LECTURE 23

09/10/2023

NOTE: In general, a manifold can be specified as (M, G)

↓

A topological manifold is simply a set of points

↓

when we add a 'metric' to this ⇝ we get a Metric manifold

↓

this is then infinitely differentiable

We define spacetime (mathematically) as a C^∞ (M, G) manifold.

{ However we are not concerned with this }

NOTE: Previously we have seen that geometry can be equated with matter-energy through:

$$\boxed{\underbrace{R_{ab} - \frac{1}{2} g_{ab} R}_{G_{ab}} = K\, T_{ab}}$$

The EINSTEIN FIELD EQUATIONS

(and we know:

$$\nabla_a G^a_b = 0)$$

⇓

* This implies that:

$$\nabla_a T^a_b = 0$$

Now, we want to know what this T_{ab} is.

113] We saw for a scalar field:

$$\mathcal{L} = -\partial_a \phi \partial^a \phi - V(\phi)$$

$$\mathcal{H} = p\dot{q} - \mathcal{L}$$

Now, $q \to \phi$

$p \to \dfrac{\partial \mathcal{L}}{\partial(\partial_a \phi)}$

Thus we get: $T^a_b = -(\Pi^a \partial_b \phi - \delta^a_b \mathcal{L})$

Here we will see another approach for obtaining T_{ab}

{by considering T_{ab} as the "source of gravity" }

114] In EM, we saw: $\mathcal{L} = -\frac{1}{4} F_{ab}F^{ab} - A_k \overset{\nearrow \text{source}}{J^k}$

$$\nabla_k F^k_l = 4\pi J_l$$

Now, in GR we write the action as:

$$S = \frac{1}{2K}\int d^4x \sqrt{-g}\, R + \int d^4x \sqrt{-g}\, \mathcal{L}_m$$

{ here since $\mathcal{L} = \mathcal{L}(g, \partial g, \partial^2 g)$

↳ thus g_{ab} is the dynamical variable }

Varying the action (w.r.t. g):

$$\delta S = \frac{1}{2K}\int d^4x\, \underbrace{\delta_g(\sqrt{-g}\, R)}_{} + \overbrace{\int d^4x\, \delta(\sqrt{-g}\,\mathcal{L}_m)}^{\delta S_m}$$

this can be then written as:

$G_{ab}\, \delta g^{ab} \sqrt{-g}$

{ since: $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$ }

* { NOTE: we know that: $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\, g_{ab}\, \delta g^{ab}$ }

NOTE: There are many ways of performing this variation

↓

one such way is the

* Palatini variation ⇝ here (Γ, g) are treated as independent variables

{ i.e. we ignore $\Gamma \sim \partial g$ }

↓

here they use the fact that $R \sim \partial\Gamma, \Gamma$

(which makes this variation much easier)

here we use ← $\delta g = 0,\ \delta\Gamma = 0$

Ques: (** for MAJOR exam)

Show that varying $\mathcal{L}_p$ and $\mathcal{L}_q$ gives the same E.O.M.:

$$\mathcal{L}_p(q, \dot{q}, \ddot{q}) = \mathcal{L}_q(q, \dot{q}) - \frac{d}{dt}(pq)$$

{ here we have to fix $\delta p = 0$ at the boundary }

115] We have:

$$\underbrace{\delta S_m}_{\text{for 'matter'}} = \int d^4x\, \delta(\sqrt{-g}\,\mathcal{L}_m) \underset{\text{(defined as)}}{\equiv} -\frac{1}{2}\int d^4x\, T_{ab}\,\delta g^{ab}\sqrt{-g}$$

↓ i.e. in other words

We define T_{ab} as:

$$\boxed{T_{ab} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{ab}}}$$

from this definition we can see that T_{ab} is symmetric in $a \leftrightarrow b$ $\Leftarrow$ which is what we want so that $\underbrace{G_{ab}}_{\text{this is also symmetric}} = K T_{ab}$

{ * However, if we did the same for EM we may NOT necessarily get a symmetric T_{ab}:

$$\mathcal{L} = -\partial_a\phi\,\partial^a\phi - V(\phi)$$
$$= -g^{ab}\partial_a\phi\,\partial_b\phi - V(\phi)$$

$$T^a_b = -(\underbrace{\pi^a\partial_b\phi}_{\text{due to this part it is NOT symmetric}} - \delta^a_b\mathcal{L})$$

However it can be "made" symmetric by adding:

$$\bar{T}^a_b = T^a_b + \Lambda^a_b \Rightarrow \nabla_a\Lambda^a_b = 0$$

which is done such that

(There is such a thing, called the * Rosen-Bellifante Tensor which fixes T^a_b to make it symmetric)}

But in general this is pretty useless!

116] By combining the two actions:

$$\delta S = \frac{1}{2K}\int d^4x\, G_{ab}\, \delta g^{ab}\sqrt{-g} - \frac{1}{2}\int d^4x\, T_{ab}\, \delta g^{ab}\sqrt{-g}$$

$$= \int d^4x \left(\frac{1}{2K} G_{ab} - \frac{1}{2} T_{ab}\right) \delta g^{ab} \sqrt{-g}$$

Thus, for $\delta S = 0, \delta g^{ab} = 0 \Rightarrow$ we get: $\boxed{G_{ab} = K\, T_{ab}}$

The EINSTEIN EQUATIONS

117] We want to show that $\nabla_a T^a_b = 0$

$\downarrow$

if this is true then it should NOT change under an arbitrary coord. transformation

$$A_m = \int d^4x \sqrt{-g}\, L_m$$

Now taking $x^a \longrightarrow \bar{x}^a = x^a + \xi^a$

$$(\delta g^{ab} = \nabla^a \xi^b + \nabla^b \xi^a)$$

$$\therefore \delta A_m = \int d^4x\, \delta(\sqrt{-g}\, L_m)$$

{ Thus, also: $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\, g_{ab}\, \delta g^{ab}$ $= -\sqrt{-g}\, g_{ab} \nabla^a \xi^b$ }

$$= \int d^4x \left[\sqrt{-g}\, \delta L_m + L_m\, \delta(\sqrt{-g})\right]$$

this becomes $\rightarrow$ $\{\bar{L}_m(\bar{x}) = L_m(x)\}$

$$= \int d^4x \left[-\sqrt{-g}\, g_{ab} \nabla^a \xi^b L_m - \sqrt{-g}\, \xi^a \nabla_a L_m\right]$$

