22.01.2019

Statistical Methods in AI (CSE/ECE 471)

Lecture-7:

- A short detour back to NB
- Linear Regression
- Logistic Regression

Ravi Kiran

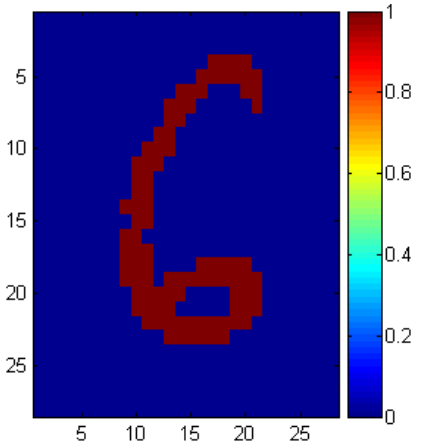
Center for Visual Information Technology (CVIT), IIIT Hyderabad

Announcements

- A2 is due Feb 2, 11.59 pm
- SMAI Mid-1 will be on Feb 7 (Thursday)
  - Syllabus: Lec 1 – Lec 8 (this week's Friday lecture)

A short detour back to Naïve Bayes

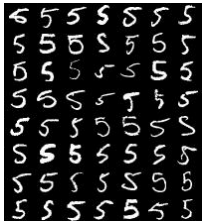
$$\arg\max_Y P(Y|X_1,\ldots,X_n)$$



Likelihood Prior

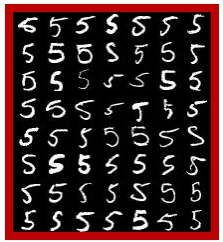
$$P(Y|X_1,\ldots,X_n) = \frac{P(X_1,\ldots,X_n|Y)P(Y)}{P(X_1,\ldots,X_n)}$$

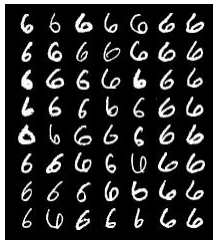
Normalization Constant



$$P(X_1,\ldots,X_n|Y) = \prod_{i=1}^{n} P(X_i|Y)$$

Generative v/s Discriminative Models





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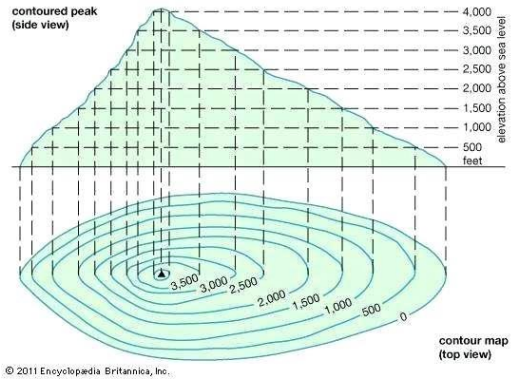
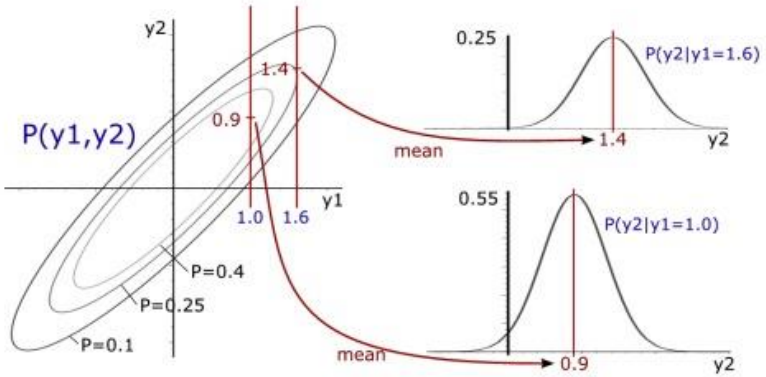
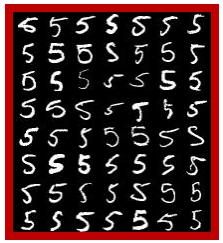
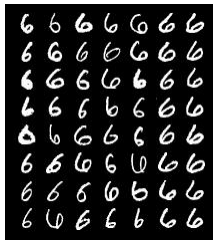




Generative v/s Discriminative Models





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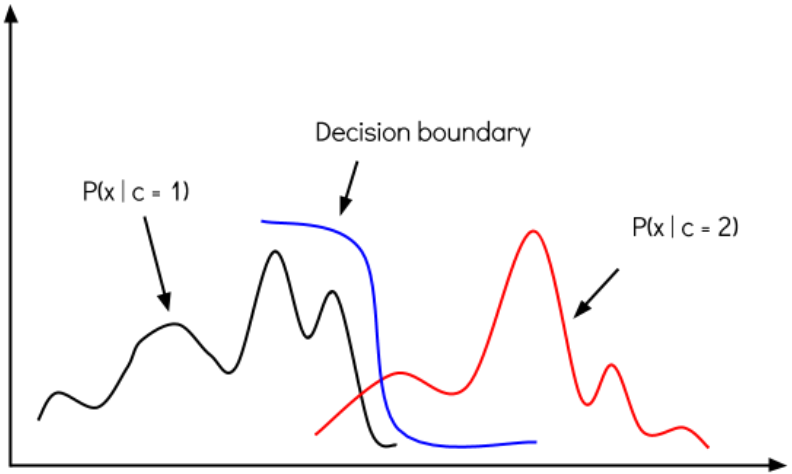
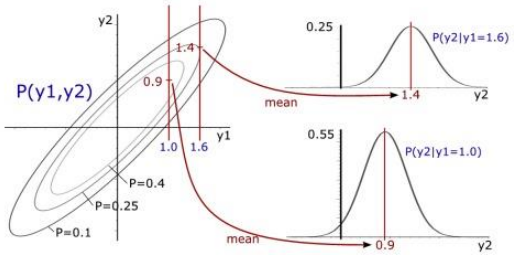
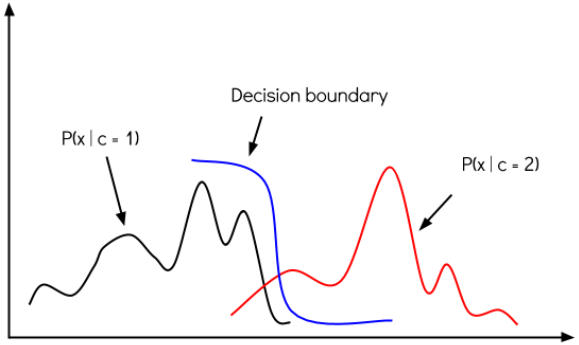


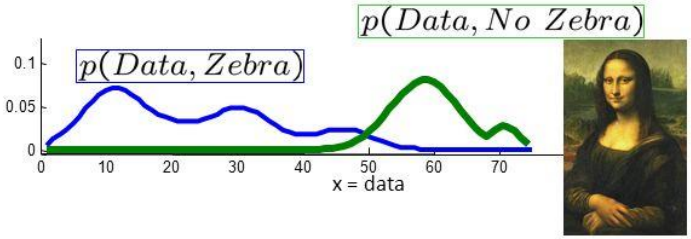


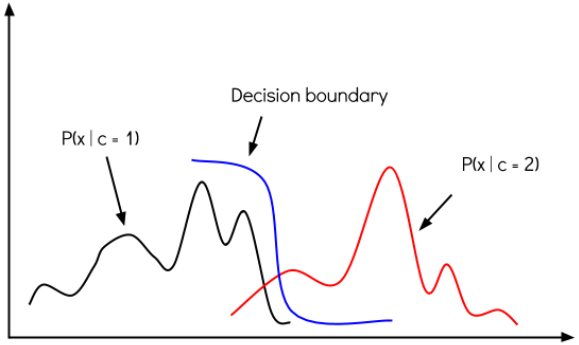


Discriminative vs. generative

- Generative model

(*The artist*)





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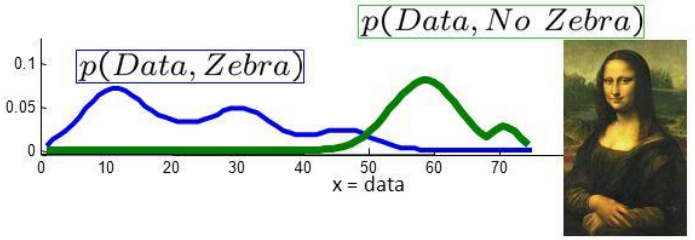
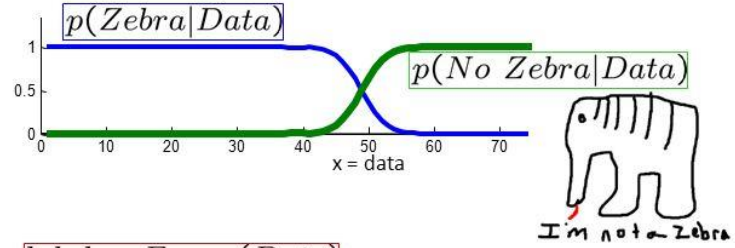
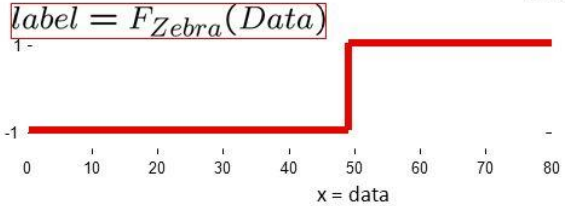


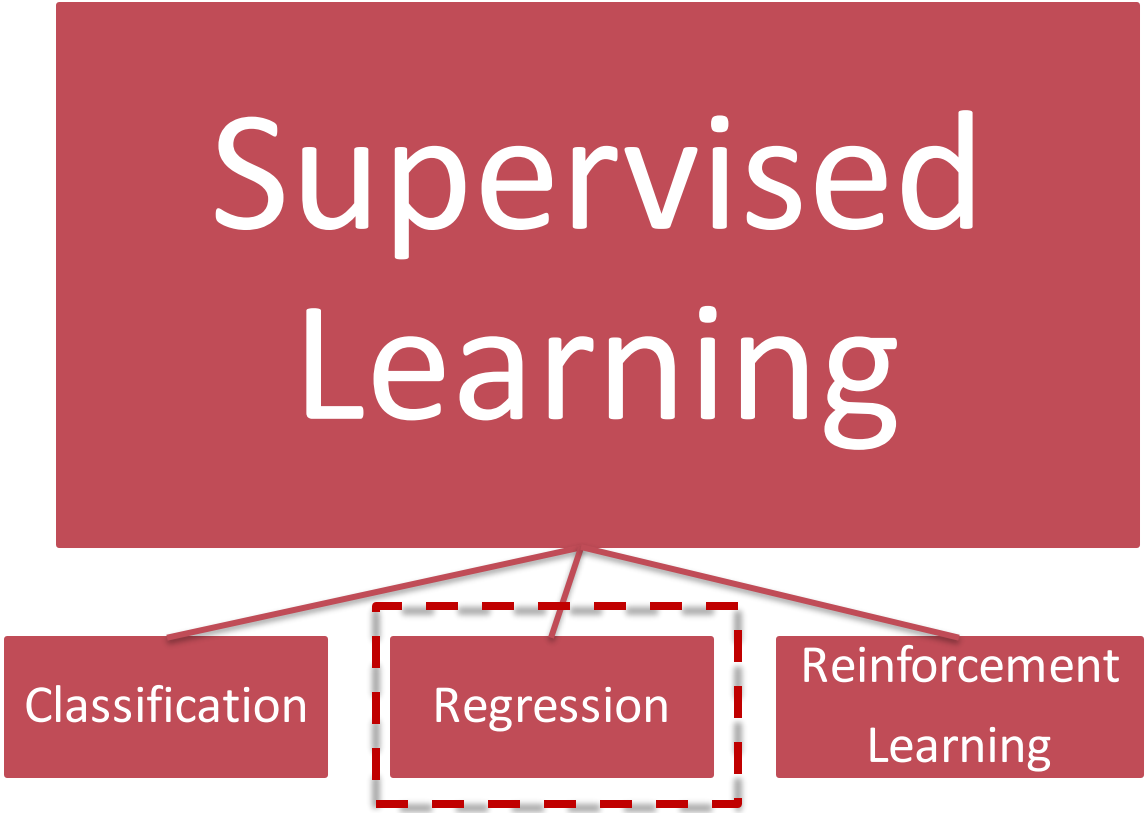
- Discriminative model

(The lousy painter)



