MTH 301 - Analysis – I

IIT KANPUR

Instructor: Indranil Chowdhury

Odd Semester, 2025-26

Assignment 3

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1. Consider $A = \{y \in M : d(x, y) \leq r\}$. Show that A is a closed set. Give an example of a metric space and A such that A is not the closure of $B_r(x)$.
2. Give an example of an infinite closed set in $\mathbb{R}$ containing only irrationals. Is there an open set consisting entirely of irrationals?
3. Given $S, T \subset M$ and $S \subset T$. Prove that $S^0 \subset T^0$ and $\bar{S} \subset \bar{T}$.
4. (a) Show that any finite subset of a metric space (M, d) is closed.
   (b) If M is an infinite set. Show that there exists an open set U such that both U and U^C are infinite.
5. Consider the space $(\ell_p, \|\cdot\|_p)$ with the norm $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$ where $x = (x_1, x_2, \cdots)$.
   (a) Show the Cauchy-Schwarz inequality: $\sum_i |x_i||y_i| \leq \|x\|_p \|y\|_p$.
   (b) Show the Minkowski's inequality: Let $1 < p < \infty$, show $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.
   (c) Show the Hölder inequality: Let $1 < p < \infty$, if $\frac{1}{p} + \frac{1}{q} = 1$, then $\sum_i |x_i||y_i| \leq \|x\|_p \|y\|_q$.
   (d) Show thatbthe set $A = \{x \in \ell_2 : |x_n| \leq 1/n, n = 1, 2, \cdots\}$ is a closed set in ℓ_2. But that $B = \{x \in \ell_2 : |x_n| < 1/n, n = 1, 2, \cdots\}$ is not an open set. [Hint: Check $B_\epsilon(0)$]
6. Consider $(\mathbb{R}^2, d)$ and $(\mathbb{R}^2, \rho)$ are metric spaces with $d(x, y) = \sqrt{\sum_{i=1}^{2} |x_i - y_i|^2}$ and $\rho(x, y) = \sum_{i=1}^{2} |x_i - y_i|$. Show that the circle $S = \{(x, y) : x^2 + y^2 = 1\}$ is a closed set in $(\mathbb{R}^2, d)$ and $(\mathbb{R}^2, \rho)$.
7. Let A, B are subsets of a metric space (M, d).
   (a) If $A \subset B$, prove that $A^0 \subset B^0$ and $\bar{A} \subset \bar{B}$.
   (b) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Give and example which showas that the last inclusion is proper.
   (c) Is it true that $A^0 \cup B^0 = (A \cup B)^0$? Justify.
8. Given A be asubset of (M, d), show that
$$M = A^0 \cup (A^C)^0 \cup \mathrm{bdry}(A).$$
9. A set that is simultaneously open and closed is called a **clopen** set. If U is a non-trivial (apart from $\emptyset, \mathbb{R}$) open subset of $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$, show that $\overline{U}$ is strictly bigger than U. In other words, $(\mathbb{R}, d)$ has no non-trivial clopen set.

10. Let A be a subset of (M, d). A point $x \in M$ is called a **limit point** of A if for every $\epsilon > 0$ we have $A \cap (B_\epsilon(x) \setminus \{x\}) \neq \emptyset$.

   (a) If x is a limit point of A, show that every neighborhood of x contains infinitely many points of A.

   (b) If A' is the set of all limit points of A, show that $\overline{A} = A \cup A'$.

   (c) Show that $A' \subset A$ iff A is closed.