(since L_m is scalar, so $\partial_a L_m$ can be written as $\nabla_a L_m$)

$$= -\int_V d^4x \sqrt{-g}\, \nabla_a (\xi^a L_m)$$

$\Updownarrow$ which can be converted to a surface integral

$$= -\int_{\partial V} d^3x \sqrt{h}\, n_k \xi^k L_m = 0$$

(since ξ^k vanishes on the boundary)

118] we thus can write:

$$\delta A_m = -\frac{1}{2}\int d^4x \sqrt{-g}\, T_{ab}\, \delta g^{ab}$$

$$= -\int d^4x \sqrt{-g}\, T^c_b\, \nabla_c \xi^b$$

$$= -\int d^4x \sqrt{-g}\left[\nabla_c (T^c_b \xi^b) - \nabla_c (T^c_b)\, \xi^b\right]$$

$$= -\int_V d^4x \sqrt{-g}\, \nabla_c (T^c_b \xi^b) + \int d^4x \sqrt{-g}\, \nabla_c T^c_b\, \xi^b$$

$$= \underbrace{-\int d^3x \sqrt{h}\, n_c T^c_b \xi^b}_{0} + \int d^4x \sqrt{-g}\, \underbrace{\nabla_c T^c_b}\, \xi^b$$

(this part vanishes) i.e.

thus for satisfying $\delta A_m = 0 \implies$ we get the result $\boxed{\nabla_c T^c_b = 0}$

* { i.e. both $\nabla_a G^a_b$ and $\nabla_a T^a_b$ independently can be shown to be zero }

LECTURE 24

10/10/2023

* **NOTE:** Reference : Sergei Winitzki : "Advanced General Relativity"
[for Advanced understanding; NOT required]

NOTE: when we found the geodesic equation
↓
we used the action having τ

$$A = -m\int \underset{\sim}{d\tau} = -m\int\sqrt{-ds^2} = -m\int\sqrt{-g_{ab}\,dx^a dx^b}$$

↓
this is applicable for timelike freely falling observers
{which gives the geodesic eq^n.}

{For lightlike, we saw another action called the Einbein action
↓
and later on for quantizing this we get BRST Quantization
{i.e. for a relativistic point particle} }

NOTE: In classical mechanics we know $\underset{\sim}{p} = m\underset{\sim}{\dot{x}}$
↓
but alternatively we can write a lagrangian:

$$\frac{1}{2}m\dot{x}^2 + \lambda(p - \dot{x})$$

and from here obtain the same eq^{n}s
⇓
Reference : "Classical Electrodynamics", by J. Schwinger
{this was not published by Schwinger himself as he himself wasn't happy with it, and hadn't completed it. Later on his students compiled it and published}

119] We know now that:

$$R_{ab} - \frac{1}{2}R g_{ab} = \kappa T_{ab}$$

{By Newtonian limit, and comparing with: $\nabla^2\phi = 4\pi G_N \rho$ $\Rightarrow$ we get: $\kappa = \dfrac{8\pi G_N}{c^4}$ }

NOTE: Here, G_N is determined by the Eotvos experiment

120] Vacuum Solution

for $T_{ab} = 0$

we get that $G_{ab} = 0$

$$\underbrace{R_{ab} - \frac{1}{2} R g_{ab}}_{G_{ab}} = \kappa T_{ab}$$

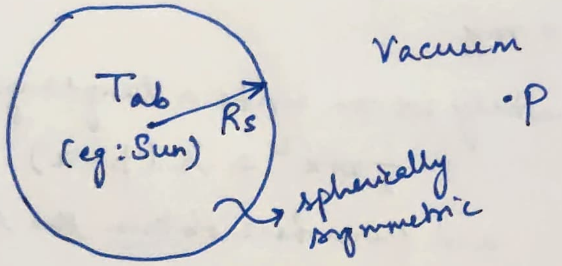
(since $R = -\kappa T \Rightarrow R = 0$)

thus $R_{ab} = 0$ as well

But, $R_{abcd} \neq 0$ and the Weyl tensor: $C_{abcd} \neq 0$

121] Spherical Symmetry

The approach will be: metric: $g_{ab} \longrightarrow \Gamma^{a}_{bc} \longrightarrow R, R_{ab}, R_{abcd}, C_{abcd}$



The most general spherically symmetric metric is:

$$ds^2 = \alpha(t,r)\, dt^2 + \beta\, dr^2 + \gamma\,(d\theta^2 + \sin^2\theta\, d\phi^2) - \mu\, dr\, dt$$

{ here: $\alpha, \beta, \gamma, \mu$ are all functions of r, t }

this metric is spherically symmetric bcz at const. t surface we simply get: $\underbrace{\beta\, dr^2 + \gamma\,(d\theta^2 + \sin^2\theta\, d\phi^2)}_{\text{which is spherically symmetric}}$

* $t = $ const surface: this is isotropic or spherically symmetric

↓ to meet this condition

we eliminate dt^2 and $dt\,dr$ terms at const. t

* Diagonalize: i.e. $\mu = 0$
  and $\gamma = R^2(r)$ { here we can specifically use r^2 }
* A and B are only functions of r

This gives us the metric:

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

* NOTE: Here 'r' should NOT be thought of as the "distance"

↓

instead it is actually the "Aerial distance":

$$r = \sqrt{\frac{A}{4\pi}}$$

{ we can compare this with:

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

here there is No $A(r)$ or $B(r)$

↓

i.e. this case is flat

(and we can transform it to $-dt^2 + dx^2 + dy^2 + dz^2$)

↓

But with $A(r)$ and $B(r)$ there is NO way to transform to a flat metric }

122] For this metric

↓

the lagrangian will be:

$$\mathcal{L} = A\dot{t}^2 - B\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$$