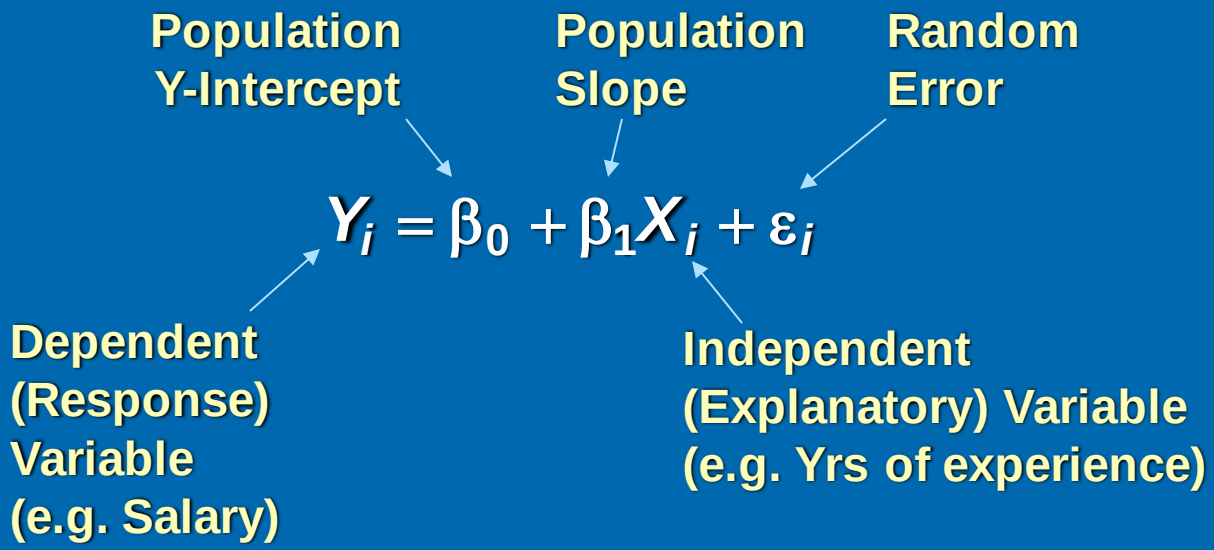
- Classification function



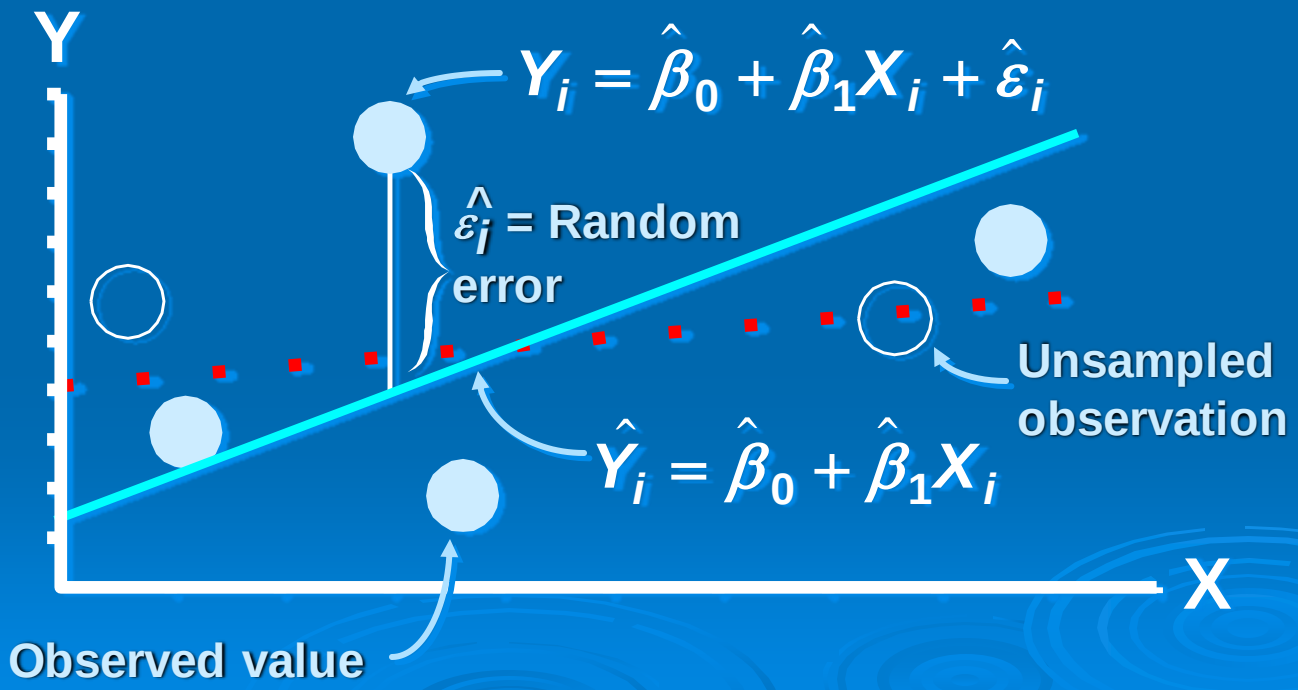

Supervised Learning
Classification
Regression
Reinforcement Learning

Linear Regression Model

- 1. Relationship Between Variables Is a Linear Function



Sample Linear Regression Model



Least Squares

- 1. ‘Best Fit’ Means Difference Between Actual Y Values & Predicted Y Values is a Minimum. *But* Positive Differences Off-Set Negative ones. So square errors!

$$\sum_{i=1}^{n}\left(Y_i - \hat{Y}_i\right)^2 = \sum_{i=1}^{n}\hat{\varepsilon}_i^2$$

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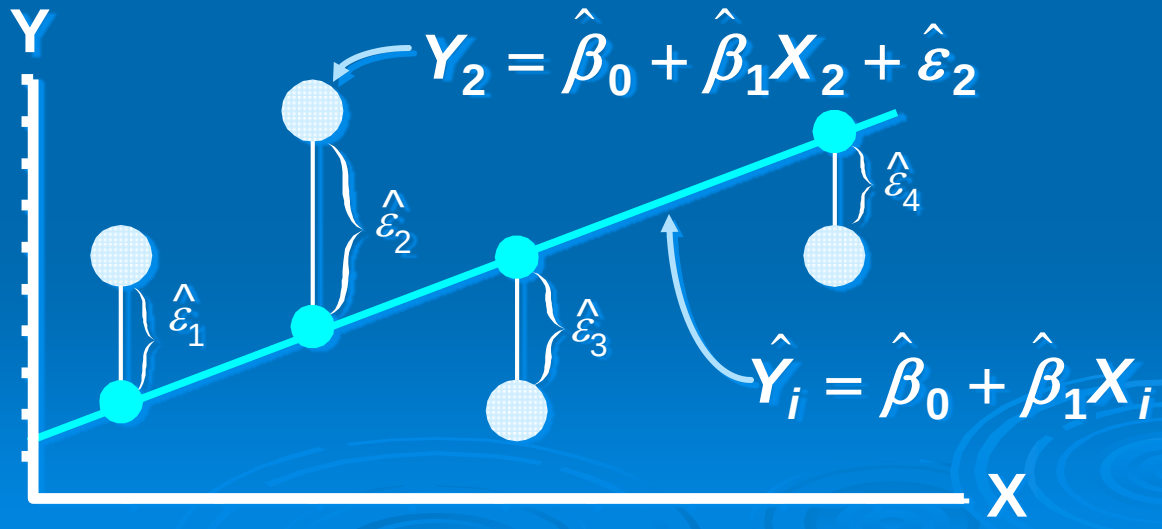
$$\sum_{i=1}^{n}\left(Y_i - \hat{Y}_i\right)^2 = \sum_{i=1}^{n}\hat{\varepsilon}_i^2$$

- 2. LS Minimizes the Sum of the Squared Differences (errors) (SSE)

Least Squares Graphically

LS minimizes $\sum_{i=1}^{n} \hat{\varepsilon}_i^2 = \hat{\varepsilon}_1^2 + \hat{\varepsilon}_2^2 + \hat{\varepsilon}_3^2 + \hat{\varepsilon}_4^2$



Coefficient Equations

- Prediction equation

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

- Sample slope

$$\hat{\beta}_1 = \frac{SS_{xy}}{SS_{xx}} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

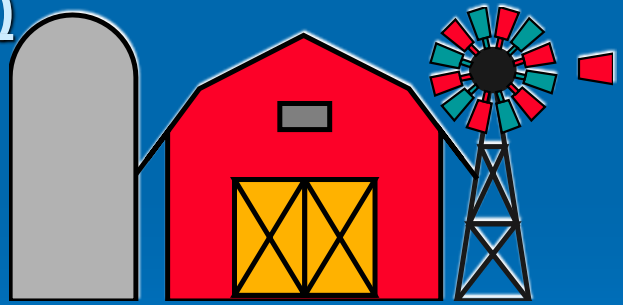
- Sample Y - intercept

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

Parameter Estimation Thinking Challenge

- You're a Vet epidemiologist for the county cooperative. You gather the following data:
- **Food (lb.)** **Milk yield (lb.)**

Food (lb.)	Milk yield (lb.)
4	3.0
6	5.5
10	6.5
12	9.0



- What is the **relationship** between cows' food intake and milk yield?

Coefficient Interpretation Solution*

- 1. Slope ($\hat{\beta}_1$)
  - Milk Yield (Y) Is Expected to Increase by .65 lb. for Each 1 lb. Increase in Food intake (X)

