LECTURE 5

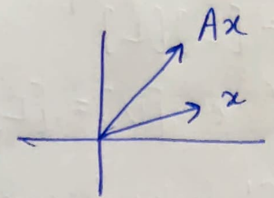
24/01/2023

19] Eigenvectors & Eigenvalues

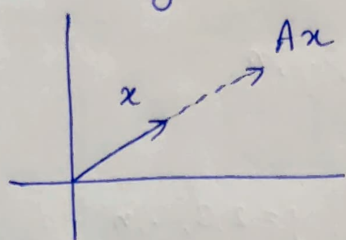
(Eigen: German word for 'Characteristic')

$$A\{x\} = \{b\}$$

↑ input ↑ output



Special case: when we get



(only 'scaling' & No rotation)

$\Rightarrow$ i.e. $[A]\{x\} = \lambda\{x\}$

(Ax is parallel/same dir^n to x)

Then we say, x is an eigenvector

If $\{x\} \neq 0$, then λ is the eigenvalue

We have:

$$[A]\{x\} = \lambda\{x\}$$

$$\Rightarrow \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= \lambda_1 x_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= \lambda_2 x_2 \\ \vdots \qquad\qquad \ddots \qquad\qquad &\vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= \lambda_n x_n \end{aligned}$$

Thus, we can say:

$$\begin{bmatrix} a_{11}-\lambda_1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda_2 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda_3 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}-\lambda_n \end{bmatrix} \{x\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \left([A] - \lambda[I]\right)\{x\} = 0$$

Which implies that, if $\{x\} \neq \{0\}$

then $|A - \lambda I| = 0$ ← Gives us the characteristic Eq^n.

eg: $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

$(A - \lambda I) \Rightarrow \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow \lambda^2 + 7\lambda + 6 = 0$ → characteristic polynomial

$\Rightarrow \lambda = -6, -1$

For $\lambda = -6$: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{matrix} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{matrix}\right\} \Rightarrow x_1 = -2x_2$

eg: $x_1 = 1,\ x_2 = -\frac{1}{2}$ (one sol^n as example)

For $\lambda = -1$: One such sol^n: $\{x\} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

* { NOTE: Linear ODE
of the form where powers of the derivatives are 1

eg: $\dfrac{dx}{dt} + \dfrac{d^2x}{dt^2} + tx = 0$

But not: $\left(\dfrac{dx}{dt}\right)^2 + x = 0$ }

Now, we can solve an ODE system for example:

$\left(x_1' = \dfrac{dx_1}{dt},\ x_2' = \dfrac{dx_2}{dt} \right)$

$x_1' = x_1 + 3x_2$

$x_2' = 3x_1 + x_2$

$\Rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

If we treat this like an Eigenvalue eq^n, we can say:

Assume $x = x_0 e^{\lambda t}$

$\Rightarrow x' = x_0 \lambda e^{\lambda t} = [A]x$

$\Rightarrow x_0 e^{\lambda t}([A] - \lambda[I]) = 0$

$\Rightarrow x_0 \left([A] - \lambda [I] \right) = 0$

Taking determinant:

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0 \qquad \left(\text{Let say: } x_0 = \begin{bmatrix} a \\ b \end{bmatrix} \right)$$

$\Rightarrow \lambda = 4, -2$

For $\lambda = 4$: $\left.\begin{aligned} -3a + 3b = 0 \\ 3a - 3b = 0 \end{aligned}\right\} \; a = b = c_1$

For $\lambda = 2$: $\left.\begin{aligned} 3a + 3b = 0 \\ 3a + 3b = 0 \end{aligned}\right\} \; a = -b = c_2$

Thus, in general, we can write the solution as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{+4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

NOTE: From the 2 rules of Linear Transformations:

(If T is a linear transformation,

$$T(c\vec{v} + d\vec{w}) = T(c\vec{v}) + T(d\vec{w}) = cT(\vec{v}) + dT(\vec{w})$$)

1. Additivity $\Rightarrow T(\vec{u} + \vec{w}) = T(\vec{u}) + T(\vec{w})$
2. Scalar Multiplication $\Rightarrow T(c\vec{v}) = cT(\vec{v})$

eg: To check $T(\vec{v}) = \|\vec{v}\|$

But, $T(c\vec{v}) = cT(\vec{v})$

is NOT satisfied when c is $-$ve

$\therefore T$ is NOT a linear transformation

LECTURE 6

27/01/2023

20] Linear Transformations

Input vector $\xrightarrow{\text{Transformation}}$ Output Vector

$$[A]\{v\} = \{w\}$$

$$\Rightarrow \underbrace{T(\vec{v})}_{\text{linear}} = [A]\{v\}$$

eg: if $\vec{v} \to c\vec{v}$

$$T(c\vec{v}) = cT(\vec{v})$$

The rules for linear transformation are:

① $T(\vec{u} + \vec{w}) = T(\vec{u}) + T(\vec{w})$

② $T(c\vec{v})$

Thus, $T(c\vec{u} + d\vec{w}) = cT(\vec{u}) + dT(\vec{w})$

21] What do we need?

