LECTURE 14

* **NOTE:** Paddy, "Sleeping Beauties in Theoretical Physics"
[Reference] — Chapter: when catterpillar becomes a butterfly
(to understand how coulomb field becomes a radiative field)

NOTE: we previously saw that
↓
we can go to a Local 'inertial' frame
↓
where all effects of gravitation disappear

70]
* **NOTE:** eg: In Newtonian force case:

$$m\ddot{x}^a = F^a$$

we can make a transformation: $z^a = z^a(x^i)$

Then, $dz^a = \dfrac{\partial z^a}{\partial x^k} dx^k$

or

$$\dot{z}^a = \frac{\partial z^a}{\partial x^k}\dot{x}^k$$

$$\therefore \Rightarrow \quad m\ddot{z}^a = m\frac{\partial z^a}{\partial x^k}\ddot{x}^k + \underbrace{m\frac{\partial z^a}{\partial x^k \partial x^l}\dot{x}^k\dot{x}^l}$$

$$\Rightarrow \quad \tilde{F}^a = \frac{\partial z^a}{\partial x^k}F^k + p^a$$

→ this "extra" (fictitious / pseudo) force
* comes up simply due to making a coordinate transformation

* **NOTE:** when we will take the metric
↓
up to first order (i.e. zeroth & first order)
↓
it will be flat
↓
but from second order onwards we will see curvature effects.

71] For a sphere:

$$dl^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

eg: On the North Pole

For $\theta \sim 0° \Rightarrow \phi = 0, \phi = \pi/2$

Here:

$$dl^2 \sim d\theta^2 + \theta^2 d\phi^2$$

* we can here make use of the coord. transformation:

$$\xi_1 = \theta \cos\phi$$
$$\xi_2 = \theta \sin\phi$$

$$\Rightarrow \xi_1^2 + \xi_2^2 = \theta^2 \quad ; \quad \frac{\xi_2}{\xi_1} = \tan\phi$$

Hence:

$$d\theta = \frac{\xi_1 d\xi_1 + \xi_2 d\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}} \quad , \quad d\phi = \frac{\xi_1 d\xi_2 - \xi_2 d\xi_1}{\xi_1^2 + \xi_2^2}$$

$$dl^2 = d\theta^2 + \theta^2 d\phi^2$$
$$= \underbrace{(d\xi_1)^2 + (d\xi_2)^2 + O(\xi^2 d\xi^2)}_{}$$

from this we can see that:

$$g_{ab}(P) = \delta_{ab} \quad , \quad \partial_c g_{ab}(P) = 0$$

[technically we should have used α & β] { NOTE: Here since everything is spatial: a & $b = 1,2,3$ }

* Such coordinates are called

Riemann Normal Coordinates (RNC)

72] Generalizing the above idea:

$$x'^a = f(x^a)$$

$$\Rightarrow x'^a = B^a_k x^k + C^a_{kl} x^k x^l + \ldots$$

($B, C, \ldots$ are the taylor series expansion coeff's)

Thus:

$$g_{ab}(x'^a) = g_{ab}\Big|_0 + \partial_c g_{ab}\Big|_P x^c + \partial_c \partial_d g_{ab}\Big|_P x^c x^d + \ldots$$

{ NOTE: We would want to make a transformation that can make as many terms zero as possible }

choose B^a_k $\Rightarrow$ such that $g_{ab}\big|_0 = \eta_{ab}$

$\downarrow$ N^2 choices

$\hookrightarrow \frac{N(N+1)}{2}$ (no. of conditions to satisfy)

rotations, lateral boost

$(3R + 3L)$ i.e. $\boxed{6}$

Thus as $N^2 > \frac{N(N+1)}{2}$ so we can make this selection AND $\Rightarrow$ we will be left with:

$N^2 - \frac{N(N+1)}{2} = \frac{N(N-1)}{2}$ degrees of freedom

choose C^a_{kl} $\Rightarrow$ such that $\partial_c g_{ab}\big|_P = 0$

{ NOTE: By analogy:

ϕ $\quad \nabla\phi \quad \nabla^2\phi$ $\rightsquigarrow$ this is responsible for tidal acceleration

$\uparrow$ Potential $\quad \uparrow$ Force

$\underbrace{g_{ab} \quad \partial g}_{\text{these 2 can be set}}$ $\quad \partial^2 g$ $\Rightarrow$ * But this term remains (curvature term) }

locally flat metric

such that there is no gravity

---

for selecting C^a_{kl} $\Rightarrow$ to make $\partial_c g_{ab}\big|_P = 0$

$\downarrow$ $\frac{N^2(N+1)}{2}$ $\quad \hookrightarrow \frac{N^2(N+1)}{2}$

thus the selection can be made

Now, to choose D^a_{klm} $\Rightarrow$ such that $\partial_c \partial_d g_{ab}\big|_P = 0$

symmetric in 3 dimensions

$N \times (?)$ $\quad \hookrightarrow \frac{N^2(N+1)^2}{4}$ (no. of conditions)