Coefficient Interpretation Solution*

1. Slope ($\hat{\beta}_1$)
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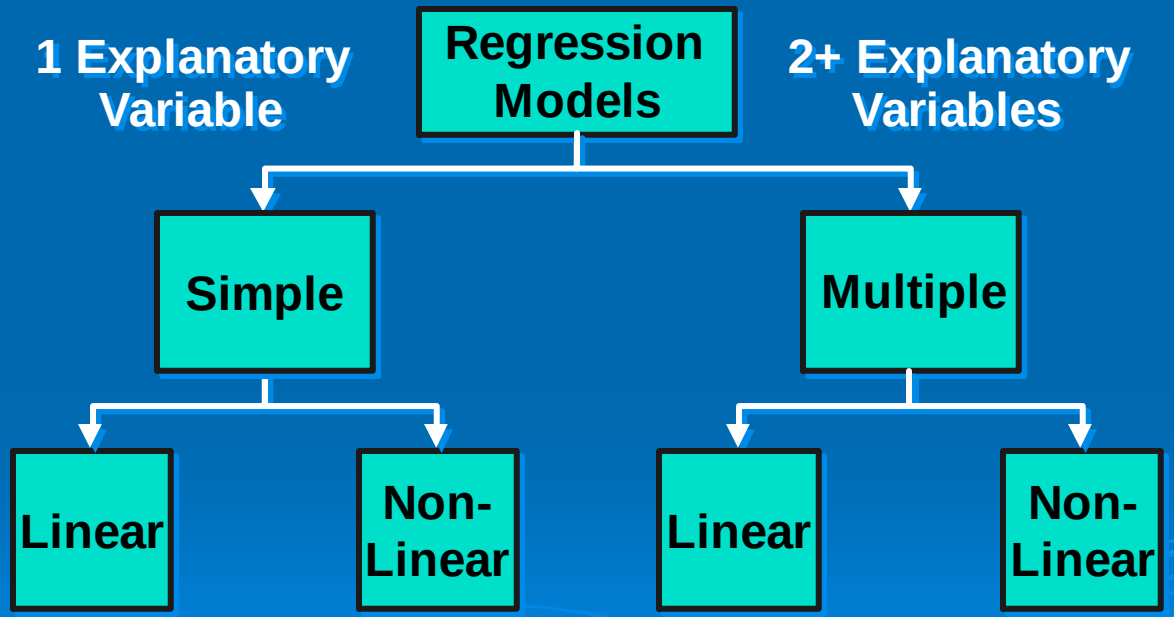
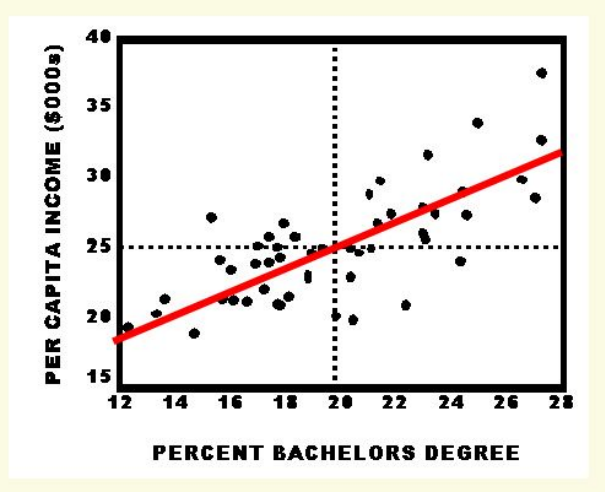
2. Y-Intercept ($\hat{\beta}_0$)
   - Average Milk yield (Y) Is Expected to Be 0.8 lb. When Food intake (X) Is 0

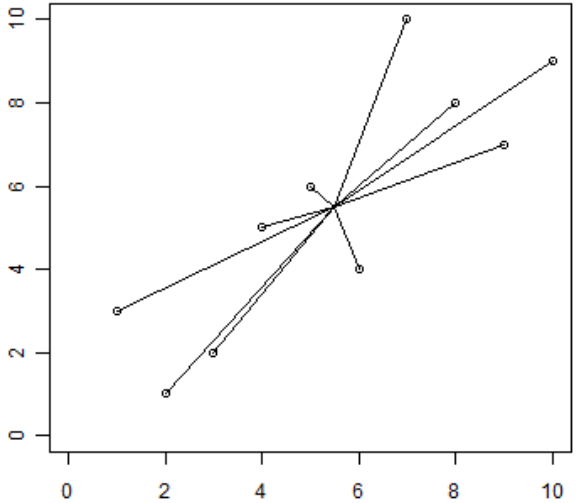
Types of Regression Models



Interpretation of coefficients

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \qquad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \qquad \hat{\beta}_1 = \frac{SS_{xy}}{SS_{xx}} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$





Regression – Error measures

$$MAE = \frac{1}{n}\sum\left|y - \hat{y}\right|$$

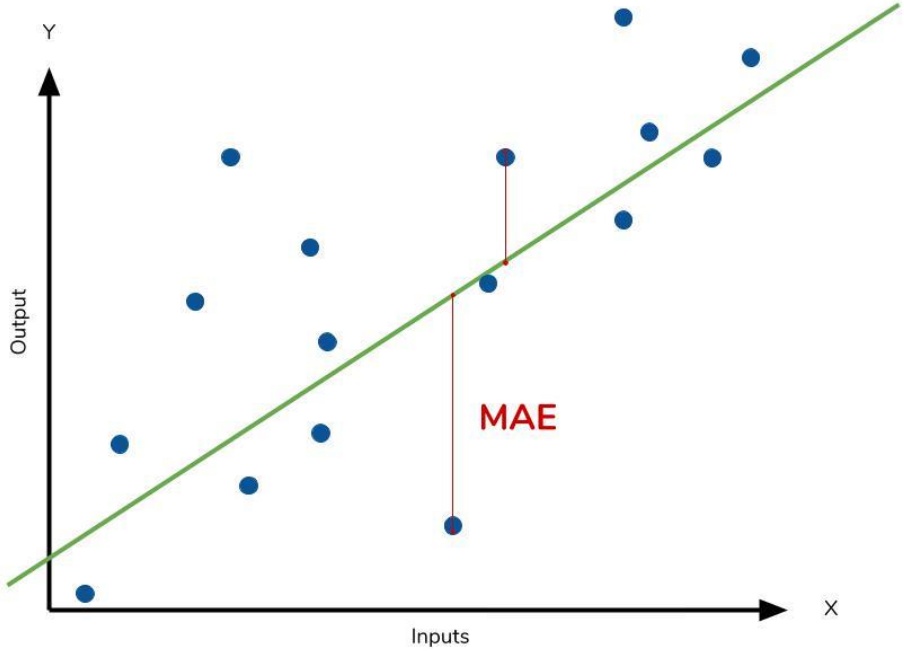
Divide by the total number of data points

Predicted output value

Actual output value

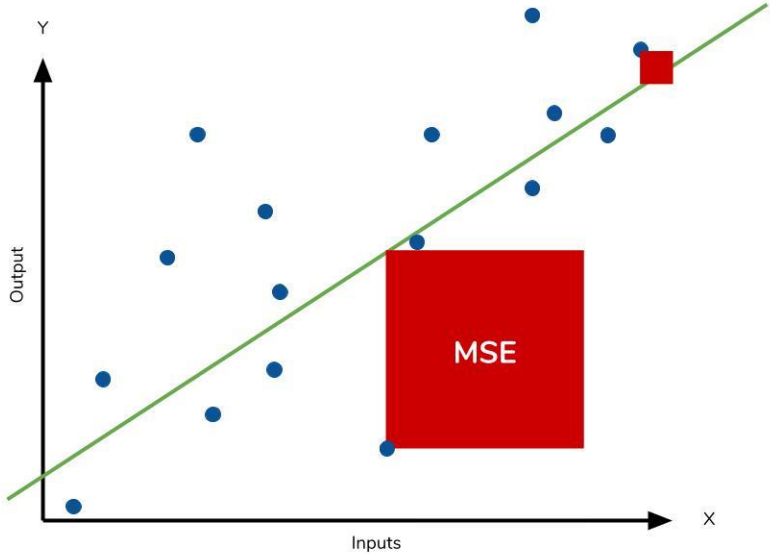
Sum of

The absolute value of the residual

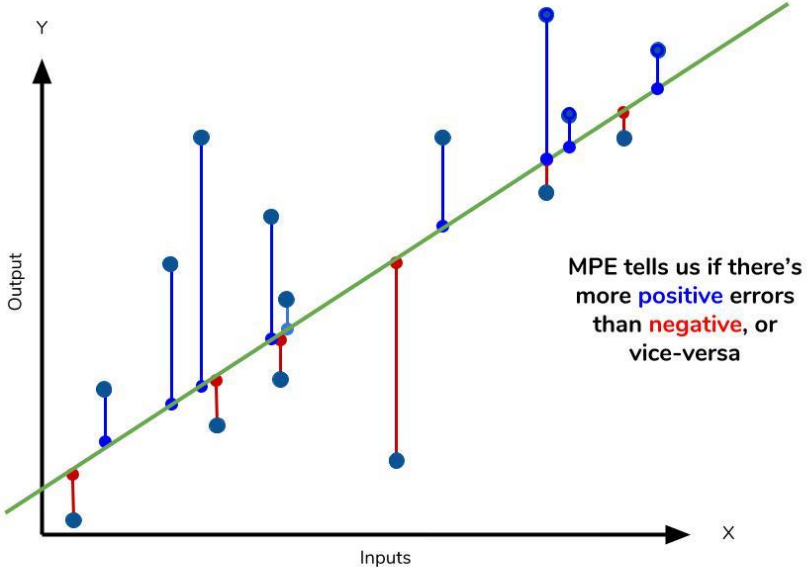


Regression – Error measures

$$MSE = \frac{1}{n} \Sigma \underbrace{\left(y - \widehat{y} \right)^2}_{\text{The square of the difference between actual and predicted}}$$

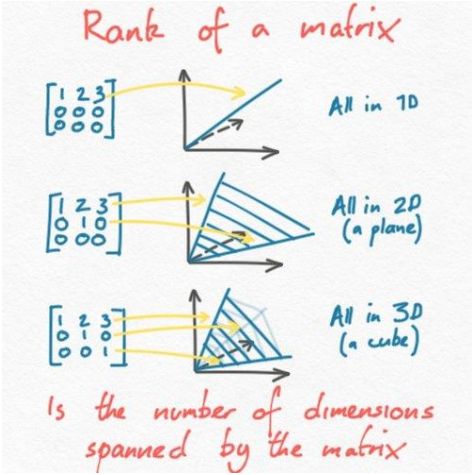


$$MPE = \frac{100\%}{n} \Sigma \left(\frac{y - \hat{y}}{y} \right)$$



Linear Algebra in 1 slide

- Matrices – Square, Rectangular
- Matrix ops – Transpose, Inverse
- Rank of a matrix



Linear Regression – Matrix Form

Linear Regression – Matrix Form

Consider the model

$$Y = X\beta + \epsilon$$

where $Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$ $X = \begin{pmatrix} 1 & X_{11} & X_{12} & \dots & X_{1p} \\ 1 & X_{21} & X_{22} & \dots & X_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} & \dots & X_{np} \end{pmatrix}$ $\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$ $\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$

Based on this model we get the following expansion for the first subject:

$$Y_1 = \beta_0 + \beta_1 X_{11} + \beta_2 X_{12} + \ldots + \beta_p X_{1p} + \epsilon_1$$

Then using matrix calculus we find that the least squares estimate for β is given by

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

Hence, the least squares regression line is $\hat{Y} = X\hat{\beta}$.

Linear Regression – Matrix Form - Issues

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

Hence, the least squares regression line is $\hat{Y} = X\hat{\beta}$.

