MTH 301 - Analysis – I

IIT KANPUR

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Assignment 8

** Let $\mathcal{R}[a, b]$ be the set of Riemann integrable functions. **

1. Prove that if $f \in \mathcal{R}[a, b]$ then f is bounded on $[a, b]$. What about the converse part?

2. If $f \in \mathcal{R}[a, b]$ and $|f(x)| \leq M$ for all $x \in [a, b]$. Show that $\left| \int_a^b f(x)\, dx \right| \leq M(b - a)$.

3. Let $g : [0, 1] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2, & x \text{ rational,} \\ x^3, & x \text{ irrational.} \end{cases}$$

   Investigate whether g is Riemann integrable on $[0, 1]$. If it is, compute the integral; if not, justify.

4. Let $g(x) = 0$ in $x \in [0, 1] \cap \mathbb{Q}$ and $g(x) = \frac{1}{x}$ in $x \in [0, 1] \cap \mathbb{Q}^c$. Determine $g \in \mathcal{R}[0, 1]$ or not?

5. If $f \in \mathcal{R}[a, b]$ and $c \in \mathbb{R}$ we define g on $[a + c, b + c]$ by $g(y) = f(y - c)$. Prove that $g \in \mathcal{R}[a + c, b + c]$ and $\int_{a+c}^{b+c} g = \int_a^b f$.

6. Suppose f is continuous function on $[a, b]$. Define $F(x) := \int_a^x f(y)\, dy$ for $a \leq x \leq b$. Show that F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

7. Evaluate the limit
$$\lim_{n \to \infty} \sum_{k=0}^{n} \frac{n}{k^2 + n^2}$$
   by interpreting it as a Riemann sum. Show your steps.

8. Let f and g are continuous on $[a, b]$. If $\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$ then show that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

9. Let $g(x) = \sin\left(\frac{1}{x}\right)$ for $x \in (0, 1]$ and $g(0) = 0$. Show that $g \in \mathcal{R}[0, 1]$.

10. If f is continuous on $[a, b]$, show that there exists $c \in [a, b]$ such that we have $\int_a^b f(x)\, dx = f(c)(b - a)$.

11. Let f be continuous on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$. Define $M_n = \left(\int_a^b f^n \right)^{\frac{1}{n}}$. Show that $\lim M_n = \sup \{f(x) : x \in [a, b]\}$.

12. Show that there does not exists a continuously differentiable function f on $[0, 2]$ such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for $0 \leq x \leq 2$.

13. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f(y)\,dy = \int_x^1 f(y)\,dy$ for all $x \in [0, 1]$. Show that $f(x) = 0$ for all $x \in [0, 1]$.