Consider a tensor in N dimensions which is symmetric in S indices

eg: B_{ij} (for 2 dimensions & 2 symmetric)

$\hookrightarrow B_{00}, B_{01}, B_{11}$ are the independent indices

B_{00} $\quad \bullet\bullet\,|$

B_{01} $\quad \bullet\,|\,\bullet$

B_{11} $\quad |\,\bullet\bullet$

this is described by
the no. of solutions of:

$$\#_0 + \#_1 = 2$$

the no. of ways
can be thought of as
juggling 2 balls & 1 partition
(i.e. 3 objects in total):

$$\frac{3!}{\underbrace{2!}_{\text{2 identical balls}}\ \underbrace{1!}_{\text{1 identical partition}}}$$

$$\underbrace{\#_0 + \#_1 + \dots + \#_{N-1}}_{\text{For N bins}} = S$$

here we get:

$$\frac{(N+S-1)!}{(N-1)!\,S!}$$

For the case of 3 dimensions $(0,1,2)$:

$$\frac{(N+3-1)!}{(N-1)3!} = \frac{(N+2)!}{(N-1)!3!}$$

$$= \frac{(N+2)(N+1)N}{6}$$

Thus for D^a_{klm} and for $\partial_c \partial_d g_{ab}|_p = 0$

$$\frac{N^2(N+2)(N+1)}{6} < \frac{N^2(N+1)^2}{4}$$

(no. of choices) conditions to be satisfied

(for $N=4$)

Thus there are fewer choices than reqd. conditions

$\Downarrow$

* Hence, the genuine curvature cannot be set to zero

i.e. $\partial_c \partial_d g_{ab}|_p \neq 0$

73] We know:

$$A^{k'} = \underbrace{\frac{\partial x^{k'}}{\partial x^{l}}}_{L^{k'}_{l}} A^{l} \quad ; \quad B_{l'} = \underbrace{\frac{\partial x^{k}}{\partial x^{l'}}}_{L^{k}_{l'}} B_{k}$$

So now we can write:

$$A^{l'} B_{l'} = \frac{\partial x^{l'}}{\partial x^{k}} A^{k} \frac{\partial x^{i}}{\partial x^{l'}} B_{i}$$

$$= \underbrace{\frac{\partial x^{l'}}{\partial x^{k}} \frac{\partial x^{i}}{\partial x^{l'}}} A^{k} B_{i}$$

$$= \delta^{i}_{k} A^{k} B_{i} = A^{k} B_{k}$$

* $$T^{a'b'\cdots}_{i'j'\cdots} = \left(\frac{\partial x^{\prime a}}{\partial x^{p}} \frac{\partial x^{\prime b}}{\partial x^{q}} \cdots\right)\left(\frac{\partial x^{k}}{\partial x^{i'}} \frac{\partial x^{m}}{\partial x^{j'}} \cdots\right) T^{pq\cdots}_{km\cdots}$$

<u>NOTE</u>: Now we'll use g_{ab} instead of η_{ab} for all purposes (including raising & lowering of indices)

* $$ds^2 = g_{i'j'} dx^{i'} dx^{j'} = g_{i'j'} \frac{\partial x^{i'}}{\partial x^{k}} \frac{\partial x^{j'}}{\partial x^{l}} dx^{k} dx^{l} = g_{kl} dx^{k} dx^{l}$$

Hence we get $\Rightarrow$ $$\boxed{g_{i'j'} = \frac{\partial x^{k}}{\partial x^{i'}} \frac{\partial x^{l}}{\partial x^{j}} g_{kl}}$$

* <u>NOTE</u>: There is <u>NO preferred coordinate transformation</u> in General Relativity

LECTURE 15

28/08/2023

74] We know now that we should write:

$$ds^2 = g_{ab}\, dx^a dx^b$$

↳ also we know that we CANNOT make second-order derivative vanish

(i.e. flatness (locally) is achievable only upto first order)

75] Now we will study motion of particles on the spacetime:

* NOTE: $\frac{d}{d\tau} \equiv u^c \partial_c$

Thus we saw earlier: $\underbrace{u^a \partial_a u^c}_{} = 0 = \frac{du^c}{d\tau}$; $x^{c'} = f(x^k)$

↓

* However this is not generally covariant

↓

Hence we need to change this

76] The action can be written as:

$$\mathcal{A} = -m \int d\tau \qquad \text{where, } d\tau = (-g_{ab}\,dx^a dx^b)^{1/2} = (-ds^2)^{1/2}$$

