MTH 301 - Analysis – I

IIT KANPUR

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Assignment 1

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1. Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that $s_0 = \sup A$

   if and only if (iff)

   s_0 is an upper bound for A, and for every $\varepsilon > 0$, there exists $a \in A$ such that $a > s_0 - \varepsilon$. Further, show that there exists a sequence $\{x_n\}_n \subset A$ such that $\lim_{n\to 0} x_n = s_0$

2. Show that $\{x_n\}$ converges to $x \in \mathbb{R}$ iff every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a further subsequence $\{x_{n_{k_j}}\}$ that converges to x.

3. Suppose that $a_n \geq 0$ and that $\sum_{n=1}^{\infty} a_n < \infty$.

   (i) Show that $\liminf\limits_{n\to\infty} na_n = 0$.

   (ii) Give an example showing that $\limsup\limits_{n\to\infty} na_n > 0$ is possible.

4. Show that $x_n = \left(1 + \frac{1}{n}\right)^n$ is strictly increasing, $y_n = \left(1 + \frac{1}{n}\right)^{n+1}$ is strictly decreasing, and $2 \leq x_n < y_n \leq 4$. Further prove that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = e$, where $2 < e < 3$.

5. Calculate $\limsup$ and $\liminf$ of the following sequences:

   (a) $(\cos \frac{n\pi}{4})^{(-1)^n}$,

   (b) $\left(1 + \frac{(-1)^n}{n}\right)^n$.

   (c) $2^{n\cos(\frac{2n\pi}{3})}$,

   (d) $a_n = \begin{cases} \frac{n}{n+1} & n \text{ is odd,} \\ \frac{1}{n+1} & n \text{ is even,} \end{cases}$

6. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in $\mathbb{R}$. Prove the following inequalities:

   $$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \;\leq\; \liminf_{n\to\infty}(x_n + y_n) \;\leq\; \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

   $$\leq\; \limsup_{n\to\infty}(x_n + y_n) \;\leq\; \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

   Further, give an example to show that

   $$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \;\neq\; \liminf_{n\to\infty}(x_n + y_n).$$

7. let $2 \leq p \in \mathbb{N}$ and $x \in [0, 1]$. Show that there is a sequence $\{a_n\}_n \subset \mathbb{N} \cup \{0\}$ with $a_n < p$ such that x is represented by a convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{p^n}$.

8. If $a > -1$ and $a \neq 0$, then show that $(1 + a)^n > 1 + na$ for any integer $n > 1$. *(Bernoulli's Inequality)*

9. Let $\{a_n\}_n$ be a bounded sequence of real numbers. Denote

$$A = \left\{a : \lim_{k\to\infty} a_{n_k} = a, \text{ where } \{a_{n_k}\}_k \text{ subsequence of } \{a_n\}_n\right\}.$$

Show that $\limsup_{n\to\infty} a_n = \sup A$ and $\liminf_{n\to\infty} a_n = \inf A$.

10. Show that there is a equivalence between $(0,1)$ and $[0,1]$.

11. Let A be any uncountable set, and $B \subset A$ be a countable subset of A. Prove that $|A| = |A - B|$.

12. (a) Let $A = \{0,1\}^{\mathbb{N}}$ be the set of all possible sequences of 0's and 1's. Prove that A is uncountable.

    (b) Find a bijection between A and $\mathcal{P}(\mathbb{N})$.