PH3915
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Simple Discounting, Compounding and Decision Criteria

Simple Compounding/Discounting

Let, P = the principal (beginning amount)

r = the interest rate

V_n = the value of the amount at the end of year n,

Assume, P = \$100 (e.g. a bond) and r = 10%

Then, $V_1 = P + r \cdot P = P(1 + r) = \$110$

$V_2 = V_1(1 + r) = P(1 + r)(1+r) = P(1 + r)^2 = 121$

$V_6 = V_5(1 + r) = P(1 + r)(1 + r)(1 + r)$

$(1 + r)(1 + r) = 177$

or, $P(1 + r)^6 = V_6$

$V_n = V_{n-1}(1 + r) = P(1 + r)^n$

General formula for simple compounding is

$V_n = P(1 + r)^n$

Discounting formula reverses this process and is

$P = V_n/(1 + r)^n$

Annuities

If V_n = the value of the amount at the end of year n, and

If $V_1 = V_2 = V_3 = \ldots = V_n$, we have a series of of fixed payments called an annuity.

When this is the case, the present value formula for discounting [$P = V_n / (1 + r)^n$] , is transformed into

$$P = V\left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \cdots + \frac{1}{(1+r)^n}\right]$$

or the present discounted value of an annuity.

Note: This is equivalent to

$$P = \sum_{t=0}^{n} \frac{V_t}{(1+r)^t}$$

Where all V_t are equal.

Or,

$$P = V_t \sum_{t=0}^{n} \frac{1}{(1+r)^t}$$

Table A: Simple Discounting Example

$P = V_n / (1+r)^n = V_5 / (1.09)^5$

$= \$500/(1.09)(1.09)(1.09)(1.09)(1.09)$

$= \$500/1.5386 = \$324.97$

Alternatively, using the table, where r = 9%, and n=5 years we find 0.6499. Simply multiple 0.6499 x 500 = $324.97.

Thus, the table give us

$$\frac{1}{(1+r)^n}$$

For various r and n values, as 0.6499 = 1/1.5386.

Table B: Present Value of Annuities

$$P = V\left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^6}\right]$$

$$= 500\left[\frac{1}{1.06} + \frac{1}{(1.06)^2} + \frac{1}{(1.06)^3} + \cdots + \frac{1}{(1.06)^6}\right]$$

= 500 [4.917] = $2,458.50, the present discounted value of the annuity; i.e, 4.917 is from the table where, r = 6% and n =6 years.

Thus, table B gives us $\sum_{t=0}^{n} \frac{1}{(1+r)^t}$

As if V = 1.

This value is then multiplied by the real value of V to yield P.

Problem 1

Assume we have the opportunity to invest in a new activity and we have the following information.

REVENUE		COSTS
$R_0 = \$0$	(TODAY)	$C_0 = \$0$
$R_1 = 100$	(YEAR 1)	$C_1 = 50$
$R_2 = 300$	(YEAR 2)	$C_2 = 200$
$R_3 = 400$	(YEAR 3)	$C_3 = 400$
$R_4 = 100$	(YEAR 4)	$C_4 = 200$

R = 12%

Given $P = V_n/(1+r)^n$, find the present value of revenue and then of costs. Determine the present value of profit.

<u>Solution to Problem 1</u>

Use Table A:

R_0 = \$0	C_0 = \$0
R_1 = 100 (.8929) = 89.29	C_1 = 50= (.8929) = 44.65
R_2 = 300 (.7972) = 239.16	C_2 = 200 (.7972) = 159.44
R_3 = 400 (.7118) = 284.72	C_3 = 400 (.7118) = 284.72
R_4 = 100 (.6355) = <u>63.55</u>	C_4 = 200 (.6355) = <u>127.10</u>
\$676.72	\$615.91

Profit (discounted) = \$676.72 - \$615.91 = \$60.81

Problem 2

$R = 12\%$

REVENUES	COSTS
R_0 = \$0	C_0 = \$1000
R_1 = 100	C_1 = 600
R_2 = 300	C_2 = 600
R_3 = 500	C_3 = 600
R_4 = 800	C_4 = 600
R_5 = 800	C_5 = 700
R_6 = 800	C_6 = 200
R_7 = 800	C_7 = 200
R_8 = 800	C_8 = 200
R_9 = 800	•
•	•
•	•
•	•
R_{20} = 800	C_{20} = 200

Find the present value of revenues, costs and profits, utilizing Tables A and B.

