Indian Institute of Technology Kanpur

Introduction to Probability Theory (MSO205)
End Semester Examination

Time: 8 AM to 10 AM **Date:** November 25, 2025 **Maximum Marks:** 30

Name: **Roll Number:**

Question 1: Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. symmetric random variables with $E(X_1^{10}) < \infty$ and center of symmetry $= 0$. Suppose that

$$U_n = \frac{X_1 + \cdots + X_n}{X_1^5 + \cdots + X_n^5} \quad \text{(pointwise equality)},$$

and U_n converges in distribution to U as $n \to \infty$. Derive the probability density function of U with full justification.

Points : 8

Question 2: Let (X, Y) be a random vector with the joint probability density function

$$f_{X,Y}(x, y) = \frac{1}{\pi} 1_{\{x^2+y^2<1\}}.$$

Let (X_1, Y_1) and (X_2, Y_2) be independent random vectors identically distributed with (X, Y). Suppose that $D = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}$. Compute $E(D^2)$ with complete justification.

Points : 8

Question 3: Let $X_1, \ldots, X_n$ be a collection of i.i.d. random variables distributed uniformly over $(0, \theta)$, where $\theta > 0$. Suppose that $X_{(i:n)}$ is the i-th order statistic of $X_1, \ldots, X_n$, where $i = 1, \ldots, n$. Does $X_{(n-13:n)}$ converge to θ in probability as $n \to \infty$? Justify your answer.

Points : 7

Question 4: Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that $E(X_n) = 0$ for all $n \in \mathbb{N}$ and $E(X_n^2) = 1$ for all $n \in \mathbb{N}$, and $|\text{Covariance}(X_n, X_m)| \leq \delta(|n - m|)$, where $\delta(k) \to 0$ as $k \to \infty$. Suppose that $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges in probability to some constant c as $n \to \infty$. Find the value of c with full justification?

Points : 7