MTH 301 - ANALYSIS - I
IIT KANPUR
Instructor: Indranil Chowdhury
Odd Semester, 2025-26

End-Semester Examination | 17th November | 5:00pm - 8:00pm

****Follow the instructions provided in 2nd page****

1. (a) Determine (with mathematical rigor) the cardinality of all possible english essays having at most 2025 finite-words, where a *finite-word* is a finite string of the english alphabets (a-z, A-Z).
   (b) What would be the cardinality of the same set if the english alphabet would consist of a single element, called **m**?

   6

2. Consider the metric space $\ell_\infty = \{\mathbf{x} = (x_1, x_2, \cdots, x_n, \cdots) : \sup_{n\in\mathbb{N}} |x_n| < \infty\}$ equipped with the norm $\|\mathbf{x}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$. Show that every sequence in the following set
$$H = \{\mathbf{x} \in \ell_\infty : |x_n| \leq \frac{1}{2^n} \text{ for each } n \in \mathbb{N}\}$$
has a *convergent subsequence*. Justify whether H is also compact. **6**

3. (a) Let $f : (M, d) \to (N, \rho)$ be a *homeomorphism*. Show that M is *separable* (having a countable and dense subset) if and only if N is *separable*.
   (b) Justify whether there is a countable and dense subset of $C[0, 1]$.

   4 + 4

4. Let (M, d) be a metric space. M is *disconnected* if and only if there exists a *continuous* function between M onto $\{0, 1\}$. **4**

5. Let $\mathbf{y} := (y_1, y_2, \cdots) \in \ell_\infty$ and define $g : (\ell_2, \|\cdot\|_2) \to (\ell_2, \|\cdot\|_2)$ by
$$g(\mathbf{x}) = (x_1y_1, \cdots, x_ny_n, \cdots).$$
Show that g is *uniformly continuous*. **4**

6. Let F and K be disjoint, nonempty subsets of a metric space (M, d) with F closed and K compact. Show that $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\} > 0$. Show that this may fail if we assume only that F and K are disjoint closed sets. **6**

7. Let M be a compact metric space, and let $\{f_n\}$ be an equicontinuous sequence in $C(M)$. Show that $A = \{x \in M : \{f_n(x)\} \text{ converges}\}$ is a closed set in M. **6**

8. Let $\mathcal{R}[a,b]$ be the set of Riemann integrable functions equipped with a metric $d(f,g) = \int_a^b |f(x)-g(x)|dx$. Note that $C[a,b]$ is a dense subset of $(\mathcal{R}[a,b],d)$. If for some $f \in \mathcal{R}[a,b]$ we have $\int_a^b x^n f(x)dx = 0$ for each $n = 0,1,2,\ldots,$ then show that $f = 0$ for all x in $[a,b]$ except a measure zero set. **7**

9. (a) Suppose that $f_n : [a,b] \to \mathbb{R}$ is a sequence of differentiable functions satisfying $|f_n'(x)| \leq 2$ for all n and x. Prove that some subsequence of $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent.

   (b) Let $\{f_n\}_{n\in\mathbb{N}}$ be an equicontinuous sequence in $C^1[a,b]$. Justify whether the sequence of derivatives $\{f_n'\}_{n\in\mathbb{N}}$ is uniformly bounded or not.

   5 + 3

10. (a) Let (M,d) be a compact metric space. Justify whether M is always *separable.*

    (b) Justify whether there is a continuous and open map $f : [0,1] \to [-3,3]$ such that $f(0) = f(1) = 2$.

    (c) Construct an example of closed bounded subset of ℓ_∞ which is NOT totally bounded.

    (d) Justify whether $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ converges uniformly on $[0,1]$.

    3 + 2 + 3 + 2

<u>Instructions:</u>

- A separate **Answer-Booklet** will be provided.
- Write your **NAME** (in CAPITAL LETTERS) and Roll Number on each designated page of the answer booklet. If either your name or roll number is missing from any of the required spaces, the booklet will **NOT** be graded.
- You are allowed to write answers only in the designated portions of the Answer-Booklet. Each question has a predefined space for its solution, and the solutions written in that assigned area will only be evaluated.
- **No marks** will be awarded for any answer that is not supported with proper justification.