- N samples, p-dimensional (what if p > N ?)
- Complexity of matrix inversion (what if N very large ?)
- Collinearity

- Linear Regression ➔ Linear in coefficients and NOT variables

- A second-order model (quadratic model):

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \epsilon$$

  - β_1: Linear effect parameter.
  - β_2: Quadratic effect parameter.

kth order polynomial model in one variable

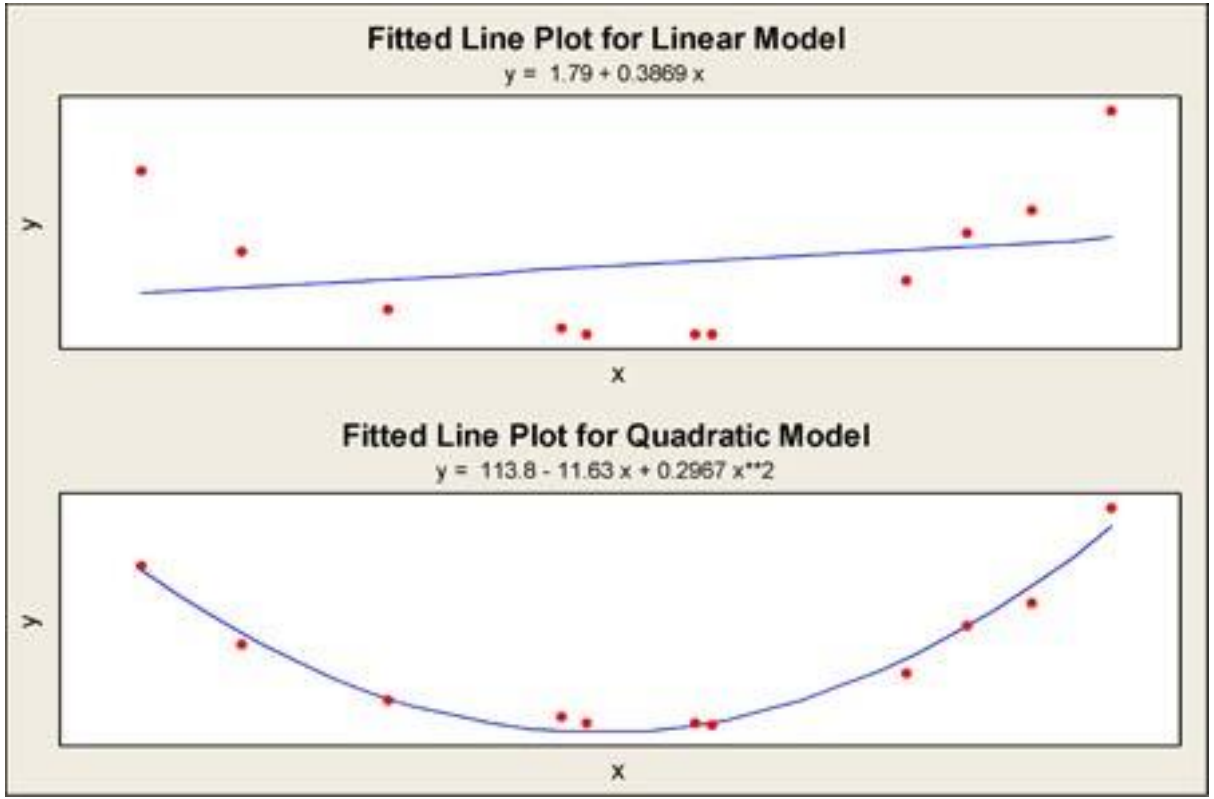
$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_k x^k + \epsilon$$

A quadratic polynomial regression function

$$Y_i = \beta_0 + \beta_1 X_i + \beta_{11} X_i^2 + \varepsilon_i$$

where:

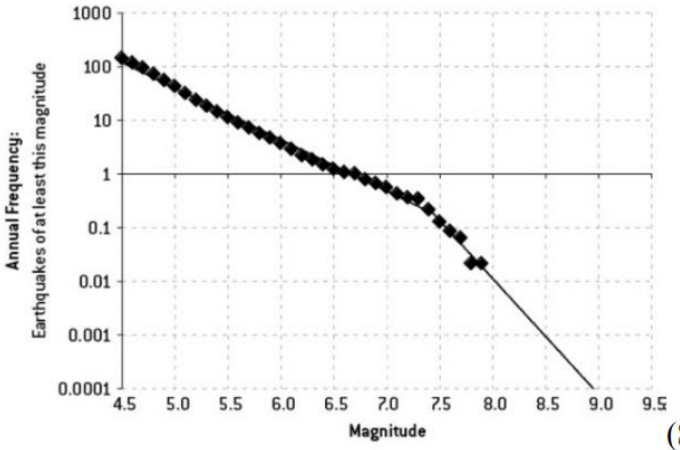
- Y_i = amount of immunoglobin in blood (mg)
- X_i = maximal oxygen uptake (ml/kg)
- typical assumptions about error terms ("INE")


Fitted Line Plot for Linear Model
y = 1.79 + 0.3869 x
y
x
Fitted Line Plot for Quadratic Model
y = 113.8 - 11.63 x + 0.2967 x**2
y
x

- Linear Regression → Linear in coefficients and NOT variables

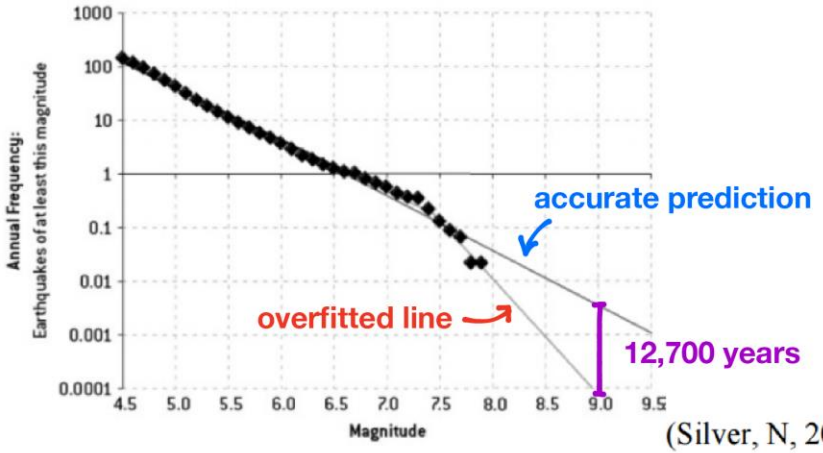
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^m \\ 1 & x_2 & x_2^2 & \dots & x_2^m \\ 1 & x_3 & x_3^2 & \dots & x_3^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^m \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_n \end{bmatrix},$$

FIGURE 5-7C: TŌHOKU, JAPAN EARTHQUAKE FREQUENCIES
CHARACTERISTIC FIT



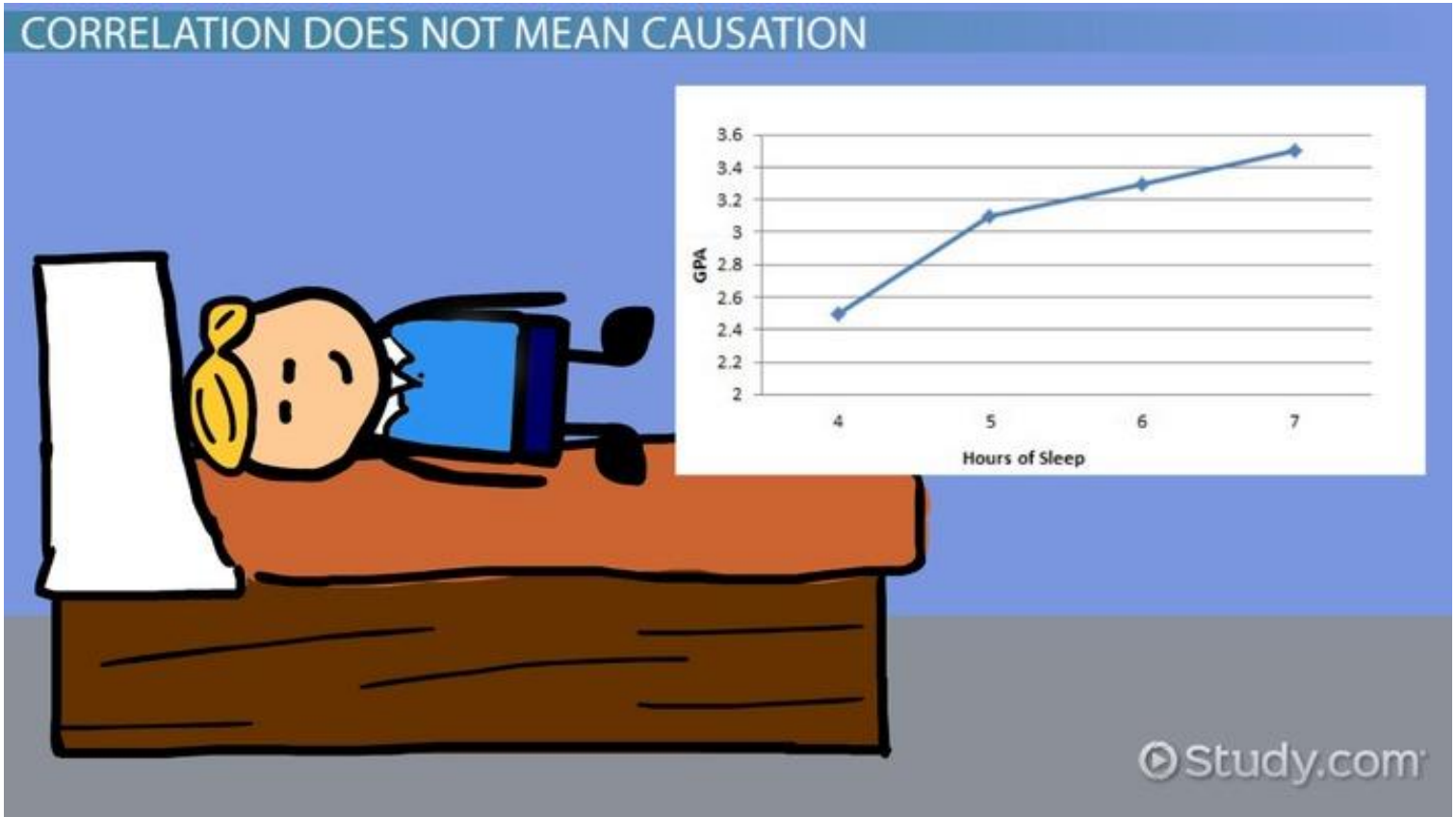
(Silver, N, 2012)

FIGURE 5-7C: TŌHOKU, JAPAN EARTHQUAKE FREQUENCIES
CHARACTERISTIC FIT

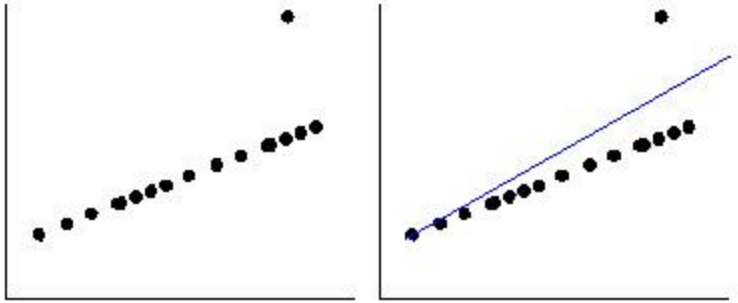


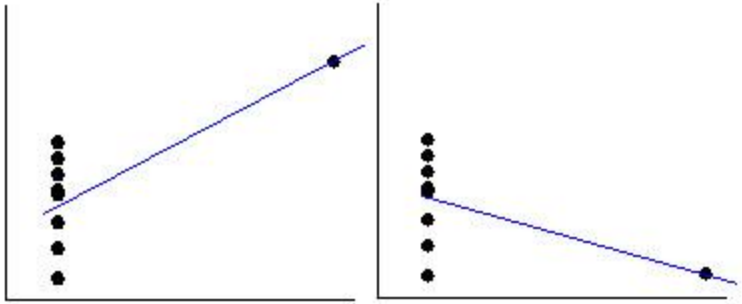
(Silver, N, 2012)

Careful: X may not be causing y !