↓

from here we can calculate the Christoffel symbols

* $\Gamma^t_{rt} = \Gamma^t_{tr} = \frac{1}{2}\frac{A'}{A}$ $\xrightarrow[\text{from which we get}]{}$ R_{rr}, R_{rt}, R_{tt} ; $R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$

* $R_{ab} = 0$, $R = 0$

$$\Rightarrow \frac{1}{rB}(A'B + B'A) = 0 \Rightarrow \frac{d}{dr}(AB) = 0 \Rightarrow \text{ i.e. } AB = \text{const. (everywhere) for } r > R_s$$

But we also know that :

$$A(r) \longrightarrow 1 \quad \text{as } r \to \infty$$
$$B(r) \longrightarrow 1 \quad \text{as } r \to \infty$$

(since metric should become flat at infinity)

$\Downarrow$

thus this fixes that : $AB = 1$

or

$$\boxed{A = 1/B}$$

* $R_{\theta\theta} = 0$

$\Rightarrow A - 1 + rA' = 0$

$\Rightarrow \quad A(r) = 1 + \frac{c}{r}$

Thus, we can now write :

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where, $f(r) = 1 + \frac{c}{r}$

We also know, that

$$g_{00} \sim (1 - 2\phi) \quad , \quad \phi = \frac{GM}{r}$$

$\Downarrow$

from here we can fix the value for c

$\downarrow$ i.e.

$$c = -2\,GM$$

(this has units of length

$\downarrow$

so we divide by c^2)

Hence, we get the final form:

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + r^2 d\Omega^2}$$

SCHWARZSCHILD METRIC

123] We can't make the Schwarzschild metric flat

↓

But we can write it in an alternative form :

$$r = \rho \left(1 + \frac{MG}{2\rho}\right)^2$$

here: $$ds^2 = -\frac{(1 - M/2\rho)^2}{(1 + M/2\rho)^2} dt^2 + \left(1 + \frac{M}{2\rho}\right)^4 [d\rho^2 + \rho^2 d\Omega^2]$$

{ called the Isometric Schwarzschild metric }

↓

* this metric is conformally flat at $t = \text{const.}$ surfaces

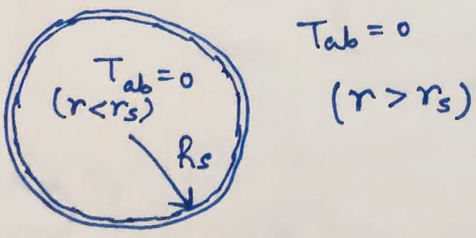
{ NOTE: $\tilde{g}_{ab} = \Omega^2(x)\, \eta_{ab}$

↓

is called conformally flat

(i.e. $h_{\alpha\beta} = \Omega^2(x)\, \delta_{\alpha\beta}$) }

124] For the case of spherical shell :



Since this is also spherically symmetric

↓

the metric we obtained with $f(r)$ should still be applicable

↓

However, here it should NOT blow up at $r = 0$

↓

Instead here we will simply get that $\boxed{C = 0}$ for $r < R_s$

(which gives a flat spacetime

⇓

i.e. there is no gravity inside)

{ just as we observe in Newtonian gravity }

LECTURE 25

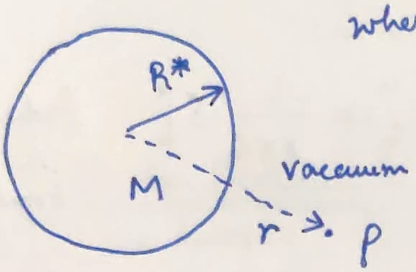
13/10/2023

NOTE: Previously we saw the Schwarzschild Metric (valid for the region outside radius of star), R_*

125] $$ds^2_{r>R_*} = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

where, $f(r) = 1 - \frac{2M}{r}$ $\left(\text{or } 1 - \frac{2GM}{c^2 r}\right)$

using units: $G = 1$, $c = 1$



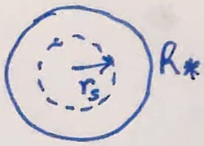
$= 1 - \frac{r_s}{r}$

such that: $\boxed{r_s = \frac{2GM}{c^2}} \approx 3\text{km}\left(\frac{M}{M_\odot}\right)$

Thus we can say:

$$f(r_s) = 0$$
$$f'(r_s) = 2K \neq 0$$

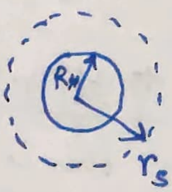
If $r_s < R_*$ then there is No problem since metric is not valid within R_* anyways



BUT, if $r_s > R_*$ $\Rightarrow$ then $\underbrace{g_{rr} \to \infty, \text{ as } r \to r_s}_{\text{the metric blows up}}$

which is something that "should NOT happen"

This situation turns out to be that of a BLACK HOLE



126] Now, $ds^2_{r>R_*} = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$

this is for spherically symmetric

So we thus find:

$$G^r_r = G^t_t, \quad G^\theta_\theta = G^\phi_\phi$$

$$\Rightarrow T^r_r = T^t_t = \frac{-\epsilon(r)}{8\pi G}, \quad T^\theta_\theta = T^\phi_\phi = \frac{-\mu(r)}{8\pi G}$$

By solving the Einstein Field Equations for this case, we get that:

$$\frac{1}{r^2}(1-f) - \frac{f'}{r} = \epsilon \quad ; \quad \nabla^2 f = -2\mu$$

An example of one solution is:

$$f(r) = 1 - \frac{a}{r} - \frac{1}{r}\int_a^r \epsilon(r)\, r^2\, dr$$

$$\{ f(a) = 0 \}$$

Now, since we can add any $T^a_b = \frac{\Lambda}{8\pi G}\delta^a_b$ (where Λ is some const.) and that would still be a solution

↓

thus, we can write a solution of the form:

$$\boxed{f(r) = 1 - \frac{c}{r} + H^2 r^2}$$

called the Schwarzschild–de Sitter Solution *

where, $H^2 = \frac{\Lambda}{8\pi G}$

{ NOTE: This is applicable only to spherically symmetric cases }

127] For the case of $c = 0$ (for inside the spherical shell)

↓

i.e. $\boxed{f(r) = 1 - H^2 r^2}$

this is called the de Sitter solution *

{and these spaces are called de Sitter spaces}

{ NOTE: In case of some charge Q additionally present:

$$f(r) = 1 - \frac{c}{r} + H^2 r^2 + \frac{Q^2}{r^2}$$ (for the Schwarzschild-de Sitter metric)

For the Schwarzschild metric ⇒ $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$ }

NOTE: We need to distinguish b/w actual singularities and "singularities" that seem to be present but are only due to a wrong metric coord. system choice

↓

i.e. if we can change coord. system & remove singularity

{ in such a case the curvature would be smooth everywhere }

However, if curvature is NOT smooth, then there is an actual singularity.

