The problem with being non-linear is:

$$(v_{avg} + \underset{\substack{\downarrow \\ \text{a fluctuating part}}}{\Delta v})(v_{avg} + \Delta v) = v_{avg}^2 + \overbrace{v_{avg}\Delta v + \Delta v \cdot v_{avg}}^{\text{these can still almost cancel out}} + (\Delta v)^2$$

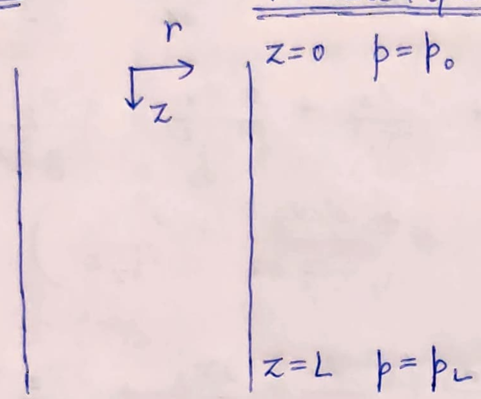
$(\Delta v)^2$ ↓ but since this is squared ↓ it is always positive ⇒ thus creates problems

{ **NOTE**: These Navier-Stokes equations are used in weather forecasting, etc. }

LECTURE-12

Vertical Pipe Flow

51]



Assumptions:

Steady, laminar, fully developed flow

Newtonian fluid

Constant ρ and μ

Intuitive velocity

$v_r = 0$, $v_\theta = 0$, $v_z = v_z(r, z)$

↓ using eq^n of continuity:

$v_z = v_z(r)$

So we have:

$$\begin{pmatrix} \tau_{rr} & \tau_{rz} & \tau_{r\theta} \\ \tau_{zr} & \tau_{zz} & \tau_{z\theta} \\ \tau_{\theta r} & \tau_{\theta z} & \tau_{\theta\theta} \end{pmatrix}$$

⇓ If we use the formulae from before, we can see that only certain components are valid

Let's see using the general eq^n of motion:

In r-dir^n: $0 = -\dfrac{\partial p}{\partial r} + \rho g_r$

In θ-dir^n: $0 = -\dfrac{\partial p}{\partial \theta} + \rho g_\theta$

In z-dir^n: $0 = -\dfrac{\partial p}{\partial z} + \rho g_z - \dfrac{1}{r}\dfrac{\partial (r\tau_{rz})}{\partial r}$

obtained by substituting: $v_r = 0$, $v_\theta = 0$ and $v_z = v_z(r)$

Initially we know: $p = p(r, \theta, z)$ but not t ($\because$ steady flow)

But since $g_r = 0 \xRightarrow[\text{first eq}^n]{\because \text{ from}} p \neq p(r)$

Similarly $g_\theta = 0 \Rightarrow p \neq p(\theta)$

Hence: $p = p(z)$ only.

$\because \tau_{rz} = -\mu \frac{\partial v_z}{\partial r} = -\mu \frac{dv_z}{dr}$ (since $v_z = v_z(r)$)

Thus, we can continue the third eq^n:

$$0 = -\frac{\partial p}{\partial z} + \rho g_z - \frac{1}{r}\frac{\partial (r \tau_{rz})}{\partial r}$$

$$\Rightarrow \quad 0 = \underbrace{-\frac{dp}{dz} + \rho g_z}_{\substack{\text{some function} \\ \text{of } z, \\ \text{say } g(z)}} \underbrace{- \frac{1}{r}\frac{\partial \left(r\left(-\mu \frac{dv_z}{dr}\right)\right)}{\partial r}}_{\substack{\text{some function} \\ \text{of } r, \\ \text{say } f(r)}}$$

This is only possible if the two are constants

$\Rightarrow$ i.e. $g(z) = c_1$ and $f(r) = -c_1$

$$\therefore \quad c_1 = -\frac{dp}{dz} + \rho g_z \Rightarrow \frac{dp}{dz} = \rho g_z - c_1$$

$$\Rightarrow p = \rho g_z z - c_1 z + c_2$$

We know that:

at $z = 0$, $p = p_0$

and at $z = L$, $p = p_L$

$$\therefore \quad p_0 = c_2$$

$$p_L = \rho g_z L - c_1 L + c_2$$

$$= \rho g_z L - c_1 L + p_0$$

$$\Rightarrow \frac{p_0 - p_L}{L} = c_1 - \rho g_z$$

$$\Rightarrow c_1 = \frac{p_0 - p_L}{L} + \rho g_z$$

We can here, for simplicity,
define a new variable $\mathcal{P}$ as:

Effective Pressure: $\boxed{\mathcal{P} \equiv p - \rho g_z z}$

So we have:
$$0 = -\frac{d\mathcal{P}}{dz} + \frac{1}{r}\frac{d(r\tau_{rz})}{dr}$$
$$\Downarrow$$
where: $-\frac{d\mathcal{P}}{dz} = -\left(\frac{dp}{dz} - \rho g_z\right)$

and similarly as before:
$$-\frac{d\mathcal{P}}{dz} = C_1 \Rightarrow \mathcal{P} = -C_1 z + C_2$$

Thus, $\mathcal{P}_0 = C_2$

and $\mathcal{P}_L = -C_1 L + C_2$
$= -C_1 L + \mathcal{P}_0$

$$\Rightarrow \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} = C_1$$

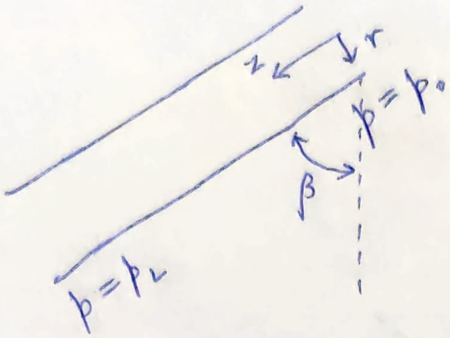
Thus, we obtain:
$$\boxed{0 = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} + \frac{1}{r}\frac{d(r\tau_{rz})}{dz}}$$

which is exactly similar to horizontal pipe eq^n.
except that instead of p, there is $\mathcal{P}$
for vertical pipe case.

Hence the
Hagen-Poiseuille eq^n.
for Vertical pipe
will be:

$$\boxed{Q = \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)}{128\mu L} D^4}$$

* This Hagen-Poiseuille eq^n form will thus also
be applicable to any pipe inclined at angle β
{$\because$ flow is still is z-dir^n only, and only the component
of g along z will matter here}



$$Q = \frac{\pi (\mathcal{P}_0 - \mathcal{P}_L) D^4}{128 \mu L}$$

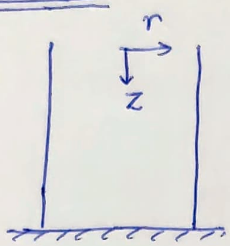
$$\mathcal{P} = p - \rho g_z z$$

$$\because g_z = g \cos\beta$$

$$\Rightarrow \mathcal{P} = p - \rho g \cos\beta \, z$$

$$\Rightarrow \therefore \mathcal{P}_0 = p_0 \text{ and } \mathcal{P}_L = p_L - \rho g \cos\beta \, L$$

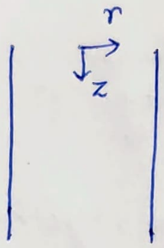
52] <u>Questions</u>

① 

$p = p_0$

$p = p_L = p_0 + \rho g L$

Here there is pressure difference
↓
But flow not taking place
↓
why ?

② 

$p = p_0$

$p_L = p_0$

Here this is NO pressure difference
↓
But flow IS taking place: why ?

The answer to both these questions lies in the concept of <u>effective pressure</u> $\Rightarrow$ { $\because$ We need to combine the effect of both pressure and gravity }

<u>In case ①</u>: $\mathcal{P} = p - \rho g z$

$$\mathcal{P}_0 = p_0 \text{ and } \mathcal{P}_L = p_L - \rho g L = p_0 + \rho g L - \rho g L = p_0$$

Thus Effective pressure difference is ZERO

Hence, flow does NOT take place $\Rightarrow$ According to the Hagen-Poiseuille eq^n.

