LECTURE 29

136] **Black Holes**

We know the schwarzschild metric is given by:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\Omega^2$$

which can be written as:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

$\left\{ \text{where, } f(r) = 1-\frac{2M}{r} \right\}$

NOTE: for $f(r=a)=0$

$\downarrow$

$\underbrace{a = r_s}_{} = 2M$ for the schwarzschild metric

$\downarrow$

this however is a "simple zero" $\Rightarrow$ $f'(a) \neq 0$ and finite

$\downarrow$

there can thus be several solutions to $f(r)=0$

137] We know that:

$\downarrow$

* at $r = r_s = 2M$, $g_{rr} \to \infty$

Also, for the case of:

$r < 2M \Rightarrow$ we see that the sign flips

i.e. here: $\left\{ \begin{array}{l} t \rightsquigarrow \text{spacelike} \\ r \rightsquigarrow \text{timelike} \end{array} \right.$

$\Downarrow$

* Thus, the metric is NO LONGER static

138] Since we see that the metric blows up (at $r = r_s$)

$\downarrow$

we can ask, is it a singular point?

* In reality, the singularity is real only if the curvature blows up $\Rightarrow$ i.e. singularity ~ curvature blowing up

{this is essentially the definition for singularity}

139] Consider the part of the metric:

$$dl^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

Taking $\mu = \sin\theta \Rightarrow d\mu = \cos\theta \, d\theta$

Hence:

$$dl^2 = \frac{d\mu^2}{1-\mu^2} + \mu^2 d\phi^2$$

here we can see that at $\mu = \pm 1$ the metric is blowing up!

{ This thus proves our point that the blowing up of the metric doesn't give us a real singularity

↓

it can change depending upon the coord. system used }

140] Now, computing the curvature for the Schwarzschild metric

↓

we can make use of:

$$\underbrace{R^{abcd} R_{abcd}}_{\text{this is also independent of choice of coordinates}} = \text{finite at } r = 2M$$

showing that there is no singularity here

↓

However we find out that $R^{abcd} R_{abcd}$ blows up at $r = 0$

↓

* Thus $r = 0$ is a real singularity

{NOTE: We get that:

$$R^{abcd} R_{abcd} = \frac{48M^2}{r^6}$$ }

141] Now, since we know that $r = 2M$ is NOT an actual singularity

↓

this thus means that there must be a coord-system where $r = 2M$ is a regular point

The original Schwarzschild metric is:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

Now, considering the metric $\underline{\text{near } r = a}$: $\{a = r_s = 2M\}$

we can write : $f(r) \approx \cancel{f(a)}^{\to 0} + f(r-a)\underbrace{f'(a)}_{2K \text{ (say)}}$

(using Taylor expansion)

Now, by defining :

$$l = r - a$$

we get : $f(r) \approx 2Kl$

Thus, the metric becomes :

$$ds^2 \approx -2Kl\,dt^2 + \frac{1}{2Kl}\,dl^2 + \underbrace{dL_\perp^2}$$

this is simply the metric for $t = $ const. & $r = $ const. surface

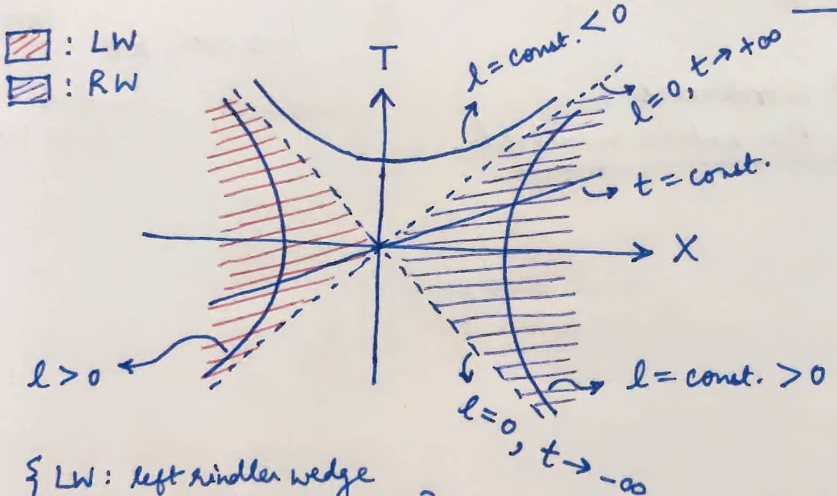
$\Downarrow$

this is the

* $\underline{\text{Rindler Metric}}$!

