GENERAL RELATIVITY AND INTRODUCTORY ASTROPHYSICS

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[PYL742]

Course Info

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1 Minor (30%)
1 Major (50%)
4 Assignments ~ they will be v. long assignments
(each with equal weightage of 20 marks)

NOTE: Problem solving is essential to the understanding of the course.

References:

1) Gravity, by Jim Hartle
2) Gravitation, by Padmanabhan

} → These 2 are the main books that'll be followed

3) Classical Theory of Fields – Vol. 2, by Landau & Lifshitz
   ↳ However, this uses the metric $(+1,-1,-1,-1)$ which is different from what we will be following in this course.
4) Relativist's Toolkit, by Eric Poisson

LECTURE - 1

24/07/2023

1] Concept of Event

refers to "occurrence of something" < where, when

: 4 numbers are needed

For event P : $P(\underbrace{x^1, x^2, x^3}_{\text{3 coord.s for "where"}}, t)$

t → for "when"

NOTE: These coord.s need NOT have same dimensions (for eg: r, θ, ϕ in case of spherical coord.s)

2] Time → it is something that a clock measures

(clock of the observer)

Anything which is periodic could be used as a clock

In general, we write coord.s as : x^0, x^1, x^2, x^3

x^0 → for time (t)

* NOTE: Coordinates must be fixed for a particular event

3] Inertial Frame

if particle is moving with a const. velocity & doesn't interact

then it will continue to move with ~~that~~ same const. velocity

* According to General Relativity

we will see that this is only an approximation → (which we use for Special Relativity)

actually we can never remove all interactions (due to gravity)

4] Principles of Special Relativity

① The laws of physics are the same in all inertial frames

② There exists a finite but non-zero speed which is an invariant quantity → i.e. if we go from one frame to another ↓ it does not change

↓

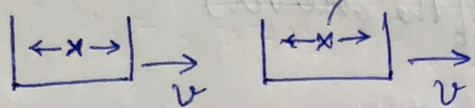
This happens to be the speed of light

The consequences of these are Special Relativity

5] Consequences of these principles

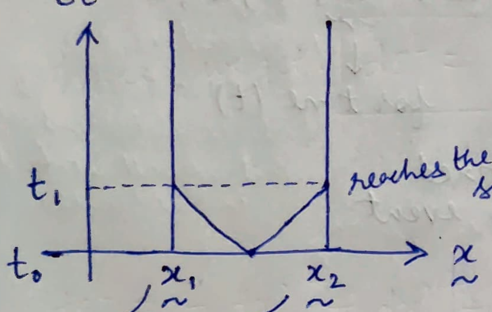
↓

i) Simultaneity is relative



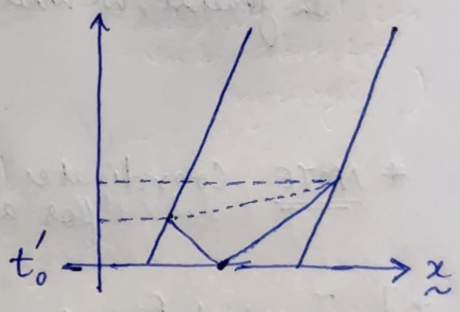
One observer is inside it, & one is outside

* { Speed of light must be same }



$\underset{\sim}{x} = (x^1, x^2, x^3)$

Viewed from within railcar



Viewed from outside railcar

(Here, they are NOT simultaneous)

ii) Another concept : "Interval"

↓

it is defined as : $ds^2 = -c^2dt^2 + \underbrace{dx^2 + dy^2 + dz^2}$

pythagorean distance b/w two points (dl^2) or $|d\underset{\sim}{x}|^2$ or $d\underset{\sim}{x}\cdot d$

in shorthand can be written as : $ds^2 = -c^2dt^2 + |d\underset{\sim}{x}|^2$

↓

this is not really a distance since it has a negative sign

This quantity is ALSO invariant :

i.e. $ds^2 = ds'^2$

* NOTE: We will follow the notation :

$\alpha \rightsquigarrow 1, 2, 3$
(greek letter)

$i \rightsquigarrow 0, 1, 2, 3$
(latin letter)

6] For two nearby events : $A(x^i)$ & $B(x^i + dx^i)$

if we connect these two with a ray of light !