Kepler Problem

128] We know:

$$ds^2_{r>r_*} = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

$$f(r) = 1 - \frac{2M}{r}$$; * Time independence and Rotational symmetry

Also,

$$g^{ab}\frac{\partial A}{\partial x^a}\frac{\partial A}{\partial x^b} = -m^2$$

For the case of $\theta = \frac{\pi}{2}$ ⇝ i.e. case of: Motion confined to a plane

↓

we get:

$$\frac{1}{f}\left(\frac{\partial A}{\partial t}\right)^2 - f\left(\frac{\partial A}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial A}{\partial \phi}\right)^2 = m^2$$

Thus we can write:

(due to time independence and rotational symmetry)

$$A = -\mathcal{E}t + L\phi + A_r(r)$$

$$\Rightarrow A = -\mathcal{E}t + L\phi + \int dr\sqrt{\frac{\mathcal{E}^2}{f^2} - \left(m^2 + \frac{L^2}{r^2}\right)\frac{1}{f}}$$

$$\left\{ \text{i.e. } \frac{\partial A}{\partial \mathcal{E}} = \text{const.}\ ,\ \frac{\partial A}{\partial L} = \text{const.} \right\}$$

⇓ ⇓

* Relating: $r \leftrightarrow t$, $r \leftrightarrow \phi$

⇓ gives time dependence ; ⇓ gives orbit

Thus, from here we get:

$$t = \frac{\varepsilon}{m} \int dr \, \frac{1}{f} \left[\left(\frac{\varepsilon}{m}\right)^2 - \left(1 + \frac{L^2}{m^2 r^2}\right) f \right]^{-1/2} \quad - ①$$

$$\phi = \int dr \left(\frac{L}{r^2}\right) \left[\varepsilon^2 - \left(m^2 + \frac{L^2}{r^2}\right) f(r) \right]^{-1/2} \quad - ②$$

129] From ①:

$$\frac{1}{f} \frac{dr}{dt} = \frac{1}{\varepsilon} \left[\varepsilon^2 - V_{eff}^2 (r) \right]^{1/2}$$

where:

$$V_{eff}^2 (r) = m^2 f(r) \left(1 + \frac{L^2}{m^2 r^2}\right) = m^2 \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{L^2}{m^2 r^2}\right)$$

Now, we can write V_{eff} as:

$$V_{eff}^2 = m^2 \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{L^2}{m^2 r^2}\right)$$

taking: $\left.\begin{aligned} x &= \frac{GM}{r} \\ k^2 x^2 &= \frac{L^2}{m^2 r^2} \end{aligned}\right]$

$\Downarrow$

we get:

$$V_{eff.}^2 = (1 - 2x)(1 + k^2 x^2)\, m^2$$
$$= m^2 (1 - 2x + k^2 x^2 - 2k^2 x^3)$$

Thus: $V_{eff.}^2 \simeq m^2$ (i.e. $V_{eff.}^2$ approaches m^2)

or

(a better approximation):

$V_{eff.}^2 \sim m^2 \left(1 - \frac{2GM}{r}\right)$ → i.e. it approaches m^2 from below *

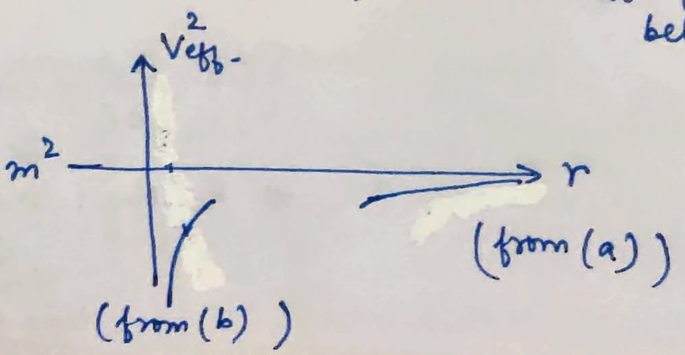
(a)

Also:

$V_{eff.}^2 (r) \sim \frac{-1}{r^3}$

(and thus $V_{eff.}^2 \to -\infty$ as $r \to 0$)

* (b)



from: $\frac{d}{dx} V^2_{eff.} = 0 = 2 + 2k^2 x_\pm - 6k^2 x^2_\pm$

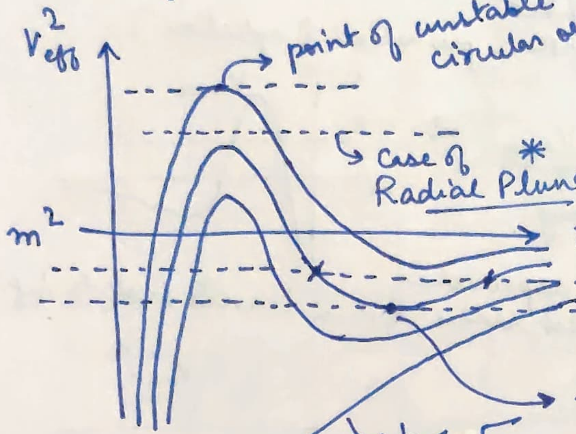
$$\Downarrow$$

$$x_\pm = \frac{1 \pm \sqrt{1 - 12/k^2}}{6} \xrightarrow{\text{thus}} \frac{12}{k^2} < 1$$

i.e. $\Rightarrow k > \sqrt{12}$

This is the Condition for existence of extrema

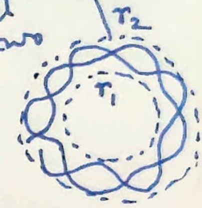
130] Thus, the plot can be of the form:



point of unstable circular orbit

Case of Radial Plunge *

at this energy level there will be two radii $\Rightarrow$



(elliptical orbit)