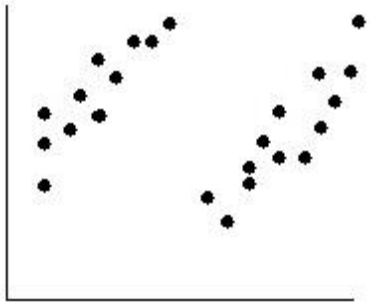


Linear Regression – Outliers

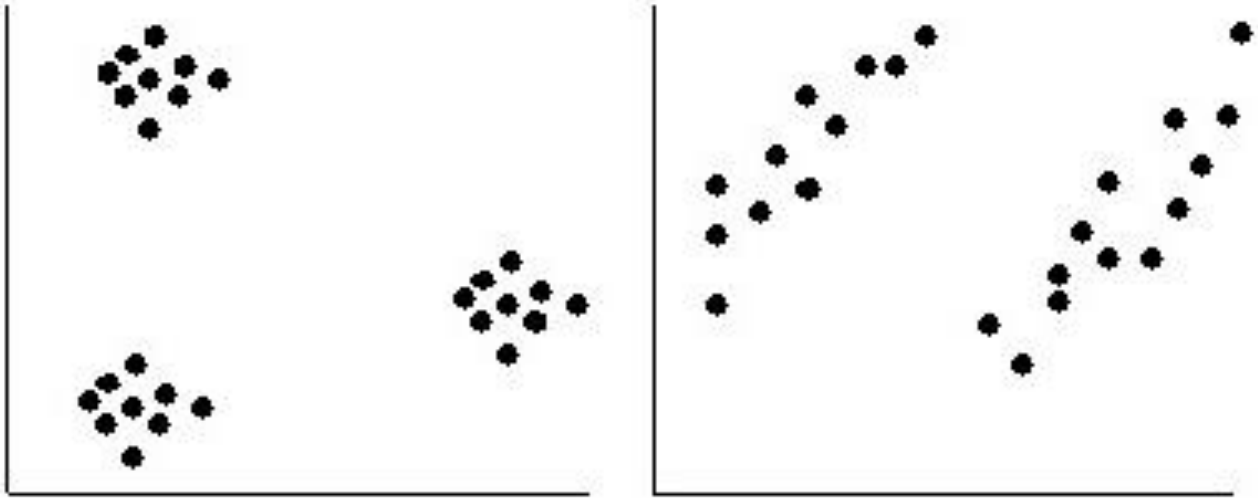




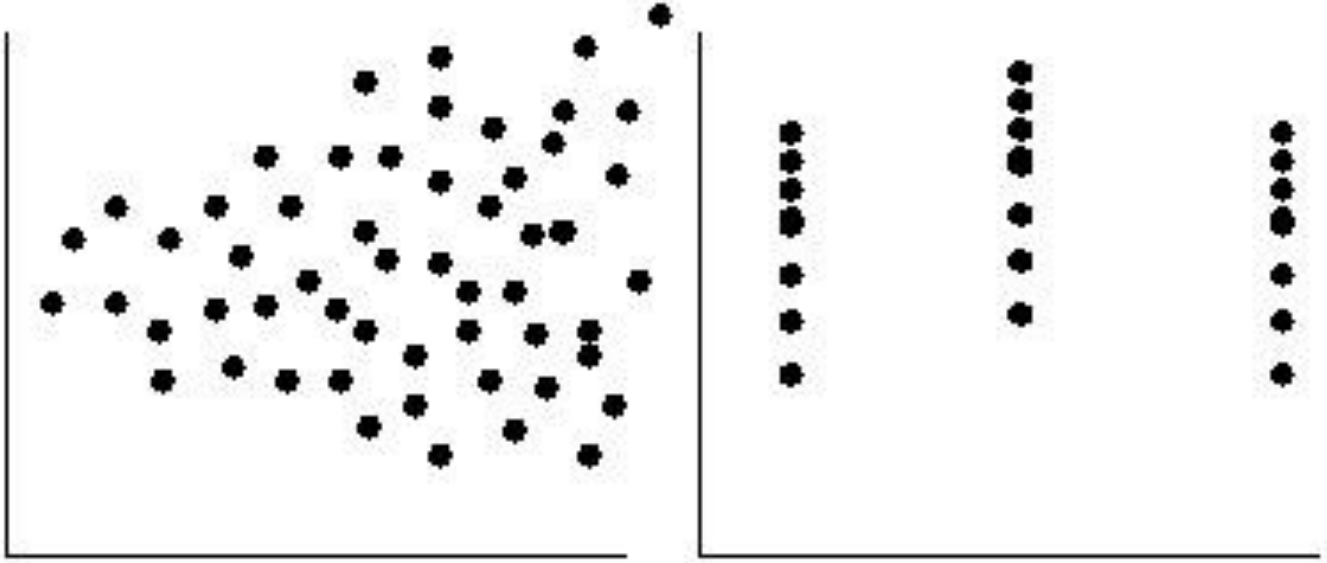
Linear Regression is problematic in many other cases



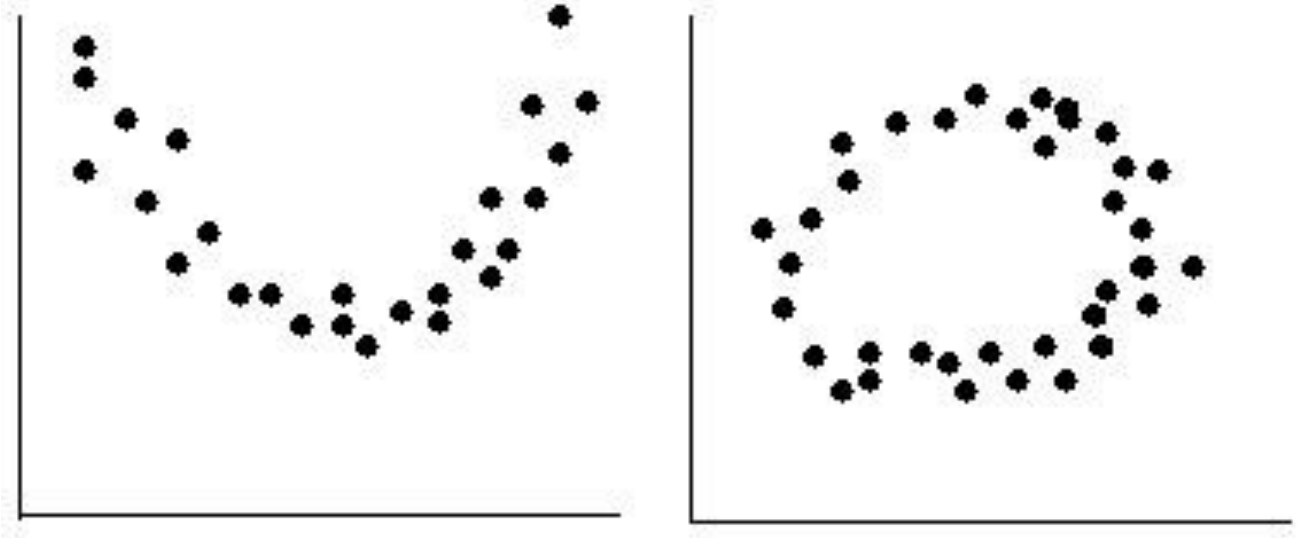
Linear Regression is problematic in many other cases



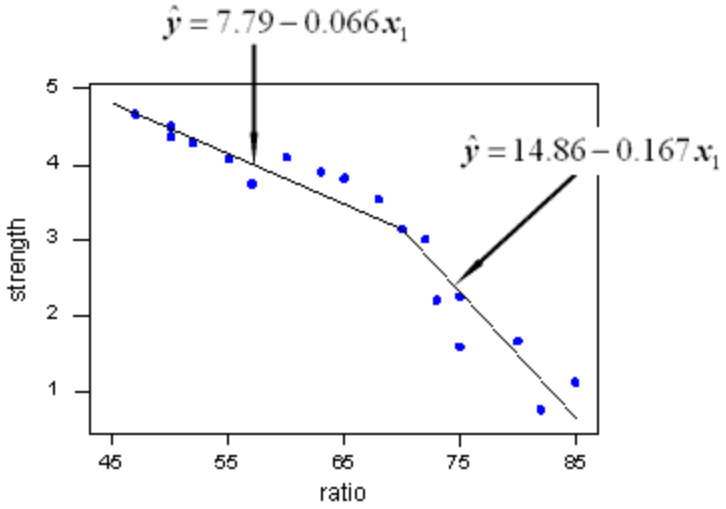
Linear Regression is problematic in many other cases

