# LECTURE 18

18/09/2023

**NOTE:** For a given metric

eg: $ds^2 = -f\,dt^2 + \frac{1}{f}dx^2$

You can write the lagrangian $\mathcal{L}$
and then use
the Euler-Lagrange eq$^n$.s
↓
from there you can read off the christoffel symbols

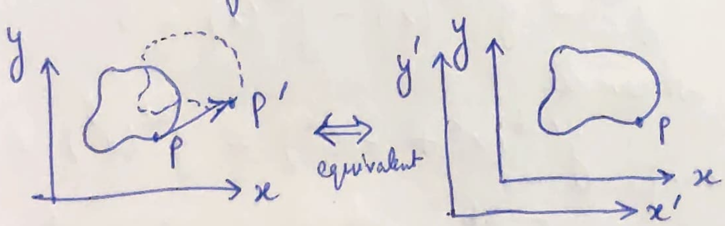
**NOTE:** $\frac{du_c}{d\lambda} = \frac{1}{2}\partial_c g_{ab} u^a u^b$

↳ here $\lambda$ must be an "affine parameter" (i.e. which transforms as $\lambda' = a + b\lambda$ )

94] Two ways of transforming coord.s

Active transformation (involves moving the body)

Passive transformation (involves shifting the coord.s)




* { Both these ways are essentially equivalent
↓
i.e. in the end, the new coord.s are the same in both }

95] Isometry of Spacetime
↓
Can we find any directions ↓ ⟶ i.e. the killing vectors

where if we transform the coord.s
↓
then the metric does not change

} ⇒ such a transformation is an isometry of spacetime (as metric is unchanged)

Making some transformation:

$$x^k \to y^k(x^l)$$

Thus, $g'^{ab}(y) = \frac{\partial y^a}{\partial x^k}\frac{\partial y^b}{\partial x^l} g^{kl}(x)$

Now, $ds^2 = g'_{ab}(y)\, dy^a dy^b$

and also, $ds^2 = g_{kl}(x)\, dx^k dx^l$

$$= g_{kl}(y)\, dy^k dy^l$$

{if metric doesn't change with this transformation}

Then:

$$\underset{\text{at } P'}{g'_{ab}(y)} = \underset{\text{at } P}{g_{ab}(y)}$$

In other words:

$$g'^{ab}(y) = \frac{\partial y^a}{\partial x^k}\frac{\partial y^b}{\partial x^l} g^{kl}(x)$$

* $[x^k \to x^k + \epsilon \xi^k(x)]$

$$= (\delta^a_k + \epsilon \partial_k \xi^a)(\delta^b_l + \epsilon \partial_l \xi^b) g^{kl}$$

$$\Rightarrow g'^{ab}(y) = g^{ab}(x) + \epsilon g^{bl}\partial_k \xi^a + \epsilon g^{al}\partial_l \xi^b + O(\epsilon^2)$$

Now $\Rightarrow \delta g^{ab} = g'^{ab}(x) - g^{ab}(x) = 0$

We can see:

{Taylor expansion} $g'^{ab}(x^k + \epsilon \xi^k) \simeq g'^{ab}(x^k) + \epsilon \xi^k \partial_k g'^{ab}$

$$\therefore \delta g^{ab} = \cancel{g^{ab}(x)} - \epsilon \xi^k \partial_k g^{ab} - \cancel{g^{ab}(x)} + g^{bk}\partial_k \xi^a + g^{al}\partial_l \xi^b = 0$$

* {we can write $\epsilon \xi^k \partial_k g'^{ab} \simeq \epsilon \xi^k \partial_k g^{ab}$ (since others terms are $O(\epsilon^2)$)}

{Because:
$g'^{ab}(x) + \epsilon \xi^k \partial_k g^{ab} = g^{ab}(x) + \epsilon g^{bk}\partial_k \xi^a + \epsilon g^{al}\partial_l \xi^b$}

Hence we have:

$$\delta g^{ab} = \underbrace{-\xi^k \partial_k g^{ab} + g^{bk}\partial_k \xi^a + g^{al}\partial_l \xi^b}_{\text{this is an alternative way to get } \xi^k(x)} = 0$$

This is used to define
a new derivative
↓
called the
"Lie Derivative"

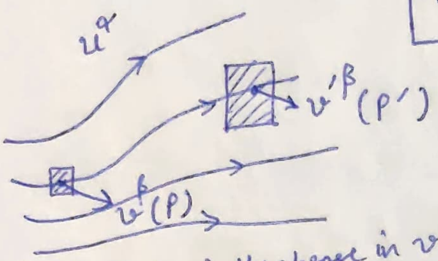
it also
turns out this is equal to:
$$\mathcal{L}_\xi g^{ab} = \nabla^a \xi^b + \nabla^b \xi^a = 0$$

$$\mathcal{L}_\xi g^{ab} \equiv -\xi^k \partial_k g^{ab} + g^{bk} \partial_k \xi^a + g^{ak} \partial_k \xi^b = 0$$

In general, the definition for Lie derivative is:

$$\boxed{\mathcal{L}_u v^k \equiv -u^l \partial_l v^k + v^l \partial_l u^k}$$

{ this is like
a directional derivative
of $v^k$ along $u$ }




this is
also called
"Lie dragging"

eg: (what is the change in $v^\beta$ with $u^\alpha$)

fluid with $u^\alpha$ speed
with ink drop in it
(with speed $v^\beta$)

