LECTURE 13 (07/03/2024)

69] We have previously discussed coord. transformations (in terms of how a point changes) $\leftarrow$ Origin shift / Change of Basis / Combined change (origin + basis)

$\downarrow$

Now we will look into Symmetry transformation

$(W, \vec{w})$ in Basis $\mathcal{B}$, Origin O

$(W', \vec{w}')$ in Basis $\mathcal{B}'$, Origin O'

So, we want to find some relation of the form:

$$(W', \vec{w}') = f\left[(W, \vec{w})\right]$$

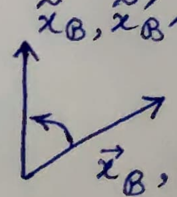
70] Consider that in $\mathcal{B}', O'$ system

$\downarrow$

a vector that was earlier $\vec{x}$ (in $\mathcal{B}, O$) will now be $\vec{x}\,' = (P, \vec{p})^{-1}\,\vec{x}$

$$\begin{array}{ccc} \underline{O,\ \mathcal{B}} & & \underline{O',\ \mathcal{B}'} \end{array}$$

$$\begin{tikzcd}
\vec{x}_{\mathcal{B}} \arrow[r, "{(P, \vec{p})^{-1}}"'] \arrow[d, "{(W, \vec{w})}"'] & \vec{x}\,'_{\mathcal{B}'} \arrow[d, "{(W', \vec{w}')}"] \\
\tilde{x}_{\mathcal{B}} \arrow[r, "{(P, \vec{p})^{-1}}"'] & \tilde{x}'_{\mathcal{B}'}
\end{tikzcd}$$



{Consider $\tilde{x}_{\mathcal{B}}$ and $\tilde{x}'_{\mathcal{B}'}$ as the vectors obtained after applying symmetry operation}

Let's say we try to reach $\tilde{x}'_{\mathcal{B}'}$ from $\vec{x}_{\mathcal{B}}$

$\downarrow$

we can take two paths to achieve this

Now, clearly :: $\tilde{x}'_{\mathcal{B}'} = (W', \vec{w}')\,\vec{x}\,'_{\mathcal{B}'}$

{ To simplify the notation we can just write without using the subscript $\mathcal{B}/\mathcal{B}'$ }

i.e. $\tilde{x}' = (W', \vec{w}')\,\vec{x}'$

$$= (W', \vec{w}')(P, \vec{p})^{-1}\vec{x}$$

Similarly, $\tilde{x}' = (P, \vec{p})^{-1}\tilde{x} = (P, \vec{p})^{-1}(W, \vec{w})\,\vec{x}$

{i.e. $\vec{x}$ is original vector in old basis;
$\tilde{x}'$ is the transformed vector in the new basis}

Equating them gives us:

$$(W', \vec{w}')(P, \vec{p})^{-1} = (P, \vec{p})^{-1}(W, \vec{w})$$

$$\Rightarrow \boxed{(W', \vec{w}') = (P, \vec{p})^{-1}(W, \vec{w})(P, \vec{p})}$$

COORDINATE TRANSFORMATION
OF MATRIX-COLUMN
REPRESENTATION OF A
SYMMETRY OPERATION

71] Simplifying this:

$$(W', \vec{w}') = (P^{-1}, -P^{-1}\vec{p})(W, \vec{w})(P, \vec{p})$$
$$= (P^{-1}W,\ P^{-1}\vec{w} - P^{-1}\vec{p})(P, \vec{p})$$
$$= (P^{-1}WP,\ P^{-1}W\vec{p} + P^{-1}\vec{w} - P^{-1}\vec{p})$$
$$= (P^{-1}WP,\ P^{-1}\{(W-I)\vec{p} + \vec{w}\})$$

If no origin shift:

$$\vec{p} = \vec{0} \Rightarrow \vec{w}' = P^{-1}\vec{w}$$
$$W' = P^{-1}WP$$

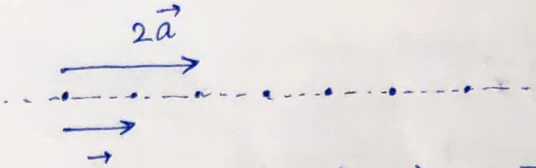
If no basis change:

$$P = I \Rightarrow W' = W$$
$$\vec{w}' = (W-I)\vec{p} + \vec{w}$$

72]    Finite objects

3m 4mm 2mm Point Groups

Infinite objects only, can show translational symmetry *



$\vec{a} + \vec{a} = 2\vec{a}$ (translational vector are being added)

If we represent this operation of translation by $\vec{a}$ as $T_{\vec{a}}$:

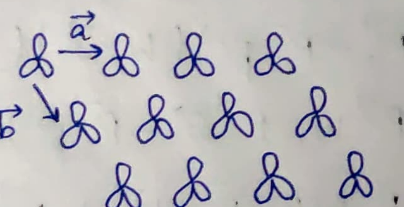
then, $T_{\vec{a}}^2 = T_{\vec{a}} T_{\vec{a}} = \vec{a} + \vec{a}$

73]

Dimensionality of Space	Dimensionality of Translations	Name of Group	No. of Groups
2	1	Frieze Group	7
2	2	Plane Group ("Wallpaper")	17

{ eg: Frieze: A A A A A

Wallpaper:

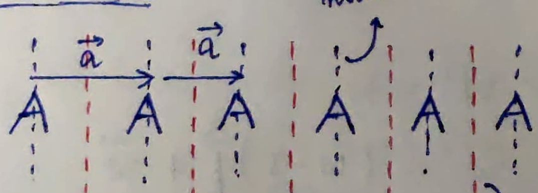


}

3	1	Rod Groups	
3	2	Layer Groups	
3	3	Space Groups	230

74] Frieze Groups

mirrors (in this example)



} mirrors in b/w two As

Mathematically, we can say:

$$T = \{ n\vec{a} \mid n \in \mathbb{Z} \}$$

is the infinite group of translations

* NOTE:
{ the groups we are now discussing involve an infinite no. of elements
↓
since they involve translations }

Thus, we can see, that if we use such an object with Frieze group

$\downarrow$

such that it contains mirror symmetry (mirrors $\perp^r$ to translation direction)

$\downarrow$

then we will observe:

$$\boxed{\text{Spacing of } \perp \text{ mirrors} = \frac{1}{2}\cdot(\text{min}^m\text{ translation}) = \frac{a}{2}}$$

{To be proved in next class}

LECTURE 14 (11/03/2024)

75] Other examples:

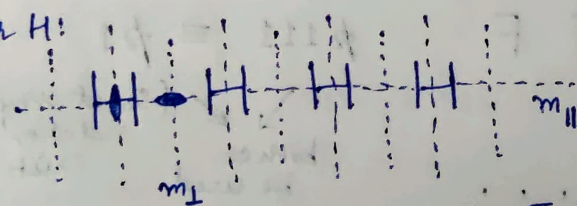
----B----B---B----B----- (mirrors parallel to translation dir^n)

(Similar for C, D and E)

But for F:

F F F F   No symmetry in addition to translation

For H:



Both $\perp^r$ and $\parallel$ mirrors as well as 2-fold at intersections

(Similarly for I and O)

For N:

N N N N   (Only 2-fold independently)

(Similarly for S)

76] Lattice, motif, unit cell and International notation for Frieze Groups

A A A A

these may be chosen as Lattice points

motif

alternatively,

A A A A

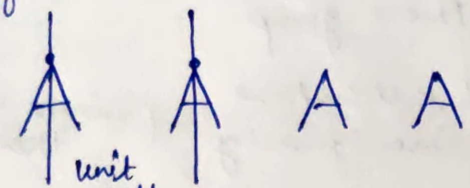
these could also be chosen as lattice points

these two lattices are the same

$\Downarrow$

* Thus, Lattice of a pattern is unique

We can form a unit cell as:



(here, it is also primitive)

$\boxed{p\,1\,m\,1}$

(1-fold ; if there was 2-fold, we use 2)

(if vertical m exists then m, else 1)

(if parallel m then m, else 1)

(IUCr notation for primitive unit cell is a cursive "p") ⇒ Type ① of Frieze *

(i.e international)

* NOTE: Primitive unit cell is that, which does NOT contain any lattice point inside it.

NOTE: In 1D translational symmetry non-primitive unit cells do not exist.

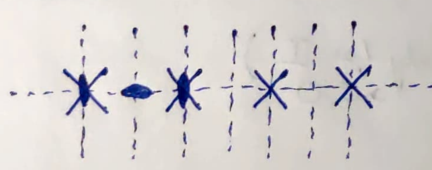
{ NOTE: "vertical m" $\xrightarrow[\text{means}]{}$ $\perp^r$ to translation direction }

eg: ---B---B--B--B--- $p\,1\,1\,m$ (Type ②)

F F F F $p\,1\,1\,1 \equiv p\,1$ (Type ③)

both can be used.

(short simpler notation exists)



$p\,2mm$ (Type ④)

N N N N $p211$ (Type ⑤)

77] <u>Possible symmetry operations in a Frieze:</u>

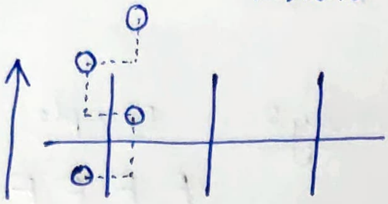
<u>Rotational Symmetry</u>: Only 2 is possible (since the rotations must map the lattice points back to the same line where they were)

• • • • •

<u>Mirror</u>: Only horizontal (∥ to translation) or vertical (⊥ to translation)

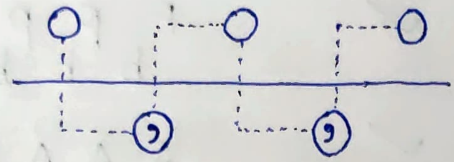
<u>Screw</u>: NOT possible in 2D space (since screw would require translation along rotation axis)

Glide:? Reflection in a mirror + translation // to the m



But Frieze does not has translation in more than 1 direction

Thus, Glide in $\perp m$ is NOT possible.