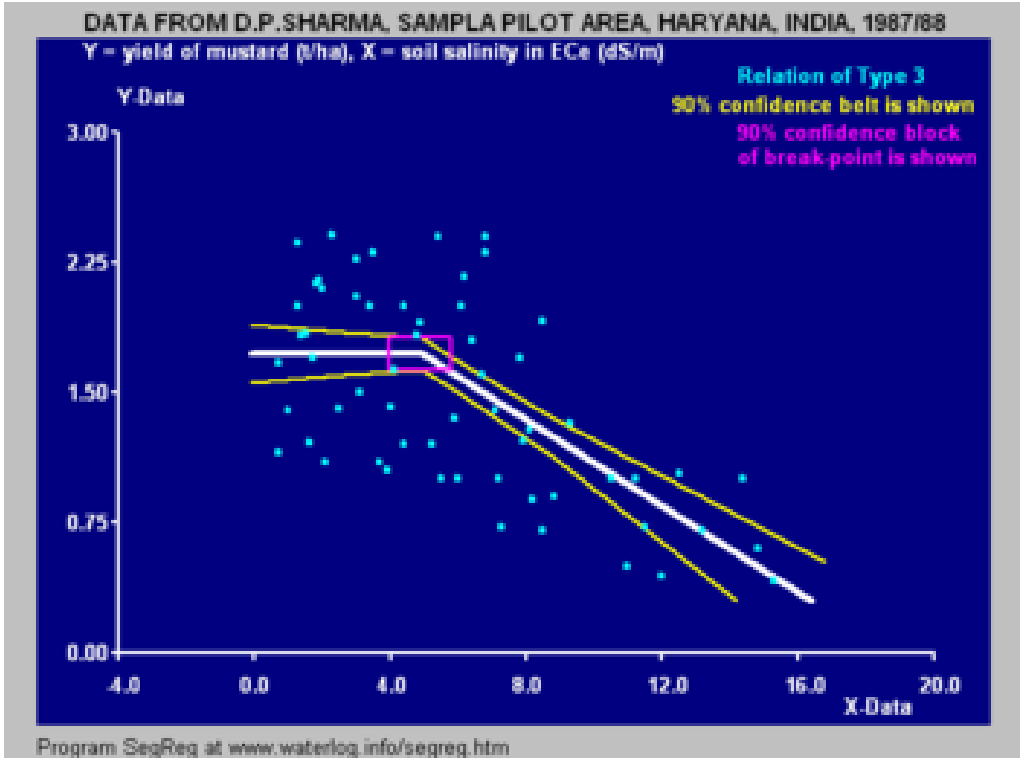


Linear Regression is problematic in many other cases

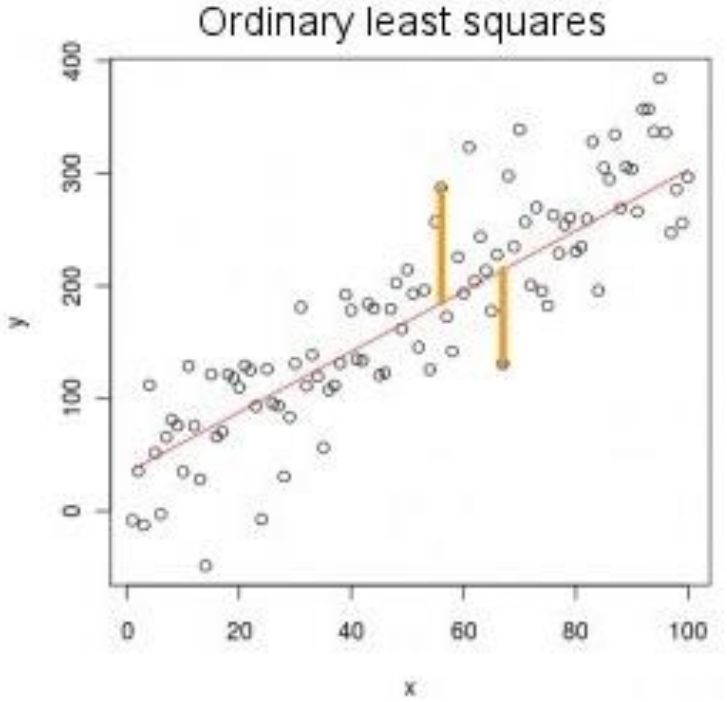


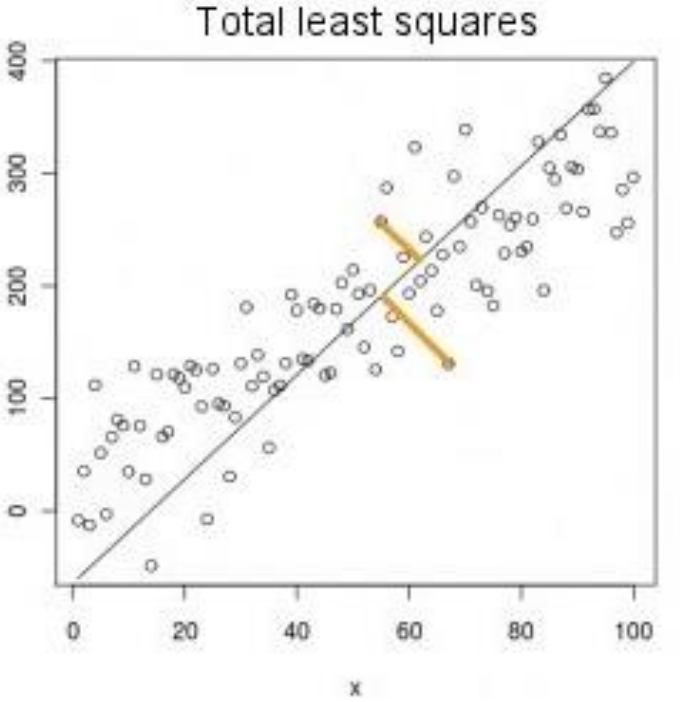
Piecewise Linear Regression



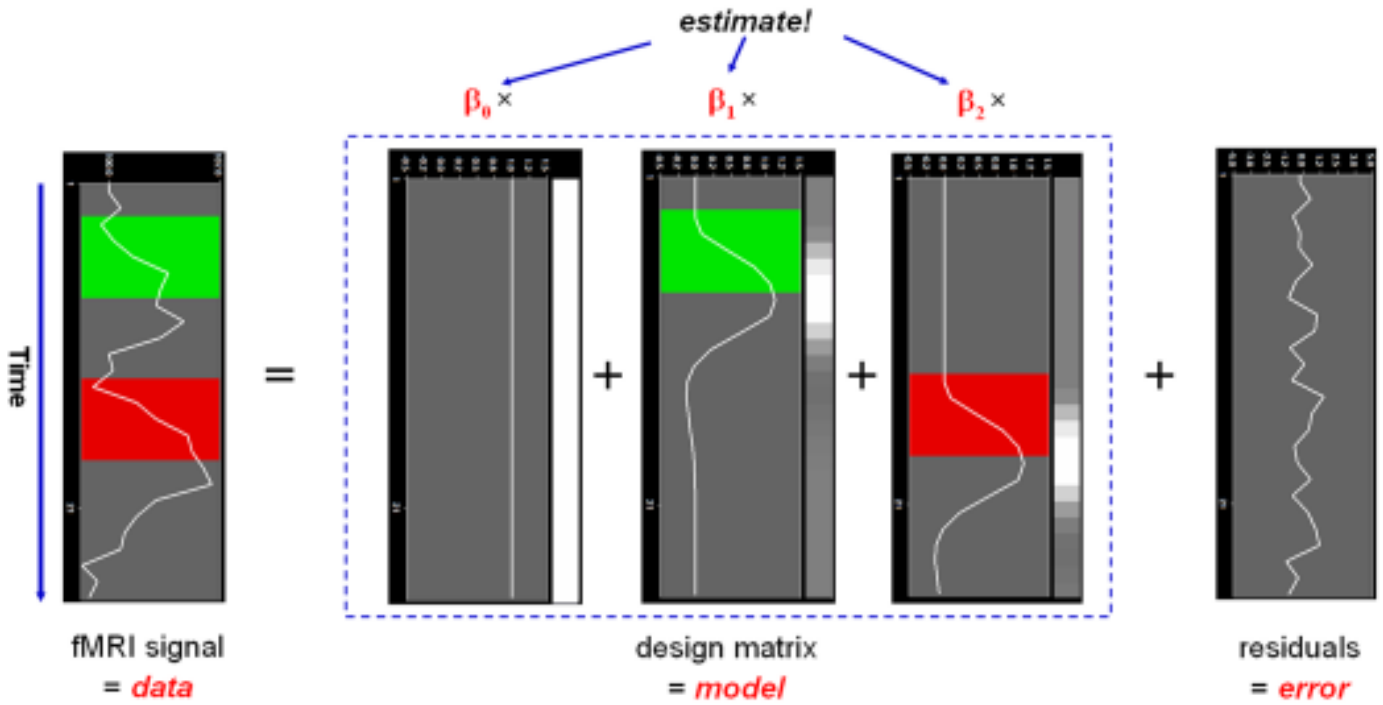


Total Least Squares
(“Errors-in-variables” model)





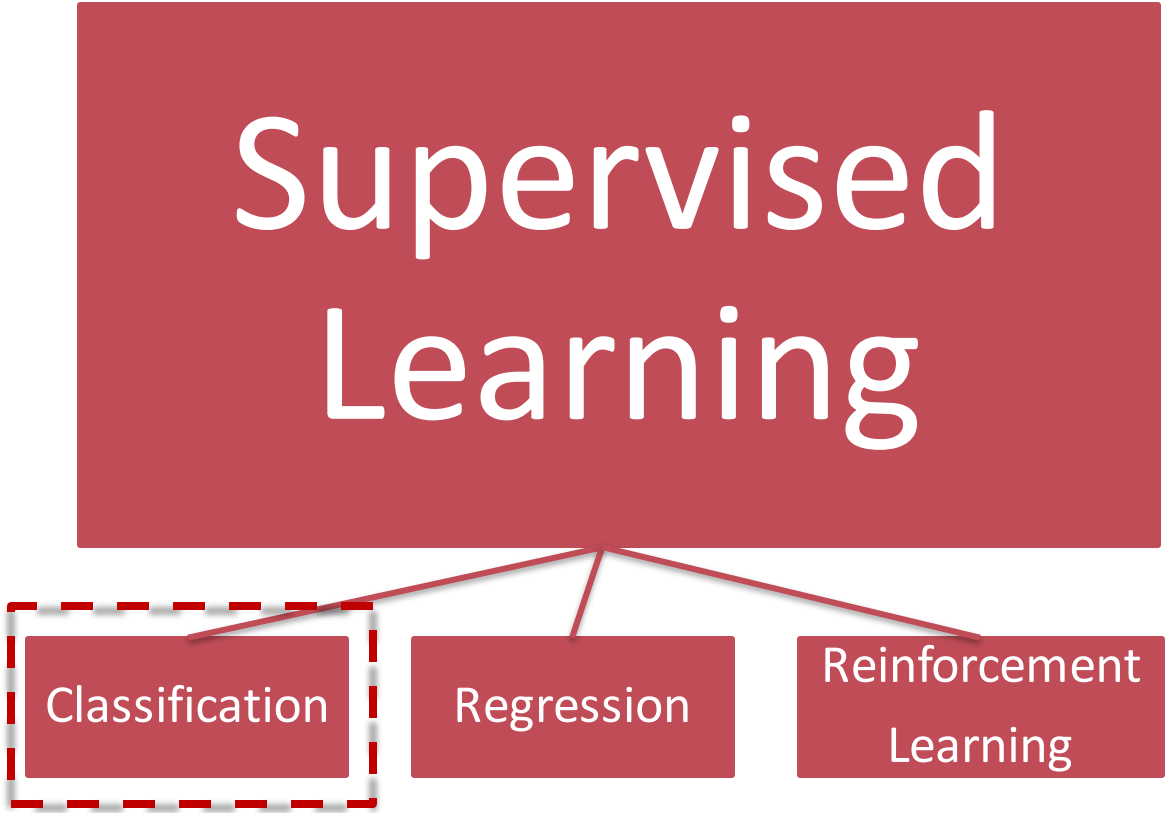
General Linear Models

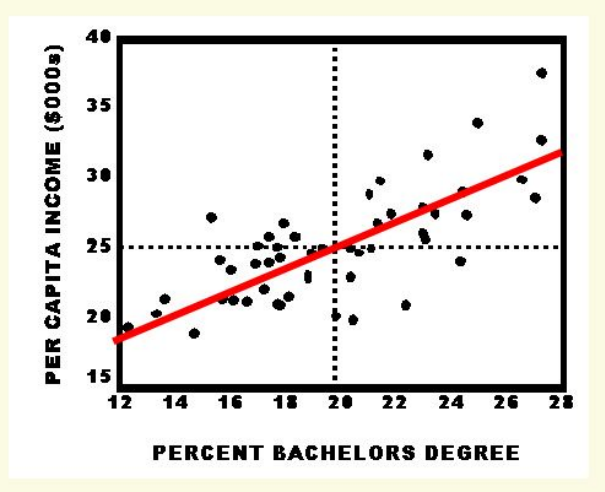


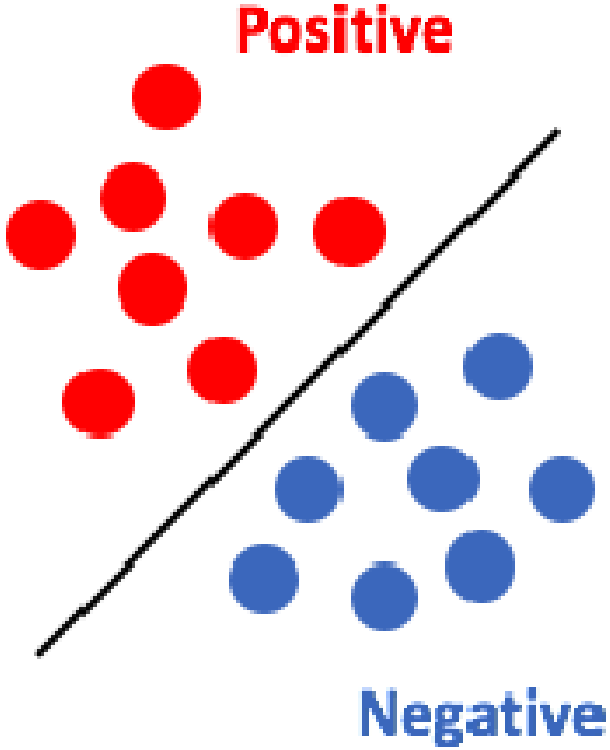
https://en.wikipedia.org/wiki/General_linear_model

