POST-MINOR-1

LECTURE-8 (13/02/2023)

30] **Scalar** : eg: T, Pressure, density, electric charge.

↓

Can be specified with just one-component

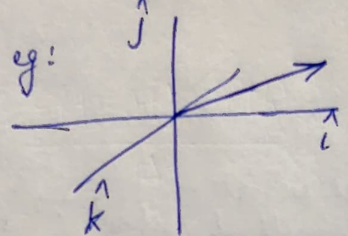
OR

"Zero" ← 0-basis vectors

↓

Scalar is also known as a Tensor of Rank 0.

31] **Vector** eg:



$(3\hat{i} + 4\hat{j} + 0\hat{k})$

↓

It can be specified with one basis vector for each component

↓

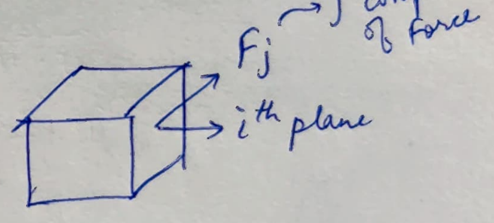
∴ A vector can be called a Tensor of Rank 1

NOTE: Rank 2 : Dyads
Rank 3 : Triads
⋮ so on..

} in general we often just refer to them as "Tensors"

32] We define stress as:

$$\text{Stress, } \sigma_{ij} = \frac{F_j}{A_i}$$



↓

$\underbrace{3 \times 3}$

∴ there are 9 components

We can, if we want, arrange them as a matrix

$$\Rightarrow \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

(NOTE: We are NOT saying that stress is a matrix

↓

Only that for now we are arranging the components in a matrix)

Here, since we need 2 basis vectors
to describe each component
$\therefore$ Stress is a
Tensor of Rank 2

No. of components $= 3^2 = 9$

$\because$ 3 dimensions being used here (but, for example in Relativity, we may use 4 dimensions) $\Rightarrow$ $\therefore$ No. of components (of Rank 2) $= 4^2 = 16$

For Rank 3, No. of components $= 3^3 = 27$; etc.

<u>NOTE:</u> <u>Notations</u>:

Vector : $\underline{v}$
Tensor : $\underline{\underline{T}}$

33] <u>Scalars</u> $\longrightarrow$ remain same after coord. transformation

<u>Vectors</u> $\longrightarrow$ Magnitude remains same in diff. coord. system ; $(v_1^2+v_2^2+v_3^2)^{1/2}$ is Invariant

<u>Tensor</u> $\longrightarrow$ Characteristic equation is same
i.e. $|T - \lambda I| = 0$ is same
i.e. $\lambda_1, \lambda_2, \lambda_3$ remains same

34] <u>There are 3 basis vectors in 3D space</u>

$\rightarrow$ <u>Orthogonal space systems</u>
- Rectangular : $\underline{e_x}, \underline{e_y}, \underline{e_z}$
- Spherical : $\underline{e_r}, \underline{e_\theta}, \underline{e_\phi}$

$\rightarrow$ <u>Generic set of cartesian unit vectors</u>:
$\underline{e_i}$ where $i = 1, 2, 3$

→ Vector can be written as:

$$v \text{ or } \underline{v} = \sum_{i=1}^{3} v_i \underline{e_i}$$

$$(\vec{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3)$$

(Later we see, Tensor: D or $\underline{\underline{D}} = \sum_{i=1}^{3} \sum_{j=1}^{3} D_{ij} \underline{e_i} \underline{e_j}$)

35] eg: $a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3$

$\downarrow$

j is the repeating index ("dummy index")

OR

"free index"

This way of writing is called Einstein Summation Convention.

eg: $a_{ij} b_j \neq a_{kj} b_j$ (i.e. Free index cannot be replaced

$\downarrow$

But dummy index CAN be replaced)

Also, no index can occur 3 or more times in a given term

eg: $a_{ii} b_{ij}$ ← NOT allowed

$a_{ij} b_j + a_{ji} b_j$ ← allowed

Further, the free indices should match on both sides of an equation:

eg: $x_i = a_{ij} b_j$: allowed

$x_i = A_{ij}$: NOT allowed

$x_j = A_{ik} u_k$: NOT allowed

36] Operations of vectors in Tensor notation

→ Scalar operation:

$$\rho \underline{v} = \rho \sum_{i=1}^{3} v_i \underline{e_i}$$

→ Addition:

$$\underline{a} + \underline{b} = \sum_{i=1}^{3} a_i \underline{e_i} + \sum_{i=1}^{3} b_i \underline{e_i}$$