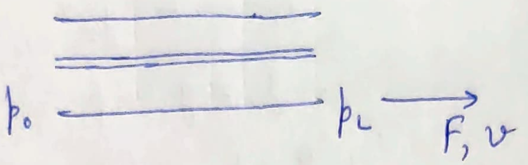
<u>In case ②</u>: $\mathcal{P}_0 = p_0$

$$\mathcal{P}_L = p_L - \rho g L = p_0 - \rho g L$$

$$\Rightarrow \mathcal{P}_0 - \mathcal{P}_L = \rho g L$$ $\Rightarrow$ Hence flow WILL take place

53] **NOTE:** We should always see carefully which forces are responsible for the flow

eg:

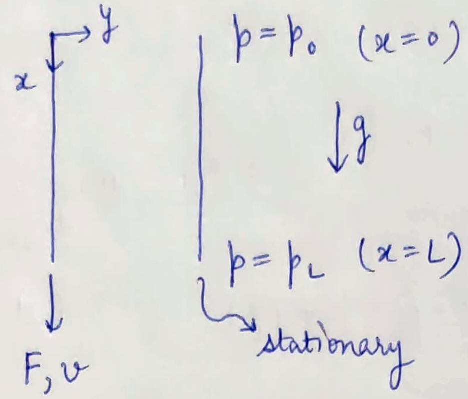


$\Rightarrow$ Here both Pressure and shear forces are driving flow

$\Downarrow$

But here this force needs to also be considered

eg:



$\Downarrow$

Hence, here all 3 are acting:
i.e. Pressure, Shear force and Gravity

NOTE: You can find the eq^{n}s for rectangular, cylindrical, etc. coordinates in the Appendix

(for eq^n of continuity, eq^n of motion, etc.)

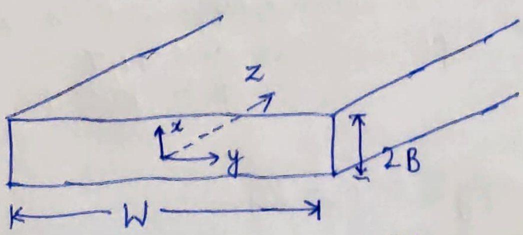
$\Downarrow$

There are 3 components for each eq^n

{ This eq^{n}s are long and may fill the whole page }

54] Often we can perform simplifications $\rightarrow$ can reduce to one independent variable

eg:



Here we can treat this as flow b/w two parallel plates

$\Downarrow$ since

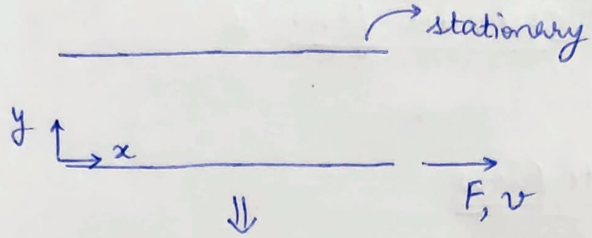
$W \gg 2B$

Thus, $v_z = v_z(x)$ only.

NOTE: For Navier-Stokes Equation (in all 3 coord. systems ⇓ rectangular, cylindrical & spherical) ⟹ can refer to Appendix

Similarly, for Newton's Law of Viscosity ⇓ each of the 9 components can be expressed in any coord. system ⟹ can refer to Appendix

55]



⇓

This pipe is theoretically supposed to be Infinitely long

↓

Hence this experiment cannot be conducted practically

NOTE! If we perform Error-Analysis on:

$$Q = \frac{\pi (P_0 - P_L) D^4}{128 \mu L}$$

Error of Q is obtained by adding up errors of $(P_0 - P_L)$ and L as well as 4 times error in D

↓

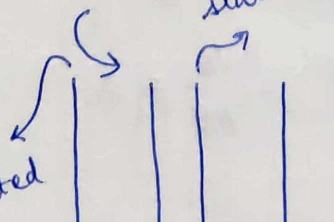
Thus, error may be large

Torque τ, Ω angular velocity

stationary

NOTE:

outer is rotated



⟸ Concentric cylinders

kept at bottom ↓ to prevent flow in z-dir"

⇓

Used to determine Viscosity

⇓

This flow is called Couette Flow *

Similarly, if we rotate the inner cylinders:

This can also be used to measure viscosity

⇓

This case of flow is called <u>Stormer flow</u> *

LECTURE-13

Couette Flow

56] <u>Assumptions</u>

1. Steady flow, fully developed flow, laminar flow
2. Newtonian fluid with constant ρ and μ

$$v_r = v_r(r, \theta, z, \cancel{t}) = 0$$
$$v_\theta = v_\theta(r, \theta, \cancel{z}, \cancel{t})$$
$$v_z = v_z(r, \theta, z, \cancel{t}) = 0$$

} Velocity profile from intuition

(NOTE: Here we can even apply Navier-Stokes eqⁿ, since ρ and μ are constant)

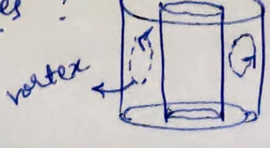
From eqⁿ of continuity:

$$\cancel{\frac{\partial \rho}{\partial t}} + \cancel{\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \cancel{\frac{\partial}{\partial z}(\rho v_z)} = 0$$

$$\Rightarrow \frac{\partial v_\theta}{\partial \theta} = 0$$

Thus, $v_\theta = v_\theta(r)$ only

*NOTE: [However, it is observed from experiment, that when the fluid is rotated (i.e. cylinder is rotated), then there is a formation of some <u>vortices</u> :



⤳ discovered by Taylor

This means our intuition for velocity profile was WRONG.

We assumed a too simplified velocity profile.
↓
it must be modified]

We took our velocity profile as:

$v_r = 0$

$v_z = 0$

$v_\theta = v_\theta(r)$

$$\begin{array}{ccc} \multicolumn{3}{c}{\left(\begin{array}{ccc} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{array}\right)} \\ \downarrow & \downarrow & \downarrow \\ r\ \text{dir}^n & \theta\ \text{dir}^n & z\text{-dir}^n \end{array}$$

Put the velocity profile, and check all 9 components
↓
and then put these 9 components into general eq.^{n} of motion

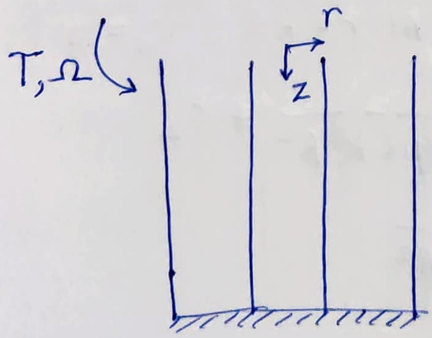
⇒ We get:

r-comp: $$-\frac{\rho v_\theta^2}{r} = -\frac{\partial p}{\partial r}$$ → will be there due to centrifugal force

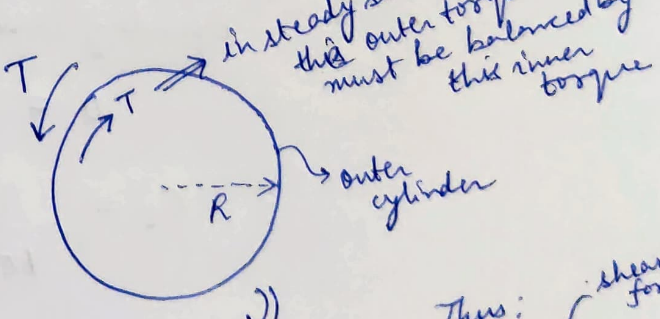
θ-comp: $$0 = \frac{1}{r^2}\frac{\partial(r^2 \tau_{r\theta})}{\partial r}$$ → only a function of r

z-comp: $$0 = -\frac{\partial p}{\partial z} + \rho g_z$$

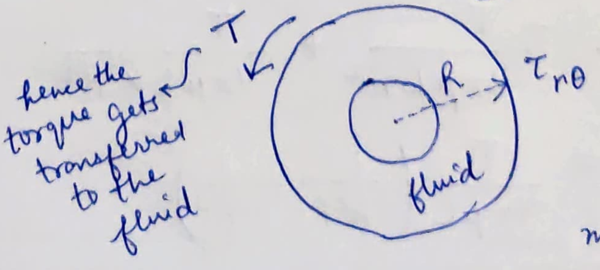
Here we have: $$0 = \frac{1}{r^2}\frac{d}{dr}(r^2 \tau_{r\theta})$$