Input basis: $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$

Output basis: $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$

$$\therefore \quad \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$$

$$T(\vec{v}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \dots + c_n T(\vec{v}_n)$$

eg: If we want $T(\vec{v}) = c_1\vec{v}_1$ (i.e. projection of $\vec{v}$ along $\vec{v}_1$)

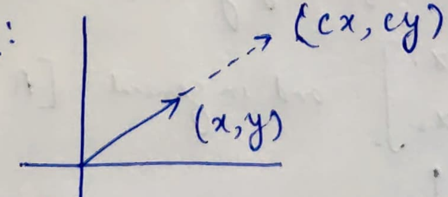
We will use:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

called, Projection Matrix

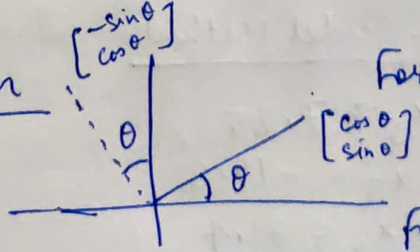
22] Different Transformations

① Stretching:



$$[A] = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

② Rotation



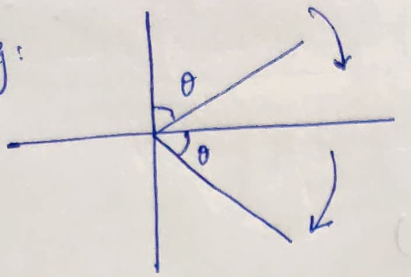
For 90° rotation:

$$[A] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

For general rotation:

$$[A] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

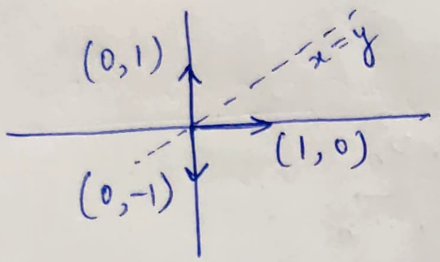
eg:



Here:

$$[A] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

③ <u>Reflection</u>:



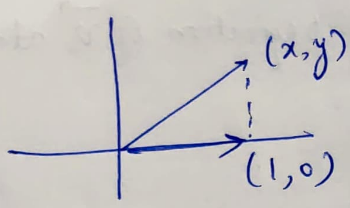
If mirror along x-axis:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

If mirror along line $x=y$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

④ <u>Projection</u>:



Projection along x-axis: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

<u>NOTE</u>: Since all these transformation are defined for the basis vectors themselves, so you can directly use them for any general vector also

eg: $[A]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and in general $[A]\begin{bmatrix} cx_1 \\ dx_2 \end{bmatrix}$

↑

x_1, x_2 basis

23] <u>Numerical Method</u> : <u>Gauss-Seidel Method</u>

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Here, we rearrange them:

(a): $x_1 = \dfrac{7.85 + 0.1x_2 + 0.2x_3}{3}$

(b): $x_2 = \dfrac{-19.3 - 0.1x_1 + 0.3x_3}{7}$

(c): $x_3 = \dfrac{71.4 - 0.3x_1 + 0.2x_2}{10}$

Checking assumed solutions:

In (a), $x_2 = x_3 = 0$ → Assumed solution

gives ⇒ $x_1 = 2.61667$

In (b), $x_1 = 2.61667$, $x_3 = 0$

⇒ $x_2 = -2.79452$

In (c), use x_1, x_2 and find x_3

⇒ $x_3 = 7.11739$ (actually wrong) → should be 7.00561

this is First Iteration

Iteration 2:

Use x_2, x_3 in (a) to find x_1

⇒ $x_1 = 2.99763$

(b): $x_2 = -2.49973$

(c): $x_3 = 7.00007$

In order to determine when to stop
↓
we compute the relative error at each step:

$$\varepsilon = \text{Relative Error (\%)} = \frac{\varepsilon_{new} - \varepsilon_{old}}{\varepsilon_{old}} \times 100\%$$