→ Dot product

$$\underline{a} \cdot \underline{b} = \left(\sum_i a_i \underline{e_i}\right) \cdot \left(\sum_j b_j \underline{e_j}\right) = \sum_i \sum_j a_i b_j (\underline{e_i} \cdot \underline{e_j})$$

Using Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & , i=j \\ 0 & , i \neq j \end{cases}$$

$$\Rightarrow \quad \underline{a} \cdot \underline{b} = \sum_i \sum_j a_i b_j \delta_{ij}$$

$$= \sum_i a_i b_i$$

or

simply $a_i b_i$

LECTURE-9 (14/02/2023)

→ Cross Product

$\vec{a} \times \vec{b} \Rightarrow$ Here we get a resultant vector $\vec{c}$ whose components are given by:

$$c_k = \sum_i \sum_j \underbrace{\epsilon_{ijk}}_{\text{called the Levi-Civita parameter}} a_i b_j$$

$$\Downarrow$$

$$\epsilon_{ijk} = \begin{cases} +1 & , \text{if } i,j,k \text{ are cyclic} \\ 0 & , \text{if any 2 indices are same} \\ -1 & , \text{if } i,j,k \text{ are anticyclic (or acyclic)} \end{cases}$$

eg: $$c_1 = \sum_i \sum_j \epsilon_{ij1} a_i b_j$$

$$c_2 = \sum_i \sum_j \epsilon_{ij2} a_i b_j$$

37] Operations for Rank 2 Tensors

→ Scalar & tensor multiplication

$$-p \underbrace{T}_{\text{unity tensor}} = -p\, \hat{e}_i \hat{e}_j \delta_{ij} \Rightarrow$$ can be shown as $$\begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

→ Dot Product of Tensors

$$\underline{\underline{a}} \cdot \underline{\underline{b}} \Rightarrow c_{ik} = a_{ij} b_{jk}$$

Here, $\underline{\underline{a}} = \sum_i \sum_j a_{ij} \underline{e_i} \underline{e_j}$

$$\underline{\underline{b}} = \sum_k \sum_l b_{kl} \underline{e_k} \underline{e_l}$$

$$\therefore \underline{\underline{a}} \cdot \underline{\underline{b}} = (a_{ij} \underline{e_i} \underline{e_j}) \cdot (b_{kl} \underline{e_k} \underline{e_l})$$

$$= a_{ij} b_{kl} \underline{e_i} \underline{e_j} \cdot \underline{e_k} \underline{e_l}$$

$$= a_{ij} b_{kl} \underline{e_i} \underline{e_l} \delta_{jk}$$ → $\therefore$ taking $k = j$

$$= a_{ij} b_{jl} \underline{e_i} \underline{e_l}$$

replacing dummy index l with k → $= a_{ij} b_{jk} \underline{e_i} \underline{e_k}$

→ Vector & Tensor Product {gives a vector}

$$\underline{a} \cdot \underline{\underline{b}} \qquad (\text{eg: } \underline{\vec{F}} = \underline{\underline{\sigma}} \cdot \hat{n})$$

$$\Downarrow$$

$$\left(\sum_i a_i \underline{e_i}\right) \cdot \left(\sum_j \sum_k b_{jk} \underline{e_j} \underline{e_k}\right)$$

$$= a_i b_{jk} \underline{e_i} \cdot \underline{e_j} \underline{e_k}$$

$$= a_i b_{jk} \delta_{ij} \underline{e_k}$$

$$= a_i b_{ik} \underline{e_k}$$

replacing k with j → $= a_i b_{ij} \underline{e_j}$

→ Double Dot Product or Scalar Product

eg: Strain energy density $= \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}$

$$\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} = \sigma_{pq} \underline{e_p} \underline{e_q} : \varepsilon_{mn} \underline{e_m} \underline{e_n}$$

$$= \sigma_{pq} \varepsilon_{mn} \underline{e_p} \underline{e_q} : \underline{e_m} \underline{e_n}$$

$$= \sigma_{pq}\, \varepsilon_{mn}\, \underline{e_p} \cdot \underline{e_n}\, \delta_{qm}$$

$$= \sigma_{pq}\, \varepsilon_{mn}\, \delta_{pn}\, \delta_{qm}$$

$$= \sigma_{pq}\, \varepsilon_{qp} \rightsquigarrow \text{which is a scalar}$$

38] <u>Gradient</u>

For a function ϕ
↓
gradient given by: $\underline{\nabla}\phi$ ⇝ which will be a vector (if ϕ is a scalar)