⇒



⇓



Thus:

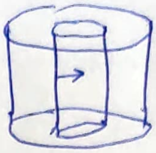
shear force

$$T = \tau_{r\theta}\Big|_{r=R} 2\pi R L \times R$$

$$= (-\tau_{r\theta})\Big|_{r=R} 2\pi R L \times R$$

momentum flux

{ **NOTE:** In case of Inner cylinder rotation + outward normal is in negative dirⁿ.



$$\text{So, } T = \underbrace{(-\tau_{r\theta})}_{\text{shear force will have negative sign}}\Big|_{r=R_{inner}} \cdots \quad \Rightarrow \quad T = \underset{\substack{\uparrow \\ \text{+ve sign}}}{+}\tau_{r\theta}\cdots$$

thus mome. flux }

[Also: Similarly | p_0 | fluid | p_L | ↑ must be balanced by same force to ensure balance of forces ⇒ Thus cylinder wall. Hence there must be an external force applied on walls of outer cylinder]

Thus, here we get:

$$\tau_{r\theta} = \frac{C_1}{r^2}$$

$$\therefore \quad T = -\frac{C_1}{R^2}\cdot 2\pi R^2 L$$

$$\Rightarrow C_1 = \frac{-T}{2\pi L}$$

$$\therefore \quad \tau_{r\theta} = -\frac{T}{2\pi L r^2}$$

We have: $\tau_{r\theta} = -\mu\left(r\frac{d}{dr}\left(\frac{v_\theta}{r}\right) + \cancel{\frac{1}{r}\frac{\partial v_z}{\partial z}}\right)$

$$\Rightarrow -\mu r \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \frac{-T}{2\pi L r^2}$$

$$\frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \frac{T}{2\pi\mu L r^3}$$

$$\frac{v_\theta}{r} = -\frac{T}{2\pi\mu L}\frac{1}{2r^2} + C_2$$

Boundary condition: At $r = kR$, $v_\theta = 0$

$$0 = -\frac{T}{4\pi\mu L k^2 R^2} + C_2$$

$$C_2 = \frac{T}{4\pi\mu L k^2 R^2}$$

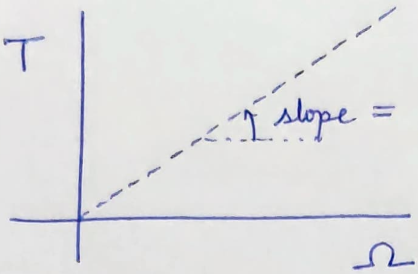
$$\therefore \frac{v_\theta}{r} = \frac{-T}{4\pi\mu L r^2} + \frac{T}{4\pi\mu L k^2 R^2}$$

$$= \frac{T}{4\pi\mu L}\left(\frac{1}{k^2R^2} - \frac{1}{r^2}\right)$$

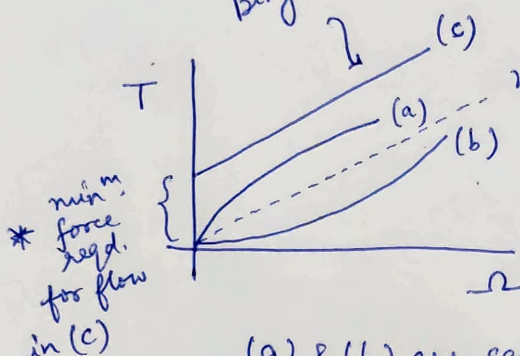
Now, at $r = R$, $\frac{v_\theta}{r} = \Omega$

$$\therefore \Omega = \frac{T}{4\pi\mu L}\left(\frac{1}{k^2R^2} - \frac{1}{R^2}\right)$$

$$\Rightarrow \boxed{T = \frac{4\pi\mu L R^2 \Omega k^2}{(1-k^2)}}$$ ← Relation b/w T and Ω



If we did NOT have a Newtonian fluid
↓
on performing this same experiment using the viscometer
↓
we would see that the plot of T vs Ω would NOT give a straight line through origin



(a) & (b) are case for Non-Newtonian fluids

(c) is also a non-newtonian fluid of a special type → called * Bingham fluids

{* Syllabus for Minor Examination will be till here }