* NOTE: It turns out that there will be a relation b/w
Lie derivative and Covariant derivative

96] We know:

$$\nabla^a \xi^b = g^{al} \nabla_l \xi^b = g^{al}(\partial_l \xi^b + \Gamma^b_{il} \xi^i)$$

The second
term can be
expanded as:
$$\frac{1}{2} g^{al} g^{bc} (-\cancel{\partial_c g_{il}} + \partial_i g_{cl} + \cancel{\partial_l g_{ic}}) \xi^i$$

by symmetry
these cancel
out

Here we can write:

$$\xi^i g^{al} g^{bc} \partial_i g_{cl} = -\xi^i \partial_i g^{ab}$$

$\Rightarrow$ this implies: $\nabla^a \xi^b = g^{al} \partial_l \xi^b - \frac{1}{2} \xi^i \partial_i g^{ab}$

and hence we obtain:

$$\boxed{\nabla^a \xi^b + \nabla^b \xi^a = -\xi^i \partial_i g^{ab} + g^{al} \partial_l \xi^b + g^{bl} \partial_l \xi^a}$$

* { NOTE: Killing vector DON'T depend
on coordinates
↓
they are simply symmetries
of spacetime }

* NOTE: An interesting article
to check out: "Varying without Varying"...
by Dawood Kothawala

# LECTURE 19
19/09/2023

**NOTE:** Extra Lecture: Wednesday (3:00 pm) {i.e. 20th sept}

**NOTE:** We will discuss curvature from 3 approaches
- 2 will be physical
- 1 mathematical

**NOTE:** We saw:

$g$ (metric) $\longrightarrow$ like potential

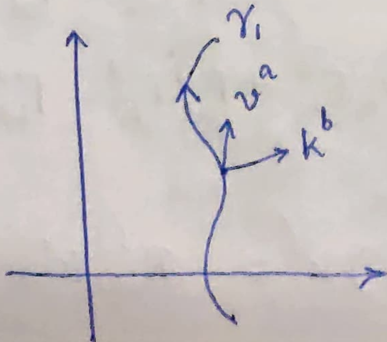
$\partial g \longrightarrow$ like force
(& this thus appears in Geodesic equation)
$\downarrow$
however we know that in local region this can be transformed such that $\partial g = 0$

Now, $\partial^2 g \longrightarrow$ relates to curvature
(* this CANNOT be made to vanish)
$\downarrow$
this is "true gravity" (i.e. it is not simply uniform acceleration) that can be made to vanish (like $\partial g$)

**NOTE:** We should use covariant derivative ($\nabla$) instead of partial derivative ($\partial$)
$\downarrow$
so that it is generally covariant

{However we cannot simply do this directly
$\downarrow$
as $\nabla g = 0 \rightsquigarrow$ so taking another derivative is pointless}
$\Downarrow$
$\therefore$ We must use ANOTHER approach

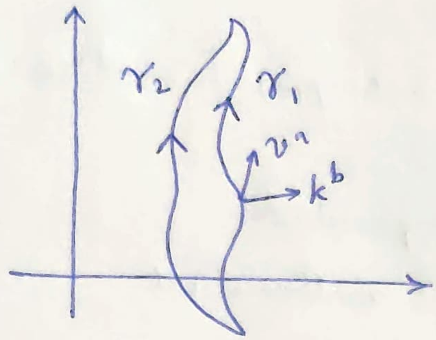
97]




Here, $v^a$ is velocity (tangent vector*) at a point along $\gamma_1$ & $k^b$ is another vector $\Longrightarrow$ Then the parallel transport is given by:

$$v^a \nabla_a k^b = 0$$

We can take 2 curves $\gamma_1$ and $\gamma_2$ with same end points:




Let's now solve the equation for each of the curves and see the difference:

$$v^a \nabla_a k^b = 0$$

$$\Rightarrow \frac{dx^a}{d\lambda}\left(\frac{\partial k^b}{\partial x^a} + \Gamma^b_{ai} k^i\right) = 0$$

i.e.

$$\frac{dk^a}{d\lambda} + \Gamma^b_{ai} k^i v^a = 0$$

Now for the solution, $k^b$ should be same for both curves, and thus should be independent of the curve:

$$\therefore \frac{dx^a}{d\lambda}\underbrace{\left(\frac{\partial k^b}{\partial x^a} + \Gamma^b_{ai} k^i\right)}_{0} = 0$$

$$\Downarrow$$

$$\partial_a k^j = -\Gamma^j_{ai} k^i$$

Hence we can write:

$$\begin{aligned}
\partial_b \partial_a k^j &= \partial_b\left(-\Gamma^j_{ai} k^i\right) \\
&= -\Gamma^j_{ai}\partial_b k^i - k^i \partial_b \Gamma^j_{ai} \\
&= -k^i \partial_b \Gamma^j_{ai} + \Gamma^j_{ai}\Gamma^i_{bl} k^l
\end{aligned}$$

$\left\{ \text{by replacing } \partial_b k^i \text{ with } \Gamma^i_{bl} k^l \right\}$