{Just as we have seen earlier}

it has a horizon * $\underline{\text{at } l = 0}$



{LW : left rindler wedge
RW : right rindler wedge}

$\Longrightarrow$ * From here we can see that the t, r coordinates are $\underline{\text{covering only a part of the spacetime}}$

$\therefore$ there exist other coordinates which cover the whole spacetime

142] We know that, in a flat spacetime :

(T, X) are coordinates given to an event by the global observer

and (t, l) are coord.s given by the accelerated observer

{NOTE : $K \to x$-direction}

From this we have seen that the transformations are given as follows, For region $|X| > |T|$: $\begin{cases} \kappa T = \sqrt{2\kappa \ell}\ \sinh(\kappa t) \\ \kappa X = \pm\sqrt{2\kappa\ell}\ \cosh(\kappa t) \end{cases}$

+ve sign for right wedge R $(X > 0)$

–ve sign for left wedge L $(X < 0)$

For region $|X| < |T|$: $\begin{cases} \kappa T = \pm\sqrt{-2\kappa\ell}\ \cosh(\kappa t) \\ \kappa X = \sqrt{-2\kappa\ell}\ \sinh(\kappa t) \end{cases}$

{here $\ell < 0$}

+ve sign for future light cone F

–ve sign for past light cone P

From here we can thus obtain the inverse transformations :

$$\ell = \frac{1}{2}\kappa\,(X^2 - T^2) \quad ; \quad \frac{T}{X} = \begin{cases} \pm \tanh \kappa t & \text{for } |X| > |T| \\ \pm \coth \kappa t & \text{for } |X| < |T| \end{cases}$$

These inertial coordinates <u>cover the entire manifold</u>.

LECTURE 30

143] { NOTE: Similarly (as we saw in the previous lecture) we can obtain the Kruskal- Szekeres coordinates }

We have:

$$ds^2 = -2\kappa l \, dt^2 + \frac{dl^2}{2\kappa l} + \underbrace{\cdots}_{\text{ignoring this part of the metric}}$$

(since we will leave it unchanged till the end)

$$\{ \text{for } -\infty < t, l < \infty \}$$

We can define:

$$x = \int \frac{dl}{\sqrt{2\kappa l}} = \frac{2}{\sqrt{2\kappa}} \sqrt{l} = \sqrt{\frac{2l}{\kappa}}$$

Thus, this implies $\Rightarrow \kappa^2 x^2 = 2\kappa l$

Hence:

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2 = -dT^2 + dX^2$$

NOTE: Previously we have seen:

$$X^2 - T^2 = \frac{2l}{\kappa} \quad ; \quad \frac{T}{X} = \tanh \kappa t$$

(for both $l<0$ and $l>0$)

i.e.

$$\kappa X = \sqrt{2\kappa l} \cosh \kappa t$$

$$\kappa T = \sqrt{2\kappa l} \sinh \kappa t$$

{alternatively $\sqrt{-2\kappa l}$ could also be used }

which can also be written as:

$$\kappa X = \kappa x \cosh \kappa t$$

$$\kappa T = \kappa x \sinh \kappa t$$

144] We have the metric in the form:

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)}$$

$$= f(r)\left(-dt^2 + dr_*^2\right) \quad -①$$

where, $dr_* \equiv \frac{dr}{f}$

In other words: $$r_* = \int \frac{dr}{f(r)}$$

which can be written also as : $$r_* = \int \frac{dl}{f(l)}$$

(be replacing $r \leftrightarrow l$ [variable of integration])

Now, we know that: $l = r - a$

and also

<u>near the horizon</u> : $f(r) \approx 2Kl$

$$\Rightarrow f(r) \approx 2K(r-a)$$

Thus : $f(l) \approx 2K(l-a)$

$$\downarrow$$

Hence: $r_* = \frac{1}{2K} \ln(2Kl)$

{ NOTE: We choose the integration constant such that we get this particular form }

Rearranging this gives us:

$$f(r) \approx \exp(2Kr_*)$$

Putting this into (1) gives us the metric as :

$$ds^2 = \underbrace{\exp(2Kr_*)}_{\substack{\text{this can be called} \\ \text{some function of } r_* \\ f(r_*)}} (-dt^2 + dr_*^2)$$

[NOTE: Here $f(r_*)$ is NOT the same function as $f(r)$]