For a scalar ρ
⇓

$$\underline{\nabla}\rho = \frac{\partial \rho}{\partial x_i} \equiv \frac{\partial \rho}{\partial x_1}\hat{e}_1 + \frac{\partial \rho}{\partial x_2}\hat{e}_2 + \frac{\partial \rho}{\partial x_3}\hat{e}_3$$

(eg: Heat flux = grad (T) × conductivity)

39] <u>Divergence</u>: $\underline{\nabla} \cdot \underline{v}$ ⇝ gives a scalar
⇓

$$\frac{\partial}{\partial x_i}\underline{e_i} \cdot v_k \underline{e_k} = \frac{\partial v_k}{\partial x_i}\, \underline{e_i} \cdot \underline{e_k}$$

$$= \frac{\partial v_i}{\partial x_i}$$

* <u>NOTE</u>: <u>Terminology</u>:

Polar vectors
Pseudovectors
Pseudotensors (or axial vectors)
↓
they transform under proper rotation but change sign under improper rotation (i.e. rotation followed by inversion)

LECTURE-10 (20/02/2023)

40] Differential Calculus

$\downarrow$

tells change in one variable w.r.t. another variable

$\downarrow$

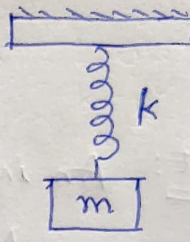
modelled by differential equations

eg: $\frac{dy}{dx} = f(x,y)$ $\Rightarrow$ Here, $y \rightarrow$ dependent variable

$x \rightarrow$ independent variable

41] A fundamental eq^n encountered in engineering: (often related to oscillating problems)

$$my'' + by' + ky = F_{ext}$$



Another example: for springs:

$$F = ma = m\frac{d^2y}{dt^2} \text{ or } my''$$

and, $F_{spring} = -ky$

$$\Rightarrow my'' = -ky$$

(similarly we can also add air damping: $-b\frac{dy}{dt} = -by'$)

Thus in general our equation is:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

3 cases: ① When $m = 0 \Rightarrow \frac{dy}{dt} = -ky$

$$\Rightarrow y = ce^{-kt} \quad \text{(decays)}$$

② When $b = 0 \Rightarrow m\frac{d^2y}{dt^{2}} = -ky \Rightarrow -w^2y = y''$

$(w^2 = k/m)$

$$\Rightarrow y = C\cos wt + D\sin wt$$

(linear combination of sines & cosines)

$\downarrow$

(periodic oscillation infinitely in vacuum)

③ when $\frac{d^2y}{dt^2} = 0 \Rightarrow y = C + Dt$

42] Now trying : $y = e^{\lambda t}$ (in our diff. eqⁿ above)

$$\Rightarrow m\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + k e^{\lambda t} = 0$$

$$\Rightarrow m\lambda^2 + b\lambda + k = 0$$

(then solve for λ)

NOTE: For solving higher order DEs
↓
trying $y = e^{\lambda t}$ is quintessential

NOTE: An important D.E. for modelling fluids:
Navier-Stokes eqⁿ: $\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = \frac{-\nabla P}{\rho} + \nu \nabla^2 \underline{u}$

43] Types of D.E. : ODE, PDE

$y(x) \longrightarrow \frac{dy}{dx} = f(x, y) \longrightarrow$ ODE

(y → dependent; x → independent variable)

$$\frac{dy}{dx} + \frac{dz}{dx} = f(x, y) + z^2$$

y & z : 2 dependent variables

x : 1 independent variable

$y(x, t) \longrightarrow \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial t^2} = mc \longrightarrow$ PDE

44] Characteristics :

① Order: The highest derivative

eg: $\frac{dy}{dx} - \frac{d^2y}{dx^2} = 1 + 5x$: Order 2

$$\frac{d^3x}{dt^3} + \cos x = \frac{dy}{dt}$$: Order 3

② **Degree** : power of highest order derivative

$$\frac{d^3x}{dt^3} + \left(\frac{dy}{dt}\right)^2 + \cos x = 0$$: Degree 1

$$\left(\frac{dy}{dt}\right)^2 + 5xy = 8y^2$$: Degree 2

③ **Linearity** :

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_0(x)\,y = f(x)$$

Here : (a) y & all its derivatives should have power 1.