NOTE : $\left\{ \begin{array}{l} x^i = (x^0, x^1, x^2, x^3) \\ x^\alpha = (x^1, x^2, x^3) \end{array} \right\}$

* where, $\boxed{x^0 = ct}$

Then, $d\underset{\sim}{x} = c\, dt$

i.e. $\Rightarrow$ $\underset{\sim}{x}_2 - \underset{\sim}{x}_1 = c(t_2 - t_1)$

and similarly : $\underset{\sim}{x}'_2 - \underset{\sim}{x}'_1 = c(t'_2 - t'_1)$

$\therefore$ we obtain :

$$\underbrace{ds^2}_{-c^2dt^2 + |d\underset{\sim}{x}|^2} = 0 = \underbrace{ds'^2}_{-c^2dt'^2 + |d\underset{\sim}{x}'|^2}$$

(since there is no c' + c is fixed everywhere & that is what leads to this result that we see here)

7] Proof for Invariance of ds

We can say :

$$ds'^2 = \overset{\text{some taylor expansion}}{f(ds^2)}$$
$$= \alpha + \beta\, ds^2$$

Now, since if $ds = 0$, ds' must also be 0
$\Rightarrow$ α must be 0

$$\therefore \quad ds'^2 = \beta\, ds^2$$

Let's say they are connected by some velocity $\vec{v}$:

$$\therefore \quad ds'^2 = \beta(v)\, ds^2$$

(and it will NOT depend on dir^n, only on magnitude)

∴ We can write :

$$ds'^2 = \beta(v_1)\, ds^2$$

Now consider another frame (″) moving with $\vec{v}_2$ w.r.t. original frame $\Rightarrow \therefore\ ds''^2 = \beta(v_2)\, ds^2$

Also, we get (we can write) :

$$ds''^2 = \beta(v_{12})\, ds'^2$$

Hence, we have : $\beta(v_{12}) = \underbrace{\dfrac{\beta(v_1)}{\beta(v_2)}}$

v_{12} MUST depend on angle/direction

But this side has no information of angle

$\left(\because\ v_{12} = \sqrt{v_1^2 + v_2^2 + 2\underset{\sim}{v_1} \cdot \underset{\sim}{v_2}}\right)$

$\Rightarrow$ Thus, this means : $\boxed{\beta = 1}$

(or technically, any constant
↓
but that const. can then be absorbed anyways)

Hence, we get : $\boxed{ds^2 = ds'^2}$

(i.e. the interval ds is invariant)

8] $ds^2 = -dx^{0\,2} + dx^{1\,2} + dx^{2\,2} + dx^{3\,2}$

in short this can be written as :

$= \eta_{ab}\, dx^a dx^b$ ⤳ follows Einstein summation convention

↓ i.e.

$= \sum_{a,b} \eta_{ab}\, dx^a dx^b$

↳ however we don't need to show this summation (as per the convention)

(* NOTE: indices should ONLY repeat twice, if they do so at all)

* This is called the Metric

(it tells the distance b/w points in space)

{ Here, since latin letters are used, so a, b are from 0, 1, 2, 3 }

η_{ab} : Matrix
$\downarrow$
$4 \times 4 = 16$ components
$\downarrow$
It will be a Symmetric matrix
$\downarrow$
$\therefore$ So only 10 independent components are actually there

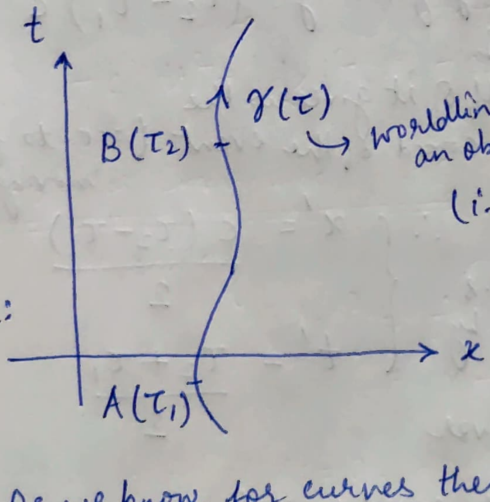
i.e. $$ds^2 = \eta_{00}(dx^0)^2 + \eta_{01}\, dx^0 dx^1 + \eta_{02}\, dx^0 dx^2 + \eta_{03}\, dx^0 dx^3 + \eta_{10}\, dx^1 dx^0 + \eta_{11}(dx^1)^2 + \eta_{12}\, dx^1 dx^2 + \dots$$ (and so on)

Here, $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$
$\downarrow$
i.e. only a diagonal matrix (with zero everywhere else) $\Rightarrow$ $\begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}$

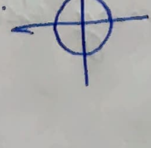
eg: For ordinar planar case we know:
$$dl^2 = dx^2 + dy^2$$
$\hookrightarrow$ Here, metric is simply kronecker delta : $\delta^{\alpha}_{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

9]

For a moving observer:



$\hookrightarrow$ worldline of an observer (i.e. trajectory of an observer)

As we know, for curves there are parameters:
eg:  $(x - x_c)^2 + (y - y_c)^2 = a^2$
$\Downarrow$
$$\left.\begin{aligned} x - x_c &= a\cos\theta \\ y - y_c &= a\sin\theta \end{aligned}\right\}$$ parameterized form

