End Sem Exam: MTH 207M

Time: 5 PM to 7 PM

Instructions: You may use the results from the lectures, provided they are stated clearly and appropriately.

Problem 1:

(a) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Find a minimum norm g-inverse of A. (5 Marks)

(b) Let $A \in \mathbb{R}^{n \times n}$ be partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} is an $r \times r$ block with $\text{rank}(A_{11}) = r$ and $\text{rank}(A) = r$. Then show that there exists a matrix $X \in \mathbb{R}^{r \times (n-r)}$ such that

$$A_{12} = A_{11}X \quad \text{and} \quad A_{22} = A_{21}X,$$

and consequently

$$A_{22} = A_{21}A_{11}^{-1}A_{12}.$$

(5 Marks)

Problem 2: Answer the following questions with reference to the linear model $\mathbb{E}(y_1) = \beta_1 + \beta_2$, $\mathbb{E}(y_2) = 2\beta_1 - \beta_2$, $\mathbb{E}(y_3) = \beta_1 - \beta_2$, where y_1, y_2, y_3 are uncorrelated with a common variance σ^2.

1. Find two different linear functions of y_1, y_2, y_3 that are unbiased for β_1. Determine their variances and the covariance between the two. (4 Marks)
2. Find two linear functions that are both unbiased for β_2 and are uncorrelated. (4 Marks)
3. Write the model in terms of the new parameters $\theta_1 = \beta_1 + 2\beta_2$, $\theta_2 = \beta_1 - 2\beta_2$. (2 Marks)

Problem 3: Consider the set $\mathcal{S}$ of all 2×2 real symmetric matrices.

1. Find an orthonormal basis for the set $\mathcal{S}$. (4 Marks)
2. Find the projection of $Y = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ on the set $\mathcal{S}$. (6 Marks)

Problem 4: Consider the residual sum of squares

$$RSS = \sum_{i=1}^{n} (y_i - \widehat{y}_i)^2$$

where $\widehat{y}_i$ is the i-th fitted value based on the standard regression model

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + \epsilon_i; \quad i = 1, \ldots, n,$$

where ϵ_i are independently normally distributed with mean 0 and variance σ^2.

(a) Let the RSS be written in a quadratic form $\mathbf{y}^T\mathbf{A}\mathbf{y}$ where $\mathbf{y}^T = (y_1, \ldots, y_n)$. Find $\mathbf{A}$? (2 Marks)

(b) Verify whether $\mathbf{A}$ is idempotent. Find the rank of $\mathbf{A}$. (2+2 Marks)

(c) Find the mean and variance of RSS. (2+2 Marks)