(b) Each coefficient depends only on x

eg: $$(x+1)\frac{dy}{dx} - y = 0$$: linear

$$\frac{dy}{dx} + xy - y^2 = 0$$: Non-linear

45] **Solution of D.E.s**

① **Explicit sol^n** : $y = f(x)$

eg: $$\frac{1}{x^3}\frac{dy}{dx} - 4y^2 = 0$$

$$\Rightarrow \frac{dy}{4y^2} = x^3\,dx$$

$$\Rightarrow y = -\frac{1}{x^4} + C$$

② **Implicit sol^n** : $f(x, y) = 0$

eg: $$\frac{dy}{dx} = \frac{-y}{x+y} \Rightarrow (x+y)\,dy = -y\,dx$$

$$\Rightarrow xy + \frac{y^2}{2} + \dots = 0$$

{ Here, separation of variables is not possible }

46] Exact ODEs

$$z = f(x,y)$$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Homogeneous : $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$ — ①

sol^n : $f(x,y) = c$

① : $M(x,y)\,dx + N(x,y)\,dy = 0$

Eq^n ① will be exact, if we get:

$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ * (condition for exactness)

Here, $M = \frac{\partial f}{\partial x}$, $N = \frac{\partial f}{\partial y}$

eg: $\underbrace{[y^2\cos x - 3x^2y - 2x]}_{M}\,dx + \underbrace{[2y\sin x - x^3 + \ln y]}_{N}\,dy = 0$

$\frac{\partial M}{\partial y} = 2y\cos x - 3x^2$, $\frac{\partial N}{\partial x} = 2y\cos x - 3x^2$

∴ It is an Exact ODE.

47] In general we see the following method:

① $\frac{\partial f}{\partial x} = M(x,y) \Rightarrow f(x,y) = \int M\,dx$

② $\frac{\partial f}{\partial y} = N \Rightarrow$ Take derivative of $f(x,y)$ wrt y & equate this to N

③ Find $g(y)$

④ $f(x,y) = c$

eg: ① $f(x,y) = \int (y^2\cos x - 3x^2y - 2x)\, dx$

$\Rightarrow f(x,y) = y^2\sin x - x^3y - x^2 + g(y)$ — Ⓐ

② $\dfrac{\partial f(x,y)}{\partial y} = 2y\sin x - x^3 + g'(y) = N$

$\Rightarrow 2y\sin x - x^3 + g'(y) = 2y\sin x - x^3 + \ln y$

$\Rightarrow g'(y) = \ln y$

$\Rightarrow g(y) = y\ln y - y + c$

from Ⓐ : $\boxed{y^2\sin x - x^3y - x^2 + y\ln y - y = C}$

is the final result.

48] Inexact ODE

eg: $(x+y)\,dx + x\ln x\, dy = 0$

$M = x+y \quad , \quad N = x\ln x$

$\Rightarrow \dfrac{\partial M}{\partial y} = 1 \quad , \quad \dfrac{\partial N}{\partial x} = \ln x + 1 \longrightarrow \therefore$ It is Inexact

To make it exact we multiply it by 'r' (some function) :

$$\underbrace{rM}_{M^*}\, dx + \underbrace{rN}_{N^*}\, dy = 0 \quad — ①$$

For making ① exact : $\dfrac{\partial (rM)}{\partial y} = \dfrac{\partial (rN)}{\partial x}$

$\Rightarrow r\dfrac{\partial M}{\partial y} + M\dfrac{\partial r}{\partial y} = r\dfrac{\partial N}{\partial x} + N\dfrac{\partial r}{\partial x}$

$\Rightarrow r\underbrace{\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)}_{T} = -\left(M\dfrac{\partial r}{\partial y} - N\dfrac{\partial r}{\partial x}\right)$

$\Rightarrow rT = -\left(M\dfrac{\partial r}{\partial y} - N\dfrac{\partial r}{\partial x}\right)$

* Here $\begin{cases} \text{If } T \text{ depends only on } x \Rightarrow r = e^{\int T/N\, dx} \\ \text{If } T \text{ depends on only } y \Rightarrow r = e^{\int -T/M\, dy} \end{cases}$