{NOTE: whether this orbit will be complete, or will precess, is determined from (2)}

for this energy level, a single stable circular orbit exists

$k \leq \sqrt{12}$ (no extremum)

Thus, the x^3 term in:

$$V^2_{eff.} = m^2(1 - 2x + k^2x^2 \boxed{- 2k^2x^3})$$

$\downarrow$

* this is the GR CORRECTION to Newtonian gravity

{this same correction also explains the precession of Mercury's orbit}

{because it turns out that if we take $c \to \infty$, then this term would disappear}

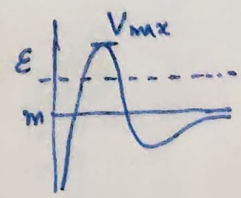
* In general:

for $k > \sqrt{12}$ $\downarrow$ there is one maximum & one minimum

for $k = \sqrt{12}$ $\downarrow$ one single extremum (i.e. both extrema merge into one)

for $k < \sqrt{12}$ $\downarrow$ There is NO extrema

* NOTE: Radial Plunge: In the case when $m < \mathcal{E} < V_{max}$



$\downarrow$

a particle from infinity will reach a radius of closest approach

$\downarrow$

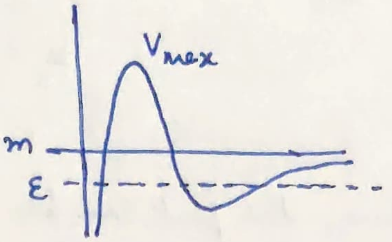
and then "plunge" back to infinity

SELF-NOTES

* {NOTE: V_{max} changes by varying k }

All cases are as follows :

For $m < V_{max}$: * if $\mathcal{E} < m$ ⇒



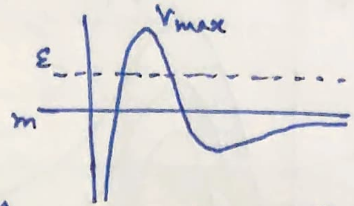
↓

(i.e. r_1 & r_2) ← there are 2 turning points

↓

i.e. particle will orbit in elliptical orbit (with a perihelion & aphelion)

* if $m < \mathcal{E} < V_{max}$ ⇒



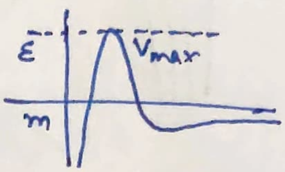
↓

only one turning point

↓

Radial plunge : particle approaches closest radius, & then travels back to infinity

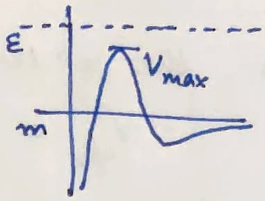
* if $\mathcal{E} = V_{max}$ ⇒



↓

unstable circular orbit

* if $\mathcal{E} > V_{max}$ ⇒



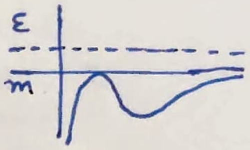
↓

particle will spiral towards the origin/center, $r = 0$ {i.e. fall towards the center}

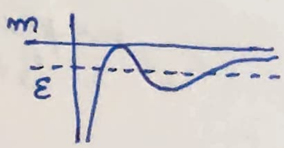
this is in contrast to Newtonian gravity (where an object with $L \neq 0$ can never reach $r = 0$)

For $m = V_{max}$:

- if $\mathcal{E} > m$ ⇒ particle will fall to origin



- if $\mathcal{E} < m$ ⇒ particle will be in a bound orbit

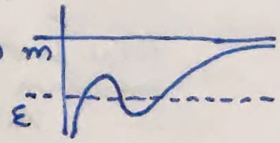


- if $\mathcal{E} = m$ ⇒ unstable circular orbit

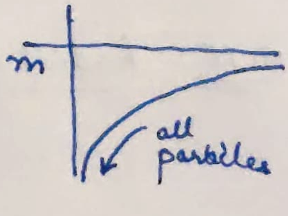
For $V_{max} < m$: if $\mathcal{E} > V_{max}$ ⇒ particle falls to origin

if $\mathcal{E} = V_{max}$ ⇒ unstable circular orbit

if $\mathcal{E} < V_{max}$ ⇒ bound orbit ⇒



if $k \leq \sqrt{12}$ ⇒ There are no turning points ⇒



↓

all particles (irrespective of $\mathcal{E}$) will fall to origin

[* **REFERENCE** : T. Padmanabhan (pg: 330-331)]

LECTURE 26

14/10/2023

NOTE: Difference b/w Transporting a vector (parallel transport) & Lie derivative :

$u_a \nabla^a v^b$
(or $u^a \nabla_a v^b$)
↓
Here you take some vector v^b along which you find the change of u^a

$\mathcal{L}_u v^a$
↓
Here you are finding the change in vector v^a along some vector field (given by u here)

NOTE: We will later discuss another way of obtaining energy-momentum tensor

{The previous method we discussed was by varying the matter lagrangian}

from momentum & pressure density
↓
* This cannot be done through a lagrangian

(i.e. Fluids cannot be described through a lagrangian
↓
this is bcz properties like pressure, temperature, etc. are not microscopic properties (they are
↓ EMERGENT properties)
so lagrangian cannot be constructed for it)

131] Previously, we wrote from the Hamilton-Jacobi eq^n.s :

$$\phi = \int dr \frac{L}{r^2} \left[\mathcal{E}^2 - \left(m^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2GM}{r}\right) \right]^{-1/2}$$

Using this we will now try to find the perihelion-shift.