Also, $\delta(d\tau^2) = 2\,\delta(d\tau)\,d\tau$

$\Rightarrow$ and, $\therefore \delta(d\tau) = \frac{1}{2\,d\tau}\delta(d\tau^2) = \frac{1}{2\,d\tau}\delta(-g_{ab}\,dx^a dx^b)$

{ NOTE: $\delta\mathcal{A} = -m\int \delta(d\tau)$ }

↓

$$= \frac{-1}{2\,d\tau}\delta(g_{ab})\,dx^a dx^b - \frac{1}{d\tau} g_{ab}\, dx^a\, \delta(dx^b)$$

$$\Rightarrow \quad = -\frac{1}{2}\partial_c(g_{ab})\,\delta x^c\, u^a u^b\, d\tau - \underbrace{g_{ab} u^a d(\delta x^b)}_{u_b\, d(\delta x^b)}$$

$$= -\frac{1}{2}\partial_c(g_{ab})\, u^a u^b\, \delta x^c\, d\tau \; \underbrace{- u_b\, d(\delta x^b)}_{-d(u_b \delta x^b) + \frac{du_c}{d\tau}\delta x^c d\tau}$$

Thus,

$$\delta A = -m \int -d(u_b \delta x^b) + \left[\frac{du_c}{d\tau} - \frac{1}{2}\partial_c g_{ab} u^a u^b\right] \delta x^c d\tau$$

From this we get the Hamilton-Jacobi eq^n.:

* $$g^{ab}\frac{\partial A}{\partial x^a}\frac{\partial A}{\partial x^b} = -m^2$$

(similar to that for flat spacetime
↓
except η^{ab} becomes g^{ab})

Further we can solve it as:

$$\delta A = m\,(u_b \delta x^b)\Big|_1^2 + m\int \underbrace{\left[\frac{du_c}{d\tau} - \frac{1}{2}\partial_c g_{ab} u^a u^b\right]}_{\text{from this we get}} \delta x^c d\tau$$

⇓

$$\boxed{\frac{du_c}{d\tau} = \frac{1}{2}\partial_c g_{ab} u^a u^b}$$

GEODESIC EQUATION

From here we can see

* that if the metric (g_{ab}) does NOT depend on the coordinates
↓
then $\partial_c g_{ab} = 0$
& we get back $\frac{du_c}{d\tau} = 0$

** { NOTE: Ques: Find the eq^n of straight line for $dl^2 = dr^2 + r^2 d\theta^2$ → this is in flat space
↓
imp. for MINOR
{ you can take the Lagrangian $\mathcal{L} = m\dot{r}^2 + mr^2\dot{\theta}^2$ } }

77] The geodesic equation can be written in a better form as well :

We know: $\dfrac{du_c}{d\tau} = \dfrac{d}{d\tau}(g_{ac}u^a) = \dfrac{1}{2}(\partial_c g_{ab})u^a u^b$

$$\Rightarrow g_{ac}\frac{du^a}{d\tau} + u^a\frac{dg_{ac}}{d\tau} = \frac{1}{2}\partial_c g_{ab}u^a u^b$$

$$\left(\text{here } \frac{d}{d\tau} \equiv u^b\partial_b\right)$$

$$\therefore \Rightarrow g_{ac}\frac{du^a}{d\tau} + u^a u^b\partial_b g_{ac} = \frac{1}{2}\partial_c g_{ab}u^a u^b$$

since this is symmetric

thus only symmetric part of this will contribute

Hence $\Rightarrow$ $g_{ac}\dfrac{du^a}{d\tau} + \dfrac{1}{2}u^a u^b(\partial_b g_{ac} + \partial_a g_{bc}) = \dfrac{1}{2}\partial_c g_{ab}u^a u^b$

So we thus have :

$$g_{ac}\frac{du^a}{d\tau} = -\frac{1}{2}(-\partial_c g_{ab} + \partial_b g_{ac} + \partial_a g_{bc})u^a u^b$$

Multiplying both sides with g^{kc} :

$$\frac{du^k}{d\tau} = -\frac{1}{2}g^{kc}[-\partial_c g_{ab} + \partial_b g_{ac} + \partial_a g_{bc}]u^a u^b$$

OR

$$\frac{du^k}{d\tau} + \underbrace{\frac{1}{2}g^{kc}[-\partial_c g_{ab} + \partial_b g_{ac} + \partial_a g_{bc}]}_{\text{this is a 3-index object which can be defined as:}}u^a u^b = 0$$

$$\boxed{\frac{du^k}{d\tau} + \Gamma^k_{ab}u^a u^b = 0}$$

Alternate form of GEODESIC EQUATION

78] Here, we see:

$$\Gamma^k_{ab} = \frac{1}{2} g^{kc} (-\partial_c g_{ab} + \partial_a g_{bc} + \partial_b g_{ac})$$

called

* <u>Christoffel Symbols</u>

or also

Affine connections

or

Levi-Civita connections

* NOTE! Γ^k_{ab} is a 3-index object but <u>NOT a tensor</u>

Since it doesn't transform that way

$$\Big(\text{i.e. } \Gamma^{a'}_{b'c'} = L^{a'}_{i} L^{j}_{b'} L^{k}_{c'} \Gamma^{i}_{jk} + \underbrace{\ldots}_{\text{these extra terms also exist}} \Big)$$

