letters and roll number inside the boxes.

Solution of Q1 :

(a) Consider the set of alphabet

$$\Phi = \{a, b, \dots, z, A, B, \dots\} \Rightarrow |\Phi| = 52$$

or $\Phi = \{a, b, \dots, z\}$. $\Rightarrow |\Phi| = 26$.

(Depends on whether considering small letter and capital letters differently or ~~a~~ not).

A finite word is a finite string of those alphabets. Therefore for a k-length word, there will be $|\Phi|^k$ numbers of possible ~~...~~ words.

Further length of a 'finite word' can be any finite integer $k \in \mathbb{N}$.

~~Hence there will be~~ If W is the set of all possible 'finite words', then.

~~$W \Rightarrow$ union of k-leg~~

$$W = \bigcup_{k=1}^{\infty} \{\text{set of } k\text{-length words}\}.$$

As countable union of finite cardinality set is countable, we get

$|W|$ has countable cardinality.

Now for an essay ~~contad~~ containing at most 2025 words $= \bigcup_{N=1}^{2025} \{\text{set consisting exactly } N \text{ numbered word essay}\}$.

As W is countable ;

~~set containg~~

An essay having exactly N-words can be written in countably infinite ways $(\approx |W|^N)$

As finite union of countable set is countable,

we get that the cardinality of the set of essays having at most 2025 words is countably infinite.

(b) Now given $\Sigma = \{m\}$.

Then the set W as defined previously is of the form

$$W = \bigcup_{k=1}^{\infty} \underbrace{m \cdots m}_{k^{th}\ \text{time}}$$

Then $|W| = |\mathbb{N}| \longrightarrow$ countable infinite.

Therefore argued by previous part, we get the cardinality of the set of essays having at most 2025 words is countably infinite.

• Write your name cleanly in **CAPITAL** letters and roll number inside the boxes.

Solution of Q2 : Consider the sequence $\{x^k\} \subseteq H$.

Note: $x^k = (x_1^k, x_2^k, \ldots)$ such that $|x_n^k| < \frac{1}{2^n}$ $\forall n \in \mathbb{N}$.

~~Then for any fixed n, we have~~

~~$|x_n^k| < \frac{1}{2^n}$ for all $k \in \mathbb{N}$.~~

~~As $\{x_n^k\}_{k \in \mathbb{N}}$ is a bounded sequence in $\mathbb{R}$,~~

~~there is a subseq~~

Note that as $|x_1^k| < \frac{1}{2}$ for all $k \in \mathbb{N}$;

there is a subseq $\{x_1^{k(1)}\}$ which is converging.

Further consider the seq $\{x_2^{k(1)}\}$ and note

$$|x_2^{k(1)}| < \frac{1}{2^2};$$

then there is a converging subseq of $\{x_2^{k(1)}\}$,

namely $\{x_2^{k(2)}\} \subseteq \{x_2^{k(1)}\}$.

We continue the process and get a converging

subseq of $\{x_n^{k(n)}\} \subseteq \{x_n^{k(n-1)}\}$.

Note that $|x_n^{k(n)}| < \frac{1}{2^n}$ $\forall k(n)$.

Denote $k_j = k(j)$ and consider the

subsequence $\{x^{k_j}\}_{j \in \mathbb{N}} \subseteq H$.

Let $\varepsilon > 0$. Then there exists $N > 0$ such that $\frac{1}{2^N} < \varepsilon$.

Further we consider for $j, m \in \mathbb{N}$ the difference in l_∞-norm

$$\| x^{k_j} - x^{k_m} \|_\infty = \sup_{n \in \mathbb{N}} | x_n^{k_j} - x_n^{k_m} |$$

$$\leq \sup_{n \leq N} | x_n^{k_j} - x_n^{k_m} | + \sup_{n \geq N} \left(|x_n^{k_j}| + |x_n^{k_m}| \right)$$

Now. by construction, as k_j's are subpart of k_p for any $p<j$; we have that

$$| x_n^{k_j} - x_n^{k_m} | < \varepsilon \quad \text{for any } j, m \geq N_1 \text{ and } n \leq N_1$$

[Justification, note that $\{x_j^{k_j}\}$ are converging subseq. Also by constructions $\{x_p^{k_j}\}_{k_j \in \mathbb{N}} \subseteq \{x_p^{k_p}\}_{k_p \in \mathbb{N}}$ for any $p<j$;].

