Question 1: Let $\mathrm{U}_1, \mathrm{U}_2, \cdots, \mathrm{U}_n, \mathrm{U}_{n+1}$ be a collection of $(n+1)$ labelled urns. Each of the first n urns contains 3 white balls and 7 black balls. The $(n+1)$-th urn U_{n+1} contains 4 white balls and 6 black balls. One urn is selected at random, and from the selected urn, two balls are drawn without replacement. It is observed that both balls drawn are black. The conditional probability that the selected urn was U_{n+1}, given that both drawn balls were black, is $\frac{1}{50}$. Determine the value of n. **Points : 5**

Solution:

Let A_i be the event that urn U_i is selected, and let B be the event that both balls drawn are black.

Given that:

$$\mathbb{P}(A_{n+1} \mid B) = \frac{1}{50}$$

Using Bayes' Theorem:

$$\mathbb{P}(A_{n+1} \mid B) = \frac{\mathbb{P}(B \mid A_{n+1}) \cdot \mathbb{P}(A_{n+1})}{\sum_{i=1}^{n+1} \mathbb{P}(B \mid A_i) \cdot \mathbb{P}(A_i)}$$

Since all urns are equally likely to be selected:

$$\mathbb{P}(A_i) = \frac{1}{n+1} \quad \text{for all } i = 1, \ldots, n+1$$

- For $i = 1, \ldots, n$ (urns with 3 white and 7 black balls):

$$\mathbb{P}(B \mid A_i) = \frac{{}^7C_2}{{}^{10}C_2} = \frac{\frac{7\times 6}{2}}{\frac{10\times 9}{2}} = \frac{42}{90} = \frac{7}{15}$$

- For $i = n+1$ (urn with 4 white and 6 black balls):

$$\mathbb{P}(B \mid A_{n+1}) = \frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$$

$$\mathbb{P}(A_{n+1} \mid B) = \frac{\frac{1}{n+1} \cdot \frac{1}{3}}{n \cdot \left(\frac{1}{n+1} \cdot \frac{7}{15}\right) + \frac{1}{n+1} \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{n \cdot \frac{7}{15} + \frac{1}{3}} = \frac{5}{7n+5}$$

$$\therefore \quad \frac{5}{7n+5} = \frac{1}{50} \implies \boxed{n = 35}$$

Question 2: Let $\{E_i : i \in \mathbb{N}\}$ be a collection of mutually independent events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Construct a collection $\{A_i : i \in \mathbb{N}\}$ of events in the following way: for each $i \in \mathbb{N}$, make a choice for A_i between E_i or E_i^c. For positive integers $k \geq 2$ and distinct indices $i_1, i_2, \cdots, i_k$, show that $\{A_{i_1}, A_{i_2}, \cdots, A_{i_k}\}$ is a collection of mutually independent events.

Points : 7

Solution:

Given, $\{E_i : i \in \mathbb{N}\}$ is a collection of mutually independent events in a probability space $(\Omega, F, \mathbb{P})$. By definition $\{E_{i_1}, E_{i_2}, \cdots, E_{i_k}\}$ is also a collection of mutually independent events then for all $k \in \{2, 3, \cdots\}$ and indices $1 \leq i_1 < i_2 < \cdots < i_k$, we have

$$\mathbb{P}\left(\bigcap_{j=1}^{k} E_{i_j}\right) = \prod_{j=1}^{k} \mathbb{P}\left(E_{i_j}\right). \tag{$*$}$$

Now we construct a collection $\{A_i : i \in \mathbb{N}\}$ of events in such a way that for each $i \in \mathbb{N}$, choosing A_i in between E_i or E_i^c.

$\boxed{\textbf{Case 1: } k = 2}$ Each of $\{E_{i_1}, E_{i_2}\}$,$\{E_{i_1}^c, E_{i_2}\}$, $\{E_{i_1}, E_{i_2}^c\}$ and $\{E_{i_1}^c, E_{i_2}^c\}$ are collections of mutually independent events.