79] Since we can write $\frac{du^k}{d\tau}$ as $u^a \partial_a u^k$

$$\therefore \; u^a \partial_a u^k + \Gamma^k_{ab} u^a u^b = 0$$

$$\Rightarrow \; u^a (\partial_a u^k + \Gamma^k_{ab} u^b) = 0$$

Here we can see:

$$\partial_{a'} u^{b'} = L^{i}_{a'} \partial_i (L^{b'}_{k} u^k)$$

$$= L^{i}_{a'} L^{b'}_{k} \partial_i u^k + \underbrace{L^{i}_{a'} u^k \partial_i L^{b'}_{k}}_{\text{due to this term we see that}}$$

* $\partial_a u^k$ does NOT transform as a tensor

Thus $\Gamma^k_{ab} u^b$ also DOES NOT transform as a <u>tensor</u>

We can now define :

$$\nabla_a u^b = \underbrace{\partial_a u^b}_{\text{normal derivative}} + \underbrace{\Gamma^b_{ca} u^c}_{\text{correction}}$$

called

* <u>Covariant derivative</u>

From this we get :

$$u^a \left(\partial_a u^k + \Gamma^k_{ab} u^b \right) = 0$$

$$\therefore \Rightarrow \boxed{u^a \nabla_a u^k = 0}$$

* most compact form of <u>GEODESIC EQUATION</u>

* i.e. eqⁿ of <u>straight line</u> (in GR)

<u>NOTE</u>: $\nabla_a u^b$ <u>is a tensor</u> since it transforms that way :

* $$\nabla_{a'} u^{b'} = L^i_{a'} L^{b'}_j \nabla_i u^j$$

<u>NOTE</u>: Conceptually :

$$\underbrace{u^a \nabla_a}_{\text{this is like Directional derivative}} \underbrace{u^k}_{\text{This is like tangent to curve}} = 0$$

i.e. the directional derivative of tangent to curve is zero

(since the geodesic is essentially a straight line in curved spacetime)

<u>NOTE</u>: Remember how to write covariant derivative :

$$\nabla_k u^l = \partial_k u^l + \underbrace{\Gamma^l_{ak} u^a}_{\text{correction term}}$$

← dummy index included

80] * Also :

$$\underbrace{\nabla_i (A^k B_k)}_{\partial_i (A^k B_k)} = (\nabla_i A^k) B_k + A^k \nabla_i B_k$$

$$\partial_i (A^k B_k) = (\partial_i A^k + \Gamma^k_{ij} A^j) B_k + A^k \nabla_i B_k$$

$$\Rightarrow \underline{B_k(\partial_i A^k)} + A^k(\partial_i B_k) = (\underline{\partial_i A^k} + \Gamma^k_{ij} A^j) B_k + A^k \nabla_i B_k$$

Rearranging

$$\therefore \Rightarrow \quad A^k \nabla_i B_k = A^k \partial_i B_k - \Gamma^p_{ik} B_p A^k$$

$$= (\partial_i B_k - \Gamma^p_{ik} B_p) A^k$$

$$\therefore \Rightarrow \boxed{\nabla_i B_k = \partial_i B_k - \Gamma^p_{ik} B_p}$$ ~ for general vector B

* In general for tensors (2-index) we can prove: [H.W. to prove this]

$$\boxed{\nabla_k T^a_b = \partial_k T^a_b + \Gamma^a_{pk} T^p_b - \Gamma^p_{bk} T^a_p}$$

From here it can be proven that for the metric tensor, the covariant derivative is zero !

$$* \quad \boxed{\nabla_k g_{ab} = 0}$$ called METRICITY CONDITION

{* All theories of gravity (including this one, i.e. General Relativity)
↓
are obtained by using an action that is generally covariant
↓
The GR action (which we took) is just the simplest most general action
↓
if we use other actions, then we will get alternate theories }

{* NOTE: In GR, the Christoffel symbol is symmetric
↓
and thus there is NO Torsion

But alternative theories could have non-symmetric, Γ i.e ⇒ they will then have torsion : $T^k_{ab} = \Gamma^k_{ab} - \Gamma^k_{ba} \neq 0$ }