Therefore;
Taking $N_0 = \max\{N, N_1\}$ we get for any $j, m \geq N_0$

$$\| x^{k_j} - x^{k_m} \|_\infty \leq \sup_{n \leq N_1} | x_n^{k_j} - x_n^{k_m} | + \sup_{n > N_1} \left(|x_n^{k_j}| + |x_n^{k_m}| \right)$$

$$\leq \varepsilon + \frac{1}{2^N} + \frac{1}{2^N} \leq 3\varepsilon.$$

Hence $\{x^{k_j}\}$ is cauchy. Now as l_∞ is complete. we get $\{x^{k_j}\}$ is a convergent subsequence.

⊙ H is compact by equivalent criteria of sequential compactness.

Solution of Q3 : Given $f : (M,d) \to (N,\rho)$ be a homeomorphism.

$\Rightarrow$ First assume that N is separable.

Let $D_N \subseteq N$ such that D_N is countable and dense.

As f is a homeomorphism (byjective) we note that $f^{-1}(D_N)$ is also countable.

Aim to show $f^{-1}(D_N)$ is dense in M.

Let $x \in M$ and $\varepsilon > 0$. Again as f^{-1} is continuous. $f(B_\varepsilon(x))$ is open in N; and $f(x) \in f(B_\varepsilon(x))$

Then $\exists\, \delta > 0$ s.t $B_\delta(f(x)) \subseteq f(B_\varepsilon(x))$. Hence by density $B_\delta(f(x)) \cap D_N \neq \phi$.

Therefore $f(B_\varepsilon(x)) \cap D_N \neq \phi$.

And hence, $B_\varepsilon(x) \cap f^{-1}(D_N) \neq 0$.

As $x \in M$ and $\varepsilon > 0$ are arbitrary; we ge that $f^{-1}(D_N)$ is a countable dense subset of M.

Hence M is separable.

$\Leftarrow$ Given M is separable. Let $D_M \subseteq M$ be a countable dense subset of M. Then $f(D_M)$ is a countable subset in ~~M~~ N.

To show $f(D_M)$ is dense in N.

For any $y \in N$ and $\varepsilon > 0$, $f^{-1}(B_\varepsilon(y))$ is open in M ~~and~~, hence by previous argument

$$f^{-1}(B_\varepsilon(y)) \cap D_M \neq \emptyset.$$

Therefore $B_\varepsilon(y) \cap f(D_M) \neq \emptyset$.

Hence $f(D_M)$ is a countable and dense subset of N.

(b) By Weirstrass approximation theorem we know for $f \in C[0,1]$ and $\varepsilon > 0$; $\exists$ P_n polynomial of degree n such that

$$\|f - P_n\|_\infty < \varepsilon$$

Therefore $P := \{P_n \in C[0,1] : P_n \text{ are polynomials of degree } n\}$.

Further consider $Q := \{q_n \in C[0,1] : q_n \text{ are polynomials with rational co-efficient}\}$

Now choose $f \in C[0,1]$ and $\varepsilon > 0$.

$\exists$ $P_n \in P$ such that $\|f - P_n\| < \varepsilon/2$

Let $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$.

By density of $\mathbb{Q}$ in $\mathbb{R}$ we get that $\exists$ $q_i \in \mathbb{Q}$ s.t

$$|q_i - a_i| < \frac{\varepsilon}{(n+1)2} \quad \forall i = \{0, \ldots, n\}.$$

~~Then~~ ~~...~~ Take $q_n = q_0 + q_1 x + \cdots + q_n x^n$.

Then $\|f - q_n\|_\infty \leq \|f - P_n\|_\infty + \|P_n - q_n\|_\infty$.

$$\leq \varepsilon/2 + \sum_{i=0}^{n} |a_i - q_i| |x|^i \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Hence Q is dense in $C[0,1]$. Further note that Q is a countable set by definition.

Hence $C[0,1]$ is separable.

Solution of Q4 :

$\Rightarrow$ Assume $f: M \to \{0,1\}$ is a continuous onto map.

As $\{0\}$ and $\{1\}$ are open sets in $\{0,1\}$ with discrete metric. We get that $f^{-1}(0)$ and $f^{-1}(1)$ are open sets.

Note that $M = f^{-1}(0) \cup f^{-1}(1)$ and $f^{-1}(0) \cap f^{-1}(1) = \phi$.

Hence, $f^{-1}(0)$ and $f^{-1}(1)$ are disconnection of M.

Therefore M is disconnected.

$\Leftarrow$ Assume M is disconnected.

There exists two open sets A and B of M such that $M = A \cup B$ and $A \cap B = \phi$.

