MTH 301 - Analysis – I

IIT KANPUR

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Assignment 6

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1. Let (M, d) be a complete metric space. Prove that every infinite totally bounded subset of M has a limit point in M. What will happen if the assumption "totally bounded" is replaced by "bounded"? Does the conclusion still hold?

2. Define the map $T : C[0,1] \to C[0,1]$ by $(Tf)(x) = \int_0^x f(t)\,dt\ x \in [0,1]$. Prove that T is not a strict contraction, while $T^2 (:= T \circ T)$ is a strict contraction on $C[0,1]$ with the supremum norm. Also, determine the fixed point of T.

3. Show that each of the hypothesis of the contraction mapping principle is necessary by finding examples of a space M and a map $f : M \to M$ having no fixed point where
(a) M is incomplete (but f is still a strict contraction).
(b) f satisfies only $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$ (but M is still complete).

4. Let (M, d) and (N, ρ) be metric spaces and let $f : M \to N$ be uniformly continuous. Show that if (x_n) is a Cauchy sequence in M, then $(f(x_n))$ is a Cauchy sequence in N.

5. Let $f : (0,1) \to \mathbb{R}$ be uniformly continuous. Prove that $\lim_{x \to 0^+} f(x)$ exists. Conclude that f is bounded on $(0,1)$.

6. Let $f : (0,1) \to \mathbb{R}$ be continuous, and suppose both $f(0^+)$ and $f(1^-)$ exist. Define $F : [0,1] \to \mathbb{R}$ by $F(0) = f(0^+)$, $F(1) = f(1^-)$, and $F(x) = f(x)$ for $0 < x < 1$. Show that F is uniformly continuous on $[0,1]$.

7. Let ℓ^2 and ℓ^1 be equipped with their usual norms. Define $f : \ell^2 \to \ell^1$ by $f\big((x_n)_{n\geq 1}\big) = \big\{\frac{x_n}{n}\big\}_{n\geq 1}$. Show that f is uniformly continuous (you may first verify that f is well-defined).

8. Consider the (right) shift operator $T : \ell_\infty \to \ell_\infty$ given by $T(x) = (0, x_1, x_2, \dots)$ for $x = (x_1, x_2, \dots)$ in ℓ_∞. Let $K = \{x \in \ell_\infty : \lim_{n\to\infty} x_n = 1\}$. Prove that:
   (a) K is a closed subset of ℓ_∞.
   (b) $T(K) \subset K$;
   (c) T is an isometry on K (i.e. $\|Tx - Ty\|_\infty = \|x - y\|_\infty$ for all $x, y \in K$), but T has no fixed point in K.

9. Let $\{f_n\}$ and $\{g_n\}$ be real-valued functions on a set X, and suppose the $\{f_n\}$ and $\{g_n\}$ converges uniformly on X. Show that $\{f_n + g_n\}$ converges uniformly on X. Give an example showing that $\{f_n g_n\}$ need not converge uniformly on X (although it will converge pointwise, of course).

10. Let $f_n : \mathbb{R} \to \mathbb{R}$, and suppose that $\{f_n\}$ converges uniformly to 0 on every closed, bounded interval $[a, b]$. Does it follow that $\{f_n\}$ converges uniformly to 0 on $\mathbb{R}$? Justify.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous, and define $f_n(x) := f\left(x + \frac{1}{n}\right)$. Show that f_n uniformly converges to f on $\mathbb{R}$.

12. Let (M, d) and (N, ρ) be metric spaces, and let $f, f_n : M \to N$ such that $\{f_n\}$ converges uniformly to f on M. Show that $D(f) \subset \bigcup_{n=1}^{\infty} D(f_n)$, where $D(f)$ is the set of discontinuities of f.

13. **(Dini's Tehorem)** Let M be a compact metric space, and suppose that the sequence $\{f_n\}$ in $C(M)$ increases pointwise to a continuous function $f \in C(M)$; that is $f_n(x) \leq f_{n+1}(x)$ for each n and x, and $f_n(x) \to f(x)$ for each x. Prove that the convergence is actually uniform. The same is true if $\{f_n\}$ decreases pointwise to f.

14. Let $C^1([a,b])$ be the vector space of all functions $f : [a,b] \to \mathbb{R}$ having a continuous first derivative on $[a,b]$. Show that $C^1([a,b])$ is complete under the norm $\|f\|_{C^1} = \max\limits_{a \leq x \leq b} (|f(x)| + |f'(x)|)$.

15. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ converges for all $|x| \leq 1$, but that the convergence is not uniform.

16. Let $0 \leq g_n \in C([a,b])$. If $\displaystyle\sum_{n=1}^{\infty} g_n$ converges pointwise to a continuous function on $[a,b]$, show that $\displaystyle\sum_{n=1}^{\infty} g_n$ converges uniformly on $[a,b]$.

17. Investigate the convergence properties of the series
$$\sum_{n=1}^{\infty} x^n(1-x) \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n x^n (1-x)$$
in the interval $[0,1]$. Justify whether those series are pointwise and uniform convergent