This appears to be possible in a Frieze

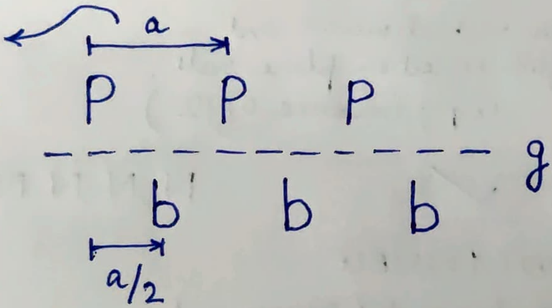
{* NOTE: "ꝰ" symbol is used to denote right-handedness} ↓ as per International notation

{ NOTE: Actually the symbols for glide and mirror differ as:

glide - - - - - -
mirror ———————— }

Translation of Frieze group

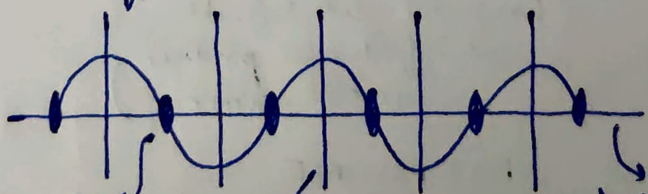
eg:



(equal to the glide translation)

$p11g$ (Type ⑥)

Another last type can be seen as follows:



$p2mg$ (Type ⑦)

78] <u>All possible Frieze groups</u>

$$p \; \frac{}{\begin{matrix}1\\2\end{matrix}} \; \frac{}{\begin{matrix}1\\m_{\perp}\end{matrix}} \; \frac{}{\begin{matrix}1\\m_{\parallel}\\g\end{matrix}}$$

$\therefore$ Possibilities $= 2 \times 2 \times 3$
$= 12$ possibilities

We will now try to understand why we didn't see all 12 :

	Group	Seen previously?	Example
①	$p111 \equiv p1$	✓	F F F F
②	$p11m$	✓	B B B B
③	$p11g$	✓	P b P b P b
④	$p1m1$	✓	A A A A
⑤	$p1mm$	NOT POSSIBLE (since two m's will lead to 2-fold, and not 1)	
⑥	$p1mg$	NOT POSSIBLE (again vertical mirror and glide reflection plane will imply existence of 2)	
⑦	$p211$	✓	N N N N
⑧	$p21m$	NOT POSSIBLE (since horizontal mirror and 2-fold will produce vertical mirror)	
⑨	$p21g$	NOT POSSIBLE (horizontal glide reflection and 2-fold will produce vertical mirror)	
⑩	$p2m1$	NOT POSSIBLE (since vertical mirror and 2-fold will produce horizontal mirror)	

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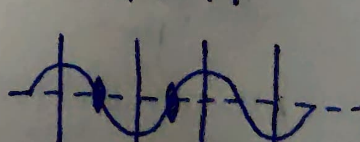
	Group	Seen previously?	Example
⑪	$p2mm$	✓	H H H H
⑫	$p2mg$	✓	

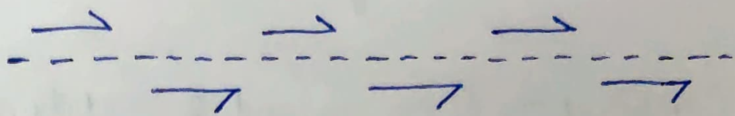
Thus we see there are;

$12 - 5 = \boxed{7 \text{ Frieze groups}}$ (which is what we wanted to prove)

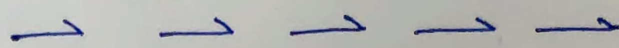
79] Conway notation of Frieze Groups

Given by
John Horton Conway

1. Walk
($p11g$)

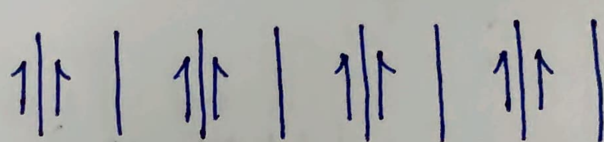


2. Hop
($p111 \equiv p1$)

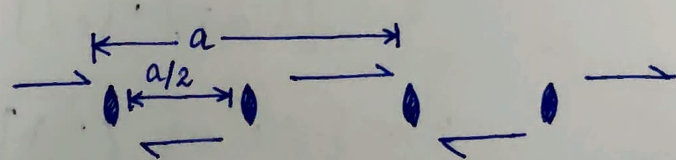


3. Jump
($p11m$)

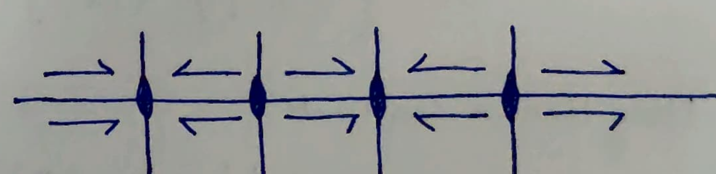
4. Sidle
($p1m1$)



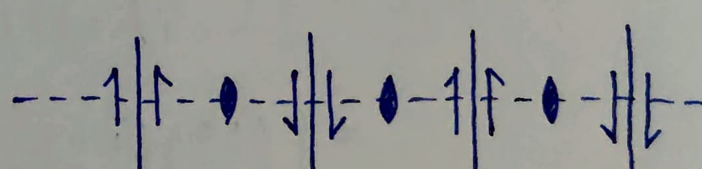
5. Dizzy Hop
($p211$)