Define $f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$

Now the opensets of $\{0,1\}$ are $\phi, \{0\}, \{1\}, \{1,2\}$

Note that ~~for any~~ $f^{-1}(\phi) = \phi$, $f^{-1}(\{0\}) = A$, $f^{-1}(\{1\}) = B$ and $f^{-1}(\{0,1\}) = M$, all are open sets.

Hence f is continuous.

Solution of Q5 : Given $y \in l_\infty$. Then $\|y\|_\infty < M$ for som $M > 0$.

Now for any $x, z \in l_2$ we have

$$\|g(x) - g(z)\|_{l_2}^2 = \sum_{n=1}^{\infty} |(x_i - z_i) y_i|^2$$

$$\leq \sum_{n=1}^{\infty} |y_i|^2 |x_i - z_i|^2$$

$$\leq M^2 \sum_{n=1}^{\infty} |x_i - z_i|^2$$

$$= M^2 \|x - z\|_{l_2}^2.$$

Hence for any $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M}$ and if $\|z - x\|_{l_2} < \delta = \frac{\varepsilon}{M}$, we get

$$\|g(x) - g(z)\|_{l^2} < \varepsilon$$

As δ is independent of $x \in l_2$ we get g is uniformly continuous.

Solution of Q6 :

(a) Assume that $d(F,K) = 0$.

Then by definition, $d(F,K) = \inf\{d(x,y) : x \in F, y \in K\}$ we get for any $n \in \mathbb{N}$ that

$$d(x_n, y_n) < \frac{1}{n} \quad \text{for some } x_n \in F \text{ and } y_n \in K.$$

Further as $\{y_n\} \subseteq K$ and K compact; $\{y_n\}$ has a convergent subseq, say $\{y_{n_k}\} \subseteq \{y_n\}$ such that

$$y_{n_K} \longrightarrow y \in K.$$

Now consider

$$d(x_{n_K}, y) \leq d(x_{n_k}, y) + d(y_{n_k}, x_{n_k})$$

for an $\varepsilon > 0$, $\exists N_\varepsilon > 0$ s.t $\frac{1}{N_\varepsilon} < \varepsilon$, hence

$$d(x_{n_k}, y_{n_k}) < \varepsilon/2 \quad \forall n > \frac{1}{N_\varepsilon}$$

and as $y_{n_k} \to y$ we get N_ε^1 such that

$$d(y_{n_k}, y) < \varepsilon/2$$

Therefore $d(x_{n_k}, y) < \varepsilon \quad \forall n \geq \max\{N_\varepsilon, N'_\varepsilon\}$

Hence $x_{n_k} \longrightarrow y$.

But as F is closed; $y \in F$; then

$y \in F \cap K$, this is a contradiction

as F and K are assumed to be disjoint.

Hence $d(F,K) > 0$.

(b) Consider $F = \mathbb{N}$.

and $K = \{ n + \frac{1}{n} : n \in \mathbb{N} \setminus \{1\} \}$.

Note that F and K are closed sets

and $F \cap K = \phi$.

But $d(F, K) = \inf \{ d(x,y) : x \in F, y \in K \}$

$$\cancel{\leq \inf \{ d(n, n+\tfrac{1}{n}) : n \in \mathbb{N} \}}$$

$$\leq d(n, n+\tfrac{1}{n}) \quad \forall n \in \mathbb{N} \setminus \{1\}$$

$$= \frac{1}{n} \longrightarrow 0.$$

Hence $d(F, K) = 0$. (An)

Solution of Q7 :

Let $\varepsilon>0$. by equicontinuity of $\{f_n\}$ we have $\delta_\varepsilon>0$ such that

$$|f_n(x)-f_n(y)|<\varepsilon \quad \text{for } d(x,y)<\delta_\varepsilon .$$

To show that A is closed, let $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ such that $x_n \longrightarrow x$.

As $x_n \in A$, we know that $\{f_k(x_n)\}_{k\in\mathbb{N}}$ is a convergent sequence for each fixed $n\in\mathbb{N}$.

Therefore $\{f_k(x_n)\}_{k\in\mathbb{N}}$ is Cauchy sequence.

