MTH 301 - Analysis – I

IIT KANPUR

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Assignment 4

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1. Let (M, d) and (N, ρ) be two metric spaces. Let $f : M \to N$ be an isometry, i.e.

$$\rho(f(x), f(y)) = d(x, y) \qquad \text{for all } x, y \in M.$$

   Show that f is continuous. Show that the natural inclusion $x \mapsto (x, 0)$ from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ is an isometry.

2. Suppose $f : (M, d) \to (N, \rho)$ is continuous and $g : (N, \rho) \to (P, \sigma)$ is continuous. Show that $g \circ f : M \to P$ is continuous.

3. Let (M, d) be a metric space and $A \subseteq M$ closed. Show that the distance function

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

   is continuous on M.

4. Show that the unit circle $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ is not homeomorphic to $[0, 1]$.

5. Consider the function

$$f : (0, 1) \to \mathbb{R}, \qquad f(x) = \tan\left(\pi x - \tfrac{\pi}{2}\right).$$

   Prove that f is a homeomorphism between (with the usual metric) $(0, 1)$ and $\mathbb{R}$.

6. A set $A \subseteq M$ is said to be **dense** in M if $\overline{A} = M$.

   (a) Show that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

   (b) If A is dense, show that $(A^C)^0 = \emptyset$.

   (c) Let $f : (M, d) \to (N, \rho)$ be continuous and onto, and let A be a dense subset of M. Show that $f(A)$ is dense in N.

7. If E is a connected subset of M, and if A and B are disjoint open sets in M with $E \subseteq A \cup B$, prove that either $E \subseteq A$ or $E \subseteq B$.

8. If $A \subseteq B \subseteq \overline{A} \subseteq M$, and if A is connected, show that B is connected. In particular, $\overline{A}$ is connected.

9. *We say that f is an **open map** (or, **closed map**) if $f(U)$ is open (or, closed) set in N whenever U is open (or, closed) set in M; that is, f maps open (or, closed) sets to open (or, closed) sets.*

   Give examples of a continuous map that is not open and an open (or, closed) map that is not continuous.

10. * *Let (X, d) be a metric space. We say that X is* ***path connected*** *if for every pair of points $x, y \in X$, there exists a continuous map*

$$\gamma : [0,1] \to X$$

such that $\gamma(0) = x$ and $\gamma(1) = y$. The map γ is called a path *in X from x to y.*

Let $C = \{(x, \sin(1/x)) : x \in (0,1]\} \cup \{(0,y) : -1 \leq y \leq 1\} \subset \mathbb{R}^2$. Prove that C is connected but not path-connected.