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eg:
$$T = -\ln x$$
$$N = x \ln x$$
$$T/N = -1/x$$
$$\therefore r = \int e^{T/N} dx$$
$$= \int e^{-1/x} dx$$

$$\therefore \ln r = \int -\frac{1}{x} dx$$
$$\Rightarrow \ln r = -\ln x + c$$
$$= \ln\left(\frac{1}{x}\right) + c$$
$$\Rightarrow r = 1/x$$

$$(x+y)\,dx + x \ln x \, dy = 0$$
$$\underbrace{\left(1 + \frac{y}{x}\right)}_{M^*} dx + \underbrace{\ln x}_{N^*} \, dy = 0$$

① $f(x,y) = \int M^* \, dx = \int \left(1 + \frac{y}{x}\right) dx = x + y \ln x + g(y)$

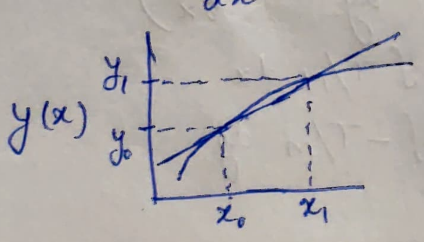
② $\frac{\partial f}{\partial y} = N^* \Rightarrow \ln x + g'(y) = N^* = \ln x$

$$g'(y) = 0 \Rightarrow g(y) = c_1$$
$$\therefore \text{Sol}^n: f(x,y) = c$$
$$\Rightarrow x + y \ln x = C$$

49] <u>Numerical Method</u>

<u>Euler's method (first order method)</u> : IVP (initial value problem)

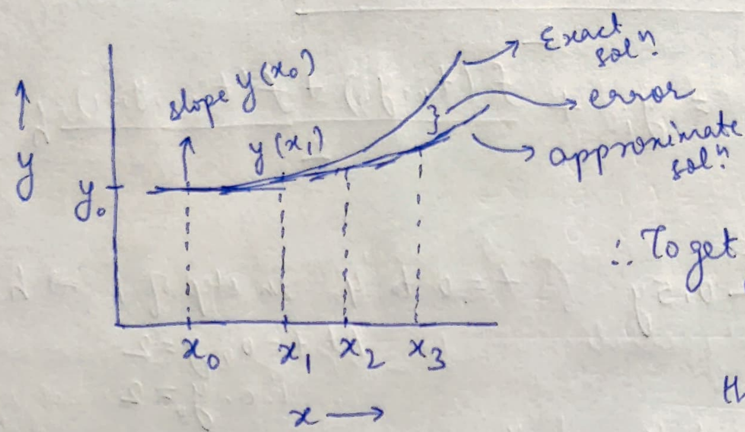
$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0$$



$$\underset{\phi}{\text{Slope}} = \frac{y_{i+1} - y_i}{\underbrace{x_{i+1} - x_i}_{h}} \Rightarrow y_{i+1} - y_i = \phi h$$

$\therefore \; y_{i+1} = y_i + \phi h$

$$\Rightarrow y_{i+1} = y_i + h \frac{dy_i}{dt_i} + \frac{h^2}{2!} \frac{d^2 y_i}{dt_i^2} + \dots$$



$\therefore$ To get more accurate soln's
$\downarrow$
we need to ensure that our **step size is small**

eg: $\frac{dy}{dx} = x^2, \; y_0 = 1, \; h = 0.3$

$\Rightarrow \left(y = \frac{x^3}{3} + 1 \right.$ will be the exact soln $\left. \right)$

i	x_i	y_i	y_i^{exact}	$\varepsilon_r = \frac{y_i - y_i^{exact}}{y_i^{exact}}$ (%)
0	0	1	1	
1	0.3	1.0	1.009	
2	0.6	1.027	1.072	
3	0.9	1.135	1.243	
4	1.2	1.378	1.576	

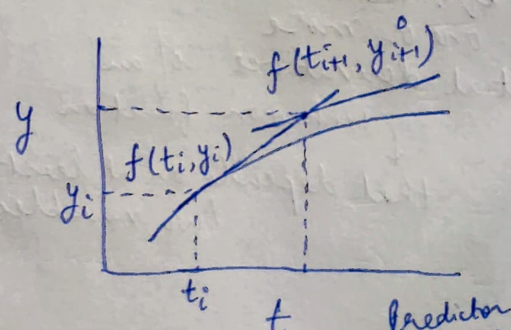
50] Heun's method

(Predictor – Corrector algorithm)

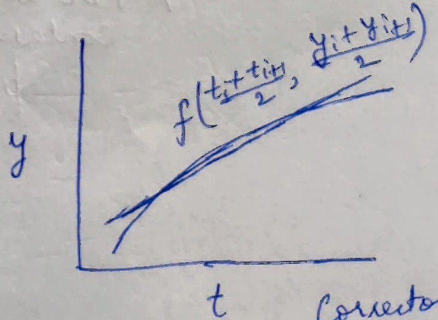
Steps:

→ determine the derivatives at 2 intervals : one at the beginning & one at the end

→ The derivatives are then averaged to obtain a better estimate.