* Thus we have: $\partial_b \partial_a k^j = -k^i \partial_b \Gamma^j_{ai} + \Gamma^j_{ai}\Gamma^i_{bl} k^l$

Also,
$$\begin{aligned}
(\partial_a \partial_b - \partial_b \partial_a) k^j &= -k^i \partial_a \Gamma^j_{bi} + \Gamma^j_{bl}\Gamma^l_{ai} k^i \\
&\quad + k^i \partial_b \Gamma^j_{ai} - \Gamma^j_{al}\Gamma^l_{bi} k^i \\
&= -\left(\partial_a \Gamma^j_{bi} - \partial_b \Gamma^j_{ai} + \Gamma^j_{al}\Gamma^l_{bi} - \Gamma^j_{bl}\Gamma^l_{ai}\right) k^i
\end{aligned}$$

This can be written as a 4-index object:

$$(\partial_a \partial_b - \partial_b \partial_a) k^j = -R^j_{abi} k^i$$

where:

$$\boxed{R^j_{\ abi} \equiv \partial_a \Gamma^j_{ib} - \partial_b \Gamma^j_{ia} + \Gamma^j_{al}\Gamma^l_{bi} - \Gamma^j_{bl}\Gamma^l_{ai}}$$

(this is second order derivative in $g$)

this object is a tensor

For the parallel transport to be unique for the two curves, we require

$$(\partial_a\partial_b - \partial_b\partial_a) k^j = -R^j_{\ abi} k^i = 0$$

$$\Rightarrow \text{ i.e. } R^j_{\ abi} = 0$$

* (we WILL get zero from LHS as $\partial_a \partial_b$ commute)

However, we know that in genuinely curved spacetime this cannot be made to vanish.

* Thus, two different curves will NOT have the same parallel transport

{i.e. the solution to parallel transport depends on the curve}

which means that our assumption that we can find parallel transport independent of curve is WRONG

and the difference can be used to know that the spacetime has a curvature

and this is characterized by $R^j_{\ abi}$

{NOTE: To remember:

$$R_{ab} \equiv \partial_a \Gamma_b - \partial_b \Gamma_a + \Gamma_a \Gamma_b - \Gamma_b \Gamma_a$$

We then throw in the spacetime indices $i$ & $j$

$$R^i_{\ abj} \equiv \partial_a \Gamma^i_{jb} - \partial_b \Gamma^i_{ja} + \Gamma^i_{al}\Gamma^l_{bj} - \Gamma^i_{bl}\Gamma^l_{aj}$$ }

* NOTE: In Electrodynamics: $F_{ab} = \partial_a A_b - \partial_b A_a$

it doesn't have

$A_a A_b - A_b A_a$

(since that would cancel out)

as they are commutative

thus EM is an Abelian Gauge theory

But in GR the $\Gamma_a\Gamma_b - \Gamma_b\Gamma_a$ does NOT cancel out (it is Non-commutative)

* $\Rightarrow$ thus, GR is Non-Abelian Gauge Theory

98] We can write:

$$\nabla_a \nabla_b v^k = \partial_a (\nabla_b v^k) + \Gamma^k_{la} \nabla_b v^l - \underbrace{\Gamma^l_{ba} \nabla_l v^k}_{\text{this term is symmetric in } a \leftrightarrow b}$$

(and so when we subtract $\nabla_a \nabla_b (\cdots)$ and $\nabla_b \nabla_a (\cdots)$ it will cancel out)

$$* \Rightarrow \nabla_a \nabla_b v^k = \underbrace{\partial_a \partial_b v^k}_{\text{symmetric in } a \leftrightarrow b} + \partial_a (\Gamma^k_{pb} v^p) + \Gamma^k_{la} \partial_b v^l + \Gamma^k_{la} \Gamma^l_{pb} v^p - (\cdots)$$

opening this gives this & another term which is symmetric with

symmetric in $a \leftrightarrow b$

$$= \underbrace{\{a \leftrightarrow b\}}_{\text{all symmetric terms}} + (\partial_a \Gamma^k_{pb}) v^p + \Gamma^k_{la} \Gamma^l_{pb} v^p$$

Using this we can obtain:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) v^k = (\partial_a \Gamma^k_{pb} - \partial_b \Gamma^k_{pa} + \Gamma^k_{la} \Gamma^l_{pb} - \Gamma^k_{lb} \Gamma^l_{pa}) v^p$$

$$= R^k_{\ pab} v^p$$

Hence:

$$\boxed{R^k_{\ pab} \equiv \partial_a \Gamma^k_{pb} - \partial_b \Gamma^k_{pa} + \Gamma^k_{aq} \Gamma^q_{bp} - \Gamma^k_{bq} \Gamma^q_{ap}}$$