Similarly, here τ is our parameter

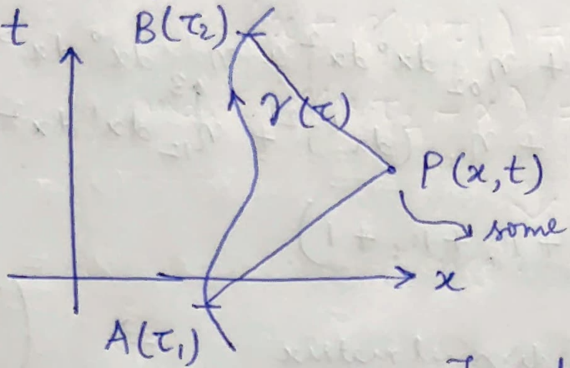
↓

we take τ as the time measured by clock of the observer

↓

it is called "Proper Time"

10] $\therefore$ The trajectory is specified by :



$$x_t = f(\tau)$$
$$y_t = g(\tau)$$

In order to know about this event, a "light clock" is used

↓

i.e. observer sends a beam of light at $\tau = \tau_1$, and it comes back at $\tau = \tau_2$

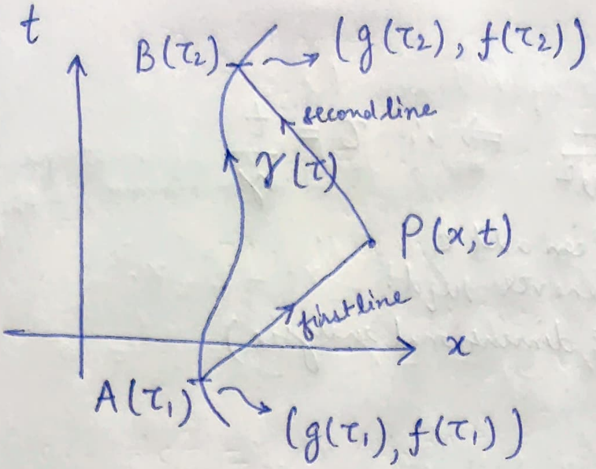
Thus, distance travelled by light is simply given by: $(\tau_2 - \tau_1)\,c$

↓

& half of it gives us the spatial coord. of the event (acc. to frame of moving observer)

$\therefore$ Coord.s given by moving observer
$$\begin{cases} x' = \dfrac{c(\tau_2 - \tau_1)}{2} \\[2ex] t' = \dfrac{\tau_2 + \tau_1}{2} \end{cases}$$

Now, we want to know the coord.s x, t given by the stationary observer.



$$\therefore\ (x - f(\tau_1)) = c(t - g(\tau_1))$$
↳ eqn. of first line

$$(f(\tau_2) - x) = -c(g(\tau_2) - t)$$
↳ eqn. of second line

Now, we already know:

$$x' = \frac{(\tau_2 - \tau_1)c}{2}$$

$$t' = \frac{\tau_1 + \tau_2}{2}$$

and also now we have:

$$(x - f(\tau_1)) = c(t - g(\tau_1))$$
$$(f(\tau_2) - x) = -c(g(\tau_2) - t)$$

11] For the case that observer is moving <u>with constant v:</u>

<u>On trajectory γ:</u> $$ds^2 = \underbrace{-c^2 d\tau^2}_{\text{as per moving observer}} = \underbrace{-c^2 dt^2 + dx^2}_{\text{as per stationary observer}}$$

$$\Rightarrow\ d\tau = dt\left(1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\right)^{1/2}$$
$$= dt\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

Thus, $\tau = \int dt \underbrace{\sqrt{1-v^2}}$ ⇐ (here c is omitted for simplicity) {i.e. using units such that $c = 1$}

⇓

Since this expression $(1-v^2)$ is always < 1

↓

* ∴ Moving clocks tick slower than <u>stationary</u> ones

Thus for this case :

if we define $\gamma = \dfrac{1}{\sqrt{1-v^2}} \Rightarrow \therefore\ \tau = \dfrac{t}{\gamma}$

(NOTE: you can add c wherever reqd. using dimensional analysis)

$\therefore$ We get :

$$(x - \gamma v \tau_1) = c\,(t - \gamma \tau_1)$$

and, $$(\gamma v \tau_2 - x) = -c\,(\gamma \tau_2 - t)$$

$$\Downarrow$$

Hence : $$\left.\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma(t - vx) \end{aligned}\right\} \text{LORENTZ TRANSFORMATIONS}$$

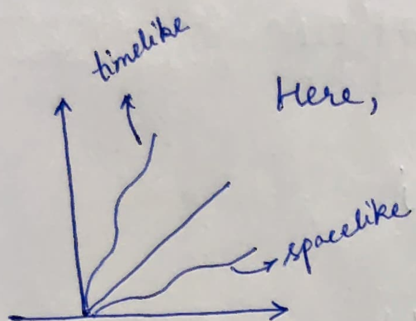
LECTURE-2

25/07/2023

NOTE: We have seen before:

For $A(x^i)$ & $B(x^i + dx^i)$

$$ds^2 = -dt^2 + \underbrace{dx^2 + dy^2 + dz^2}_{dl^2 = |d\underset{\sim}{x}|^2 \text{ or } d\underset{\sim}{x} \cdot d\underset{\sim}{x}}$$

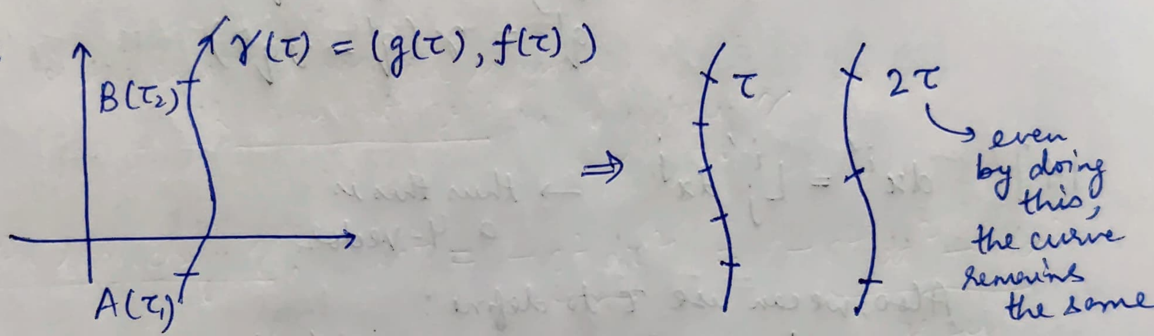


Here, $ds^2 = 0$: light like curve (i.e. $dl^2 > 0$)
or null

$ds^2 < 0$: time like curve

$ds^2 > 0$: space like curve

NOTE:



This is called "Reparameterization invariant symmetry"

* **NOTE:** C is a constant in ALL frames (inertial as well as non-inertial)

NOTE: We saw (and this holds true IN GENERAL) → not only for inertial frame

$$ds^2 = -dt^2 + dx^2 = -d\tau^2$$

$$\Rightarrow \quad d\tau = (-ds^2)^{1/2}$$

and, $\tau = \int dt \sqrt{1 - v^2(t)}$ (where, $v = \frac{dx}{dt}$)

NOTE: For Uniform velocity: $v = $ const. $\Rightarrow$ i.e. $\frac{dx}{dt} = v \Rightarrow x = vt$

$$\therefore \boxed{t = \gamma\tau, \; x = \gamma v \tau}$$

12] We can write for multiple spatial coord.s :

$$x^{i'} = L^{i'}_{j}\, x^{j} \quad \Rightarrow \text{Lorentz Transformations}$$

(they are linear)

$\therefore$ Here $\Rightarrow$ $x^{0'} = t' = L^{0'}_{0} x^{0} + L^{0'}_{1} x^{1} + L^{0'}_{2} x^{2} + L^{0'}_{3} x^{3}$

where $L^{0'}_{0} = \gamma$, $L^{0'}_{1} = -\gamma v$, $L^{0'}_{2} = L^{0'}_{3} = 0$

$$x^{1'} = x' = L^{1'}_{0} x^{0} + L^{1'}_{1} x^{1} + L^{1'}_{2} x^{2} + L^{1'}_{3} x^{3}$$

13] **4-vector** : $\underset{\sim}{q} = q^{\alpha} = (q^{1}, q^{2}, q^{3})$

$\hookrightarrow q^{i} = (q^{0}, \underset{\sim}{q})$ and $q^{i'} = L^{i'}_{j}\, q^{j}$

i.e. $q^{0'} = \gamma(q^{0} - v q^{1})$

$q^{1'} = \gamma(q^{1} - v q^{0})$

14] $dx^{i'} = L^{i'}_{j}\, dx^{j}$ $\rightarrow$ thus this is a 4-vector

Also, we can use τ to define :

$u^{i} = \dfrac{dx^{i}}{d\tau}$, $u^{i'} = L^{i'}_{j}\, u^{j}$ $\Rightarrow$ this is called 4-velocity

Here, $u^{i} = (u^{0}, \underset{\sim}{u}) = \left(\dfrac{dx^{0}}{d\tau}, \dfrac{d\underset{\sim}{x}}{d\tau}\right) = (\gamma, \gamma \underset{\sim}{v})$ or $\gamma(1, \underset{\sim}{v})$

$= \left(\dfrac{dt}{d\tau}, \dfrac{dt}{d\tau}\dfrac{d\underset{\sim}{x}}{dt}\right)$

15] Further, we can take :

$p^{i} = m u^{i} = (p^{0}, \underset{\sim}{p})$, $p^{i'} = L^{i'}_{j}\, p^{j}$

* mass of particle is the SAME (it does NOT change)

Thus, $p^{i} = m u^{i} = m(\gamma, \gamma \underset{\sim}{v}) = (\gamma m, \gamma m \underset{\sim}{v}) = \left(\dfrac{m}{\sqrt{1-v^2}}, \dfrac{m v}{\sqrt{1-v^2}}\right)$

