MTH 301 - Analysis – I

IIT KANPUR

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Assignment 5

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1. Let D be a dense subset of a metric space (M, d), and suppose that every Cauchy sequence from D converges to some point of M. Prove that (M, d) is complete.

2. Prove that (M, d) is complete if and only if, for each $r > 0$, the closed ball $\{y \in M : d(x, y) \leq r\}$ is complete.

3. Denote $S = \big\{\{x_n\}_{n\in\mathbb{N}} : x_k \in \mathbb{R} \text{ and } x_k = 0 \text{ for all but finitely many } k \in \mathbb{N}\big\}$, which is a vector sub-space of ℓ_∞. Show that S is not complete under the $\|\cdot\|_\infty$ norm, i.e. $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.

4. Prove that (M, d) is compact if and only if every infinite subset of M has a limit point.

5. **(Pre-compactness)** If A is a totally bounded subset of a complete metric space (M, d), show that $\overline{A}$ is compact in M. In such case, A is called pre-compact.

6. If $f : \mathbb{R} \to \mathbb{R}$ is both continuous and open, show that f is strictly monotone.

7. Show that M is compact if and onlyif $\mathcal{G}$ is a collection of open sets in M with $M \subseteq \cup_{G_\alpha\in\mathcal{G}} G_\alpha$, then there are finitely many $G_1, G_2, \cdots G_k \in \mathcal{G}$ such that $M \subseteq \cup_{i=1}^k G_i$.

8. (a) Let F and K be disjoint, nonempty subsets of a metric space (M, d) with F closed and and K compact. Show that $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\} > 0$.

   (b) Show that this may fail if we assume only that F and K are only disjoint closed sets.