* This is called the RIEMANN CURVATURE TENSOR

{ from here we can see that it is a tensor

↓

as the LHS $(\nabla_a \nabla_b - \nabla_b \nabla_a) v^k$ is a tensor

and on RHS $v^p$ is a 4-vector

$\Rightarrow \therefore R^k_{\ pab}$ must be a 4-index tensor }

* NOTE: $R^a{}_{bcd}$ : it is antisymmetric in $c \leftrightarrow d$

$$R_{abcd} = g_{ak} R^k{}_{bcd}$$

↳ this is antisymmetric in $a \leftrightarrow b$

also, $ab \leftrightarrow cd$ are symmetric

* NOTE: The no. of components of $R_{abcd}$ in $N$ dimension is:

$$\frac{N^2(N^2-1)}{12}$$

NOTE: $R^k{}_{pab}$ helps describe a genuinely curved spacetime

↓

if $R^k{}_{pab} = 0 \Rightarrow$ then spacetime is genuinely flat

↓

But for genuinely curved spacetime, there is NO WAY to make $R^k{}_{pab}$ zero

NOTE: For a spacetime with constant curvature (eg: a sphere) $\Rightarrow$ the expression for $R$ becomes much simpler

(we can describe it directly in terms of $g$ + the derivatives of $\Gamma$ won't be needed)

{ it might have been the case that the early universe had a near constant curvature

↓

this is described in terms of the spectral index (where spectral index of 1 means constant curvature)

⇓

the value of spectral index for early universe was $\sim 0.96$ (the deviation of course is relevant, otherwise things would have been quite boring!) }

NOTE: At infinite distances away (i.e. $r \to \infty$) $\Rightarrow$ the value of $R \to 0$ (the spacetime becomes flat at infinity)

{ in general $R$ somewhat depends on: $1/r^6$ }

NOTE: In general we set up Lagrangians in classical mechanics as

$$\mathcal{L}(q, \dot{q})$$

↓

since the eq$^n$.s of motion from this have second-order which follows what we observe in nature

↓

(& thus $\ddot{q}$ in $\mathcal{L}$ is not required) ↗ as that would give third order E.O.M.

However, it turns out that there exists a special class of Lagrangians where $\mathcal{L}(q, \dot{q}, \ddot{q})$ still gives second order E.O.M. only

↓

* It turns out, in GR the Lagrangian is of such a kind (where the second derivative of metric is taken in Lagrangian)

↓

Einstein chose the simplest kind [by taking the curvature tensor itself ⇝ which is second-derivative of $g$ )

# LECTURE 20

25/09/2023

**NOTE:** We will discuss an intuitive way to approach Curvature ↓ through "Tidal forces"

**NOTE:** Tidal forces ↓ are not direct gravitational forces ⤳ otherwise the Earth should be rotating around the moon

↓ instead they are due to residual forces

(since moon's tidal forces are greater than the Sun)

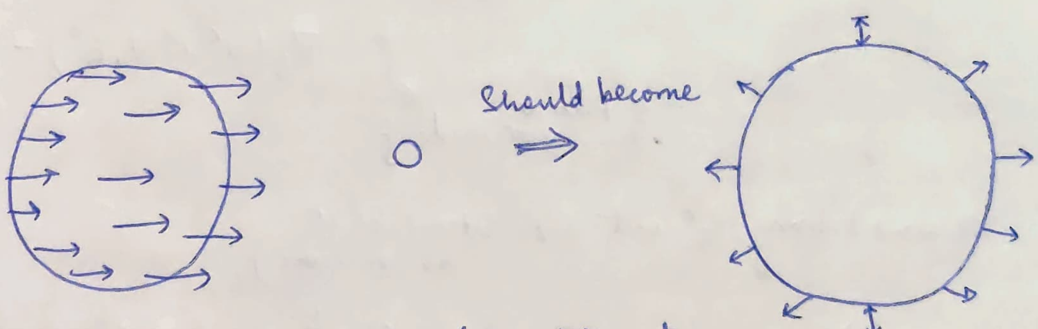
{i.e. related to gradient of the force}

↓

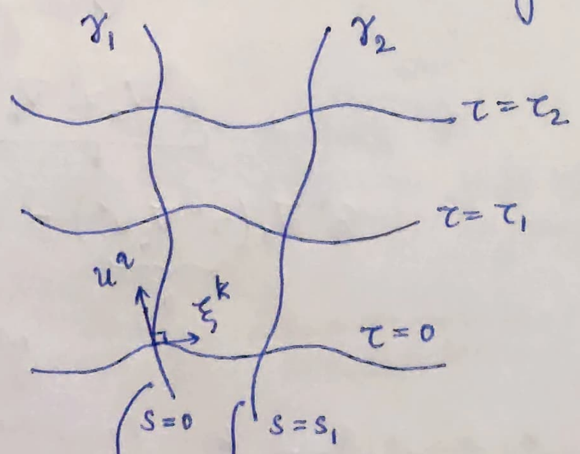
∴ varies by $1/r^3$

99] We can subtract out the "central force" from the overall force at every point

* {Newtonian Picture}




100] Considering two curves (two different geodesics)




two geodesics (they will have different $u^a$)

We take a vector $\xi^k$ $\perp^r$ to $u^a$, and "shoot" it towards the second curve ↓ this acts like the * separation vector

We want to know how the separation b/w the two geodesics changes as we move along the curve (i.e. this is the "tidal force" concept)

We can define $\xi^a$ from: $\mathcal{L}_u \xi^b = 0 = u^a \partial_a \xi^b - \xi^b \partial_a u^b$

("called connecting vector")

i.e. $\Rightarrow$ $\mathcal{L}_u \xi^b = u^a \nabla_a \xi^b - \xi^a \nabla_a u^b$

(this can be done as the $\Gamma$ terms will cancel out)