81] Remember that :

$$* \quad \nabla_a A^b = \partial_a A^b + \Gamma^b_{ka} A^k$$

and also :

$$* \quad \underbrace{\Gamma^a_{bc}}_{} = \frac{1}{2} g^{ak}\left(-\partial_k g_{bc} + \partial_b g_{kc} + \partial_c g_{kb}\right)$$

this has

$\frac{N^2(N+1)}{2}$ components $\rightsquigarrow$ we can thus obtain

$\Downarrow$ $\qquad \Gamma(P) = 0$

$\frac{16 \times 5}{2} = \boxed{40}$ components

* Ques : Consider the lagrangians :
[H.W.]

$$\mathcal{L} = \dot{r}^2 + r^2 \dot{\theta}^2$$

$$\mathcal{L} = \dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2$$

Vary these lagrangians, find the equation of motion, and from that find the non-zero Γs

NOTE: In Non-relativistic approximation :

$$\frac{du_c}{d\tau} = \frac{1}{2} \partial_c g_{ab} u^a u^b$$

$g_{00} \sim -(1+2\phi)$

$g_{\alpha\beta} = \delta_{\alpha\beta}$

[Weak field gravity]

Here : $d\tau \sim dt$

$u^0 = 1$

$u^\alpha = v^\alpha$

$$\Rightarrow \therefore \frac{dv^\alpha}{dt} = \frac{1}{2} \partial_\alpha g_{00} = -\partial_\alpha \phi$$

$$\Rightarrow \boxed{\frac{d\underset{\sim}{v}}{dt} = -\underset{\sim}{\nabla}\phi}$$

which gives back Newtonian gravity (showing that we are on the right track !)

*[H.W.] : Find the geodesic equation in (2+1)-dimensional spacetime.
{for MINOR}

LECTURE 16

29/08/2023

NOTE: We have seen:

* $ds^2 = g_{ab}\, dx^a dx^b$
* $\frac{du_c}{d\tau} = \frac{1}{2} \partial_c g_{ab} u^a u^b$

$\Rightarrow$ alternatively, $\frac{du^k}{d\tau} + \Gamma^k_{ab} u^a u^b = 0 \overset{\text{i.e.}}{\Rightarrow} \frac{d^2x^k}{d\tau^2} + \Gamma^k_{ab} \frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = 0$

where:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ak}(-\partial_k g_{bc} + \partial_b g_{kc} + \partial_c g_{kb})$$

also, alternatively:

$\Rightarrow u^a \nabla_a u^b = 0$

where:

$$\nabla_k u^l \equiv \partial_k u^l + \Gamma^l_{pk} u^p$$

$\therefore$ and, $\nabla_k u_l = \partial_k u_l - \Gamma^p_{kl} u_p$

NOTE: This also gives us:

* $\nabla_k g_{ab} = 0$

* **NOTE:** The christoffel symbol is essentially related to "∂g" {Conceptually}

i.e. $\Gamma \sim \partial g \rightarrow$ "forces"

Sidenote: Problems you can work on in Physics (for Projects) or otherwise

CM : Celestial Mechanics
H-J Theory
Non-linear Dynamics

Fluid Mechanics - L/O limit
MHD
Plasma Oscillations

Quantum Mech. : TDSE Numerically
J-C Model (Jaynes-Cummings Model)
Path Integrals

Optics : Caustics
Paraxial Optics
↳ Q.M.

NOTE: Compute things for following metrics: (i.e. finding all things as we did, leading to geodesic eq^n)

$$ds^2 = -\rho^2 dt^2 + d\rho^2$$

$$ds^2 = -dt^2 + a^2(t)\,dx^2$$

$$dl^2 = \underbrace{\frac{dx^2}{1-x^2} + dy^2}_{\text{same as}}$$ ~> this is essentially a 2-sphere

$$\Rightarrow \quad d\theta^2 + \sin^2\theta\, d\phi^2$$

$$dl^2 = dr^2 + r^2 d\theta^2$$

For: $x^2+y^2+z^2=1 \Rightarrow dx^2+dy^2+dz^2 \rightsquigarrow \mathbb{R}^3$

$$x^2+y^2-z^2=-1$$

$$\underbrace{x^2+y^2+z^2+w^2=1}$$

$\Downarrow$ i.e.

$$dl^2 = dx^2+dy^2+dz^2+dw^2 \rightsquigarrow \mathbb{R}^4$$

82] We write:

$$\mathcal{A} \propto \int \sqrt{-\frac{dx_a}{d\lambda}\frac{dx^a}{d\lambda}}\, d\lambda$$ (for some reparameterized form using λ)

Then, $\tau = f(\lambda)$; $u^k = \frac{dx^k}{d\tau}$; $v^k = \frac{dx^k}{d\lambda}$

Then! we get $\frac{d}{d\tau} = \frac{1}{f'(\lambda)}\frac{d}{d\lambda}$ or $\frac{1}{f'}\frac{d}{d\lambda}$