{* **NOTE:** In order to be valid for massless particles like photons ↓ Simply take $m = 0$ }

Take $u = 1/r$

$$\Rightarrow du = -\frac{dr}{r^2}$$

$$\therefore \left(\frac{du}{d\phi}\right)^2 = \frac{1}{L^2}\left[\frac{\mathcal{E}^2}{c^2} - (m^2c^2 + L^2u^2)\left(1 - \frac{2GM}{c^2}u\right)\right]$$

$\downarrow$ differentiating this again w.r.t. ϕ :

$$\underbrace{\frac{d^2u}{d\phi^2} + u}_{\substack{\text{free particle in polar}\\ \text{coordinates}\\ (r,\phi)}} = \underbrace{\frac{GMm^2}{L^2}}_{\substack{\text{Newtonian}\\ \text{gravity}}} + \underbrace{\frac{3GM}{c^2}u^2}_{\substack{\text{GR}\\ \text{correction}}}$$

{ **NOTE!** Since we write geodesic eq^n.s as:

$$\frac{du_c}{d\tau} = \frac{1}{2}\partial_c g_{ab} u^a u^b$$

$\downarrow$

metric is independent of time and there is rotational symmetry

$\downarrow$

Thus: $u_0 = \text{const.}$, $u_\phi = \text{const.}$

$$\left\{ u_k = g_{ka}u^a = g_{ka}\frac{dx^a}{d\tau} \right\}$$

But, $\dot{\phi} = \dfrac{u_\phi}{u_0}$ }

132] From the above expression:

$$\frac{3GM}{c^2}u^2 \times \frac{L^2}{GMm^2} \sim \left(\frac{L^2u^2}{m^2}\right)\frac{1}{c^2} \sim \frac{v^2}{c^2}$$

$\hookrightarrow$ dimensions are of v^2/c^2

This is a **Non-linear term** *

{i.e. if we wrote a linear theory of gravity (or if we assumed: $g_{ab} = \eta_{ab} + h_{ab}$) then we wouldn't get this effect }

$\Longleftarrow$ it turns out that the effect of this relativistic correction MAY NOT be LESS

$\downarrow$

i.e. if we take $r = r_0 = \text{const.}$ (for some orbit) and $u_0 = 1/r_0$

* then $\dfrac{GM}{c^2 r_0^2}$ may not be small

133] <u>Case 1</u> : <u>Nearly circular orbit</u>

i.e. to compare & collect terms easily later on

$$u_0 = \frac{1}{r_0}$$

* { NOTE: Actually ϵ is only written as a "Book-keeping" device → we don't need it }

Let's say : $u = u_0 + \underbrace{u_1 \epsilon}_{\text{some small perturbation}}$

$$\text{Then, } \epsilon \frac{d^2 u_1}{d\phi^2} + u_0 + \epsilon u_1 = \frac{GMm^2}{L^2} + \frac{3GM}{c^2}(u_0^2 + \overbrace{u_1^2 \epsilon^2}^{\text{neglected}} + 2u_0 u_1 \epsilon)$$

$$\Rightarrow O(\epsilon^0) : u_0 = \frac{GMm^2}{L^2} + \frac{3GM}{c^2} u_0^2$$

(comparing non-ϵ terms)

↳ from here we can solve for u_0

$$\Rightarrow O(\epsilon) : \frac{d^2 u_1}{d\phi^2} + u_1 = \frac{6GM}{c^2} u_0 u_1$$

$$\frac{d^2 u_1}{d\phi^2} + \underbrace{\left(1 - \frac{6GM}{c^2} u_0\right)}_{\omega^2} u_1 = 0$$

⇓

$$u_1 = A \cos\left[\left(1 - \frac{6M}{c^2} u_0\right)^{1/2} \phi\right]$$

Thus we can now write :

(by ignoring ϵ ↓ since it wasn't reqd. anyways)

$$\frac{1}{r} = \frac{1}{r_0} + A \cos\left[\left(1 - \frac{6M}{c^2} \frac{1}{r_0}\right)^{1/2} \phi\right]$$

$$\text{for } \phi_c = \underbrace{2\pi\left(1 - \frac{6GM}{c^2} \frac{1}{r_0}\right)^{-1/2}}_{\text{this will give us the } \cos 2\pi \text{ term}}$$

↓

Hence : $\phi_c - 2\pi \Rightarrow$ this is the <u>shift precession</u> *

$$= 2\pi\left[\left(1 - \frac{6GM}{c^2} \frac{1}{r_0}\right)^{-1/2} - 1\right]$$

$$\approx 2\pi\left(\cancel{1} + \frac{3GM}{c^2 r_0} - \cancel{1}\right) = \boxed{\frac{6GM}{c^2 r_0} \pi}$$

if we calculate this for Mercury ↓ it becomes ~ 43 arcseconds per century

* NOTE: in reality the precession of mercury is larger, due to effects from Jupiter & Saturn in addition to the Sun

LECTURE 27
16/10/2023

Reference : "General Relativity" by Kenyon
→ can refer to chapter 8 for tests of General Relativity

*** NOTE** : Perihelion shift of different planets
(* here the observed values are modified so as to only consider influence of Sun, and NOT other planets)

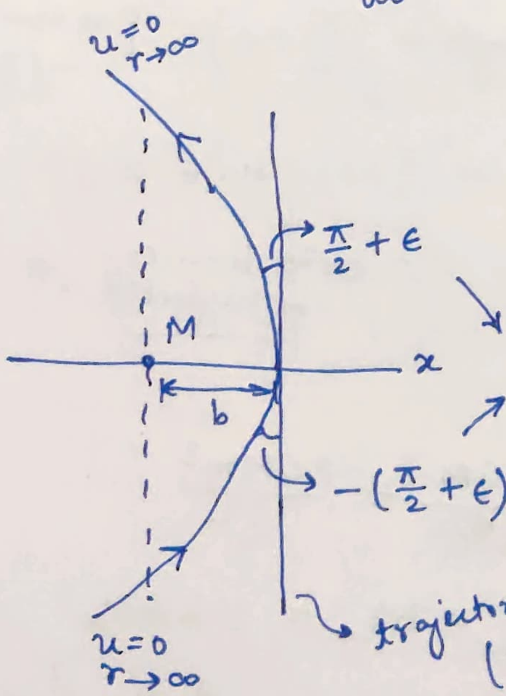
(in arcsec / century)

