vec{v} = 0$

If $\vec{u}$ is linearly dependent on $\vec{v}$, then $c \neq 0$

In general:

For, $c_1\vec{a_1} + c_2\vec{a_2} + \dots + c_m\vec{a_m} = 0$

If $c_1 = c_2 = \dots = c_m = 0$

then $\vec{a_1}, \vec{a_2}, \dots, \vec{a_m}$ are linearly independent.

{ But if any one of the coeff.s is non-zero then the system of vectors is linearly dependent }

eg: $\vec{a_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{a_2} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \vec{a_3} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$

$$c_1\vec{a_1} + c_2\vec{a_2} + c_3\vec{a_3} = 0$$

$$c_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 2 & -2 & 6 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This canbe solved using Gauss Elimination

We get $\Rightarrow$ $c_1 = 0$
$c_2 = 0$
$c_3 = 0$

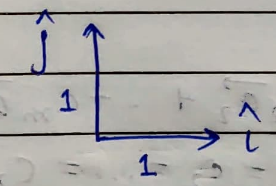
Thus, $\vec{a_1}, \vec{a_2}, \vec{a_3}$ are linearly independent.

17] **Basis Vectors / Unit Vectors**

$\hat{i}, \hat{j}, \hat{k} \longrightarrow$ unit vectors

Basis: Defined by the no. of independent (linearly) vectors in a vector space.

Dimensionality: Max^{m} no. of linearly independent vectors.



$$x\hat{i} + y\hat{j} = 0$$

$\cancel{\text{If } x \neq 0} \rightarrow \hat{i} = \frac{-y}{x}\hat{j}$

$\downarrow$

But not possible as $\hat{i} \perp \hat{j}$

If $y \neq 0 \Rightarrow$ Again, orthogonality not satisfied

If $x \neq 0, y = 0 \Rightarrow x\hat{i} = 0$. (Not possible)

If $y \neq 0, x = 0 \rightarrow y\hat{j} = 0$. (Not possible)

$\therefore$ $\hat{i}$ and $\hat{j}$ are linearly independent

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NOTE: Also, in 2D space, the dimensionality is 2.

Similarly, $\hat{i}, \hat{j}$ and $\hat{k}$ are linearly independent

But, if we check for: $\hat{i}, \hat{j}, \hat{i}+\hat{j}+\hat{k}$

$$\Rightarrow x\hat{i} + y\hat{j} + z(\hat{i}+\hat{j}+\hat{k}) = 0$$

$$\Rightarrow \hat{i}(x+z) + \hat{j}(y+z) + z\hat{k} = 0$$

If independent:

$$x+z=0$$
$$y+z=0$$
$$z=0$$

18] **Inner Product (Dot Product)**

eg: $\vec{a_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$, $\vec{a_2} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}_{3\times 1}$

Inner product is defined by: $(\vec{a_1}, \vec{a_2})$

$$\Rightarrow \vec{a_1} \cdot \vec{a_2} = \vec{a_1}^T \vec{a_2}$$

$$= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Rules it follows:

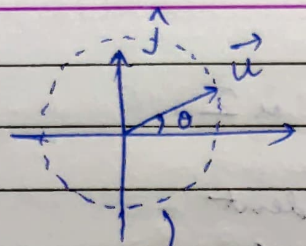
1. Linearity: For all scalars, $q_1, q_2, \ldots$ and $\vec{a}, \vec{b}, \vec{c}$ in V space

$$(q_1\vec{a} + q_2\vec{b}, \vec{c}) = q_1(\vec{a}, \vec{c}) + q_2(\vec{b}, \vec{c})$$

2. Symmetry: $(\vec{a}, \vec{b}) = (\vec{b}, \vec{a})$
3. Positive definiteness: $(\vec{a}, \vec{a}) \geq 0$

and, $(\vec{a}, \vec{a}) = 0$ iff $\vec{a} = \vec{0}$

Also, $\|\vec{a}\| = \sqrt{(\vec{a}, \vec{a})}$



taking $\vec{u}$ anywhere in this circle

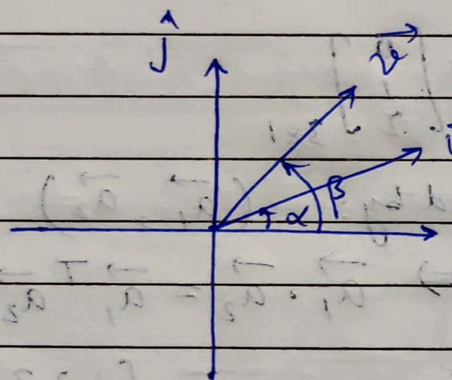
We know that any vector lying in the vector space can be written as a linear combination of the basis vectors:

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \text{or} \quad \vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{or} \quad \vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\|\vec{u}\| = \sqrt{\cos^2\theta + \sin^2\theta}$$
$$= 1 \leadsto \text{in all cases}$$

Now, let's say:



$$\beta - \alpha \equiv \theta$$

$$\vec{u} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$
$$= \cos(\alpha - \beta)$$
$$= \cos\theta$$

In general we thus see:

$$\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta}$$

or

$$\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \cos\theta$$