MTH 301 - Analysis – I

IIT KANPUR

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Assignment 7

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1. Let $\{x_i\}$ be the sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i^k$ exists for every $k = 0, 1, 2, \ldots$. Show that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i)$ exists for every $f \in C([0,1])$.

2. Prove that uniformly bounded collection of functions is also pointwise bounded. Give an example of a collection of functions that is pointwise bounded but not uniformly bounded.

3. Let M be a compact metric space. Prove that an equicontinuous subset of $C(M)$ is pointwise bounded if and only if it is uniformly bounded.

4. Let M be a compact metric space, and let $\{f_n\}$ be an equicontinuous sequence in $C(M)$. Show that $C = \{x \in M : \{f_n(x)\} \text{ converges}\}$ is a closed set in M.

5. For K and α fixed, denote $C_K^\alpha := \{f \in C[0,1] : |f(x) - f(y)| < K|x-y|^\alpha \text{ for } x, y \in [0,1]\}$. Show that $\{f \in C_K^\alpha : f(0) = 0\}$ is a compact subset of $C([0,1])$.

6. Let $\{f_n\}$ be a sequence in $C([a,b])$ with $\|f_n\|_\infty \leq 1$ for all n and define $F_n(x) := \int_a^x f_n(t)\,dt$. Show that some subsequence of $\{F_n\}$ is uniformly convergent.

7. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous and Lipschitz in its second variable, i.e. $|F(r,s) - F(r,t)| \leq K|s-t|$. Define $T : C([a,b]) \to C([a,b])$ by $(Tf)(x) = \int_a^x F(t, f(t))\,dt$.
   (a) Show that T is continuous. Consequently, T achieves a minimum on any compact set in $C[a,b]$.
   (b) Show that T maps bounded sets into equicontinuous sets.

8. Show that there is a *linear isometry* from $C[0,1]$ onto $C[a,b]$ that maps polynomials to polynomials.

9. Let $B_n(f)$ be the *Bernstein Polynomial of* f. Show that $B_n(x^2) \to x^2$ uniformly as $n \to \infty$.

10. Let $f \in C^1([a,b])$ be continuously differentiable and let $\epsilon > 0$. Show that there is a polynomial p such that $\|f - g\|_\infty < \epsilon$ and $\|f' - g'\|_\infty < \epsilon$. Conclude that $C^1([a,b])$ is separable.

11. Construct a sequence of polynomials that converge uniformly on $[0,1]$ but whose derivatives fail to converges uniformly.

12. If $f \in C([a,b])$ and if $\int_a^b x^n f(x)\,dx = 0$ for each $n = 0, 1, 2, \ldots$ then show that $f = 0$.