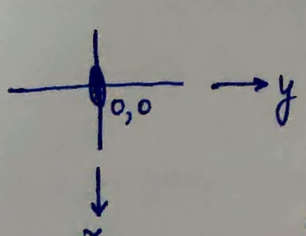
6. Dizzy Jump
($p2mm$)



7. Dizzy Sidle
($p2mg$)



80] Two $\perp$ m result in a 2 at the point of intersection



$m_x = \begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}$, $m_y = \begin{pmatrix} 1 & 0 \\ 0 & \bar{1} \end{pmatrix}$

Thus, $m_x m_y = \begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \bar{1} \end{pmatrix} = \begin{pmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{pmatrix}$

So, $\text{Det}(m_x m_y) = +1 \Rightarrow$ Rotation

Also, $\text{Trace}(m_x m_y) \cdot \text{Det}(m_x m_y) = 2\cos\theta$

$\Rightarrow -2 \times 1 = 2\cos\theta \Rightarrow \cos\theta = -1 \Rightarrow \theta = 180^\circ$

* Hence, this is a 2-fold rotation

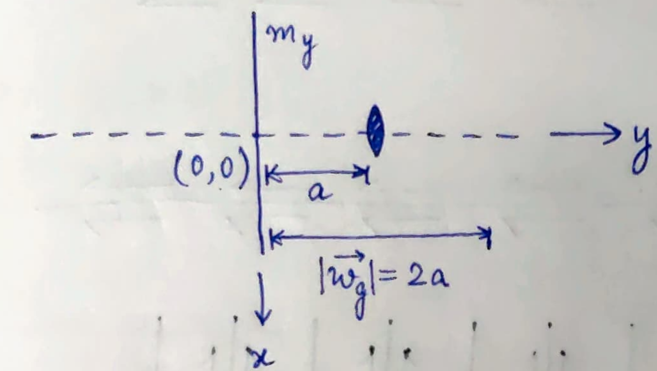
So we get that:

$$m_x m_y = m_y m_x = 2$$

Also, $2 m_x = m_y$

and, $2 m_y = m_x$

81] Combination of an m and 2 not lying on m



$$m_y = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \bar{1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$2 = \left\{ \begin{pmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{pmatrix}, \begin{pmatrix} 0 \\ 2a \end{pmatrix} \right\}$$

Using this we can write:

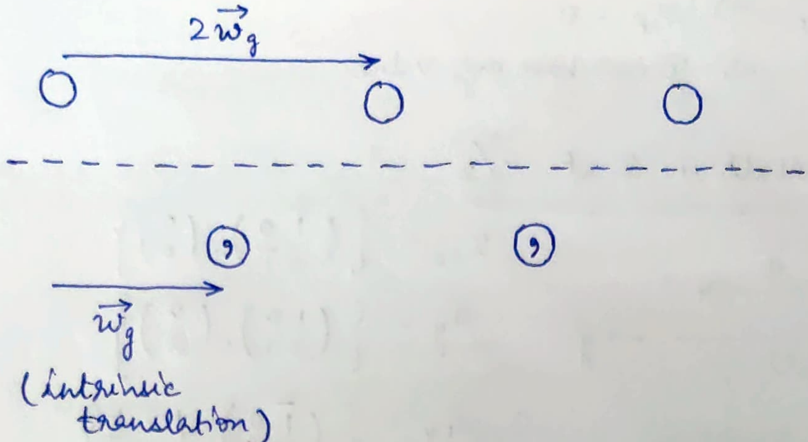
$$m_y 2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \bar{1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \left\{ \begin{pmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{pmatrix}, \begin{pmatrix} 0 \\ 2a \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \bar{1} \end{pmatrix} \begin{pmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \bar{1} \end{pmatrix} \begin{pmatrix} 0 \\ 2a \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}}_{W}, \begin{pmatrix} 0 \\ -2a \end{pmatrix} \right\}$$

Thus, $\mathrm{Det}\left[\begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix} \right] = -1 \Rightarrow$ Mirror, glide

(NOT rotoinversion $\rightarrow$ since that cannot occur in 2D)



Now, $(m_y 2)^2 = \left\{ \begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2a \end{pmatrix} \right\} \left\{ \begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2a \end{pmatrix} \right\}$

$$= \left\{ \begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -2a \end{pmatrix} + \begin{pmatrix} 0 \\ -2a \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 \\ -4a \end{pmatrix}}_{\neq \vec{0}} \right\}$$

$\rightsquigarrow$ hence this is glide *

$$\boxed{\vec{w}_g = \frac{1}{2}\begin{pmatrix} 0 \\ -4a \end{pmatrix} = \begin{pmatrix} 0 \\ -2a \end{pmatrix}}$$

Next we can find $\vec{w}_\ell$:

$$\vec{w}_\ell = \vec{w} - \vec{w}_g$$
$$= \begin{pmatrix} 0 \\ -2a \end{pmatrix} - \begin{pmatrix} 0 \\ -2a \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Reduced operation $(W, \vec{w}_\ell)$

$$\Rightarrow \left\{ \begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

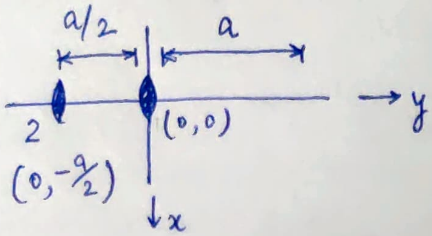
Fixed points of the reduced operation:

$$\begin{pmatrix} x_F \\ y_F \end{pmatrix} = \left\{ \begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \begin{pmatrix} x_F \\ y_F \end{pmatrix} = \begin{pmatrix} -x_F \\ y_F \end{pmatrix}$$