Hence, for $\varepsilon>0$, $\exists K_n>0$ such that

$$|f_k(x_n)-f_m(x_n)|<\varepsilon \quad \forall k,m \geq K_n . \longrightarrow (i)$$

Further; as $x_n \to x$, for $\delta>0$, $\exists N_\delta>0$ such that

$$d(x,x_n)<\delta \quad \forall n\geq N_\delta .$$

So, by equicontinuity $\therefore$ for every x_n with $n\geq N_\delta$

$$|f_k(x)-f_k(x_n)|<\varepsilon . \longrightarrow (ii)$$

Finally, to show $\{f_k(x)\}_{k\in\mathbb{N}}$ cauchy, we take

$$|f_k(x) - f_m(x)| \leq |f_k(x) - f_k(x_{N_\delta})| + |f_k(x_{N_\delta}) - f_m(x_{N_\delta})|$$
$$+ |f_m(x_{N_\delta}) - f_m(x)|$$

Now, for any $k, m \geq K_{N_\delta}$ we get by (ii)

$$|f_k(x) - f_k(x_{N_\delta})| < \varepsilon \text{ and } |f_k(x) - f_k(x_{N_\delta})| < \varepsilon$$

Also, by (i) we have for any $k, m \geq K_{N_\delta}$

$$|f_k(x_{N_\delta}) - f_m(x_{N_\delta})| < \varepsilon.$$

Therefore

$$|f_k(x) - f_m(x)| < 3\varepsilon \quad \text{for all } k, m \geq K_{N_\delta}.$$

Hence $\{f_k(x)\}_{k \in \mathbb{N}}$ is cauchy in $\mathbb{R}$. Therefore, $\{f_k(x)\}_{k \in \mathbb{N}}$ is convergent.

This implies $x \in A$.

Hence A is a closed set.

Solution of Q8 : Given $\mathcal{C}[a,b]$ is a dense subset of $\mathcal{R}[a,b]$.

Assume $g \in \mathcal{R}[a,b]$.

Let $\varepsilon > 0$. then $\exists f_\varepsilon \in \mathcal{C}[a,b]$ such that

$$\int_a^b |f_\varepsilon(x) - g(x)|\, dx < \varepsilon.$$

Further; By Weirstrass Approximation theorem we have polynomial P_{n_ε} such that

$$\| P_{n_\varepsilon} - f_\varepsilon \|_\infty < \varepsilon.$$

~~Now consider~~ $\cancel{\int_a^b P_{n_\varepsilon} g}$

As $\int_a^b x^n g(x) = 0 \quad \forall n$, we note that $\int_a^b P_{n_\varepsilon}(x) g(x) dx = 0$

Further

$$\left| \int_a^b \left(g^2(x) - P_{n_\varepsilon}(x)\, g(x) \right) dx \right|$$

$$\leq \left| \int_a^b \left(g^2(x) - f_\varepsilon(x)\, g(x) \right) dx \right| + \left| \int_a^b f_\varepsilon(x) g(x) - P_{n_\varepsilon}(x) g(x) \right|$$

$$\leq \|g\|_\infty \int_a^b |g(x) - f_\varepsilon(x)|\, dx + \|g\|_\infty \|g - f_{n_\varepsilon}\|_\infty (b-a).$$

$$\leq \|g\|_\infty\, \varepsilon + (b-a)\, \|g\|_\infty\, \varepsilon$$

Therefore, we find $\int_a^b g^2(x)\, dx \leq \|g\|_\infty \varepsilon + \|g\|_\infty\, \varepsilon (b-a)$ for any $\varepsilon > 0$.

Hence $\int_a^b g^2(x)\,dx = 0$.

Further, we aim to show that $g = 0$ every where except on a set of measure zero.

Note that $\{x \in [a,b] : g^2(x) > 0\} \subseteq \bigcup_{n=1}^{\infty} \{x \in [a,b] : g^2(x) > \frac{1}{n}\}$.

Now, as $\{x \in [a,b] : g^2(x) > \frac{1}{n}\} \subseteq [a,b]$ we get

$$0 = \int_a^b g^2(x)\,dx \geq \int_{\{x \in [a,b] : g^2(x) > \frac{1}{n}\}} g^2(x)\,dx$$

$$\geq \frac{1}{n} \int_{\{x \in [a,b] : g^2(x) > \frac{1}{n}\}} dx.$$

This implies measure of the set $\{x \in [a,b] : g^2(x) > \frac{1}{n}\}$ is zero.

As $\{x \in [a,b] : g^2(x) > 0\}$ is a subset of countable union of measure zero set. we get that

measure of $\{x \in [a,b] : g^2(x) > 0\}$ is zero.

Therefore $g = 0$ everywhere in $[a,b]$ except on a set of measure zero.

9. (a) The question is actually not true.

Consider $f_n : [a,b] \to \mathbb{R}$ given by.

$$f_n(x) = n \quad \forall x \in [a,b].$$

Continuous function vanishing derivative.