Using this we can write the geodesic eq^n as:

$$\frac{d}{d\lambda}\left(\frac{1}{f'}\frac{dx^k}{d\lambda}\right) + \frac{\Gamma^k_{ab}}{f'}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} = 0$$

$$\Rightarrow \frac{d^2x^k}{d\lambda^2} + \Gamma^k_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} = \frac{f''}{f'}\frac{dx^k}{d\lambda}$$

Defining, $\frac{f''}{f'} = g(\lambda)$

we get: $$\boxed{v^a \nabla_a v^k = g(\lambda)\, v^k}$$ GEODESIC EQUATION (in reparametrized form)

Here if we take

$$\underbrace{\bar{\lambda} = a\lambda + b}_{\substack{\Downarrow \\ \text{then it is} \\ \text{called an}}} \Rightarrow v^a \nabla_a v^k = 0$$

such that

* "affine parameter"

* NOTE: This is useful for light wave!

$$k^b \nabla_b k^a = 0 \quad , \; k^a = (k, 0, 0, k)$$

where $k_a k^a = 0$

{ k is the tangent vector to the light wave }

83] It turns out now that d^4x will NOT be invariant

+

instead

$$\sqrt{-g}\, d^4x = \sqrt{-g'}\, d^4x'$$

{ we will see how this happens }

* NOTE: We will use the property: (identity)

$$\underset{\substack{\downarrow \\ \text{trace}}}{Tr} (\ln \underset{\substack{\downarrow \\ \text{some matrix}}}{M}) = \ln (\underset{\substack{\downarrow \\ \text{determinant}}}{\det} M)$$

{ log of matrix is found using Taylor expansion }

84] Derivative of Determinant of Metric Tensor

Taking $g = \det(g_{ab})$

Now, $$\frac{\delta g}{g} = g^{ab}\, \delta g_{ab}$$

$$\Rightarrow \frac{\partial_i g}{g} = g^{ab}\, \partial_i g_{ab} \overset{\text{or}}{\Rightarrow} \frac{1}{2} \frac{-\partial_i g}{-g} = \frac{1}{2} \underbrace{g^{ab}\, \partial_i g_{ab}}_{\text{this is like}}$$

$Tr(M^{-1} \delta M)$

$\therefore$ We can say:

$$\boxed{\partial_i \ln \sqrt{-g} = \frac{1}{2} g^{ab}\, \partial_i g_{ab}}$$

* NOTE: $\nabla_i (\phi) = \partial_i \phi$

85] We now want to obtain $\partial_k g^{ab}$ in terms of $\partial_k g_{ab}$:

we know, $\partial_i (\delta^k_l) = 0$

$$\Rightarrow \partial_i (g^{ka} g_{al}) = 0$$

$$\Rightarrow g^{bl} g_{al} (\partial_i g^{ka}) = -(\partial_i g_{al}) g^{ka} g^{bl}$$

Thus, $\delta^b_a \partial_i g^{ka} = -g^{ka} g^{bl} \partial_i g_{al}$

$$* \Rightarrow \boxed{\partial_i g^{kl} = -g^{ka} g^{lb} \partial_i g_{ab}}$$

86] Next we want to be able to find Γ^k_{ak}:

{ NOTE: we want this bcz it is used in:

$$\nabla_i A^i = \partial_i A^i + \Gamma^i_{pi} A^p \}$$

we have:

$$\Gamma^k_{ak} = \frac{1}{2} \underbrace{g^{kp}}_{\substack{\text{this is} \\ \text{symmetric} \\ \text{in } k \leftrightarrow p}} (\underbrace{-\partial_p g_{ak} + \partial_a g_{pk} + \partial_k g_{ap}}_{\substack{\text{antisymmetric in} \\ k \leftrightarrow p}})$$

$$= \frac{1}{2} g^{kp} \partial_a g_{pk}$$

Hence using our previous result $\Rightarrow \boxed{\Gamma^k_{ak} = \partial_a \ln \sqrt{-g}}$

87] Finally using this we can write:

$$\nabla_i A^i = \partial_i A^i + \Gamma^i_{pi} A^p$$

$$= \frac{1}{\sqrt{-g}} \sqrt{-g}\, \partial_i A^i + \frac{\partial_p(\sqrt{-g})}{\sqrt{-g}} A^p$$

$$= \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, \partial_p A^p + \partial_p (\sqrt{-g}) A^p \right)$$

Hence we finally obtain the required result:

$$\boxed{\nabla_p A^p = \frac{1}{\sqrt{-g}} \partial_p \left(\sqrt{-g}\, A^p\right)}$$