$x_F = -x_F \Rightarrow x_F = 0$

$y_F = y_F \Rightarrow y$ can have any value

82] <u>2 and $\vec{a}$ result in 2 at $\vec{a}/2$</u>



$$2_{00} = \left\{ \begin{pmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$a_y = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix} \right\}$$

$$2_{00}\, a_y = \left\{ \underbrace{\begin{pmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{pmatrix}}_{W}, \begin{pmatrix} 0 \\ -a \end{pmatrix} \right\}$$

$\text{Det } W = +1 \Rightarrow$ Rotation

$2\cos\theta = -2 \Rightarrow \theta = 180° \Rightarrow$ 2-fold rotation

Fixed point:

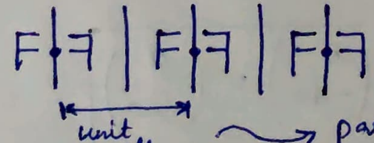
$$\begin{pmatrix} x_F \\ y_F \end{pmatrix} = (2_{00}\, a_y)\begin{pmatrix} x_F \\ y_F \end{pmatrix} = \left\{ \begin{pmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{pmatrix}, \begin{pmatrix} 0 \\ -a \end{pmatrix} \right\} \begin{pmatrix} x_F \\ y_F \end{pmatrix} = \begin{pmatrix} -x_F \\ -y_F - a \end{pmatrix}$$

$$\Rightarrow \left. \begin{array}{l} x_F = -x_F \\ y_F = -a - y_F \end{array} \right\} \Rightarrow x_F = 0,\ y_F = -\frac{a}{2}$$

Hence shown.

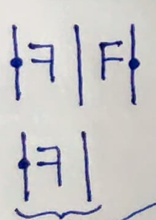
*{NOTE: Similarly, m and $\vec{a}$ result in m at $\vec{a}/2$ }

83] <u>Asymmetric unit</u>:



p1m1 ≡ Sidle

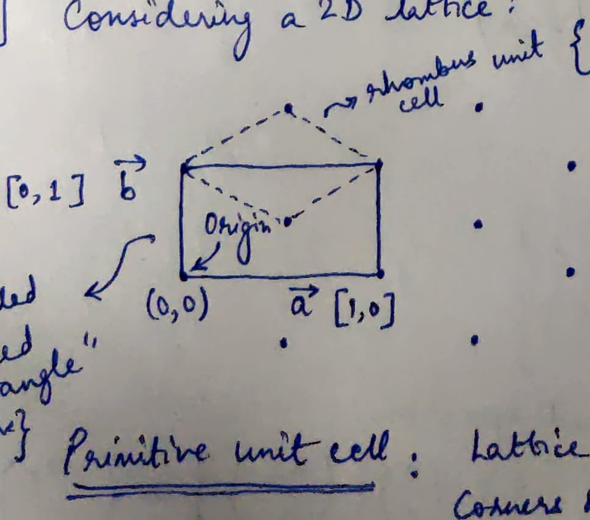
unit cell ⟶ part of the pattern which can generate the whole pattern by TRANSLATIONS *



asymmetric unit ⇒ Smallest part of the pattern which can generate the entire pattern if ALL symmetric operations are applied.

LECTURE 16 (18/03/2024)

84] Considering a 2D lattice:



{this is the primitive unit cell}

⇓

* Here we see that all lattice points can be indexed by <u>integers</u>

<u>Primitive unit cell</u>: Lattice points only at the corners of a parallelogram (2D) or a parallelopiped (3D).

The basis vectors are primitive basis vectors.
All lattice points have <u>integral components</u>.

<u>Non-primitive unit cell</u>: Lattice points at its corners
+
Additional non-corner lattice points

The basis vectors are non-primitive.

Some lattice points will have <u>fractional components</u>.

85] Classification of 2D periodic patterns on the basis of symmetry

↓

Translational periodicity puts restriction on symmetry

↓

There are only 17 different types of symmetry possible

86] Possible rotational symmetry of a plane group

Consider an n-fold centre in the pattern

What are the possible values of n for a plane group?

↓

Let us choose a lattice point and the origin at the n-fold centre

{ <u>NOTE</u>: We use n-fold "centre" instead of n-fold axis, since we are in 2D here }

↓

Let us introduce primitive basis vectors $\vec{a}, \vec{b}$.

{ Now, all lattice points will have integral components. }

Let W represent an n-fold rotation matrix:

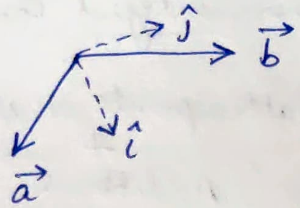
$$\tilde{x} = W \vec{x}$$

(Integers) Image of lattice point after rotation.