16] We define the following: $q_i = \eta_{ij} q^j$ $\Rightarrow$ this is called
* Lowering the index

$\therefore \quad u_i u^i = \eta_{ij} u^j u^i$

$= -(u^0)^2 + |\underset{\sim}{u}|^2$

$= -\gamma^2 + \gamma^2 |\underset{\sim}{v}|^2$

$\therefore \Rightarrow \boxed{u^a u_a = -1}$

Similarly, $\boxed{p^a p_a = -m^2}$

In general, we say, the norm of 4-vector is:

$$q^i q_i = \eta_{ij} q^i q^j = -(q^0)^2 + |\underset{\sim}{q}|^2$$

Also, since $q^i = (q^0, \underset{\sim}{q}) \Rightarrow q_i = (-q^0, \underset{\sim}{q})$

NOTE: For null (or light-like vector) $\Rightarrow$ norm is 0
For time-like $\Rightarrow$ norm is < 0
For space-like $\Rightarrow$ norm is > 0

($\therefore$ To be accurate, metric tells us what the "norm" is)

17] If we define the following: $M^{ij} = A^i B^j$

also, $A^{i'} = L^{i'}_j A^j$ and $B^{i'} = L^{i'}_j B^j$

and if the transformation for M^{ij} is defined as:

$$M^{i'j'} = A^{i'} B^{j'}$$

$$\Rightarrow \boxed{M^{i'j'} = L^{i'}_k L^{j'}_m A^k B^m}$$

$\Rightarrow$ then we call M
* 2-tensors

And, In general:

$$T^{i'j'k'\cdots} = L^{i'}_a L^{j'}_b L^{k'}_c \cdots T^{abc\cdots}$$

Similarly, we take the following definition:

$$M^i_j = \eta_{jk} M^{ki}$$

and also, $M_{ab} = \eta_{ai}\eta_{bj} M^{ij}$

{ NOTE: $(L^{a'}_b)(L^c_{a'}) = \delta^c_b$ }

{ We will continue using L in G.R., but it won't represent Lorentz Transformation, & instead will be for General coordinate transformations }

NOTE: $MM^{-1} = I$

18] We also define:

$$q_{a'} = (L^i_{a'})q_i$$

Thus, we can write:

$$T^{a'b'c'\ldots}_{p'q'r'\ldots} = (L^{a'}_i L^{b'}_j L^{c'}_k \ldots)(L^l_{p'}, L^m_{q'}, L^n_{r'}, T^{ijk\ldots}_{lmn\ldots})$$

Also: $T^{ab}\,T^{ij}\,\eta_{ai}\eta_{bj}$

$$= T_i^{\,b}\,T^i_{\,b}$$

$$= T_i^{\,b}\,\eta_{bj}\,T^{ij}$$

$$= T_{ij}\,T^{ij}$$

contraction

19] Some more important identities:

$A_{ij} = -A_{ji}$ ← where A is antisymmetric tensor

For a symmetric tensor (S): $S_{ij} = S_{ji}$

also, $S^{ij}A_{ij} = 0$

$$\left\{ S^{ij}A_{ij} = \frac{1}{2}S^{ij}A_{ij} + \frac{1}{2}S^{ij}A_{ij} = \frac{1}{2}S^{ji}A_{ji} + \frac{1}{2}S^{ij}A_{ij} = -\frac{1}{2}S^{ij}A_{ij} + \frac{1}{2}S^{ij}A_{ij} = 0 \right\}$$

LECTURE-3

28/07/2023

NOTE: We know:

$$ds^2 = \eta_{ab}\, dx^a dx^b = \eta_{i'j'}\, dx^{i'} dx^{j'} = ds'^2$$

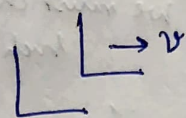
$$\Rightarrow \eta_{ab}\, dx^a dx^b = \eta_{i'j'} L^{i'}_a L^{j'}_b\, dx^a dx^b$$

$$\Rightarrow \boxed{\eta_{i'j'} = L^a_{i'} L^b_{j'} \eta_{ab}}$$

$\{ *$ **NOTE:** $L^{a'}_b \neq L^a_{b'} \}$

↓

We can think of inverse Lorentz transform as:



$\Rightarrow$ Inverse is for going in $-v$ dirⁿ. (i.e. $L(-v)$)

it is simple here since this is only 1-D case

Lor. Trans. gives relation for:

$x' = \gamma(x - vt)$

$t' = \gamma(t - vx)$

$x = f(x', t')$

$t = g(x', t')$

20] We have:

* $M^{i'j'} = L^{i'}_a L^{j'}_b M^{ab}$

* $\eta_{ab} q^a q^b = \eta_{a'b'} q^{a'} q^{b'}$

$= q^a q_a = q^{j'} q_{j'}$

{since, we know the definition: $q_a \equiv \eta_{ab} q^b$ }

* $q_{a'} = L^j_{a'} q_j$ and $q^{j'} = L_k^{\ j'} q^k$

NOTE: Historically we call:

$q^a, M^{ik}, R^{abc}, S^{abcd}, \ldots \rightarrow$ Contravariant tensors

$q_a, M_{ik}, R_{abc}, \ldots \rightarrow$ Covariant tensors

$\underbrace{\text{* However these terms here do NOT have any mathematical meaning}}$

21] * $M^a_{\ c} = M^{ab}\eta_{bc}$

$\Rightarrow M_{dc} = M^{ab}\eta_{bc}\eta_{ad}$

<u>NOTE</u>: In order to obtain trace, just multiply δ^a_b at the end:

eg: For $M^a_{\ c} \Rightarrow M^a_{\ c}\delta^c_a = M^a_{\ a} = \text{Trace}(M)$

or

$M^c_{\ c}$

22] For some Scalar function $\phi(t, \underset{\sim}{x})$

* $\phi(x^i) \xrightarrow[\text{means}]{} \phi'(x') = \phi(x)$

i.e. in the new coord system
we must change the function to some ϕ'
such that the value remains
the same

Now, $\partial_{i'}\phi'(x^{i'}) \equiv \dfrac{\partial\phi'(x^{i'})}{\partial x^{i'}} = \dfrac{\partial x^k}{\partial x^{i'}}\dfrac{\partial\phi(x^k)}{\partial x^k}$

this can be any x^a (but it will be unprimed)

since $dx^{\alpha'} = L^{\alpha'}_{\ i}dx^i$

$$\Rightarrow \frac{\partial}{\partial x^{i'}} = \frac{\partial x^k}{\partial x^{i'}}\frac{\partial}{\partial x^k} = L^k_{\ i'}\frac{\partial}{\partial x^k}$$

Hence, this means:

$$\boxed{\partial_{i'} = L^k_{\ i'}\partial_k}$$

<u>NOTE</u>: Using $\partial_0\phi, \partial_{\underset{\sim}{x}}\phi$:

$$\partial_k\phi = \left(\frac{\partial\phi}{\partial t}, \underset{\sim}{\nabla}\phi\right)$$

23] We can thus use $\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{\nabla}\phi, \nabla\cdot\underset{\sim}{A}, \nabla\times\underset{\sim}{A}, \underset{\sim}{A}\times\underset{\sim}{B}$:

In general: Divergence is $\partial_i q^i = \partial_0 q^0 + \nabla\cdot\underset{\sim}{q}$

$$= \frac{\partial q^0}{\partial t} + \nabla\cdot\underset{\sim}{q}$$

However, curl by itself does NOT make sense outside of 3-dimensions

(Since the cross product does NOT generalise to the 4th dimension)

Instead if we consider the definition

$$\underset{\sim}{A} \times \underset{\sim}{B} = A^{\alpha} B^{\beta} \epsilon_{\alpha\beta\gamma}$$

this is a pseudovector

(and ignore the unit vector)

where $\epsilon_{123} = 1$ cyclic, $\epsilon_{213} = -1$ anti-cyclic, $\epsilon_{113} = 0$

Similarly $\Rightarrow$ $q^i p^j \epsilon_{ijkl} = A_{kl}$ $\rightsquigarrow$ these are anti-symmetric tensor (since $A_{kl} = -A_{lk}$)

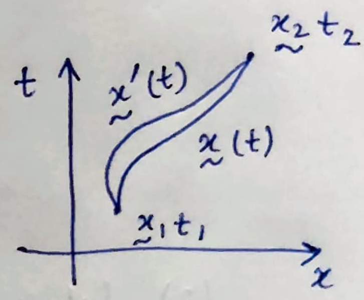
but, $A^{ij} B^{mn} \epsilon_{ijmn}$ $\rightsquigarrow$ this is a pseudoscalar

24] The Action Principle

The dynamics of any particle simply follows from a quantity called the "action"

The action, $A[\underset{\sim}{x}(t) ; \underset{\sim}{x}_2, t_2, \underset{\sim}{x}_1, t_1]$

the action is a function of the end point and a functional of the path



$$A = \int_{t_1}^{t_2} \mathcal{L}(x(t), \dot{x}(t))\, dt$$

$\mathcal{L}$ → the lagrangian

* Extremum (we then extremize the path)

$$\underbrace{A[\underset{\sim}{x}(t) + \delta x(t), 2, 1]}_{\text{slightly shifted from original path}} - A[\underset{\sim}{x}(t), 2, 1] = \delta A = 0$$

this gives the trajectory

Now, $\delta A = \int_{t_1}^{t_2} \delta L \, dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x^\alpha} \delta x^\alpha + \frac{\partial L}{\partial \dot{x}^\alpha} \delta \dot{x}^\alpha \right) dt$