Generalized Linear Models

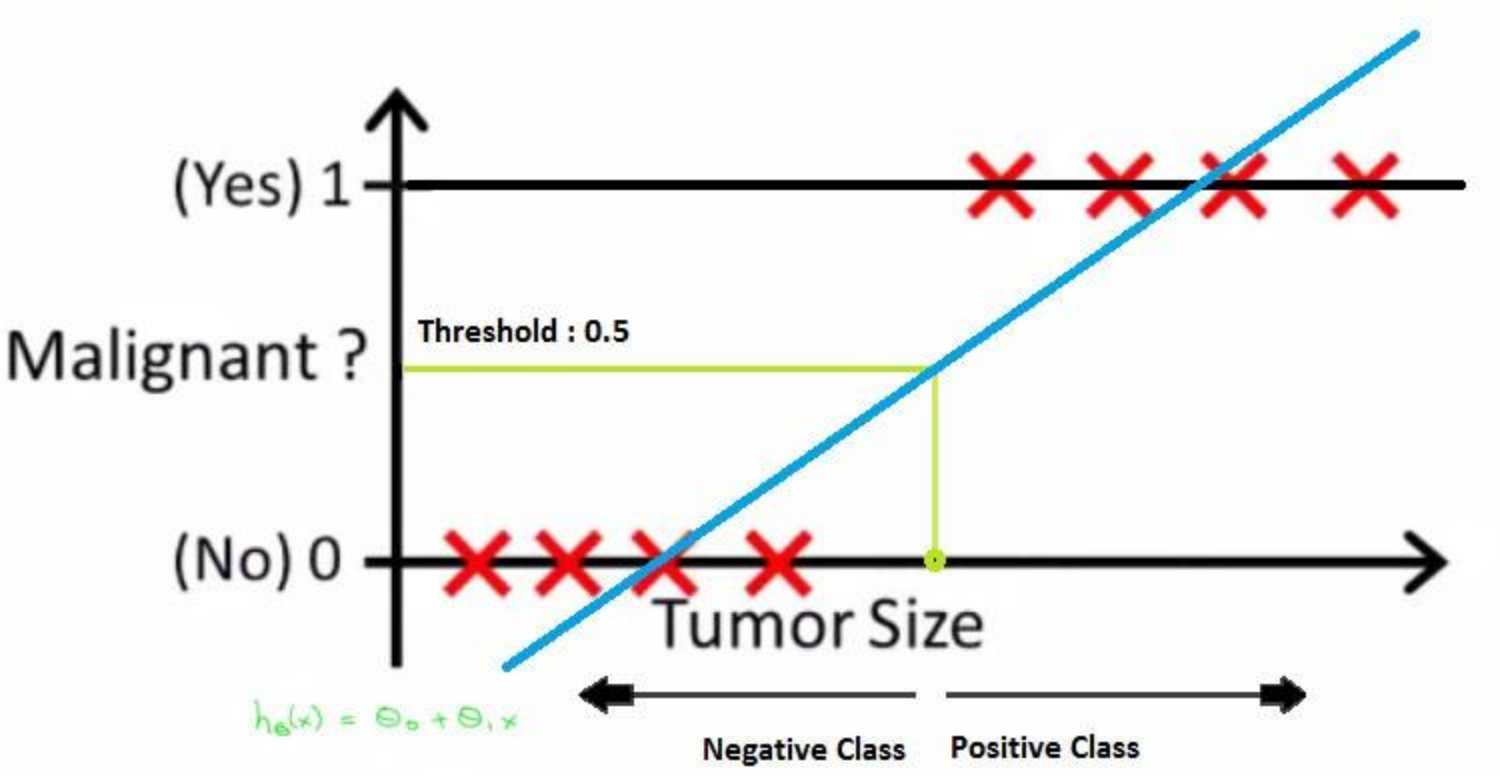
- For bounded or discrete data
  - Positive quantities (e.g. prices, populations)
  - Varying over a large scale (log-normal, Poisson distribution)
  - Categorical data (Bernoulli / Binomial / Multinomial)
  - Ordinal data (e.g. ratings – Ordered logit)


Supervised Learning
Classification
Regression
Reinforcement Learning

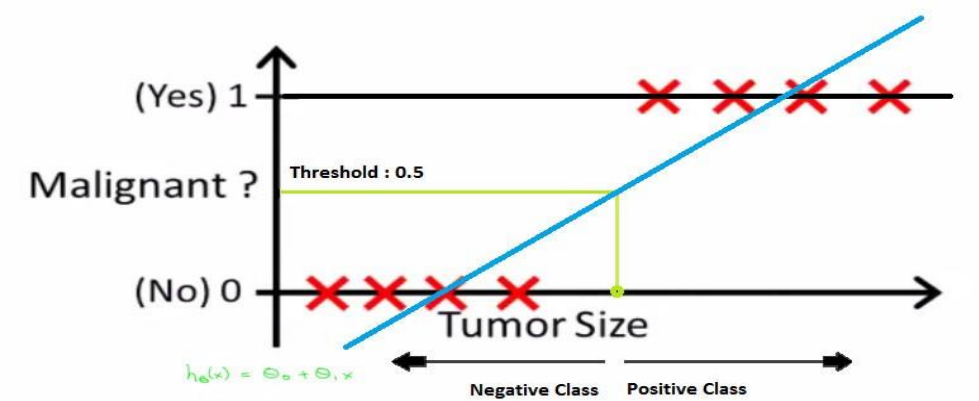

PER CAPITA INCOME ($000s)
40
35
30
25
20
15
12
14
16
18
20
22
24
26
28
PERCENT BACHELORS DEGREE

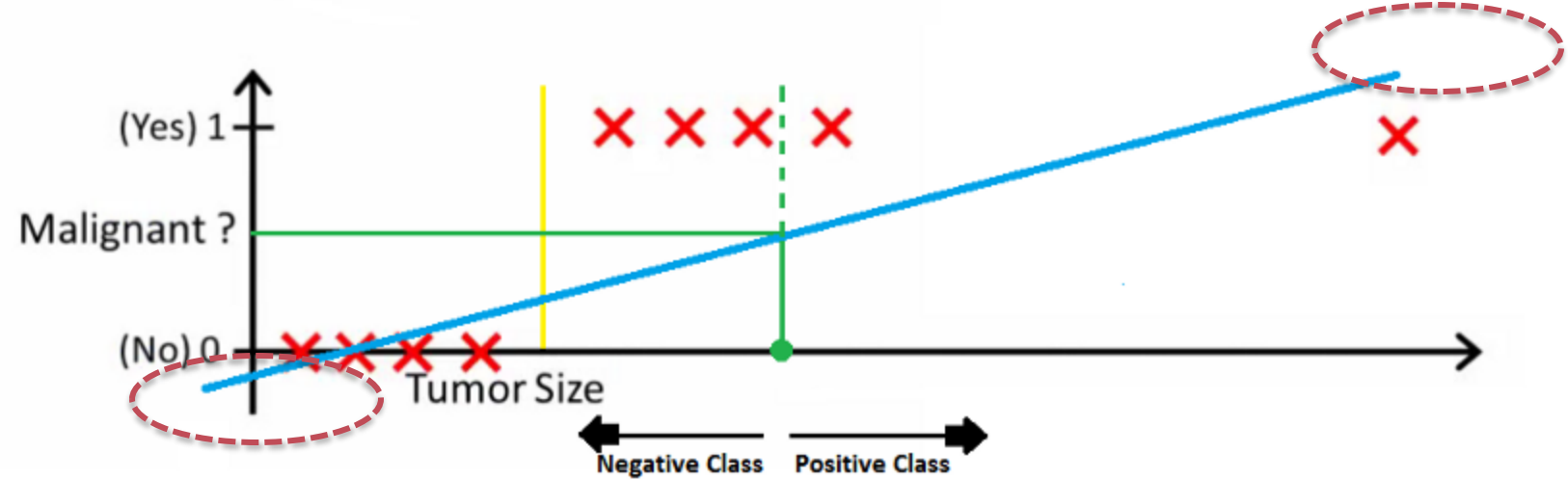

Positive
Negative

Re-using linear regression ?

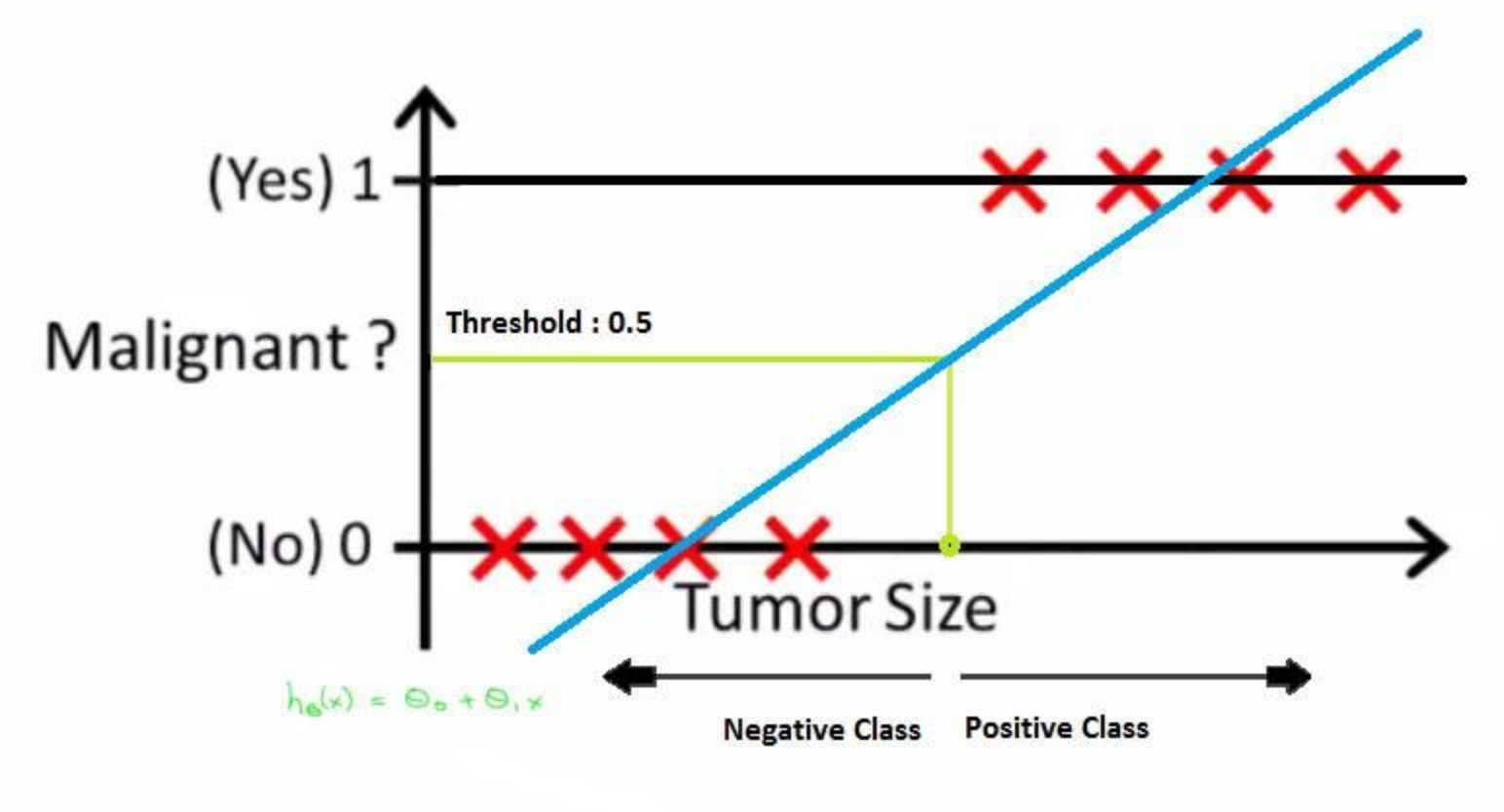


Re-using linear regression ?



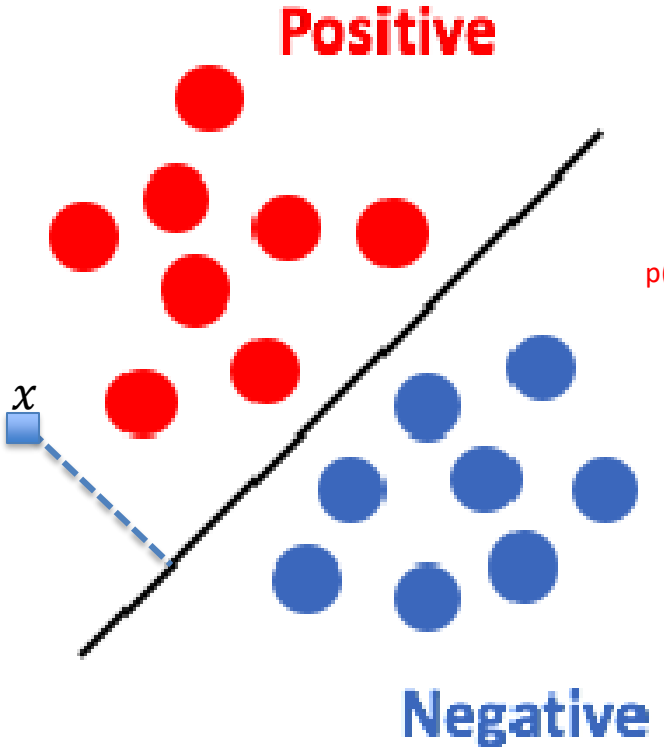


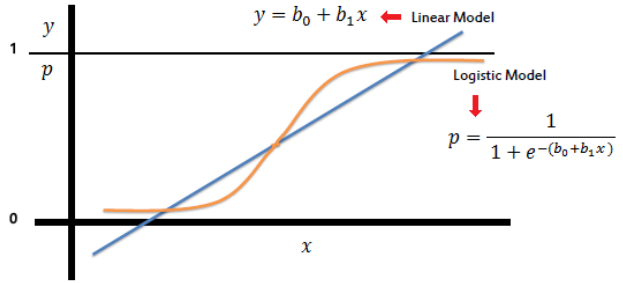
What we really want – a step function !



