MTH 301 - ANALYSIS - I
IIT KANPUR
Instructor: Indranil Chowdhury
Odd Semester, 2025-26

Quiz 4	Max Marks: 25	6:30pm - 7:45pm

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1. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous and Lipschitz in its second variable, i.e. $|F(r,s) - F(r,t)| \leq K|s-t|$ for all $r \in \mathbb{R}$. Define $T : C([0,1]) \to C([0,1])$ by $(Tf)(x) = \int_0^x F(t, f(t))\, dt$.

   (a) Justify whether T is always a strict contraction or not.

   (b) Show that T maps bounded sets into equicontinuous sets. [3+4]

2. Let $\{x_i\}_{i\in\mathbb{N}} \subset (0,1)$ be a sequence such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n x_i^k$ exists for every $k \in \mathbb{N}$. Show that
$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^n \frac{1}{2+\sin(x_i)} = 0.$$
[4]

3. Let $\{g_n\}_{n\in\mathbb{N}} \subset C([a,b])$ and for each $n \in \mathbb{N}$ we have $g_n(x) \geq -\frac{1}{n^2}$ for every $x \in [a,b]$. If $\sum_{n=1}^\infty g_n$ converges pointwise to a continuous function on $[a,b]$, show that $\sum_{n=1}^\infty g_n$ converges uniformly on $[a,b]$. [3]

4. Let $(\mathbb{R}, d)$ be the usual metric spaces, and $f, f_n : \mathbb{R} \to \mathbb{R}$ be such that $\{f_n\}$ converges uniformly to f on $\mathbb{R}$. Show that $D(f) \subset \bigcup_{n=1}^\infty D(f_n)$, where $D(f)$ is the set of discontinuous points of f. [5]

5. (a) Let $f : [-13, 13] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} x^4 - 13, & x \in \mathbb{Q}, \\ 13x^2 - 49, & x \notin \mathbb{Q}. \end{cases}$$
Justify whether f is Riemann integrable on $[-13, 13]$.

   (b) Give short Justifications, using results from the class, to check whether the following functions $f, g : [0,1] \to \mathbb{R}$ are Riemann integrable or not:
$$f(x) = \begin{cases} 1, & x = 0, \\ 1 - \frac{1}{n}, & x \in (\frac{1}{n+1}, \frac{1}{n}]. \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1, & x = 0, \\ n - \frac{1}{n}, & x \in (\frac{1}{n+1}, \frac{1}{n}]. \end{cases}$$
[3 + 1.5 + 1.5]