We use the following "trick":
$$\delta\left(\frac{dx^\alpha}{dt}\right) = \frac{d}{dt}\delta x^\alpha$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x^\alpha} \delta x^\alpha + \frac{\partial L}{\partial \dot{x}^\alpha} \frac{d}{dt}(\delta x^\alpha) \right) dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x^\alpha} \delta x^\alpha + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha} \delta x^\alpha \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) \delta x^\alpha \right] dt$$

$\therefore$ We get, $\delta A = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) \right] dt \, \delta x^\alpha + \left. \frac{\partial L}{\partial \dot{x}^\alpha} \delta x^\alpha \right|_{t_1}^{t_2}$

Ways of solving this:

(1) End pts. are fixed & $\delta A = 0$:

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = 0$$

or

$$\frac{\partial L}{\partial x^\alpha} - \frac{dp_\alpha}{dt} = 0 \qquad \left(\text{where, } p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha}\right)$$

(2) Let end pts. be free:

$$\frac{\partial A}{\partial x^\alpha} = \frac{\partial L}{\partial \dot{x}^\alpha} = p_\alpha \Rightarrow \text{* \underline{Hamilton–Jacobi equation}}$$

{ NOTE: We can make a canonical transformation from $(q, p) \longrightarrow (q', p')$ such that Hamiltonian $= 0$

Then: $\dot{q} = \underbrace{\{q, H\}}_{\text{poisson brackets}} = 0$, $\dot{p} = \{p, H\} = 0$

NOTE: $\begin{cases} \text{Classical Mechanics, by Kibble} \\ \text{Classical Mechanics, by Landau-Lifshitz} \end{cases}$

(References for Classical Mechanics revision)

then we can write :

$$\frac{\partial A}{\partial t} + H\left(q, \frac{\partial A}{\partial x^\alpha}\right) = 0$$

eg: For a free particle : $H = \frac{p^2}{2m} = \frac{1}{2m}\left(\frac{\partial A}{\partial x}\right)^2$

taking $A = -E(t) + f(x)$

$$\Rightarrow -\dot{E} + \frac{1}{2m}\left(\frac{df}{dx}\right)^2 = 0$$

LECTURE-4

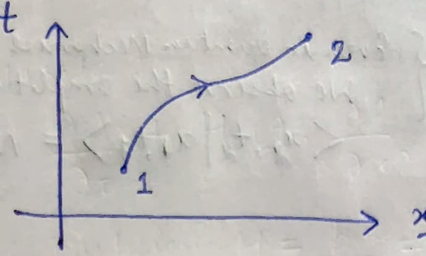
31/07/2023

This lagrangian is to obtained depending on the problem

NOTE: We have seen :

The action, $A[\tilde{x}(t)\,;\, x_2, t_2\,;\, x_1, t_1] = \int_{t_1}^{t_2} \mathcal{L}(\tilde{x}, \tilde{v})\, dt$

$\mathcal{L}$ could also depend on t, but here we haven't taken it to be explicitly dependent on t



Then, $\delta A = \int_{t_1}^{t_2} \left[\frac{d\tilde{p}}{dt} - \frac{\partial \mathcal{L}}{\partial \tilde{x}} \right] \delta \tilde{x}(t)\, dt + \tilde{p}\, \delta \tilde{x} \Big|_{t_1}^{t_2}$

(where, $\tilde{p} = \frac{\partial \mathcal{L}}{\partial \dot{\tilde{x}}}$)

Here we have two ways of solving :

(1) Fix end pts. & demand $\delta A = 0$

$$\Rightarrow \quad \frac{dp}{dt} = \frac{\partial \mathcal{L}}{\partial \tilde{x}} \quad \rightarrow \text{this is the trajectory}$$

(2) Fix trajectory to be that one which solves ELE, and vary the end pts. (Euler-Lagrange Eq^n)

$$\tilde{p} = \frac{\partial A}{\partial \tilde{x}} \quad , \quad \mathcal{H}(\tilde{x}, \tilde{p}) \longrightarrow \frac{\partial A}{\partial t} + \mathcal{H}\left(\tilde{x}, \frac{\partial A}{\partial \tilde{x}}, t\right) = 0$$

25] We consider the Lagrangian as :

$$\mathcal{L}(\tilde{v}, \cancel{x}, \cancel{t}) = \mathcal{L}(v)$$

Considering the assumptions :

Newtonian mechanics, free particle

The Galileon transformations are :

$$x' = x - Vt \qquad \Rightarrow \text{We can take} : v \rightarrow v' = v - V$$
$$t' = t$$

Now, $\mathcal{L}' = \mathcal{L} + \frac{df(v, t, x)}{dt}$ i.e. $\mathcal{L} = \mathcal{L}(v^2)$