- $\mathbb{P}(E_{i_1} \cap E_{i_2}) = \mathbb{P}(E_{i_1}) \cdot \mathbb{P}(E_{i_2})$. [From $(*)$]
- $\mathbb{P}(E_{i_1}^c \cap E_{i_2}) = \mathbb{P}(E_{i_2}) - \mathbb{P}(E_{i_1} \cap E_{i_2}) = \mathbb{P}(E_{i_2}) - \mathbb{P}(E_{i_1}) \cdot \mathbb{P}(E_{i_2}) = \mathbb{P}(E_{i_1}^c) \cdot \mathbb{P}(E_{i_2})$.
- $\mathbb{P}(E_{i_1} \cap E_{i_2}^c) = \mathbb{P}(E_{i_1}) - \mathbb{P}(E_{i_1} \cap E_{i_2}) = \mathbb{P}(E_{i_1}) - \mathbb{P}(E_{i_1}) \cdot \mathbb{P}(E_{i_2}) = \mathbb{P}(E_{i_1}) \cdot \mathbb{P}(E_{i_2}^c)$.
- $\mathbb{P}(E_{i_1}^c \cap E_{i_2}^c) = 1 - \mathbb{P}(E_{i_1} \cup E_{i_2}) = 1 - \mathbb{P}(E_{i_1}) - \mathbb{P}(E_{i_2}) + \mathbb{P}(E_{i_1} \cap E_{i_2}) = \mathbb{P}(E_{i_1}^c) \cdot \mathbb{P}(E_{i_2}^c)$.

This proves the conditions for $k = 2$, under all choices of A_{i_j}'s.

$\boxed{\textbf{Case 2: } k \geq 3}$ Let $m := \#\{j = 1, 2, \cdots, k : A_{i_j} = E_{i_j}^c\}$. Then $0 \leq m \leq k$.

- If $m = 0$, then all A_{i_j}'s are E_{i_j}'s. Independence follows from $(*)$.
- If $m = 1$, then suppose $A_{i_\alpha} = E_{i_\alpha}^c$ and $A_{i_j} = E_{i_j}, \forall j \neq \alpha$. Then

$$
\begin{aligned}
\mathbb{P}(\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \quad & \mathbb{P}\left(\left(\bigcap_{j\in\{1,2,\cdots,\alpha-1,\alpha+1,\cdots,k\}} E_{i_j}\right) \quad \bigcap \quad E_{i_\alpha}^c\right) \\
&= \prod_{j\in\{1,2,\cdots,\alpha-1,\alpha+1,\cdots,k\}} \mathbb{P}(E_{i_j}) - \prod_{j=1}^{k} \mathbb{P}(E_{i_j}) \\
&= \prod_{j\in\{1,2,\cdots,\alpha-1,\alpha+1,\cdots,k\}} \mathbb{P}(E_{i_j}) \cdot \mathbb{P}(E_{i_\alpha}^c) = \prod_{j=1}^{k} \mathbb{P}(A_{i_j}).
\end{aligned}
$$

- Assume the result holds for $m = l$. Consider $m = l + 1$. Without loss of generality, let $A_{i_\beta} = E_{i_\beta}^c$ and the remaining l complements among the other indices. Then,

$$
\begin{aligned}
\mathbb{P}\Big(\bigcap_{j=1}^{k} A_{i_j}\Big) &= \mathbb{P}\Big(E_{i_\beta}^c \cap \bigcap_{j\neq\beta} A_{i_j}\Big) \\
&= \mathbb{P}\Big(\bigcap_{j\neq\beta} A_{i_j}\Big) - \mathbb{P}\Big(E_{i_\beta} \cap \bigcap_{j\neq\beta} A_{i_j}\Big) \\
&= \Big(\prod_{j\neq\beta} \mathbb{P}(A_{i_j})\Big) - \Big(\mathbb{P}(E_{i_\beta}) \prod_{j\neq\beta} \mathbb{P}(A_{i_j})\Big) \quad \text{(by induction and independence)} \\
&= \mathbb{P}(E_{i_\beta}^c) \prod_{j\neq\beta} \mathbb{P}(A_{i_j}) \\
&= \prod_{j=1}^{k} \mathbb{P}(A_{i_j}).
\end{aligned}
$$

Thus, by induction, the formula holds for any number $0 \leq m \leq k$ of complements. Therefore, for any finite subcollection $\{A_{i_1}, \ldots, A_{i_k}\}$, we have

$$
\mathbb{P}\Big(\bigcap_{j=1}^{k} A_{i_j}\Big) = \prod_{j=1}^{k} \mathbb{P}(A_{i_j}),
$$

showing that they are mutually independent.

Question 3: Let

$$f_X(x) = \begin{cases} \dfrac{x^2}{\mathbf{c}}, & -3 < x < 3, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbf{c} > 0$ is a constant. Determine the value of '$\mathbf{c}$' for which f_X is a valid probability density function. Find the corresponding cumulative distribution function $F_X(x)$ and sketch its graph. Also compute $\mathbb{P}(|X| < 1)$ and $\mathbb{P}(X^2 \leq 4)$.