Original lattice point (Integers)

$\Longrightarrow$ This is possible only when: <u>W is itself an integer matrix</u> * ⟵ i.e. $W_{ij} \in \mathbb{Z}$

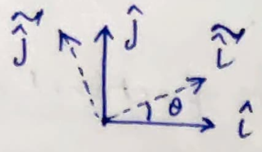
Now, let us use an orthonormal basis $(\hat{i}, \hat{j})$



$\mathcal{B} = (\vec{a}, \vec{b}) \Rightarrow$ Primitive

$C = (\hat{i}, \hat{j}) \Rightarrow$ Orthonormal

Rotations here will be as follows:



$$\tilde{i} = \cos\theta \, \hat{i} + \sin\theta \, \hat{j}$$
$$\tilde{j} = -\sin\theta \, \hat{i} + \cos\theta \, \hat{j}$$

$$W' = \begin{array}{cc} \tilde{i} & \tilde{j} \\ \end{array}$$
$$W' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\text{Trace } W' = W'_{ii} = W'_{11} + W'_{22} = \boxed{2\cos\theta}$$

Now, what will be Trace W ?

{ NOTE: We have seen, the basis transformation of a symmetry matrix:

$$W' = P^{-1} W P = Q W Q^{-1}$$

Where, P is the basis transformation matrix

& Q is the coordinate transformation matrix }

So, we know that:

$$\text{Trace } W' = W'_{ii}$$
$$= (P^{-1} W P)_{ii}$$
$$= P^{-1}_{ij} W_{jk} P_{ki} \quad \{\text{using matrix multiplication}\}$$
$$= P_{ki} P^{-1}_{ij} W_{jk} \quad \{\text{elements of matrices are numbers, and thus are commutative}\}$$
$$= (PP^{-1})_{kj} W_{jk}$$
$$= I_{kj} W_{jk}$$
$$= (IW)_{kk} = W_{kk} = \text{Trace } W$$

Thus, we have obtained that:

$$\text{Trace } W = \text{Trace } W' = 2\cos\theta$$

But then this means:

$$\boxed{2\cos\theta = \text{Integer}}$$

$\Rightarrow 2\cos\theta = k$ (say)

$\Rightarrow -1 \leq \cos\theta = \frac{k}{2} \leq 1$

$\Rightarrow -2 \leq k \leq 2$

k	$\cos\theta = \frac{k}{2}$	θ	$n = \frac{360°}{\theta}$
-2	-1	$180°$	2
-1	$-1/2$	$120°$	3
0	0	$90°$	4
$+1$	$+1/2$	$60°$	6
$+2$	$+1$	$360°$	1

$\Downarrow$

Thus, for any 2D periodic pattern,

* Only 1, 2, 3, 4 and 6-fold rotation axes are possible

{called the Crystallographic Restriction Theorem}

{* **NOTE**: We will later also see that for 3D, the only difference is that:

$$\text{Trace } W = 2\cos\theta + 1$$ }

87] Possible point groups for 2D periodic pattern

Point Group		Operations
1	=	$\{1\}$

Point Group		Operations
2	=	$\{1, 2\}$
3	=	$\{1, 3^+, 3^-\}$
4	=	$\{1, 4^+, 2, 4^-\}$
6	=	$\{1, 6^+, 3^+, 2, 3^-, 6^-\}$

Now, m can also be present as a point group and it can also combine with other rotations.

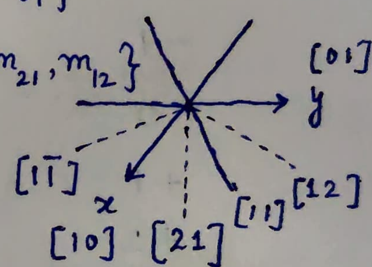
Thus, we have five more point groups:

$1 + m \equiv m = \{1, m\}$

$2 + m \equiv 2mm = \{1, 2, m_{10}, m_{01}\}$

$3 + m \equiv 3m = \{1, 3^+, 3^-, m_{1\bar{1}}, m_{21}, m_{12}\}$

$= \{1, 3^+, 3^-, m_{10}, m_{11}, m_{01}\}$



(mirrors in 3-fold may either be chosen along solid lines, or along the dotted lines)

$4 + m \equiv 4mm = \{1, 4^+, 2, 4^-, m_{10}, m_{01}, m_{11}, m_{\bar{1}1}\}$

$6 + m \equiv 6mm = \{1, 6^+, 3^+, 2, 3^-, 6^-, \underline{m, 6^+m, 3^+m, 2m, 3^-m, 6^-m}\}$

* alternative notation to write the mirrors

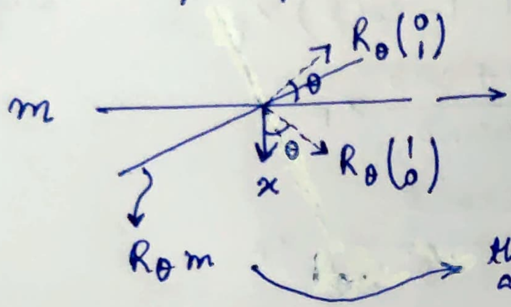
{we could have used this notation for the previous cases as well}

LECTURE 17 (21/03/2024)

88] There are 10 2D Point groups

1	m
2	$2mm$
3	$3m$
4	$4mm$
6	$6mm$

NOTE: Combination of a rotation by θ and a reflection in a m passing through the rotation point:



$$m = \begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

this is also a mirror (but at angle $\frac{\theta}{2}$ from y axis)

