ANALYSIS

Name: 

Roll No: 

- Write your name cleanly in **CAPITAL** letters and roll number inside the boxes.
- Write answers in the space provided only. Maximum marks: 20 Time: 12:00 pm - 13:15 pm.

1. Let $f : (M, d) \to (N, \rho)$ be an ~~isomorphic~~ isometric (i.e. $\rho(f(x), f(y)) = d(x, y)$) and onto map. Show that $f$ is a homeomorphism.

[5]

Answer :-

$\boxed{\text{Step 1}}$ :- We first show that $f$ is bijective:

Given $\rho(f(x), f(y)) = d(x,y)$; if $f(x) = f(y)$ then

$$\rho(f(x), f(y)) = 0 = d(x,y) \Leftrightarrow x = y.$$

Hence $f$ is a one-one mapping, and as given $f$ is onto we get $f$ is a bijective.

$\boxed{\text{Step 2}}$ :- To show $f$ is continuous:

for any $\varepsilon > 0$, take $\delta = \varepsilon$; then if $d(x,y) < \varepsilon = \delta$ we get

$$\rho(f(x), f(y)) = d(x,y) < \varepsilon.$$

Therefore, $B^d_\varepsilon(x) \subseteq f^{-1}(B^\rho_\varepsilon(f(x)))$ for any $\varepsilon > 0$.

Hence, $f$ is continuous.

$\boxed{\text{Step 3}}$ :- To show $f^{-1}$ is continuous:

First we define the map $f^{-1}(p) = x$ such that $f(x) = p$;

Then note that for any $p, q \in N$, $\exists\, x, y \in M$ such that $f^{-1}(p) = x$ and $f^{-1}(q) = q$ and

$$\rho(p,q) = d(f^{-1}(p), f^{-1}(q))$$

Now, for any $\varepsilon > 0$, take again $\delta = \varepsilon$, then for any $\rho(p,q) < \varepsilon = \delta$ we get

$$d(f^{-1}(p), f^{-1}(q)) = \rho(p,q) < \varepsilon$$

Hence $B^\rho_\varepsilon(p) \subseteq (f^{-1})^{-1}(B^d_\varepsilon(f^{-1}(p)))$ for any $\varepsilon > 0$.

2. Let $(\ell_2, \|\cdot\|_2)$ be the complete metric space where $\ell_2 = \{(x_1, x_2, \cdots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ and the norm is given by $\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}}$ for $x = (x_1, x_2, \cdots, x_n, \cdots)$.

(a) Show that the set $A = \{x \in \ell_2 : |x_n| \leq 1/n$ for $n \in \mathbb{N}\}$ is a closed set in $\ell_2$.

(b) Is $B = \{x \in \ell_2 : |x_n| < 1/n$ for $n \in \mathbb{N}\}$ an open set? Justify.

(c) Is A complete where $A$ is defined in part (a)?

[2+2+1]

2 (a) Given $A = \{x \in \ell_2 : |x_n| \leq \frac{1}{n}$ for $n \in \mathbb{N}\}$.

Let $\{x^n\}_{n\in\mathbb{N}} \subset A$ be a convergent seq in $A$. Assume $x^n \longrightarrow x \in \ell_2$. This implies $\|x^n - x\|_{\ell^2} \xrightarrow[n\to\infty]{} 0$.

Let $x^n = (x_1^n, x_2^n, x_3^n, \cdots)$ and $x = (x_1, x_2, \cdots)$

As $x^n \in A \Rightarrow |x_k^n| \leq \frac{1}{k}$ for any $k \in \mathbb{N}$.

It is sufficient to show that $x \in A$ or $|x_k| < \frac{1}{k}$ $\forall k \in \mathbb{N}$

In contrary, assume $\exists k_0 \in \mathbb{N}$ such that $|x_{k_0}| > \frac{1}{k_0}$.

Then, we can also say ~~$|x_{k_0}| \geq \frac{1}{k_0}$ $\varepsilon_0 > 0$ for some $\varepsilon_0 > 0$.~~

$|x_{k_0}| \geq \left(\frac{1}{k_0} + \varepsilon_0\right) > 0$ for some $\varepsilon_0 > 0$.

But as $|x_{k_0}^n| \leq \frac{1}{k_0}$ $\forall n \in \mathbb{N}$, we find that

$|x_{k_0} - x_{k_0}^n| \geq |x_{k_0}| - |x_{k_0}^n| \geq \frac{1}{k_0} + \varepsilon_0 - \frac{1}{k_0} = \varepsilon_0$.

But that contradicts the fact that $\left(\sum_{i=1}^{\infty} |x_i - x_i^n|^2\right)^{1/2} \to 0$

Hence, $|x_k| < \frac{1}{k}$ $\forall k \in \mathbb{N}$ and $x \in \ell_2$.

Therefore $A$ is closed. (Proved)

(b) No! B is not an openset

Note that $\bar{0} = (0, 0, 0, \cdots) \in A$.

We aim to show that for any $\varepsilon > 0$ $B_\varepsilon(\bar{0}) \notin A$

for any $\varepsilon > 0$, by ~~Ar~~ Archimedian property $\exists$ $n_\varepsilon > 0$ such that

$\varepsilon > \frac{1}{n_\varepsilon}$ ; and $n_\varepsilon \in \mathbb{N}$.