145] We can use the following substitution:

$$\left.\begin{aligned} u &= t - r_* \\ v &= t + r_* \end{aligned}\right\} \Rightarrow \therefore\ du\,dv = dt^2 - dr_*^2$$

and

$$K(v-u) = 2Kr_*$$

Thus we get:

$$ds^2 = -\exp(K(v-u))\,du\,dv$$ $\rightsquigarrow$ <u>near the horizon</u>

or in general : $ds^2 = -\underbrace{f(v-u)}_{\text{some function}}\,du\,dv$

Now, we define:

$$\kappa V \equiv \exp(\kappa v) \qquad \kappa U \equiv -\exp(-\kappa u) \Bigg\} \Rightarrow \kappa\, dV = \kappa e^{\kappa v} dv \qquad \kappa\, dU = \kappa e^{-\kappa u} du$$

$$\Downarrow$$

$$\therefore \kappa^2 UV = -e^{\kappa(v-u)}$$

Multiplying them we get:

$$\kappa^2 dU\, dV = \kappa^2 e^{\kappa(v-u)} du\, dv$$

$$\Downarrow$$

Thus we can write:

$$dU\, dV = -\kappa^2 UV\, du\, dv$$

Hence this gives us:

$$du\, dv = -\frac{dU\, dV}{\kappa^2 UV}$$

Also the function $f(v-u)$ will then change to some other function $f(U,V)$

This gives us that:

$$ds^2 = -f(v-u)\, du\, dv = \frac{f(U,V)}{\kappa^2 UV} dU\, dV$$

[NOTE: $f(v-u)$ and $f(U,V)$ are NOT the same functions]

{NOTE: Also since near the horizon: $f(v-u) \approx \exp(\kappa(v-u))$
$$= -\kappa^2 UV$$
$$\Rightarrow \frac{f}{UV} \text{ is finite at the horizon}$$
}

146] Finally using:

$$U \equiv T - X$$
$$V \equiv T + X$$

we get the metric as:

$$ds^2 = \frac{f}{\kappa^2(X^2 - T^2)}(-dT^2 + dX^2)$$

where we know:

$$\kappa X = \sqrt{2\kappa l}\, \cosh \kappa t \qquad \kappa T = \sqrt{2\kappa l}\, \sinh \kappa t \Bigg\} \Rightarrow \kappa X = e^{\kappa r_*} \cosh \kappa t \qquad \kappa T = e^{\kappa r_*} \sinh \kappa t$$

and $f(r) \approx 2\kappa l = \exp(2\kappa r_*)$ {for $|X| > |T|$ region}

147] From the Schwarzschild metric we know that:

$$f(r) = 1 - \frac{2M}{r} \quad \text{and thus} \quad a = r_s = 2M$$

Using this we can get:

$$r_* = \int \frac{dr}{f(r)} = \int \frac{r\,dr}{r-2M} = r + 2M \ln\left|\frac{r}{2M} - 1\right|$$

or

$$r + r_s \ln\left|\frac{r}{r_s} - 1\right|$$

Also, since we know that we defined: $f'(a) = 2K$

$$\Rightarrow K = \frac{1}{2} f'(2M) = \frac{1}{4M} \quad \text{or} \quad \frac{1}{2r_s}$$

148] From the values of r_* and K, we can find:

$$\exp(K r_*) = e^{\frac{1}{2r_s}\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right)}$$

$$= e^{r/2r_s} \left|\frac{r}{r_s} - 1\right|^{1/2}$$

149] We now have the coordinate transformation from (t, r) to (T, X) which are called the KRUSKAL-SZEKERES COORDINATES:

$$\text{for } r > r_s \quad \begin{cases} X = \left(\frac{r}{r_s} - 1\right)^{1/2} e^{r/2r_s} \cosh(t/2r_s) \\ T = \left(\frac{r}{r_s} - 1\right)^{1/2} e^{r/2r_s} \sinh(t/2r_s) \end{cases}$$

$$\text{for } r < r_s \quad \begin{cases} X = \left(1 - \frac{r}{r_s}\right)^{1/2} e^{r/2r_s} \sinh(t/2r_s) \\ T = \left(1 - \frac{r}{r_s}\right)^{1/2} e^{r/2r_s} \cosh(t/2r_s) \end{cases}$$