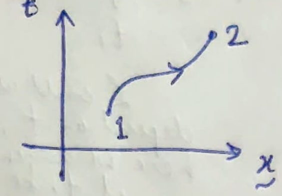
The lagrangian satisfying this is : $\mathcal{L} = kv^2$ $\left(k = \frac{m}{2}\right)$

such that it is invariant

Also, $\tilde{p} = m\tilde{v}$

So we can write the action as:

$$A[\underset{\sim}{x}(t)\,;\,2,1] = \int_{t_1}^{t_2} \frac{1}{2} m v^2 \, dt = \frac{m}{2}\frac{|\underset{\sim}{x}_2 - \underset{\sim}{x}_1|^2}{(t_2 - t_1)}$$



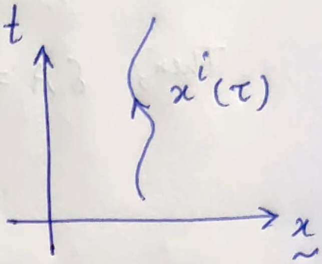
{ Even in Quantum Mechanics, the action is important. We observe the amplitude as:

$$\langle x_2, t_2 | x_1, t_1 \rangle = N(t) \exp\left(\frac{-i m |\underset{\sim}{x}_2 - \underset{\sim}{x}_1|^2}{2\hbar (t_2 - t_1)}\right) \}$$

26] In relativity, we have:

$$A[x^i(\tau)\,;\,2,1] = -K\int_{\tau_1}^{\tau_2} d\tau$$

this has been chosen, since τ is invariant



$$= -K\int_1^2 \sqrt{-ds^2} = -K\int_1^2 \sqrt{-\frac{dx_a}{d\tau}\frac{dx^a}{d\tau}}\, d\tau$$

* **NOTE:** This action is reparameterization invariant

(since $d\tau$ could be replaced by any $d\tau'$ (such as $2\,d\tau, 3\,d\tau, \ldots$) and A remains same)

$$\therefore \quad A = -K\int_{t_1}^{t_2} \sqrt{1 - v^2}\, dt$$

{ **NOTE:** Now, $\mathcal{L} = \left(\sqrt{1-v^2}\right)(-K) = -K\left(1 - \frac{v^2}{2} + \ldots\right)$

$$= -K + K\frac{v^2}{2} + \ldots \quad *$$

*[and setting $\frac{v}{c} \ll 1$ (or equivalently, $v \ll 1$ (for $c=1$))] should give us the non-relativistic Action }

$\therefore$ Taking $K = m$: $\quad A = -m\int d\tau = \boxed{-m\int_{t_1}^{t_2} \sqrt{1-v^2}\, dt}$

We know, $p^k = m u^k = \gamma m c\left(1, \frac{\underset{\sim}{v}}{c}\right) = (\gamma m c, \gamma m \underset{\sim}{v})$

$$= \left(\frac{E}{c}, \gamma m \underset{\sim}{v}\right)$$

Also, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \underset{\sim}{v}}\right) = 0 \Rightarrow \frac{d}{dt}\left(\frac{m\underset{\sim}{v}}{\sqrt{1-v^2}}\right) = 0$

{ * **NOTE:** You can use a different parameterization for $d\tau$ if required.

Eg: For light $d\tau$ can't be used

(since $ds = 0$)

But some other parameterization could still be taken. }

NOTE: We know:

$$p_a p^a = -m^2$$

$$\Rightarrow E^2 + |\underset{\sim}{p}|^2 = -m^2$$

$$\Rightarrow \boxed{E = \sqrt{|\underset{\sim}{p}|^2 + m^2}}$$

$\therefore$ For massless particles, $\boxed{E = cp}$

27] $A = -m \int \sqrt{-dx_a dx^a}$, Now: $dx_a dx^a = \eta_{ab} dx^a dx^b$

$$\delta A = -m \int \delta \sqrt{-dx_a dx^a}$$

Also: $\delta(\eta_{ab} dx^a dx^b) = \eta_{ab} \delta(dx^a) dx^b + \eta_{ab} dx^a \delta(dx^b)$

$$= -m \int \frac{1}{2} \frac{-2 dx_a \delta(dx^a)}{\sqrt{-dx^a dx_a}}$$

$$= m \int \frac{dx_a}{d\tau} \delta(dx^a)$$

$$= m \int u_a \, d(\delta x^a)$$

$$= m \int_1^2 d(u_a \delta x^a) - \frac{du_a}{d\tau} \delta x^a \, d\tau$$

{ Also, $\delta x^a(\tau_1) = \delta x^a(\tau_2) = 0$ if we fix end pts. }

$$\Rightarrow \delta A = m u_a \delta x^a \Big|_1^2 - \int_1^2 m \frac{du_a}{d\tau} \delta x^a \, d\tau$$

(1) Fix end pts. & $\delta A = 0$: $m \frac{du_a}{d\tau} = 0$

(2) $m u_a = \frac{\partial A}{\partial x^a} = p_a \Rightarrow \boxed{\frac{\partial A}{\partial x^a} \frac{\partial A}{\partial x^b} \eta^{ab} = -m^2}$ Hamilton-Jacobi Equation for Relativity