* NOTE: The action for electromagnetism would become:

$$S = -m \int \sqrt{-g_{ab}\, dx^a dx^b} + q \int A_k \, dx^k$$

we would still get:

$$m \frac{du^k}{d\tau} = q F^{kl} u_l$$

However now:

$$F_{ab} = \nabla_a A_b - \nabla_b A_a$$

but on solving this:

$$= \partial_a A_b - \cancel{\Gamma^p_{ab} A_p} - \partial_b A_a + \cancel{\Gamma^p_{ab} A_p}$$

$$= \partial_a A_b - \partial_b A_a$$

Thus (luckily) electromagnetism survives (and doesn't need to be modified for curved spacetime)

* NOTE: If we consider

$$\mathcal{L} \propto (\partial_a \phi^*)(\partial^a \phi) \quad , \text{ where } \phi \to e^{-i\Lambda(x)} \phi$$

$$\phi^* \to e^{i\Lambda(x)} \phi \quad \text{(transformation)}$$

Here the form is NOT preserved:

since: $\partial_a (e^{i\Lambda} \phi) = (\partial_a \phi) e^{i\Lambda} + i\Lambda' \phi$

However if we want interactions b/w charged particles, we must redefine:

$$\phi \to e^{-iq\Lambda(x)} \phi$$
$$\phi^* \to e^{iq\Lambda(x)} \phi$$

{q is included} which acts as charge conceptually

AND we must also define some A such that: $\partial_k \to i\partial_k - qA_k \equiv \nabla_k$

where, $A_k \to A_k + \partial_k \Lambda$

So we get: {ON NEXT PAGE}

we get: $\mathcal{L} \propto -(\nabla_k \phi^*)(\nabla^k \phi)$

and it turns out that this A_k we included is going to act as the vector potential

↓

Thus if we WANT interaction b/w charged particles, it is going to lead to electromagnetism

{ If interactions were not required, then we could just use ϕ instead of ϕ^* and write $\mathcal{L} \propto (\partial_a \phi)(\partial^a \phi)$ }

LECTURE 17

01/09/2023

NOTE: References for Hamilton-Jacobi Theory:

* Theoretical Astrophysics - Paddy (vol 1)
* Hamilton-Jacobi Theory (Notes) - by Samir Mathur (at Ohio State University)

88] We can observe symmetries
↓
to find out conservation laws

eg: for 2D space: $dl^2 = dx^2 + dy^2$

since x & y are independent
⇓
∴ This symmetry gives us:
p_x and p_y are conserved

Similarly we can write this as:

$$dl^2 = dr^2 + r^2 d\theta^2$$

(and in spacetime we include $-dt^2$ as well)

⇓
we can get conservation of energy (E) and angular momentum as well

For the case of:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

⇓
Symmetries are:
3 spatial translations
1 time translation
3 spatial rotations
& also 3 Lorentz boosts

⇒ ∴ Total 10 symmetries (which can give 10 conservation laws)

89] NOTE: We have seen:

* $\nabla_k u^l = \partial_k u^l + \Gamma^l_{kp} u^p$
* $\Gamma^k_{ik} = \partial_i \ln\sqrt{-g}$
* $\nabla_i A^i = \dfrac{1}{\sqrt{-g}} \partial_i (\sqrt{-g}\, A^i)$

* Also, remember that in Special relativity we saw that

$$\text{if } \partial_k J^k = 0 \underset{\text{this implied}}{\Rightarrow} \int_V d^4x \underbrace{(\partial_k J^k)}_{0} = \int_{\partial V} d^3x \, n_k J^k$$

{i.e. we got some Conservation law}

[in other words:

$\partial_k J^k = 0 \Rightarrow$ Conservation law]

Similarly here also:

$$\int_V d^4x \sqrt{-g} \, \nabla_k J^k = \int_V d^4x \sqrt{-g} \frac{1}{\sqrt{-g}} \partial_k (\sqrt{-g} \, J^k)$$

which should be equal to $\Rightarrow$

$$= \int_{\partial V} d^3x \, n_k \sqrt{-g} \, J^k$$

NOTE: Here:

$$ds^2 = g_{ab} \, dx^a dx^b$$

$$= g_{00} (dx^0)^2 + 2 g_{0\alpha} dx^0 dx^\alpha + g_{\alpha\beta} dx^\alpha dx^\beta$$

$\Rightarrow$ here there is NO way to tell which is exactly the spatial part {i.e. the d^3x used above} $\rightsquigarrow$ we will need to see this from the specific metric given to us

90] Hence we can write the action for Electromagnetism in curved spacetime as:

$$\mathcal{A}^{EM} = \int d^4x \sqrt{-g} \, A_k J^k - \frac{1}{4} \int d^4x \sqrt{-g} \, F_{ab} F^{ab}$$

(this action is now generally covariant)