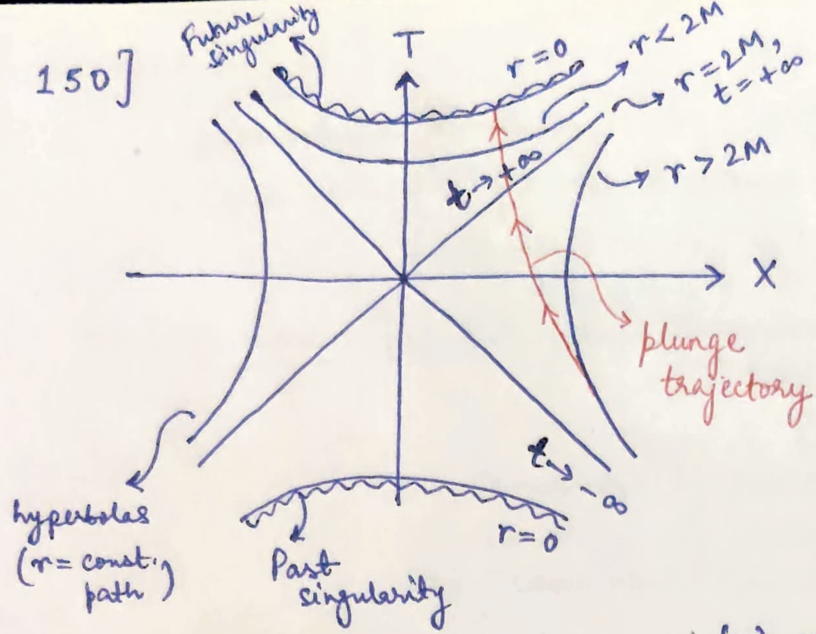
Here, we have:

$$K = 1/2r_s = 1/4M, \quad \text{also} \quad X^2 - T^2 = \left(\frac{r}{r_s} - 1\right) e^{r/r_s}$$

And the metric is finally given by:

$$\boxed{ds^2 = \frac{4r_s^3}{r} e^{-r/r_s} \left(-dT^2 + dX^2\right) + r^2 d\Omega^2}$$

which is finite at $r = r_s$



KRUSKAL DIAGRAM
OF SCHWARZSCHILD SPACETIME
FOR A BLACK HOLE

Here:

(i) Light cones are $45°$ straight lines

(ii) $r =$ const. paths are hyperbolas

(iii) $t =$ const. paths are straight lines through the origin

(iv) $X = T \iff r = r_s$ and $t = +\infty$

(v) $r =$ const., $t =$ const. surfaces are 2-spheres
↓
eg: Horizon has an area of $16\pi M^2$

(vi) The metric is not static!

LECTURE 31, 32

Discussion on Cosmology – (Lec 31)

Presentation on "The Cosmic Distance Ladder" – (Lec 32)

(PPT shared on MS Teams)

* References: (1) Extragalactic Astronomy and Cosmology, by P. Schneider [Chapter 1 and 2] → this is part of syllabus

(2) Cosmology, by David Tong (lecture notes)

(3) Cosmology, by Weinberg

(4) Observational Cosmology, by Sergeant

LECTURE 33 (03/11/2023)

Covered from the book "Extragalactic Astronomy..." by Schneider

Discussed part of Chapter 2

(Starting from 2.2.4 Photometric Distance; extinction and reddening)

NOTE: Apparent magnitude

by convention we use a scale in which 5 points lower implies 100 times brighter

$$\frac{B_2}{B_1} = 100^{\left(\frac{m_1 - m_2}{5}\right)}$$, where brightness: $B = \frac{L}{4\pi d^2}$

$$\Rightarrow m_1 - m_2 = 2.5 \log_{10}\left(\frac{B_2}{B_1}\right)$$ (L: Luminosity)

Also, colour index $= \underbrace{m_B - m_V}_{\text{sometimes this is just written as: } B-V} = 2.5 \log_{10}\left(\frac{S_V}{S_B}\right)$ {using S for brightness here, just for clarity}

* H.W. [Solve this till next class]:

Consider the metric: $$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

(i) Find the Christoffel symbols

(ii) Find G_{ab}

LECTURE 34, 35 {Till the end of Handout - 6}

10/11/2023

Discussed : de Sitter & anti-de Sitter spacetime
de Sitter → ($\Lambda > 0$)
anti-de Sitter → ($\Lambda < 0$)