Mercury	43.11 ± 0.45	43.03
Venus	8.4 ± 4.8	8.6
Earth	5.0 ± 1.0	3.8

134] **Deflection of ultra-relativistic particle**

We know:

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2} + \frac{3GM}{c^2}u^2$$



* (∴ deflection δ can be given by: $\delta = 2\epsilon$)

trajectory when M=0 (straight line)

for the case of $M=0$
↓
solution to $\frac{d^2u}{d\theta^2} + u = 0$
↓ gives
$u = \frac{1}{b}\cos\theta$
(or $r\cos\theta = b$)

We can call the general solution to original eqn. as :

$$* \quad u = \frac{1}{b}\cos\theta + v(\theta) \quad , \quad bv \ll 1 \qquad -(1)$$

$$\therefore v'' + v = \frac{GMm^2}{L^2} + \frac{3GM}{2b^2c^2}(1+\cos 2\theta)$$

$$\Rightarrow v'' + v = \underbrace{\left(\frac{GMm^2}{L^2} + \frac{3GM}{2c^2b^2}\right)}_{\alpha} + \underbrace{\frac{3GM}{2b^2c^2}}_{\beta}\cos 2\theta$$

$$\alpha + \beta\cos 2\theta$$

Considering : $v = f + \alpha$

$\Rightarrow \quad f'' + f = \beta \cos 2\theta \ \underset{\text{thus}}{\rightsquigarrow} \ f = C \cos 2\theta$

From this we get :

$$\underbrace{-4C + C}_{-3C} = \beta \Rightarrow C = -\beta/3$$

So now (1) becomes:

$$u = \frac{1}{b}\cos\theta - \frac{GM}{2b^2c^2}\cos 2\theta + \frac{GMm^2}{L^2} + \frac{3GM}{2c^2b^2}$$

Using $\cos 2\theta = 2\cos^2\theta - 1$:

$$u = \frac{1}{b}\cos\theta + \frac{2GM}{b^2c^2} - \frac{GM}{b^2c^2}\cos^2\theta + \frac{GMm^2}{L^2}$$

135] We want to find ϵ

$\downarrow$

the easiest way is to take the case of $u = 0$ (i.e. particle being far away)

$\downarrow$

and here angle is $\left(\frac{\pi}{2} + \epsilon\right)$ { or equivalently $-\left(\frac{\pi}{2}+\epsilon\right)$ }

$\downarrow$

$\therefore \cos^2\left(\frac{\pi}{2}+\epsilon\right) = \sin^2\epsilon \approx \epsilon^2$

$\downarrow$

$\therefore$ $\cos^2\theta$ term can be neglected *

Thus : (at $u=0$)

$$\delta \equiv 2\epsilon = \frac{4GM}{bc^2} + \frac{2GMm^2}{L^2}$$

Also, angular momentum : $L = bp_\infty = \gamma b m v_\infty$

So we can write δ as :

$$\left\{ \gamma = \frac{1}{\sqrt{1 - v_\infty^2/c^2}} \right\}$$

$$\delta = \frac{4GM}{bc^2} + \frac{2GM\cancel{m^2}\cancel{b}}{\gamma^2 b^{\cancel{2}} \cancel{m^2} v_\infty^2}$$

$$= \frac{4GM}{bc^2} + \frac{2GM}{bv_\infty^2}\left(1 - \frac{v_\infty^2}{c^2}\right)$$

$$= \frac{4GM}{bc^2} + \frac{2GM}{bv_\infty^2} - \frac{2GM}{bc^2}$$

$$= \frac{2GM}{bc^2} + \frac{2GM}{bv_\infty^2}$$

Hence we obtain:

$$\boxed{\delta = \underbrace{\frac{2GM}{bv_\infty^2}}_{\substack{\text{Newtonian} \\ \text{Gravity} \\ \text{(N.G.)}}} \Big(1 + \underbrace{\frac{v_\infty^2}{c^2}}_{\text{GR}}\Big)}$$

* DEFLECTION OF ULTRA-RELATIVISTIC PARTICLE (DUE TO A MASS M)

Also, for light we get: (using $v_\infty = c$)

$$\boxed{\delta_{\text{Light}} = \frac{4GM}{bc^2}}$$

* BENDING OF LIGHT

{* Newtonian gravity also predicts bending of light, but the values of NG and GR differ by factor of 2 and 4, which was used as a test to confirm GR}

⇓

* NOTE: N.G. gives bending of light only when stated as a geometric theory (using a metric)

↓

$$A = -m\int d\tau + q\int A_k\,dx^k + \lambda\int \phi\,d\tau + \int\sqrt{B_{ij}\,dx^i dx^j}$$

here if we take:

$$\underbrace{B_{ij} = \phi^2 \eta_{ij}}$$

then this term reduces down to the $\int \phi\,d\tau$ term

↓

this gives rise to the Newtonian gravity.

* Reference: "Gravitation" by Misner, Thorne, Wheeler

{* NOTE: Just as we found a deflection for light

↓

Similarly we can also calculate the escape velocity for light}

NOTE: Electrodynamics is NOT a geometric theory

↓

Since A_k is just a connection

(it is used to describe something that exists in spacetime

↓

but not the spacetime itself)

But it is not a metric

NOTE: You can read about the meaning of the concept of "Shapiro Time Delay" *

* NOTE: We discussed only the case of metric outside a star

↓

But not inside a star

↓

However, this can be done

{comes under the topic of Polytropes}

{it involves considering the density and radiation pressure

which acts against the gravity to keep the star from collapsing

↓

The curvature primarily depends on ρ and P

(but if ρ and P depend on T, the temperature effect also will come into curvature, but this is not so direct)}

* We are also NOT going to study the topic of Chandrasekhar Limit

{i.e. at what point does a star collapse}

you can check this out yourself in case interested

* NOTE: The problem of understanding whether an accelerating charged particle should emit radiation is very complex (and interesting)

eg: Should a freely falling charged particle not emit radiation?