From here we will get: (by varying the action)

$$\nabla_i F^{ik} = 4\pi J^k$$

Now, $\nabla_i F^{ik} = \partial_i F^{ik} + \Gamma^i_{pi} F^{pk} + \underbrace{\Gamma^k_{\overleftrightarrow{pi}} F^{\overleftrightarrow{ip}}}_{\text{this terms cancels out}}$

(due to antisymmetry of F^{ip})

$$= \frac{\sqrt{-g}}{\sqrt{-g}} \partial_i F^{ik} + \frac{(\partial_p \sqrt{-g}) F^{pk}}{\sqrt{-g}}$$

$$= \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} F^{ik})$$

NOTE: For a symmetric tensor:

$$\nabla_i T^{ij} = \partial_i T^{ij} + \Gamma^i_{pi} T^{pj} + \Gamma^j_{pi} T^{ip}$$

$$= \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g}\, T^{ij}) + \underbrace{\Gamma^j_{pi} T^{ip}}_{\text{this term persists}}$$

NOTE: We saw for scalar fields the Energy-momentum tensor:

$$T^a_b = -(\Pi^a \partial_b \phi - \delta^a_b \mathcal{L}) \Rightarrow T^{ab} = \int \frac{d^3 \underset{\sim}{p}}{E_{\underset{\sim}{p}}} p^a p^b f(\underset{\sim}{p})$$

Here, $T^{00} \rightsquigarrow E_{\underset{\sim}{p}}$ density

$T^{0\alpha} \rightsquigarrow \underset{\sim}{p}$ density

We saw in Special relativity:

$$\text{if } \partial_a T^a_b = 0 \Rightarrow \int_V d^4x\, \partial_a T^a_b = \int_{\partial V} d^3x\, n_a T^a_b$$

which gave us that T^a_b is conserved

But we need to check what will happen in General relativity.

91] We saw, $\nabla_i T^{ij} = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g}\, T^{ij}) + \Gamma^j_{pi} T^{ip}$

* Now, $\Gamma^j_{pi} T^{ip} = \frac{1}{2} g^{ja} (-\partial_a g_{pi} + \partial_i g_{ap} + \partial_p g_{ai}) T^{ip}$

$\Rightarrow$ on solving this gives $= T^{ab} \partial_i g_{ab}$

Hence:

$$\nabla_i T^{ij} = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g}\, T^{ij}) + T^{ab} (\partial_k g_{ab})\, g^{jk}$$

92] From this if we try to write the conservation law:

$$\int_V d^4x \sqrt{-g}\, \nabla_i T^{ij} = \int_{\partial V} d^4x\, n_i T^{ij} \sqrt{-g} + \int_V d^4x \underbrace{\sqrt{-g}\, g^{jk}\, T^{ab}\, \partial_k g_{ab}}$$

$\Downarrow$ we get this additional term which spoils our conservation law for $\nabla_k T^k_l = 0$

* But, if we can get some coordinate for which $\partial_k g_{ab} = 0$

$\downarrow$ then that term vanishes

93] If we take some quantity $\xi^i(x) \underset{\text{i.e.}}{\rightarrow} \xi_i = g_{ai} \xi^a$

and, we can take: $\nabla_i (T^{ij} \xi_j) = \underbrace{\nabla_i (T^{ij})}\, \xi_j + T^{ij} \nabla_i \xi_j$

* for divergenceless condition (i.e. $\nabla_a T^a_b = 0$) this vanishes

$\therefore \Rightarrow \nabla_i (T^{ij} \xi_j) = T^{ij} \nabla_i \xi_j$

Now, $\int_V d^4x \sqrt{-g}\, \nabla_i (T^{ij}\xi_j) = \int_{\partial V} d^3x \underbrace{(n_i T^{ij} \xi_j)}_{\text{this is a scalar}} \sqrt{-g}$

Thus from here we CAN get a conserved quantity

We now need to choose our ξ_j

Earlier just like

$\nabla_k J^k = 0 \Rightarrow$ Conserved quantity

Similarly here:

we will take: $\nabla_i (T^{ij}\xi_j) = 0$

$\left\{ \text{* i.e. we need BOTH } \nabla_a T^a_b = 0 \text{ and } \nabla_i (T^{ij}\xi_j) = 0 \right\}$

Now, from $\nabla_i (T^{ij}\xi_j) = T^{ij} \nabla_i \xi_j = 0$

$\downarrow$ (now since T^{ij} is symmetric but $T^{ij} \neq 0$)

$\Rightarrow T^{ij}(\nabla_i \xi_j + \nabla_j \xi_i) = 0$

$\Rightarrow \boxed{\nabla_i \xi_j + \nabla_j \xi_i = 0}$

This is called the KILLING EQUATION

{ it was given by Lord Killing }

* This will become:

$\underbrace{\partial_i \xi_j + \partial_j \xi_i = 0}_{\text{from here we solve for } \xi_i}$