- We want a step-function like behavior
  - But with nicer mathematical properties (e.g. like linear regression)!
- Probabilistic classification is also nice (Naïve Bayes)
- Combine all of these ?

Logistic Regression - Intuition





p(X) = p(Y=1|x) = probability that x belongs to positive class

$$\Longrightarrow p(X) = \frac{e^{(\beta_o + \beta_1 x)}}{e^{(\beta_o + \beta_1 x)} + 1}$$

$$\Longrightarrow p(e^{(\beta_o + \beta_1 x)} + 1) = e^{(\beta_o + \beta_1 x)}$$

$$\Longrightarrow p.e^{(\beta_o + \beta_1 x)} + p = e^{(\beta_o + \beta_1 x)}$$

$$\Longrightarrow p = e^{(\beta_o + \beta_1 x)} - p.e^{(\beta_o + \beta_1 x)}$$

$$\Longrightarrow p = e^{(\beta_o + \beta_1 x)}(1 - p)$$

$$\Longrightarrow \frac{p}{1 - p} = e^{(\beta_o + \beta_1 x)}$$

$$\Longrightarrow \ln(\tfrac{p}{1-p}) = \beta_0 + \beta_1 x$$

Distance of x from boundary

Maximum Likelihood

- The likelihood function is the simultaneous density of the observation, as a function of the model parameters.

$$L(\Theta) = \Pr(Data|\Theta)$$

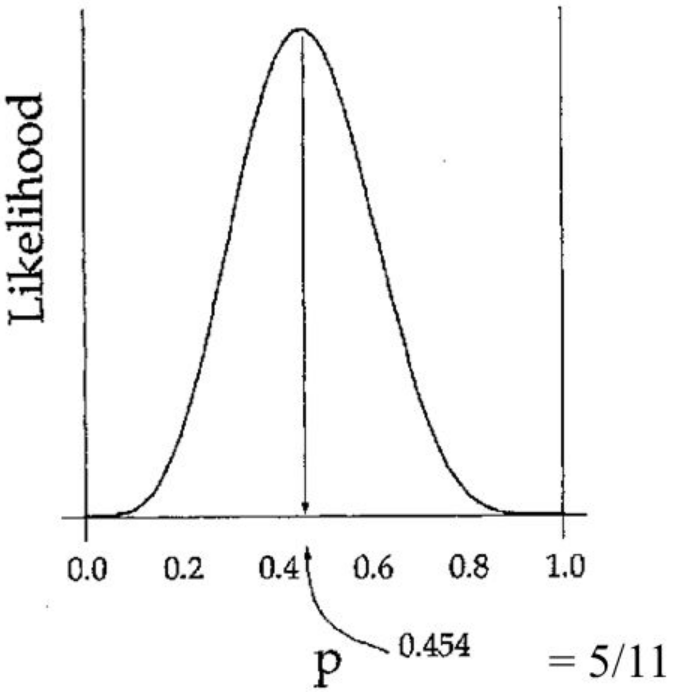
- If the observations are independent, we can decompose the term into

$$\Pr(Data \mid \Theta) = \prod_{i=1}^{n} \Pr(X_i \mid \Theta)$$

An example

- Consider the estimation of heads probability of a coin tossed *n* times
- Heads probability p
- Data = HHTTHTHHTTT
- $L(p) = \Pr(D|p) = pp(1-p)(1-p)p(1-p)pp(1-p)(1-p)(1-p) = p^5(1-p)^6$

$$L(p) = p^5(1-p)^6$$



Maximum Likelihood

$L(p) = p^5(1-p)^6$

Take the derivative of L with respect to p:

$$\frac{dL}{dp} = 5p^4(1-p)^6 - 6p^5(1-p)^5$$

Equate it to zero and solve:

$$\hat{p} = 5/11$$

Log Likelihood

$L(p) = p^5(1-p)^6$

- For computational reasons, we maximise the logarithm

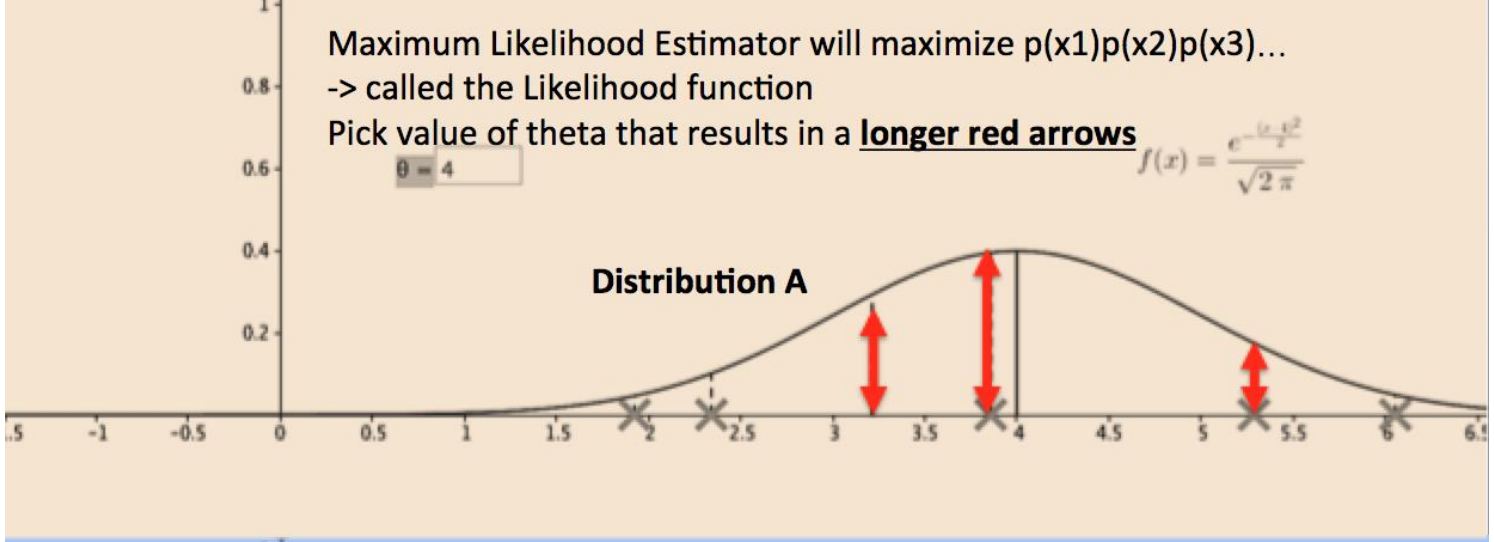
  $\ln L = 5 \ln p + 6 \ln(1-p)$

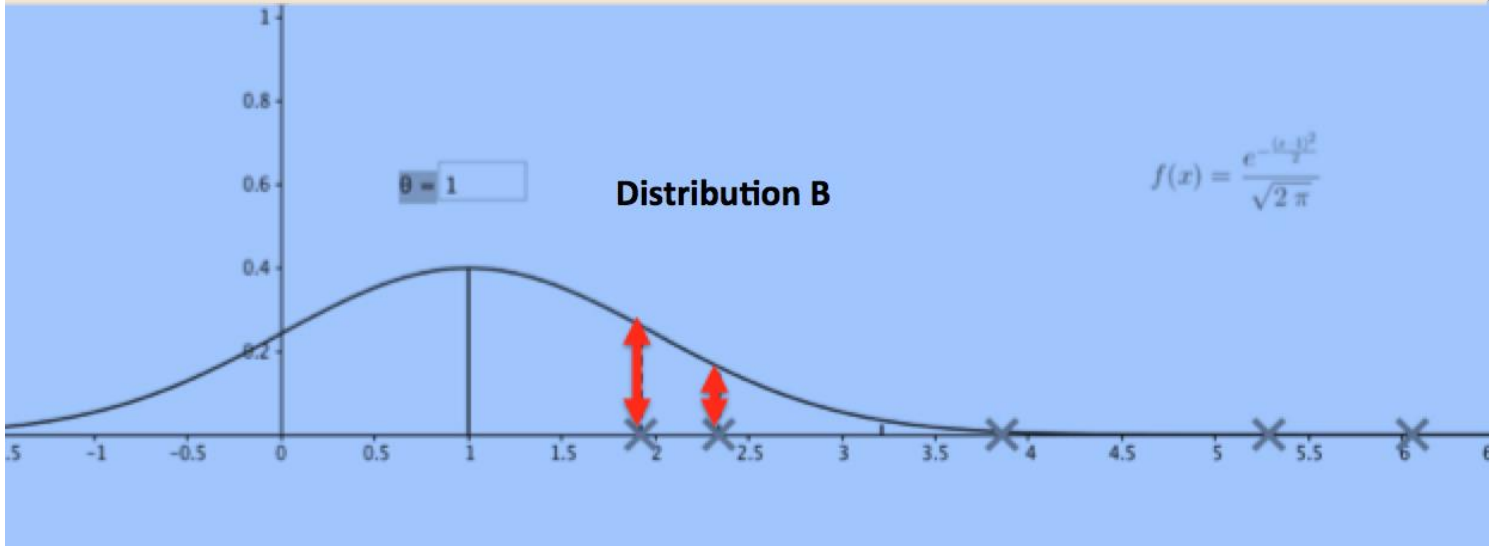
  with derivative

$$\frac{d(\ln L)}{dp} = \frac{5}{p} - \frac{6}{(1-p)} = 0$$

$$\hat{p} = 5/11$$

Maximum Likelihood Estimator will maximize p(x1)p(x2)p(x3)...
-> called the Likelihood function
Pick value of theta that results in a **longer red arrows**





Logistic Regression - Learning parameters

$$h_\theta(x) = g(\theta^T x) = \frac{1}{1 + e^{-\theta^T x}}.$$

$$p(y \mid x; \theta) = (h_\theta(x))^y \left(1 - h_\theta(x)\right)^{1-y}$$

$$\begin{aligned} L(\theta) &= p(\vec{y} \mid X; \theta) \\ &= \prod_{i=1}^{m} p(y^{(i)} \mid x^{(i)}; \theta) \\ &= \prod_{i=1}^{m} \left(h_\theta(x^{(i)})\right)^{y^{(i)}} \left(1 - h_\theta(x^{(i)})\right)^{1-y^{(i)}} \end{aligned}$$

$$\begin{aligned} \ell(\theta) &= \log L(\theta) \\ &= \sum_{i=1}^{m} y^{(i)} \log h(x^{(i)}) + (1 - y^{(i)}) \log(1 - h(x^{(i)})) \end{aligned}$$

References and Reading

- Linear Regression
  - https://en.wikipedia.org/wiki/Linear_regression
  - http://www.stat.purdue.edu/~boli/stat512/lectures/topic3.pdf (up to page 7)
- Logistic Regression
  - https://en.wikipedia.org/wiki/Logistic_regression (up to Section 6)