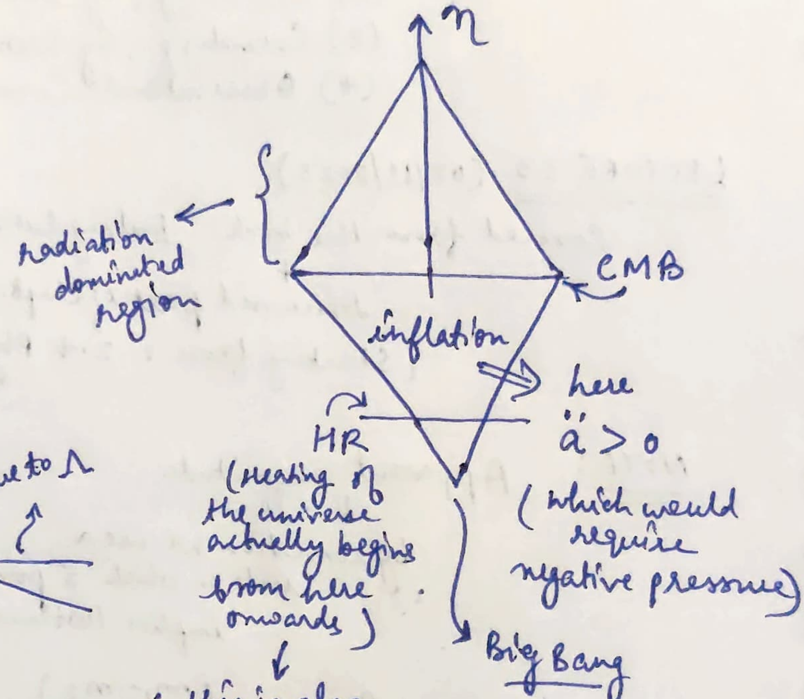
related to the metric we saw earlier:

$$ds^2 = -(1-H^2r^2)\,dt^2 + \frac{dr^2}{1-H^2r^2} + r^2 d\Omega^2$$

i.e. $f(r) = 1 - H^2 r^2$

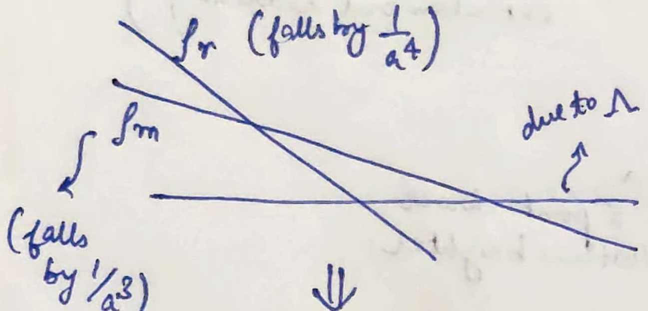
Horizon problem, CMB and inflation → using metric of the form:

$$ds^2 = -a(\eta)\left(d\eta^2 + dx^2\right)$$



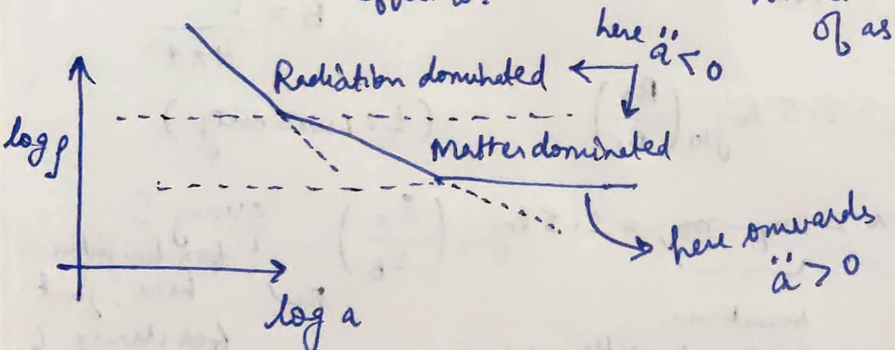
∴ this is also sometimes thought of as the "real" Big Bang

We now currently:

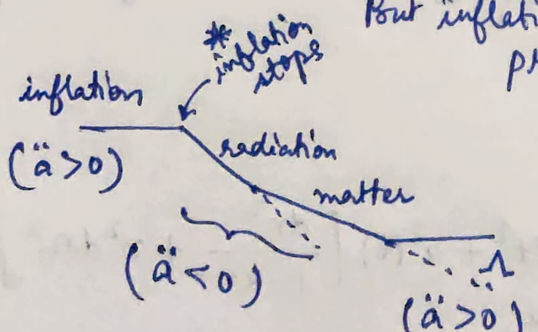


⇓

thus overall effect is:



But inflation model proposes that before the radiation dominated region (before CMB)
↓
there was an inflationary epoch
↓
where $\ddot{a} > 0$ was true again



* NOTE: One problem that is still unsolved by this model is that we currently don't have any way to explain how the inflation would stop at some point