* Thus, $u^a \nabla_a \xi^b = \xi^a \nabla_a u^b$ — (1)

and also we know: $u^a \nabla_a u^b = 0$

Now, we want to find out:

* $D_\tau^2 \xi^k \equiv u^a \nabla_a (u^b \nabla_b \xi^k)$

From (1):

$$D_\tau^2 \xi^k = u^a \nabla_a (\xi^b \nabla_b u^k)$$

$$= (u^a \nabla_a \xi^b)(\nabla_b u^k) + u^a (\nabla_a \nabla_b u^k) \xi^b$$

$$= u^a (\nabla_a \nabla_b u^k) \xi^b + \underbrace{(u^a \nabla_a \xi^b) \nabla_b u^k}$$

$$\nabla_b (u^k u^a \nabla_a \xi^b) - u^k \nabla_b (u^a \nabla_a \xi^b)$$

{this is NOT reqd.}

We want to bring $\xi^b$ out (to later take it as common):

$$\Rightarrow D_\tau^2 \xi^k = u^a (\nabla_a \nabla_b u^k) \xi^b + \underbrace{u^a \nabla_a \xi^b \nabla_b u^k}$$

$\Downarrow$ change this to:

$$\underbrace{\xi^a \nabla_a u^b \nabla_b u^k}$$

$$\xi^a \underbrace{\nabla_a (u^b \nabla_b u^k)}_{0} - \underbrace{u^b \nabla_a (\nabla_b u^k) \xi^a}$$

interchange $a \leftrightarrow b$ (since they are dummy indices)

$$= u^a (\nabla_a \nabla_b u^k) \xi^b - \xi^b u^a \nabla_b \nabla_a u^k$$

$$= u^a(\nabla_a \nabla_b u^k - \nabla_b \nabla_a u^k)\,\xi^b$$

$$= (\nabla_a \nabla_b - \nabla_b \nabla_a)u^k\,\xi^b u^a$$

Thus, we obtain:

$$\boxed{D^2_\tau \xi^k = R^k_{\ pab}\, u^p u^a \xi^b}$$

* Thus, geodesic separation is captured completely by the Riemann Curvature Tensor
+
and this effect CANNOT be made to vanish

(it will only be zero for a genuinely flat spacetime)

{ this an equation of the form:

$\ddot{Y} + Y = 0$

(where $\xi^k$ acts like $Y$) }

NOTE: We will see in the next lecture:

(we know, $R_{abcd} = g_{ap} R^p_{\ bcd}$)

Now, if we contract indices as:

$$R^c_{\ bcd} = \delta^c_p R^p_{\ bcd} \equiv R_{bd}$$

this is then called the * Ricci Tensor

$$(R^c_{\ bcd} = R^0_{\ b0d} + R^1_{\ b1d} + R^2_{\ b2d} + R^3_{\ b3d})$$

NOTE: Several such scalars (etc.) can be built
↓
eg: $R^{abcd} R_{abcd} = K$ → this is also a scalar called * Kretschmann scalar

NOTE: The purpose of building all these tensors, etc. is to ultimately build the Lagrangian for General Relativity

NOTE: Next we will also come up with a tensor that acts like the stress-energy-momentum tensor of E.M.

{ i.e. $\nabla_a T^a_b = 0$ , and here we should get some: $\nabla_a G^a_b = 0$ }

# LECTURE 21

26/09/2023

101] We would now like to look into some properties of $R^a_{\ bcd}$

where, $R^a_{\ bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ci}\Gamma^i_{bd} - \Gamma^a_{di}\Gamma^i_{bc}$

Now, in the Local inertial frame (LIF):

we have $g \sim \eta$ , $\Gamma \sim 0$

Thus: $\underbrace{R^a_{\ bcd} = \partial_c \Gamma^a_{bd} - \dots}_{\Downarrow \text{ this changes to}}$

$$R_{abcd} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc}$$

(we are considering this, instead of $R^a_{\ bcd}$)

$$= \frac{1}{2}\partial_c\left(-\partial_a g_{bd} + \cancel{\partial_d g_{ba}} + \partial_b g_{ad}\right) - \frac{1}{2}\partial_d\left(-\partial_a g_{bc} + \cancel{\partial_c g_{ba}} + \partial_b g_{ac}\right)$$

$$\therefore R_{abcd} = \frac{1}{2}\left(-\partial_c\partial_a g_{bd} + \partial_d\partial_a g_{bc} + \partial_c\partial_b g_{ad} - \partial_d\partial_b g_{ac}\right)$$

(1) * Now, we know $R_{abcd}$ is antisymmetric in $c \leftrightarrow d$ (by construction {i.e. in the beginning itself we can see this})

From here we can also see:

(2) * $R_{abcd}$ is antisymmetric in $a \leftrightarrow b$

(3) * It is block symmetric in $ab \leftrightarrow cd$

Further, we can also get that:

(4) * $R_{a[bcd]} = 0$

(or, in other words: $R_{abcd} + R_{adbc} + R_{acdb} = 0$ )

Thus, These are the 4 properties of $R_{abcd}$

102] ## # of non-zero index components of $R_{abcd}$.