Points : 2+2+2+1+1

Solution:

(i) Since f_X is a valid probability density function, so

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1.$$

Given, $f_X(x) = \frac{x^2}{c}$, for $-3 < x < 3$ and 0, otherwise,

$$\therefore \quad 1 = \int_{-3}^{3} \frac{x^2}{c} dx = \frac{1}{c}\int_{-3}^{3} x^2 dx = \frac{1}{c}\, \frac{x^3}{3}\bigg|_{-3}^{3} = \frac{1}{c}\left[\frac{27}{3} - \frac{(-27)}{3}\right] = \frac{1}{c}\Big[9 - (-9)\Big] = \frac{18}{c}$$

Hence

$$\frac{18}{c} = 1 \quad \Longrightarrow \quad \boxed{\mathrm{c} = 18}$$

(ii)

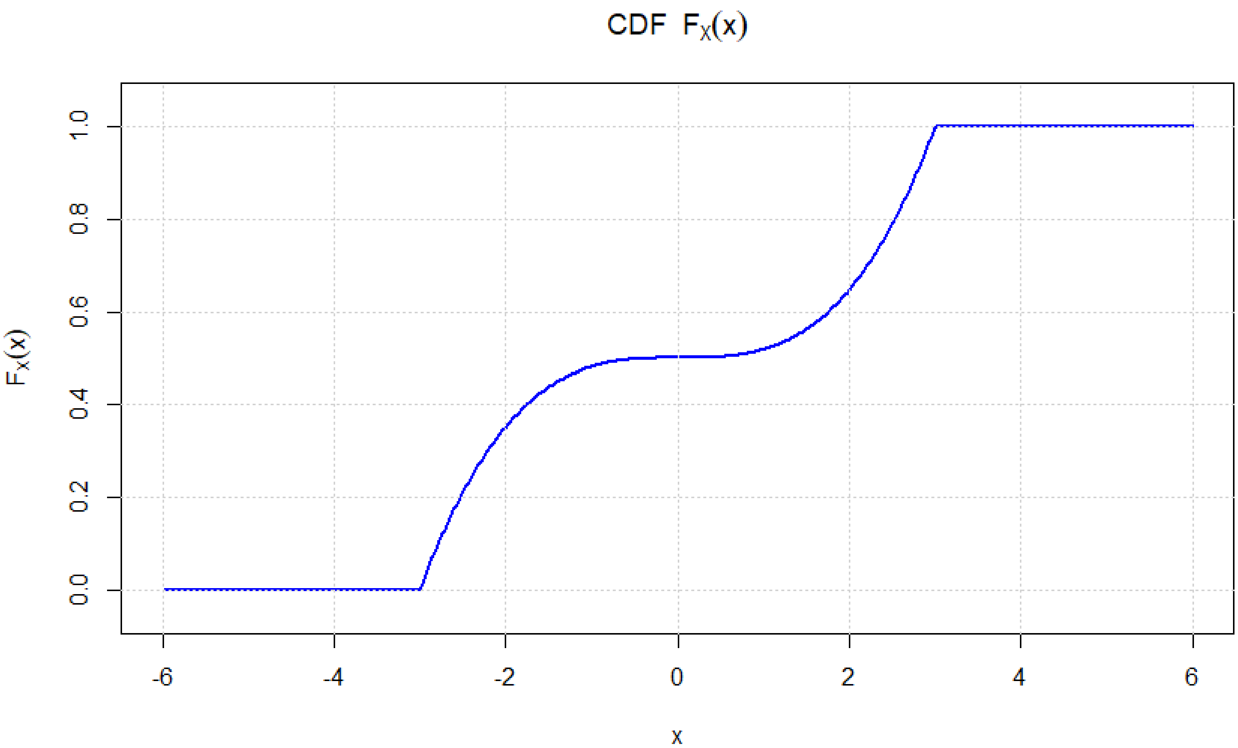
- For $x \leq -3$, clearly $F_X(x) = 0$.
- For $-3 < x < 3$,

$$F_X(x) = \int_{-3}^{x} \frac{t^2}{18}\, dt = \frac{1}{18} \cdot \frac{t^3}{3}\bigg|_{-3}^{x} = \frac{1}{54}\Big[x^3 - (-3)^3\Big] = \frac{x^3 + 27}{54}$$

- For $x \geq 3$, $F_X(x) = 1$.

$$\therefore \quad F_X(x) = \begin{cases} 0, & x \leq -3, \\ \dfrac{x^3 + 27}{54}, & -3 < x < 3, \\ 1, & x \geq 3. \end{cases}$$

(iii)



(iv)

$$P(|X| < 1) = P(-1 < X < 1) = F_X(1) - F_X(-1) = \frac{28}{54} - \frac{26}{54} = \frac{2}{54} = \frac{1}{27}.$$

$$P(X^2 \leq 4) = P(-2 \leq X \leq 2) = F_X(2) - F_X(-2) = \frac{35}{54} - \frac{19}{54} = \frac{16}{54} = \frac{8}{27}.$$