Predictor



not correct probably (see ahead)

Corrector

$$y^{\circ}_{i+1} = y_i + f(t_i, y_i)\, h \quad \leftarrow \text{Predictor eq}^n$$

$$y'_{i+1} = f(t_{i+1}, y^{\circ}_{i+1})$$

$\therefore$ Avg. slope : $\bar{y}' = \dfrac{f(t_i, y_i) + f(t_{i+1}, y^{\circ}_{i+1})}{2}$

Corrector eq^n: $y_{i+1} = y_i + \dfrac{f(t_i, y_i) + f(t_{i+1}, y^{\circ}_{i+1})}{2} h$

eg: $\dfrac{dy}{dt} = 4e^{0.8t} - 0.5y$, $t = 0$ to 4 in step of $1 \Rightarrow h = 1$

At $t=0$, $y=2$

i.e. $y_0 = 2$

t	y_{euler}	y_{Heun}	y_{true}
0	:	:	:
1	:	:	:
2	:	:	:
3	:	:	:
4	:	:	:

NOTE: There are 2 types of errors in such cases :

— Truncation error (eg: truncating a taylor series)

— Roundoff error (eg: decimal places are roundedoff upto a certain limit based on floating point precision of computer program)

local error

$\Theta(h^2)$

↓

global error

$\Theta(h)$

LECTURE 12 (03/03/2023)

51] Introduction to Partial Differential Equations (PDEs)

↓

A PDE is an eq^n. that contains one or more partial derivatives of an unknown function

↓

that depends on at least two variables ⇝ usually time & space

NOTE! Linear PDE :- If it is first degree in unknown function 'u' and its partial derivatives

* A PDE is called Homogeneous → if each of its terms contains either 'u' or one of its partial derivatives

Examples:
(of linear PDEs) We have $u(x,t)$ such that:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow \text{Wave Equation (1-D)}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow \text{Heat Equation (1-D)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \rightarrow \text{Laplace Equation (3-D)}$$

Other examples of PDEs:
→ Schrödinger eq^n (from Quantum chemistry & Quantum physics)
→ Poisson-Boltzmann eq^n
→ Fick's second law of diffusion
→ Material Science applications ← Phase transformations, Heat conduction, Diffusivity

52] Characteristics of PDEs

Quasi-linear second-order PDEs:

3 cases (will be discussed) ←

$$\underbrace{a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial^2 u}{\partial x \partial t} + c(x,t)\frac{\partial^2 u}{\partial t^2}}_{\text{linear}} = f\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right)$$

It is quasi-linear PDE bcz it is linear in 2nd order partial differentials at LHS
↓
but can be non-linear in u and first-order partial differentials of u w.r.t. x, y at RHS

{i.e. it is linear in highest order derivatives but may be non-linear for other derivatives} eg: $\frac{\partial^2 u}{\partial x^2} + u \frac{\partial^2 u}{\partial t^2} = 0$

Non-linear case example : $\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + u^2 \frac{\partial^2 u}{\partial t^2} = 0$ → 2nd order linear homogeneous PDE

In general, we can write also:

$F(x, y, u, u_x, u_y) = 0$ → linear homogeneous eq^n

<u>There are 3 cases possible</u> (for the original quasi-linear eq^n):

1. Hyperbolic : $b^2 - ac > 0$ → Wave eq^n (is not steady state)
2. Parabolic : $b^2 - ac = 0$ → Heat eq^n (reaches steady state)
3. Elliptic : $b^2 - ac < 0$ → Laplace eq^n (steady state, no evolution with time)

For some function $f(x, y)$:

* <u>NOTE</u>: $\int \frac{\partial f}{\partial x} dx \longrightarrow \ldots + \underbrace{g(y)}$

i.e. the "constant" term will be a function of y here

53] <u>Initial and Boundary Value Problems</u>

For PDEs : Initial value (IV) is in time variable & Boundary values (BV) are in space variables

— <u>ODEs</u>: we first find general sol^n
↓
then apply IV or BV to get value of arbitrary constant
↓
which gives particular sol^n