Consider an elemen $\overline{x}_\varepsilon = (0, 0, 0, \ldots, \underset{\substack{\uparrow \\ n_\varepsilon^{th} \text{ plae}}}{0}, \ldots, \underset{\substack{\uparrow \\ (n_\varepsilon^2)^{th} \text{ place}}}{\frac{1}{n_\varepsilon}}, 0, \ldots)$

Then $\|\overline{x}_\varepsilon\|_{\ell_2} = \frac{1}{n_\varepsilon} < \varepsilon$,

Hence, $\overline{x}_\varepsilon \in B_\varepsilon(\overline{0})$.

But note that as $\left|\frac{1}{n_\varepsilon}\right| \nleq \frac{1}{n_\varepsilon^2}$. $\overline{x}_\varepsilon \notin A$

Hence $B_\varepsilon(\overline{0}) \not\subseteq A$ $\forall \varepsilon > 0$.

So, A is not an open set. (Ans)

(c) Yes, as a closed subset of a metric space is complete and A is closed, then A is a complete space. (Ans)

........ Rough Work ........

Name: Roll No:

3. $(M, d)$ is a connected metric space and has at-least two elements. Show that $M$ is uncountable. [3]

Answer :- Given $(M, d)$ is a metric space. Let $x_0, y_0 \in M$ be two distinkt elements.

By definition of a metric, $d(x_0, y_0) \neq 0$.

We define a map $f: M \to \mathbb{R}$ by.

$$f(x) = d(x, y_0) \quad \text{for } x \in M.$$

We note that $f$ is a non-constant map as

$f(x_0) \neq 0$ and $f(y_0) = 0$.

Furthere, we claim that $f$ is continuous.

for any $\varepsilon > 0$, if $d(x,y) < \varepsilon$, we get

$$f(x) - f(y) = d(x, y_0) - d(y, y_0) \leq d(x,y) < \varepsilon.$$

[follows from Tri-angle Inequality]

~~By symmetry of~~ Slly, $f(y) - f(x) \leq \varepsilon$

$\Rightarrow |f(y) - f(x)| < \varepsilon$

Hence $f$ is a continuous function.

Now; as we know $M$ is connected and $f$ is a (non-constant) continuous function; $f(M)$ is connected in $\mathbb{R}$.

We know that any connected set in $\mathbb{R}$ is an interval. As $f$ is non-constant, the interval will be non-trivial as well (e.g. $[f(y_0), \sup f(x))$, $[f(y_0), \sup f(x)]$).

As a non-trivial interval ontains uncountable points $f^{-1}$ of such interval must be uncountable. Hence $M$ is uncountable.

(Proved)

4. Let $(M, d)$ be a metric space and $A \subseteq M$. Prove that if $A$ is totally bounded then (the closure) $\overline{A}$ is totally bounded. [3]

<u>Answer</u>:- Given $A \subseteq M$ is totally bounded.

for any $\varepsilon > 0$; we can find finite number of points $\{x_1, \dots, x_{n_\varepsilon}\}$ such that

$$A \subseteq \bigcup_{i=1}^{n_\varepsilon} B_{\varepsilon/2}(x_i)$$

Further; if a point $y \in \bar{A} \setminus A$; we know that

$$B_{\varepsilon/2}(y) \cap A \neq \phi.$$

let, $x_0 \in B_{\varepsilon/2}(y) \cap A$, then $x_0 \in B_{\varepsilon/2}(x_k)$ for some $k \in \{1, \dots, n_\varepsilon\}$.

$d(x_0, y) < \varepsilon/2$ and $d(x_0, x_k) < \varepsilon/2$

Then, $d(y, x_k) \leq d(y, x_0) + d(x_0, x_k)$ [By triangle Ineq.]

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Hence $\cancel{x_0 \in B_\varepsilon(x_k)}$ $y \in B_\varepsilon(x_k)$.

As $y$ is arbitrary point in $\bar{A} \setminus A$. we find that for any $y \in \bar{A} \setminus A$, there ~~exsi~~ exists $k \in \{1, \dots, n_\varepsilon\}$ such that $y \in B_\varepsilon(x_k)$; hence $\bar{A} \setminus A \subset \bigcup_{i=1}^{n_\varepsilon} B_\varepsilon(x_k)$.

Therefore, $\bar{A} \subseteq \bigcup_{i=1}^{n_\varepsilon} B_\varepsilon(x_k)$. for every $\varepsilon > 0$;

Hence $\bar{A}$ is totally bounded. (<u>Proved</u>)

5. Let $(M, d)$ be a complete metric space. A subset $A \subseteq M$ is complete if and only if $A$ is a closed set. [4]

Answer :-

$\boxed{\text{Step 1}}$ :- To show $A \subseteq M$ is complete $\Rightarrow$ A is a closed set.

For that take any convergent seq $\{x_n\} \subset A$;

Let $x_n \to y \in M$.

As $\{x_n\}$ is a convergent seq., it is a cauchy seq in A.

By completeness of A, we get $x_n \to x \in A$. And uniqueness of limit implies $x = y \in A$.

Hence every converging seq $\{x_n\} \subset A$ converges to a point in A.

Therefore, A is closed.

$\boxed{\text{Step 2}}$ :- To show A is a closed set $\Rightarrow$ A is complete.

First we take a cauchy seq $\{x_n\} \subset A$.

Now by completeness of M, $x_n \to x \in M$.

But as A is a closed set; we have $x \in A$.

Hence any cauchy seq $\{x_n\} \subset A$ is converging in A.

Therefore A is complete

(Proved)