MTH 301 - ANALYSIS - I
IIT KANPUR
Instructor: Indranil Chowdhury
Odd Semester, 2025-26

Mid-Semester Examination **18th September** **1:00pm - 3:00pm**

**** Consider τ as the usual metric in $\mathbb{R}$ given by $\tau(x, y) = |x - y|$. ****

1. Let A be any uncountable set, and $B \subset A$ be a countable subset of A. Prove that $|A| = |A - B|$. **4**

2. Denote $X = \{x_n \in \mathbb{R} : n \in \mathbb{N} \text{ and } x_k \neq x_m \text{ if } k \neq m\}$ and assume (X, d) is a metric space. Show that there exists an open set $U \subset X$ in (X, d) such that both U and $X \setminus U$ have infinite number of elements. **4**

3. Given $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in a metric space (M, d).
   (a) Show that $\{d(x_n, y_n)\}_{n\in\mathbb{N}}$ is Cauchy in $(\mathbb{R}, \tau)$.
   (b) Show that the set $A = \{x_n\}_{n\in\mathbb{N}} \cup \{y_n\}_{n\in\mathbb{N}}$ is totally bounded in (M, d).

   3 + 2

4. Consider the metric space $\ell_\infty = \{\mathbf{x} = (x_1, x_2, \cdots, x_n, \cdots) : \sup_{n\in\mathbb{N}} |x_n| < \infty\}$ equipped with the norm $\|\mathbf{x}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.
   (a) Show that $c_0 = \{\mathbf{x} \in \ell_\infty : x_n \to 0\}$ is a closed set.
   (b) Justify whether $c_0 \subseteq \ell_2 := \{\mathbf{x} = (x_1, x_2, \cdots, x_n, \cdots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$!

   4 + 2

5. Let $f : (M, d) \to (N, \rho)$ be continuous and onto, and let A be a dense subset of M. Show that $f(A)$ is dense in N. **3**

6. Consider the space of functions on $I \subseteq \mathbb{R}$ by $\mathcal{C}(I) = \{f : I \to \mathbb{R} : f \text{ is continuous on } I\}$. Also denote $d_n(f, g) = \sup_{|x|\leq n} |f(x) - g(x)|$.
   (a) Prove or disprove $(\mathcal{C}([-1, 1]), d)$ for $d(f, g) = \frac{d_1(f,g)}{1+d_1(f,g)}$ is a metric space.
   (b) Show that $(\mathcal{C}(\mathbb{R}), d)$ for $d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{d_n(f,g)}{1+d_n(f,g)}\right)$ is a metric space.
   (c) Denote $g(x) = x^2$, justify whether $g \in B_{1/4}(\mathbf{0})$, where $\mathbf{0}$ is the constant zero function in $\mathbb{R}$!
   (d) Justify whether $B_1(\mathbf{0})$ is a dense subset of $(\mathcal{C}(\mathbb{R}), d)$!

   3 + 3 + 3 + 2

7. Justify whether the statements are correct or not:

   (a) There is a homeomorphic map between the interval $[0, 1)$ to the open interval $(0, 1)$ in $(\mathbb{R}, \tau)$.

   (b) There is a homeomorphic map between the open interval $(0, 1)$ to the closed interval $[0, 1]$ in $(\mathbb{R}, \tau)$.

   (c) Consider a compact metric space (M, d) such that M has uncountable points. There exists a sequence $\{x_n\} \in M$ and a continuous map $f : (M, d) \to (\mathbb{R}, \tau)$ such that $|f(x_n)| > n$ for all $n \in \mathbb{N}$.

   (d) $\mathbb{Q} \subset \mathbb{R}$ is disconnected in $(\mathbb{R}, \tau)$.

   (e) Let $(\mathbb{R}, d_1)$ be a metric space where $d_1 \neq \tau$. Any closed and bounded set in $(\mathbb{R}, d_1)$ is compact.

   (f) Let $(\mathbb{R}, d_1)$ be a metric space where $d_1 \neq \tau$. Any closed and bounded set in $(\mathbb{R}, d_1)$ is complete.

$2 \times 6 = 12$