{NOTE: for this we don't need to take the LIF assumption}

(1) All 4 indices are the same (it will be zero)

eg: $R_{1111}$

(2) 3 indices are the same → this case is also zero

eg: $R_{1112}$

(3) 2 indices are the same

(a) Other two are also the same

{eg: $R_{1212}$}

↓

Here there are $\frac{n(n-1)}{2}$ possible choices

(b) Other two are different

{eg: $R_{1213}$}

↓

for first pair we have '$n$' choices

and the other two have $(n-1)$ and $(n-2)$ choices

↓

Thus we get: $\frac{n(n-1)(n-2)}{2}$ choices

(4) All indices are different

{eg: $R_{1234}$, $R_{1423}$, $R_{1342}$, etc...}

we can write: $3 \times \frac{n(n-1)(n-2)(n-3)}{4!}$

however this will reduce to: 2

{because of:

$R_{abcd} + R_{adbc} + R_{acdb} = 0$

if we fix one → the other 2 are directly dependent on one another

}

(i.e. in total only 2 are independent)

103] ∴ In general we can write:

$$\# = \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{2} + \frac{n(n-1)(n-2)(n-3)}{4\times 3}$$

$$= n^2(n^2-1)/12$$

This is the same no. of choices we had got remaining from our earlier discussion

↓

when we expanded taylor series of $g_{ab}$ and looked at the coefficients

{ thus $R$ specifies all the ways of describing the curvature $(\partial^2 g)$ }

104] We can see that

$R_{abcd}$ ⇒ here
$a \leftrightarrow c$
$b \leftrightarrow d$

We can define from here:

$$\boxed{R_{bd} \equiv \delta^c_a R^a_{\ bcd}}$$

this is a second rank tensor called <u>RICCI TENSOR</u> *

In addition, from here we can also define:

$$\boxed{\delta^b_a R^a_b \equiv R}$$ (i.e. $R = Tr(R_{ab})$ )

this is called the <u>RICCI SCALAR</u> *

<u>NOTE</u>: We can see that

| $n$ | # |
|---|---|
| 1 | 0 |
| 2 | 1 |

# ⇝ no. of non-zero components of $R_{abcd}$

1 | 0 → i.e. No gravity in one-dimensional space

2 | 1 → i.e. Gravity is trivial

(hence only in 3 and >3 dimensional spacetime we have actual proper components that give gravity )

105] <u>Bianchi Identity</u>

In Electrodynamics we saw:

$$\partial_a {}^{(*)}F^{ab} = 0$$

↓ i.e.

$$\partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} = 0$$

Similarly it turns out that here also we have the following:

$$* \quad \boxed{\nabla_i R^a{}_{bcd} + \nabla_d R^a{}_{bic} + \nabla_c R^a{}_{bdi} = 0}$$

The BIANCHI IDENTITY

**Proof:**

$\Big\{$ we can see that in LIF:

$$R \sim \partial\Gamma + \Gamma^2, \quad \nabla R \sim \partial^2\Gamma + 2\Gamma\partial\Gamma$$

$$\nabla \sim \partial \Big\}$$

In local inertial frame:

$$\partial_i(\partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc}) + \partial_d(\partial_i\Gamma^a_{bc} - \partial_c\Gamma^a_{bi})$$
$$+ \partial_c(\partial_d\Gamma^a_{bi} - \partial_i\Gamma^a_{bd}) = 0$$

{i.e. all terms on LHS cancel out}

106] **Contracted Bianchi Identity**

Taking the Bianchi Identity

↓

Contract $a$ and $i$:

$$\nabla_a R^{ab}{}_{cd} + \nabla_d R^{ab}{}_{ac} + \nabla_c R^{ab}{}_{da} = 0$$

↓

now, contract $b$ and $d$:

$$\nabla_a R^{ab}{}_{cb} + \nabla_b R^{ab}{}_{ac} + \nabla_c R^{ab}{}_{ba} = 0$$

{* Now, remember that: $\delta^c_a R^{ab}{}_{cd} = R^b{}_d$ and $\delta^d_b R^b{}_d = R$ }

$\Downarrow$ hence this can be written as:

$$\underbrace{2\nabla_a R^a{}_c}_{\text{from first two terms}} + (\nabla_c R^{ab}{}_{da})$$

↓ we should flip this and put a (−) sign

$$-\nabla_c R^b{}_d$$

↓ contracting $b$ and $d$

$$-\nabla_a \delta^a_c R$$

$$\Rightarrow \underbrace{2\nabla_a R^a{}_c - \nabla_a \delta^a_c R = 0}$$

from this
we can define the following:

$$\boxed{G^a_b \equiv R^a_b - \frac{1}{2}\delta^a_b R}$$

called the
EINSTEIN TENSOR
*

or

$$\boxed{G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R}$$

And, from the expression we just obtained previously,
we can thus write:

$$\boxed{\nabla_a G^a_b = 0}$$

called the
CONTRACTED
BIANCHI IDENTITY (C.B.I.)
*

{ NOTE: Later on we will see
that the effect of curvature
will be directly related with
the Stress-energy-momentum tensor as:

$$G^a_b = \kappa T^a_b$$

which gives us:

$$R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab}$$

(where $\kappa = \frac{8\pi G}{c^4}$) }

# LECTURE 22

29/09/2023

NOTE: Previously, we have defined the Curvature Tensors:

* $R^a{}_{bcd}$
* $R_{bd} = R^a{}_{bad} = \delta^c_a R^a{}_{bcd} = g^{ac} R_{abcd}$
* $R = g^{bd} R_{bd} = \delta^a_b R^b_a$

$\downarrow$

the Ricci scalar is linear in R

$\downarrow$

Similarly there can be other scalars:

eg: $R_{ab} R^{ab}$ ⇝ this is quadratic in R

* $R^{abcd} R_{abcd}$ ⇝ called the Kretschmann scalar (K)
* $\epsilon^{abcd} R_{abcd} = 0$

NOTE: In addition to this we saw the Bianchi Identity for which we had:

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \quad ; \quad \nabla_a G^a_b = 0$$

107] * $G_{ab}$ is NOT unique

$\downarrow$

i.e. we can add something to $G_{ab}$ such that $\nabla_a G^a_b$ is still zero

$\downarrow$

Since we know that $\nabla g = 0$ thus we can also add a term $\Lambda g_{ab}$ such that:

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R + \underbrace{\Lambda g_{ab}}_{} \quad ; \quad \nabla_a G^a_b = 0$$

this term contains $\Lambda$ which is called the "Cosmological Constant" *

Originally Einstein added this term to prevent an expanding universe

{the constant can be chosen, since it is not given by the theory itself}

hence, as the universe was presumed to be static at the time, this term was added

such a universe is called the "Einstein-De Sitter universe" *

Later on when it was found
out that the Universe
is actually expanding (by Hubble)

$\downarrow$

the $\Lambda g_{ab}$ term was removed

$\downarrow$

However, later on it was added back due to a "bigger" issue

$\Downarrow$

We essentially want an equation saying:

$$\underbrace{\text{Matter} = \text{Geometry}}$$

$\updownarrow$

$$K\,T_{ab} = G_{ab}$$

Now, instead of taking the entire expression to be $G_{ab}$, we can still exclude $\Lambda g_{ab}$ from it

$$\underbrace{R_{ab} - \frac{1}{2} g_{ab}}_{\substack{\text{we can} \\ \text{just call this} \\ G_{ab}}} + \Lambda g_{ab} = K\,T_{ab}$$

this can be taken to the other side

(and subtract it from $T_{ab}$)

<u>NOTE</u>: To explain the present accelerated universe, the $\Lambda$ is taken such that:

$$\Lambda L_p^2 \sim 10^{-122}$$

(where $L_p$ is the planck length $\{L_p \sim 10^{-34}\}$)

Also, since $\Lambda$ is constant the acceleration of the universe is also constant {which is what is observed as well}

* <u>NOTE</u>: For now (as we are not doing Cosmology) we will set $\Lambda = 0$

109] We wanted to find a generalization for the Newtonian equation:

$$\underbrace{\nabla^2 \phi}_{\substack{\text{this} \\ \text{comes from} \\ \text{geometry (curvature)}}} = \underbrace{4\pi G_N \rho}_{\substack{\text{this comes from} \\ \text{energy-momentum} \\ \text{tensor}}}$$

$$\Rightarrow \boxed{G_{ab} = \kappa T_{ab}}$$ called the Einstein field Equations

110] In electrodynamics we saw the action:

$$\mathcal{A} = \underbrace{-m\int d\tau}_{\mathcal{A}_{\text{particle (pl.)}}} + \underbrace{q\int A_k\, dx^k}_{\mathcal{A}_{\text{pl-EM interaction}}} \overbrace{-\frac{1}{4}\int F_{ab}F^{ab}\sqrt{-g}\, d^4x}^{\mathcal{A}^{\text{free}}_{\text{EM}}} + \underbrace{\int d^4x \sqrt{-g}\, A_k J^k}_{\mathcal{A}_{\text{Source-EM}}}$$

{ when we varied it:

w.r.t. $x^k(\tau) \xrightarrow[\text{that gave}]{}$ Lorentz Force

w.r.t. $A_k(x) \xrightarrow[\text{that gave}]{}$ Maxwell's Eq$^n$s }

⤷ i.e. $\nabla_k F^{kl} = 4\pi J^l$

The same should be true for gravity

↓

i.e. we can write the action in the form:

$$\mathcal{A} = \mathcal{A}_{GR} + \mathcal{A}_{\text{Source}}$$

In EM we had:

$$\mathcal{A}_{\text{EM}} = \mathcal{A}^{\text{free}}_{\text{EM}} + \mathcal{A}_{\text{source}}$$

(and the dynamical variable was $A_k(x)$ )

↓

we wrote $\mathcal{L} = \mathcal{L}(A, \partial A)$

⇓

but from Gauge Invariance we saw that this should just be: $\mathcal{L}(A)$

Similarly here: ⟸

$\mathcal{L}(g, \partial g)$

↓

But in local inertial frame we know $g \to \eta$ and $\partial g \to 0$

$\therefore \Rightarrow$ we should take: $\mathcal{L}(g, \partial g, \partial^2 g)$ ⇝ this will now give true gravity