— <u>PDEs</u>: Arbitrary const. is a function of x or y
↓
many arbitrary functions are possible
↓
* We do NOT find the complete sol^n;
↓
We consider the sol^n for IVP or BVP & then apply whichever condition is easy to apply to obtain final sol^n

→ which satisfies the PDE ← as well as the condition applied

NOTE: IVP → time taken as initial : For $u(x,t)$: $u(x,0)$ → i.e. initial $t=0$

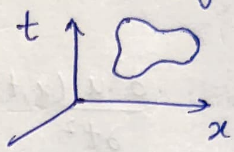
54] Boundary conditions

↓

are over space variables (cannot be described in both space & time → since time is decoupled from space)

1. Dirichlet condition : supply value of function over the entire boundary

i.e. $u(0,t)$



2. Neumann condition : supply derivative normal to the boundary

i.e. $$\left.\frac{\partial u}{\partial t}\right|_{x=y=0}$$

NOTE: In a part of the domain Dirichlet condⁿ can be applied & in the other part of domain Neumann condⁿ can be applied.

55] For second order PDEs : Cauchy condition

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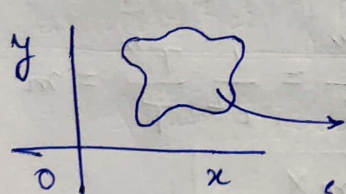
- Time is decoupled from space variables
- u and $\frac{\partial u}{\partial t}$ needs to be supplied over the entire domain of PDE

↓

initial condⁿ



Initial conditions,

{for some function: $u(t,x,y)$}

init. condⁿs ⇒ $u(0,x,y)$ and $\left.\frac{\partial u}{\partial t}\right|_{x=y=0}$

↓

(value at $t=0$)

need to be supplied inside the domain in $x-y$ plane

56] Method of separation of variables

we know that!

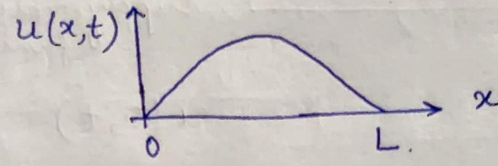
$$\frac{\partial u}{\partial x} \equiv u_x \ , \quad \frac{\partial^2 u}{\partial x^2} \equiv u_{xx} \ , \quad \frac{\partial u}{\partial x \partial t} \equiv u_{xt}$$

We take some : $X = f(x)$
$T = f(t)$

and convert eq^n into the form :

$$\underbrace{\phi(x, X', X'')}_{\text{ODE}} = \underbrace{\psi(t, T', T'')}_{\text{ODE}} = \text{separation constant}$$

57] <u>Solution of 1-D Wave Equation</u>



$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \rightarrow \text{Wave equation}$$

<u>Initial cond^n</u> : At $t=0$, $u(x,0) = f(x)$; $\frac{\partial u}{\partial t} = g(x)$; $(0<x<L)$

Boundary conditions : $u(0,t) = u(L,t) = 0$, $t>0$

<u>Separation of variables</u> : $u(x,t) = X(x)\,T(t)$

Wave eq^n becomes : $\frac{\partial^2 u}{\partial t^2} = X(x)\,T''(t)$

$\frac{\partial^2 u}{\partial x^2} = T(t)\,X''(x)$

$\Rightarrow XT'' = c^2\,TX''$

Dividing by XT $\Rightarrow$ $\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = m$ (separation constant)

$$\Rightarrow \boxed{\begin{array}{l} X'' = mX \\ T'' = c^2 m T \end{array}}$$

for $m=0$: $X''=0 \Rightarrow X(x) = ax+b$

B.C. : $X(0) \Rightarrow a(0)+b=0$

$\Rightarrow a=b=0$

for $m=k^2$: $X'' = k^2 X \Rightarrow X(x) = e^{-kx}$ ($\because$ then : $X'' = k^2 e^{-kx}$)

$X(x) = Ae^{-kx} + Be^{kx}$

$X(0) = A+B = 0 \Rightarrow A = -B$

$X(L) = -Be^{-kL} + Be^{kL} = 0$

$\Rightarrow B = 0$

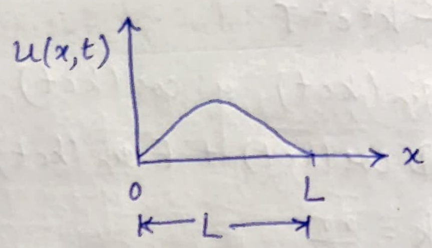
for $m = -k^2$: $X'' = -k^2 X \Rightarrow X(x) = \sin kx$

