MTH 301 - Analysis – I

IIT KANPUR

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Assignment 2

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1. Show that Cantor set $C \subset [0,1]$ contains no closed interval $[a,b]$ with $a < b$.
2. Let $C \subset [0,1]$ be the Cantor set. Prove that there exists a function $g : [0,1] \to [0,1]$ such that it is continuous at every point in $[0,1] \setminus C$ and discontinuous at every point in C.
3. Prove that *extended cantor set* $f : [0,1] \to [0,1]$ is non-decreasing.
4. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Suppose that for each $c \in I$ there exists $a, b \in \mathbb{R}$ with $a > 0$ such that $f(x) \geq ax + b$ for all $x \in I$ and $f(c) = ac + b$. Determine whether the function f is monotone. Justify your answer with a proof or counterexample.
5. Let $f : [a,b] \to \mathbb{R}$ be a function. Then, for every finite partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of the interval $[a,b]$, we denote

$$V_a^b(f) := \sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

   Show that if f is monotone, then $V_a^b(f)$ has finite value. Is the converse true?
6. Let $f : [a,b] \to \mathbb{R}$ be increasing function and let $\{x_n\}$ be an enumeration of the discontinuous points of f. Define $a_n = f(x_n) - f(x_n^-)$ and $b_n = f(x_n^+) - f(x_n)$ for each $n \in \mathbb{N}$, where $a_n = 0$ for $x_n = a$ and $b_n = 0$ for $x_n = b$.
   (a) Show that $\displaystyle\sum_{n=1}^{\infty} a_n \leq f(b) - f(a)$ and $\displaystyle\sum_{n=1}^{\infty} b_n \leq f(b) - f(a)$.
   (b) Define $h(x) = \displaystyle\sum_{x_n \leq x} a_n + \sum_{x_n < x} b_n$. Show that h is increasing and the function $g(x) = f(x) - h(x)$ is both continuous and increasing.
7. Examine whether d is a metric on X, where
   (a) $X = \mathbb{R}$ and $d(x,y) = \min\{\sqrt{|x-y}, |x-y|^2\}$ for all $x, y \in \mathbb{R}$.
   (b) $X = \mathbb{R}$ and for all $x, y \in \mathbb{R}$,

$$d(x,y) = \begin{cases} 1 + |x-y|, & \text{if exactly one of } x \text{ and } y \text{ is positive,} \\ |x-y|, & \text{otherwise.} \end{cases}$$

   (c) $X = \mathbb{R}$ and $d(x,y) = |x-y|^p$ for all $x, y \in \mathbb{R}$, where $0 < p < 1$.
8. If d is any metric on X, show that $\sigma(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric on X.

9. Let $C_b(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : \ f \text{ is continuous and bounded}\}$ and let

$$d(f,g) = \sup\left\{\left|\int_x^{x+h} (f(t) - g(t))\, dt\right| : x \in \mathbb{R},\ 0 < h \leq 1\right\}, \quad \forall\, f, g \in C_b(\mathbb{R}).$$

Show that d is a metric on $C_b(\mathbb{R})$.

10. Let A be a closed set in a metric space X and $x \in X \setminus A$. Show that there exist disjoint open sets G and H such that $x \in G$ and $A \subset H$.

11. Let (X, d) be a metric space. Show that every closed set in X is a countable intersection of open sets in X.

12. Let $d(m,n) = |m - n|$ and $\rho(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|$ for all $m, n \in \mathbb{N}$. Prove or disprove the following statements:

   (a) Every subset of $\mathbb{N}$ is open in the metric space $(\mathbb{N}, \rho)$.

   (b) Every subset of $\mathbb{N}$ is closed in the metric space $(\mathbb{N}, \rho)$.