Problem Set 1: MTH207M

Problem 1. Prove the Factorization Theorem stated below.

Theorem 1 (Rank Factorization)**.** Let $\mathbb{F}$ be a field and let $A \in \mathbb{F}^{m\times n}$ have rank r. Then there exist matrices

$$B \in \mathbb{F}^{m\times r}, \qquad C \in \mathbb{F}^{r\times n},$$

each of rank r, such that

$$A = BC.$$

Conversely, any factorization $A = BC$ with $B \in \mathbb{F}^{m\times r}$ and $C \in \mathbb{F}^{r\times n}$ satisfies $\operatorname{rank}(A) \leq r$, and if $\operatorname{rank}(B) = \operatorname{rank}(C) = r$ then $\operatorname{rank}(A) = r$.

Problem 2. Let $A \in \mathbb{R}^{n\times n}$ be partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is an $r \times r$ block with $\operatorname{rank}(A_{11}) = r$ and $\operatorname{rank}(A) = r$. Then there exists a matrix $X \in \mathbb{R}^{r\times(n-r)}$ such that

$$A_{12} = A_{11}X \qquad \text{and} \qquad A_{22} = A_{21}X,$$

and consequently

$$A_{22} = A_{21}A_{11}^{-1}A_{12}.$$

Problem 3. Find two different g–inverse of

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 0 & -2 & 4 \\ -1 & 1 & 1 & 3 \\ -2 & 2 & 2 & 6 \end{pmatrix}$$

Problem 4. Find the minimum norm solution of the system of equations

$$\begin{aligned} 2x + y - z &= 1 \\ x - 2y + z &= -2 \\ x + 3y - 2z &= 3. \end{aligned}$$

Problem 5. Find the Moore–Penrose inverse of

$$\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}.$$