* When $\varepsilon < 1\%$ ⇒ we stop

NOTE: Sometimes this method doesn't lead to converged solutions

24] Vector Calculus

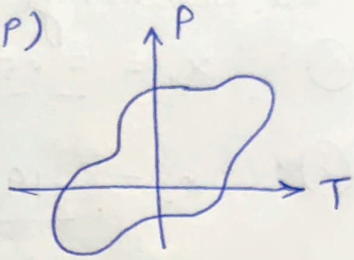
We will need:

— Vector function

— Point in the vector function

(i) Scalar Function
↓
always gives scalar value as output

eg: $f(T, P)$



this gives us a Scalar field

(eg: T, P, E, etc. are scalar fields)

(ii) Vector Function ⟶ gives value as a vector

Examples of vector fields: $\vec{F}$, $\vec{v}$

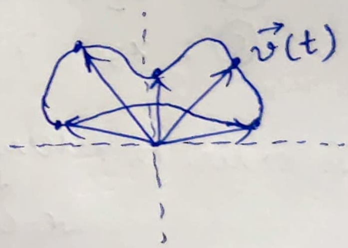
Parametric form : $x = f(t)$, $y = g(t)$, $z = h(t)$

$$\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{v}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$$

$$\therefore [x, y, z] \longrightarrow [f(t), g(t), h(t)]$$

Definition: It is a parametrically defined function where terminal points of vectors lie on a curve.



LECTURE 7

25] Vector Calculus
- Gradient
- Divergence
- Curl

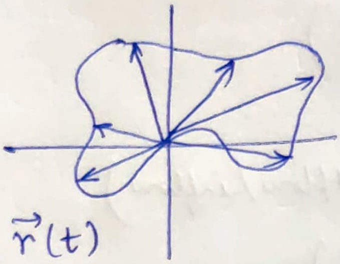
eg: $x = f(t)$, $y = g(t)$, $z = h(t)$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ ⟶ or $\vec{r} = [x, y, z]$

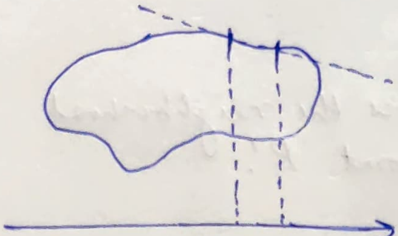
$$\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$$

or

$$\vec{r}(t) = [f(t), g(t), h(t)]$$



So we can say:



$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}\hat{i} + \frac{g(t+\Delta t) - g(t)}{\Delta t}\hat{j} + \frac{h(t+\Delta t) - h(t)}{\Delta t}\hat{k}$$

$$\vec{r}'(t) = f'(t)\,\hat{i} + g'(t)\,\hat{j} + h'(t)\,\hat{k}$$

$$\vec{r}'(t) = [f'(t), g'(t), h'(t)]$$

$$\vec{r}'(t_1, t_2) \Rightarrow \frac{\partial}{\partial t_1}(\ ), \frac{\partial}{\partial t_2}(\ )$$

26] The following rules are valid:

1. $(c\vec{v})' = c\vec{v}'$
2. $(\vec{u} + \vec{v})' = \vec{u}' + \vec{v}'$
3. $(\vec{u} \cdot \vec{v})' = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$
4. $(\vec{u} \times \vec{v})' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$

27] <u>Gradient</u> : directional derivative

for $f(x, y, z)$

$$\vec{\nabla} f(x, y, z) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(x, y, z)$$

nabla

$$\Rightarrow \boxed{\vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}}$$

Directional derivative: $D_a f = \frac{\vec{a}}{|a|} \cdot (\vec{\nabla} f)$

28] <u>Divergence</u>: $\vec{\nabla} \cdot \vec{f}$ (gives idea of outflow / inflow)
<u>Curl</u>: $\vec{\nabla} \times \vec{f}$ (gives idea of rotation)

Divergence is a measurement of how fluid enters & leaves the neighborhood of some point 'P'.

We get that for curl:

$$\vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

29] <u>Divergence of Gradient</u>

$$\vec{\nabla} \cdot (\vec{\nabla} \phi) \equiv \nabla^2 \phi$$

called Laplacian

We get that: $\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$

<u>Curl of Gradient</u>: $\vec{\nabla} \times \vec{\nabla} f$