↓

bcz the way we treat accelerating frames in Newtonian mechanics and GR is very different

{eg: This makes the study of Hydrogen atom also very complex

↓

generally local curvature doesn't vary much, but for instance, near a black hole, there will be significant effects}

LECTURE 28

17/10/2023

[Class cancelled (at the starting time of lecture itself)
↓
But we discussed some Astrophysics anyways!]

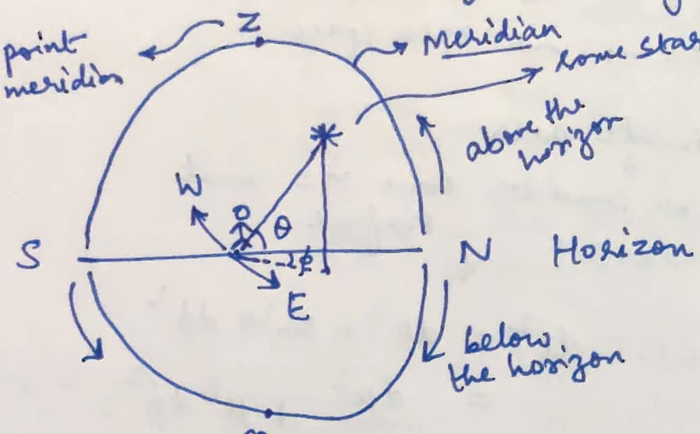
NON-LECTURE DISCUSSION

In order to specify Celestial coordinates
↓
we use the
Alt - Az coordinate system (most commonly)
↙ altitude
↘ azimuth

You stand along N-S line
↓
from the North, draw a line going straight above your head ;

AND similarly you can draw a line going straight below you

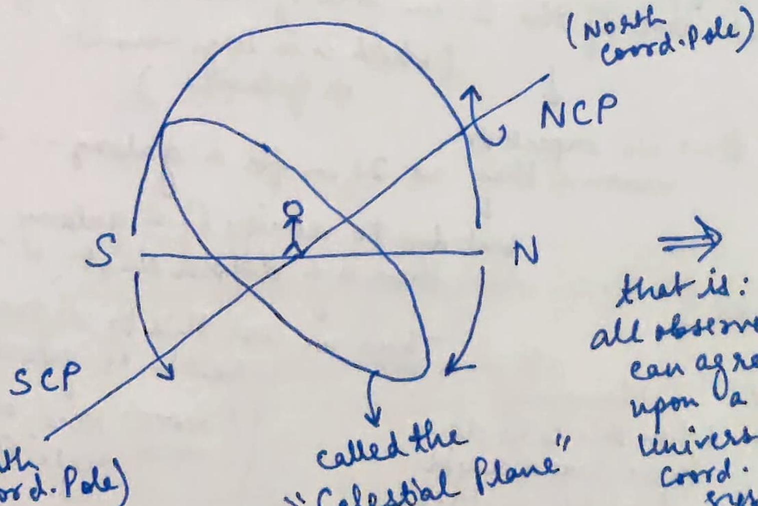
{ this line is the Meridian }
↓
and it divides the sky into West and East hemispheres



(top point of meridian called "Zenith")

(bottom point called "Nazeth")

$\theta \rightarrow$ altitude
$\phi \rightarrow$ azimuth

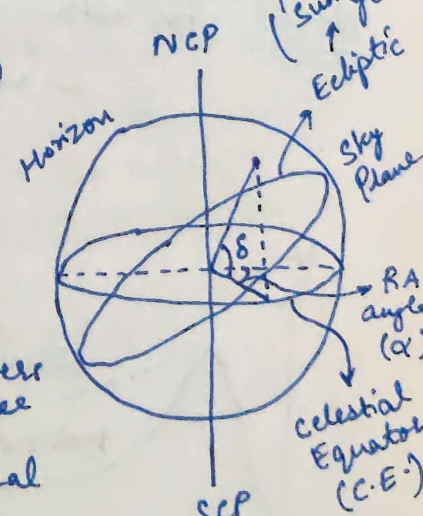


(North Coord. Pole)

(South Coord. Pole)

called the "Celestial Plane"

⟹ that is: all observers can agree upon a universal coord. system



[δ : declination angle ⇝ in degrees
RA : α ⇝ customary to write in hr, min, sec.]

generally these are fixed since star don't actually move much

but, for instance, for stars near center of milky way, they move v. fast

due to Black hole
↓
& thus α & δ change alot

NOTE: The global frame is chosen in accordance with the CMB *
(Cosmic Microwave Background)
↓
{by convention}

NOTE: To find area for some metric:

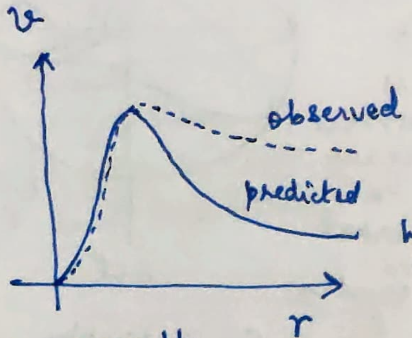
$$ds^2 = -dt^2 + a^2(t)\underbrace{\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)}_{\text{3D space}}$$

Firstly, we take $t = $ const.
↓
then "a" ⤳ called scale factor

Also:
$k = 0$ ⤳ flat space
$k = +1$ ⤳ closed space
$k = -1$ ⤳ open space

Now to calculate area
↓
we consider some $r = $ const. surface
↓
i.e. $$dl^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$
$$= \frac{d\mu^2}{1-\mu^2} + \mu^2 d\phi^2$$

NOTE: For motions of galaxies
↓
we make use of the 21 cm line of neutral hydrogen → related to parallel & antiparallel spins of e^- and p^+ in H
(which is in large amount in galaxies)
↓
thus we expect to measure this as 21 cm for a galaxy → if it is static
↓
but due to velocity of a galaxy, there is a doppler shift
↓
Thus, we use this to determine velocities of galaxies



⟸ However, when this was done there was a major diff. b/w predicted and observed curves *

(**NOTE:** Here, at the scale of galaxies
↓
only Newtonian gravity is reqd., since GR effects are negligible)

This can currently only be corrected for in: $\nabla^2\phi = 4\pi G\rho$ {Jean's Instability Criterion} → by including extra ρ called DARK MATTER