Solution to Problem 2

R_0 = \$0	C_0 = \$1000 (no discounting)
R_1 = 100 (.8929) = 89.29	C_1 = 600(3.037) = \$1822.00 to where, 3,037 is a 4 C_4 year annuity at 12% interest
R_2 = 300(.7972) = 239.16	C_5 = 700(.5674) = 397.18
R_3 = 500(.7118) = 355.90	C_6 = 200(6.811) (.5674) = 772.91 to where 6.811 is a 15 year C_{20} annuity; 0.5674 is the 5 year discount factor at 12%
R_4 = 800(7.120) (.7118) to = 4,054.42 R_0 where, 7.120 is a 17 year annuity at 12% interest, and 0.7118 is the 3 year discount factor at r = 12%	Present value of costs = \$3,992
Present value of R = \$4,738.76	

Present value of profit = \$4,738.76 - \$3,992.29 = \$747.47

Solution to Exercise

Alternative #1:

$R_0 = \$0$	$C_0 = \$100,000$
$R_1 = 6,000(.9259) = 5555.40$	$C_1 = 2000(.9259) = 1851.80$
R_2 to R_{15} $= 20,000(8.244)(.9259) = 152,662.39$	$C_2 = 5000(.8573)= 4286.50$
R_{16} to R_{30} $= 5,000(8.559)(.3152) = 13,488.98$	C_3 to C_{32} $= 4000(11.258)(.8573) = 38,605.93$
	Scrap Value $- 1,000 \times 0.0994 = -99.40$
Present Value = \$171,706.77	Present Value = \$144,644.83

Net present value = \$27,061.94

Alternative #2:

Net present value = \$35,000

Solution: Choose alternative #2

Brief Review

(1) $V_n = P(1 + r)^n$, is the simple compounding formula

(2) $P = V_n/(1 + r)^n$ or

$$P = \sum_{t=0}^{n} V_t /(1+r)^t,$$

Is the formula for present value (simple) discounting table A calculates

$$\frac{1}{(1+r)^n}$$ for you.

(2) $$P = V\left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \cdots + \frac{1}{(1+r)^n}\right]$$

or $$P = V\sum_{t=0}^{n} 1/(1+r)^t$$

Is the formula for finding the discounted present value of an annuity.

Table B calculates $\sum_{t=0}^{n} 1/(1+r)^t$ for you.

Compounding/Discounting Multiple Times per Year

$$P = V_n / (1 + {}_r/_m)^{mn}$$

Where, m = number of times per year
which we compound
n = number of years
r = discount rate

Decision Criteria

1. Net Present Value Rule (NPV):

Formula: $$NPV = A_0 + \frac{A_1}{(1+r)} + \frac{A_2}{(1+r)^n} + \cdots + \frac{A_n}{(1+r)^n}$$

Where A = benefits less costs

r = social discount rate

n = year

The net benefits (benefits less costs) in each period are discounted to the present to yield the NPV, if NPV > 0 $\rightarrow$ accept project. If NPV < 0 $\rightarrow$ reject.

When considering mutually exclusive projects, the one with the higher NPV is recommended.

Alternatively, the formula may be stated.

$$\sum_{t=0}^{n} B_t / (1+r)^t = \sum_{t=0}^{m} C_t / (1+r)^t$$

Decision Criteria

2. Benefit – Cost Ratio (B/C):

Formula:

$$\frac{\left(b_0 + \frac{b_1}{1+r} + \frac{b_2}{(1+r)^2} + \cdots + \frac{b_n}{(1+r)^n}\right)}{\left(c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + \cdots + \frac{c_m}{(1+r)^m}\right)}$$

Where b = benefits

c = costs

n = years of benefits

m = years of costs

The discounted present value of benefits is divided by the discounted present value of costs. If b/c = > accept project. If b/c = < reject.

When considering mutually exclusive projects, the one with the higher b/c is recommended.

When the NPV and b/c ratios make contradictory recommendations, the NPV rule is superior.

$$\sum_{t=0}^{n} B_t (1+r)^t \,/\, \sum_{t=0}^{n} C(1+r)^t$$

Decision Criteria

3. Internal Rate of Return Rule (IRR):

Formula: $$A_0 + \frac{A_1}{1+i} + \frac{A_2}{(1+i)^2} + \cdots + \frac{A_n}{(1+i)^n} = 0$$

Where A = benefits less cost

I = the internal rate of return

r = the social discount rate

n = years

- If r < i = > accept project
- If r > i => reject project

❖The internal rate of return that will reduce the future net benefit stream to zero is calculated and compared to the social discount rate.

❖Where the decision-criteria conflict, the NPV rule is always superior.

Numerical Example Illustrating Comparison of Decision Rules

Assume Projects 1, 2 & 3 are Mutually Exclusive
(only one will be funded)

Basic question: Which project should be funded?

Project 1

		Discount Rate	
		r = 10%	r = 6%
Decision Rule	Net Present Value	-$56,741	$29,606
	Benefit/Cost Ratio	0.92	1.04

Project 2

		Discount Rate	
		r = 10%	r = 6%
Decision Rule	Net Present Value	$383,500	$541,767
	Benefit/Cost Ratio	1.59	1.80

Project 3

		Discount Rate	
		r = 10%	r = 6%
Decision Rule	New Present Value	$126,478	$163,721
	Benefit/Cost Ratio	2.71	3.11