But $\{f_n\}$ does not have any convergent sub seq.

(b) Given $\{f_n\} \subseteq C^1[a,b]$ be equicontinuous.

For $\varepsilon > 0$, $\exists \delta > 0$ ~~independent~~ such that for all $x, y \in [a,b]$ and $n \in \mathbb{N}$ if $|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon$.

Consider the function $f_n(x) = \frac{1}{n} \sin(n^2 x)$.

Let $\varepsilon > 0$. There is N_ε s.t $\frac{2}{N_\varepsilon} < \varepsilon$.

Now $|f_n(x) - f_n(y)| \leq \frac{1}{n} |\sin(n^2 x) - \sin(n^2 y)| \leq \frac{2}{n} < \varepsilon$

$\forall n \geq N_\varepsilon$.

Further for any $n < N_\varepsilon$, by continuity, we have $\delta_n > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ for $|x-y| < \delta_n$

Take $\delta = \min\{\delta_1, \dots, \delta_{N_\varepsilon - 1}\} > 0$.

Then we get that $\forall n \in \mathbb{N}$ $|f_n(x) - f_n(y)| < \varepsilon$ whenever $|x-y| < \delta$

Therefore $\{f_n\}$ are equicontinuous.

Obviously $f_n = \frac{1}{n}\sin(n^2 x) \in C'[0, 2\pi]$.

But $f_n'(x) = \frac{n^2 \cos(n^2 x)}{n} = n\cos(n^2 x)$.

Therefore $\sup\limits_{x\in[a,b]} |f_n'(x)| = n \longrightarrow \infty$.

Hence the statement is <u>not true</u>.

Solution of Q10 :

10. (a) Given (M,d) is a compact metric space.

Then for any $k \in \mathbb{N}$, $\exists X_k = \{x_1^k, \dots, x_{n_k}^k\}$ such that

$$M = \bigcup_{i=1}^{n_k} B_{1/k}(x_i) \longrightarrow (i)$$

Consider the set $D = \bigcup_{k=1}^{\infty} X_k$. Note that D is a countable set.

We aim to show that D is dense in M.

Let $\varepsilon > 0$ and $x \in M$. There exists $N \in \mathbb{N}$ s.t $\frac{1}{N} < \varepsilon$;

Then for given N, we get by (i) that there exist $k_0 \in \{1, \dots, n_N\}$ such that

$$x \in B_{1/N}(x_{k_0}) \subset B_\varepsilon(x_{k_0})$$

Therefore $x_{k_0} \in B_\varepsilon(x)$, which implies $B_\varepsilon(x) \cap D \neq \phi$

Hence, D is dense in M and D is countable.

Therefore M is separable.

(Ans)

(b) As $f : [0,1] \to [-3,3]$ is open and continuous set, we know that f is strictly increasing or strictly decreasing function.

So if $f(0) = 2$; we can never have $f(1) = 2$, Otherwise f will not be strictly monotone.

Therefore, such function will never exist.

(Ans)

10 (c) For a totally bounded set, every sequence must have a cauchy subsequence.

Consider $e_n = (0, \dots, \underset{\substack{\uparrow \\ n^{th} \text{ place}}}{1}, 0, \dots)$.

Note that $e_n \in l_\infty$.

Now for any $n, m \in \mathbb{N}$

$$\|e_n - e_m\|_\infty = 2$$

Hence, $\{e_n\}$ can never have a Cauchy subsequence.

Note that $A = \{e_n : n \in \mathbb{N}\}$ is bounded as $\|e_n\|_\infty = 1$ $\forall n \in \mathbb{N}$ and closed as only convergent sequence in A are eventually constant sequences ~~which~~ whose limit by default belongs to A.

So, A is closed and bounded set in l_∞ which is not totally bounded.

(Ans)

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} := f(x)$ (If exists). Now for any $x \in (0,1]$ we get

Note that $f(0) = 0$;

$$\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=1}^{\infty} \frac{1}{(1+x^2)^n} = \frac{x^2}{(1+x^2)} \cdot \frac{1}{1 - \frac{1}{1+x^2}} = 1$$

We see therefore the partial sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ converges point wise to the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in (0,1]. \end{cases}$$

As the partial sums $\displaystyle\sum_{n=1}^{N} \frac{x^2}{(1+x^2)^n}$, for all $N \in \mathbb{N}$ are continuous, if partial sums converges uniformly then f must be a continuous function.

Hence $\displaystyle\sum \frac{x^2}{(1+x^2)^n}$ does not converge uniformly (Ans)