Indian Institute of Technology Kanpur

Introduction to Probability Theory (MSO205)
End Semester Examination

Time: 8 AM to 10 AM **Date:** September 19, 2025 **Maximum Marks:** 30

Name: **Roll Number:**

Question 1: Let $\Omega = (0, 1)$ be the sample space equipped with the uniform probability measure $\mathbb{P}$, such that for any interval $(a, b) \subseteq \Omega$, $\mathbb{P}((a, b)) = b - a$.Let $X : \Omega \to \mathbb{R}$ be a random variable defined by:

$$X(\omega) = -\ln(1 - \omega)$$

Calculate the probability $\mathbb{P}[X \leq \ln 2]$.

Points : 6

Question 2: Let X be a random variable with the moment generating function given by:

$$M_X(t) = e^{\frac{t^2}{2}}$$

Let $\Phi(x)$ denote the CDF of the standard normal distribution, defined as:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

Define a new random variable $Y = \Phi(X)$. Determine if the following infinite series converges:

$$\sum_{n=0}^{\infty} (\mathbb{E}[Y^n])^2$$

Points : 6

Question 3: Let X be a random variable. Prove that the L^p-norm of X, defined as $\|X\|_p = (\mathbb{E}[|X|^p])^{1/p}$, is a non-decreasing function of p for $p \in (0, \infty)$. Show that if $0 < p < k$, then:

$$(\mathbb{E}[|X|^p])^{1/p} \leq (\mathbb{E}[|X|^k])^{1/k}$$

Points : 6

Question 4: Let X and Y be a standard normal random variable.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathbb{1}_{\{x \in \mathbb{R}\}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \mathbb{1}_{\{x \in \mathbb{R}\}} + 10^7 \mathbb{1}_{\{x \in}$$

Are X and Y identically distributed? Is it true that $E[X^n] = E[Y^n]$ for all $n \in \mathbb{N}$? Justify your answer. Compute the value of the even moments, $E[X^{2m}]$ (or $E[Y^{2m}]$), for all $m \in \mathbb{N}$.

Points : 6

Question 5: Consider a continuous RV X with the p.d.f.

$$f_X(x) := \begin{cases} \frac{1}{2}, & \text{if } x \in (-1, 0), \\ \frac{1}{3}, & \text{if } x \in (0, \frac{3}{2}), \\ 0, & \text{otherwise.} \end{cases}$$

Consider the RV $Y = X^4$.

(i) Find the DF F_Y.

(ii) Find the p.d.f. f_Y.

Points : 6