89] Reflection followed by rotation

$$= R_\theta m$$
$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \bar{1} & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\therefore \text{Det}(R_\theta m) = -\cos^2\theta - \sin^2\theta = -1$$
$$\Rightarrow \text{Mirror}$$

Mirror line is the fixed points of the operation

Thus, $$\begin{pmatrix} x_F \\ y_F \end{pmatrix} = (R_\theta m) \begin{pmatrix} x_F \\ y_F \end{pmatrix}$$
$$= \begin{pmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_F \\ y_F \end{pmatrix}$$
$$= \begin{pmatrix} -\cos\theta \, x_F - \sin\theta \, y_F \\ -\sin\theta \, x_F + \cos\theta \, y_F \end{pmatrix}$$

$$\Rightarrow x_F = -\cos\theta \, x_F - \sin\theta \, y_F$$
$$y_F = -\sin\theta \, x_F + \cos\theta \, y_F$$

Both of these are the same equation!

But this is not surprising, since
we expect to NOT have a unique solution
(we expect a line, NOT a point)

$\downarrow$

thus our answer
will be an equation of a line

From the first eqⁿ:

$$x_F = -\cos\theta\, x_F - \sin\theta\, y_F$$

this itself is the eqⁿ of the mirror line

$$\Rightarrow \sin\theta\, y_F = (-1-\cos\theta)\, x_F$$

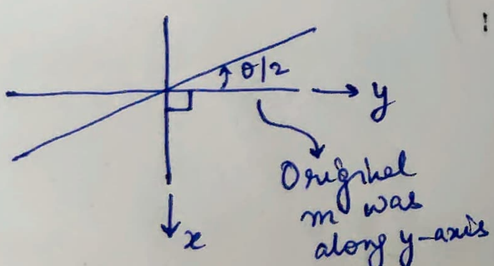
$$\Rightarrow y_F = -\frac{1+\cos\theta}{\sin\theta}\, x_F$$

$$= -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\, x_F$$

$$= -\cot\left(\frac{\theta}{2}\right) x_F$$

$$\Rightarrow y_F = +\underbrace{\tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)}\, x_F$$

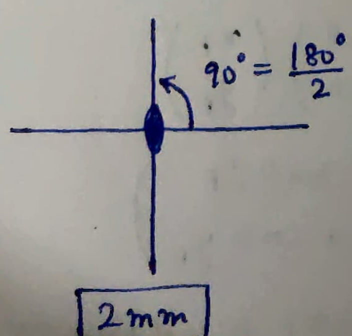
this makes sense, since an angle of $\left(90^\circ+\frac{\theta}{2}\right)$ from x-axis is an angle of $\frac{\theta}{2}$ from y-axis

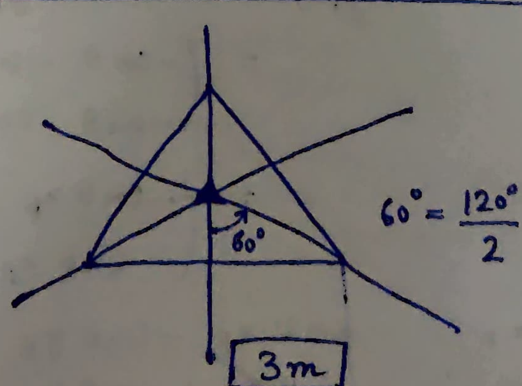


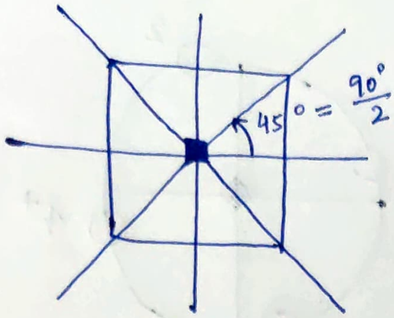
Hence we get that,

* $\boxed{\text{Resultant of an } m \text{ and } R_\theta \text{ is an } m \text{ at } \frac{\theta}{2} \text{ to the original } m.}$

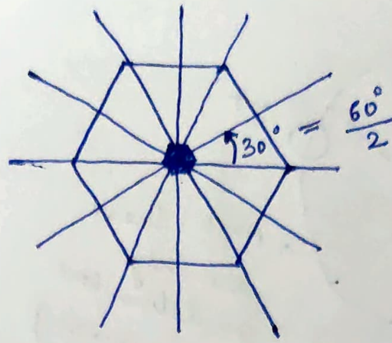
eg:







$\boxed{4mm}$



$\boxed{6mm}$

* Why 3m has only one m?

NOTE:

↓

the no. of m's in the notation is based upon

↓

the no. of m's which cannot be generated by a rotation of the other mirrors by rotational symmetry.

eg: in 2mm ⇒ vertical mirror CANNOT be obtained by 2 fold rot^n of horizontal mirror

↓

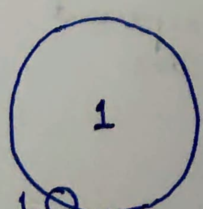
thus we need 2 m's

But in 3m ⇒ a single mirror can be rotated by 3-fold to obtain the other two

↓

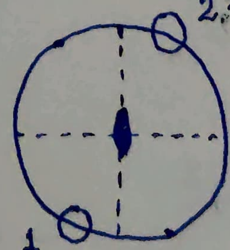
thus we just need 1 m.