Now, for lagrangians of the form

$$\mathcal{L}(q, \dot{q}, \ddot{q}) \; ; \; p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$\Downarrow$$

$$\delta \mathcal{L}(q, \dot{q}, \ddot{q}) = \underbrace{\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q}}_{\left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right]\delta q} + \underbrace{\frac{\partial \mathcal{L}}{\partial \ddot{q}}\delta \ddot{q}}_{\frac{\partial \mathcal{L}}{\partial \ddot{q}}\frac{d^2}{dt^2}(\delta q)}$$

which usually should give a third-order E.O.M. $\Leftarrow$

Thus here the $\dot{q} \underset{\text{gives}}{\rightsquigarrow}$ momentum

and $\ddot{q} \underset{\text{gives}}{\rightsquigarrow} -\nabla\phi$ (i.e. the acceleration)

* However, there are certain lagrangians for which this $\ddot{q}$ term vanishes

$\downarrow$

thus it still gives second-order Eq. of motion only

111] We can write:

$$\mathcal{A} = \mathcal{A}_{GR} + \mathcal{A}_{source}$$

Now, $\mathcal{A}_{GR}$ can be written as:

$$\mathcal{A}_{GR} = \int d^4x \sqrt{-g}\, \mathcal{L}(g, \partial g, \partial^2 g)$$

And, the standard lagrangian used here is:

$$\mathcal{A}_{GR} = \underbrace{\frac{1}{2K}\int d^4x \sqrt{-g}\, R}_{\text{this is the Einstein–Hilbert action}}$$

*

Now, if we write the full action:

$$\mathcal{A} = \mathcal{A}_{GR} + \mathcal{A}_{source}$$

$$= \int d^4x \sqrt{-g}\, \mathcal{L}(g, \partial g, \partial^2 g) + \int d^4x \sqrt{-g}\, \mathcal{L}_m$$

matter lagrangian

Thus: $\mathcal{A}^{GR} = \underbrace{\frac{1}{2K}\int d^4x \sqrt{-g}\, R}_{\text{for free gravity}} + \underbrace{\int d^4x \sqrt{-g}\, \mathcal{L}_m(\Phi, \partial\Phi)}_{\text{source}}$

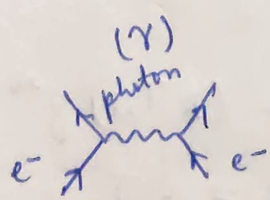
this lagrangian takes into account all matter, energy and momentum

{* NOTE: $\Phi$ here is NOT a scalar field }

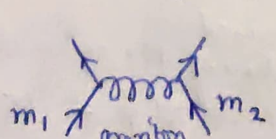
* [hence even light will be taken into account within $\mathcal{L}_m$
↓
leading to concept of bending of light ]

NOTE: Historically, Hilbert came up with this action 2 weeks before Einstein
(Einstein also discovered it independently)
↓
However, Einstein had already found 5 wrong versions of $G_{ab} = K T_{ab}$ before Hilbert
↓
and Hilbert's work was thus also inspired from Einstein.

NOTE: In Field theory, we develop ideas based on the physics we observe
↓
for eg: The Schrödinger eq$^n$. is Non-relativistic → thus the Dirac eq$^n$. was developed which was relativistic
↓
But this did not apply to $e^-$s * (as it didn't take into account spin) ⇝ { it did apply to pi-mesons though }
↓
Later QED (quantum electrodynamics) was developed which explained that $e^-$s interact & this interaction is mediated by the electromagnetic field {and the particles were photons which mediated this}




A similar idea may be thought for gravity ↳ where we start with the field theory idea that mass particles interact through some gravity particle (say, called "graviton")




* However, the problem with such theories is that <u>none of them are able to obtain the Lagrangian for the Einstein-Hilbert action</u>

112] In Electrodynamics $\rightsquigarrow$ when we work in free space (i.e. No source)

$\Downarrow$

we get Electromagnetic waves (i.e. light)

Similary, we will see the same for gravity

+

in source free case : $T_{ab} = 0$ , $T = 0$ {and thus $G_{ab} = 0$}

$\Downarrow$

and this also implies that

$R = 0$ and $R_{ab} = 0$

{ NOTE: For any tensor we can write:

$$T^a_{\ b} = T^a_{\ b}\,(\text{trace free}) + \frac{1}{4}\delta^a_{\ b} T \}$$

(i.e. breaking into trace-free and trace parts )

* However, here $R_{abcd} \neq 0$

$\Downarrow$

i.e. we can break $R_{abcd}$ into the part $R_{ab}$ and some other portion $C_{abcd}$ $\rightsquigarrow$ which is going to still cause gravity

$\Downarrow$

this $C_{abcd}$ is called the Weyl Tensor

*

* NOTE: We can write :

$$G_{ab} = \kappa T_{ab}$$

$$\Rightarrow g^{ab}\left(R_{ab} - \frac{1}{2} g_{ab} R\right) = (\kappa T_{ab})\, g^{ab}$$

$$\Rightarrow R - \frac{1}{2} R\,(4) = \kappa T$$

$$\Rightarrow \boxed{R = -\kappa T}$$

* NOTE: The value of $\kappa$ doesn't lie within the theory itself $\rightsquigarrow$ it is obtained by reducing the eq$^n$s down to $\nabla^2\phi = 4\pi G_N \rho$ form

$\Downarrow$

We then see : $\frac{\kappa}{2} = 4\pi G \rightarrow \boxed{\kappa = 8\pi G}$