($\because X'' = -k^2 \sin kx$)

$$X(x) = A\cos kx + B \sin kx$$

LECTURE 13 (13/03/2023)

58] Solution of 1-D Wave equation



c : Speed with which vibration is travelling

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} \quad — (A)$$

Initial condition: $u(x,0) = f(x)$

$$\frac{\partial u(x,0)}{\partial t} = g(x) \quad (0 < x < L)$$

Boundary condn: $u(0,t) = u(L,t) = 0 \quad (t > 0)$

Now, $u(x,t) = X(x)\,T(t) \quad — (1)$, where:

Spatial function : $X(x)$

Temporal function : $T(t)$

Putting (1) in (A):

$$\Rightarrow TX'' = \frac{1}{c^2} X T''$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -k^2 \quad \text{(separation constant)}$$

Thus: $X'' = -k^2 X$

$T'' = -k^2 c^2 T$

Now, $X(x) = A\cos kx + B\sin kx$

Spatial soln: $X(x) = \sin kx$

$X'(x) = k\cos kx$

$X''(x) = -k^2 \sin kx$

B.C.: $X(0) = X(L) = 0$

when $x=0$: $A\cos(0) + B\sin(0) = 0 \Rightarrow A = 0$

$x=L$: $X(L) = B\sin kL = 0$

$$\Rightarrow \sin kL = \sin n\pi$$

$$\Rightarrow k = \frac{n\pi}{L}$$

$$\therefore X(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad , n = 1, 2, 3, \ldots$$

<u>Temporal solⁿ</u>: $T'' = -k^2c^2T$

$$T(t) = \sin(kct) \text{ or } \cos(kct)$$

$$T(t) = D\cos(kct) + E\sin(kct)$$

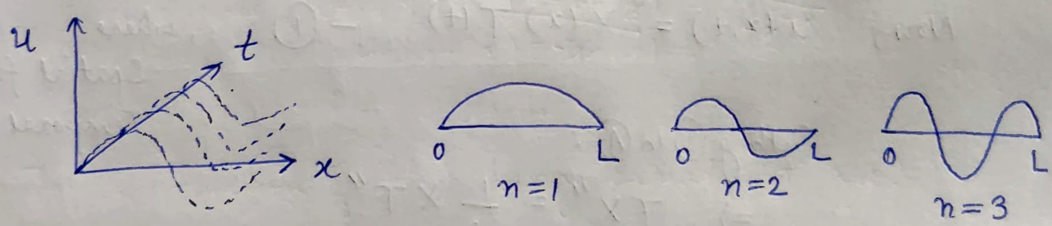
$$k = \frac{n\pi}{L}$$

Thus, $T(t) = \sum_n D_n \cos\left(\frac{n\pi ct}{L}\right) + \sum_n E_n \sin\left(\frac{n\pi ct}{L}\right)$

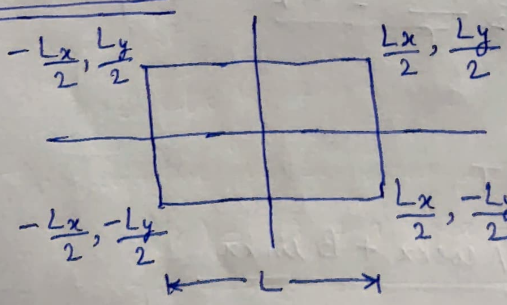
$$\therefore u(x,t) = \sum_n X_n(x)\, T_n(t)$$

$$\Rightarrow \boxed{u(x,t) = \sum_n B_n \sin\left(\frac{n\pi x}{L}\right)\left[D_n \cos\left(\frac{n\pi ct}{L}\right) + E_n \sin\left(\frac{n\pi ct}{L}\right)\right]}$$

$$H_n = B_n D_n \quad , \quad G_n = B_n E_n$$



59] <u>Wave in 2D</u>



$u(x,y,t)$

Wave equation in 2D: $\dfrac{\partial^2 u(x,y,t)}{\partial x^2} + \dfrac{\partial^2 u(x,y,t)}{\partial y^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$

Boundary conditions: $u(x,y,t) = 0$, if $x \geq \frac{L_x}{2}$ or $x \leq \frac{-L_x}{2}$

$y \geq \frac{L_y}{2}$ or $y \leq \frac{-L_y}{2}$