90] Stereographic projection



General Position

(we can think of these as points, or even as motifs)

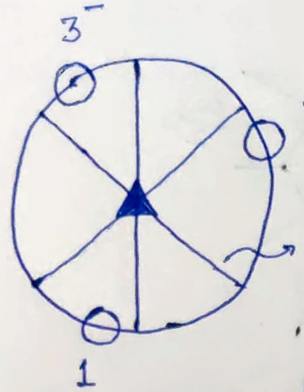


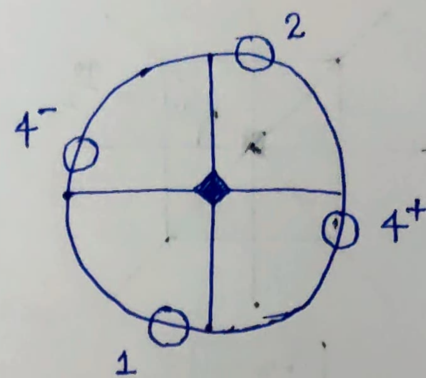
2 ⇝ just a label used by us (to show that we get this by a 2-fold operation)

General positions and symmetry operations are in one-to-one correspondence

↓

* Thus: $\boxed{\text{No. of General Positions} = \text{No. of symmetry operations in group} = \text{Order of the group}}$

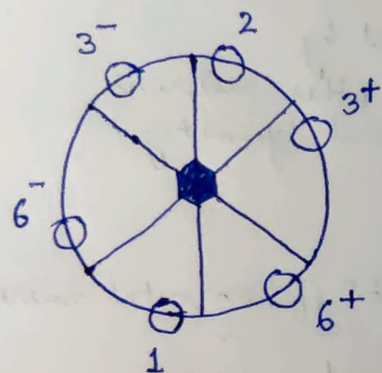




NOTE: These are only guidelines here (NOT mirrors)

↓

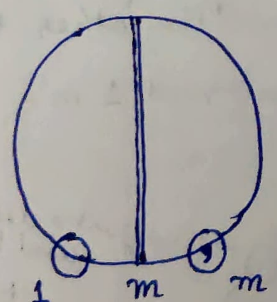
Technically, to distinguish, we can use a different colour for drawing symmetry operations



These 5 diagrams we have made are for five proper symmetry operations (since they have no mirrors)

91] Taking the case of mirror operation:

m :

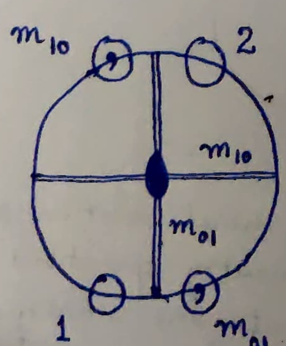


NOTE: ◯ : Right handed

⦿ : Left handed

We are using double line to represent mirror line

2mm :

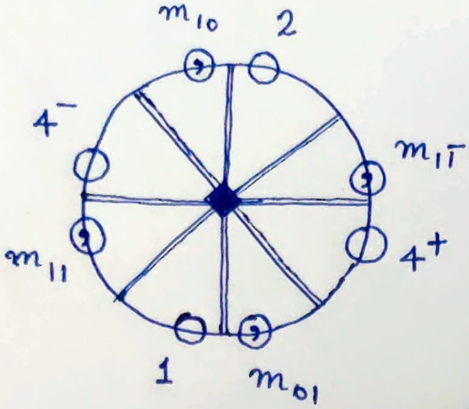


⇒ From here it is also clearly visible that $m_{01} m_{10} = 2$

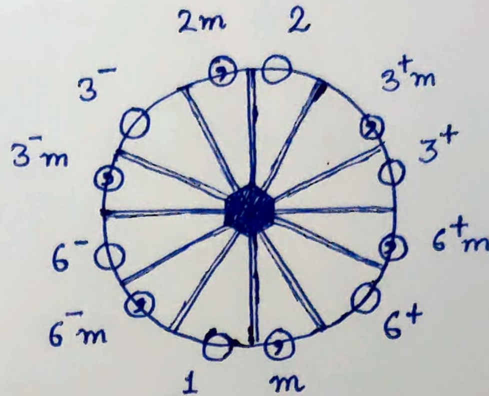
$2 m_{01} = m_{10}$

and, $2 m_{10} = m_{01}$

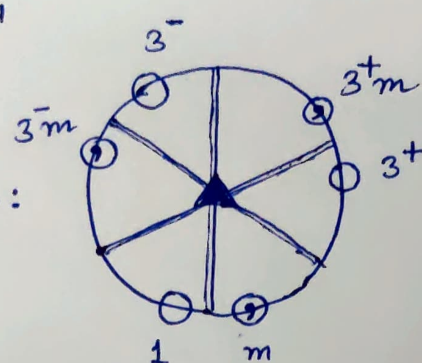
4mm:



: 6mm



3m:



<u>H.W.</u>: Establish Group-subgroup relations between point groups.

<u>LECTURE 18</u> (01/04/2024)