~~Sol~~ Solution:- Mid Semester Examination (MSO205).

1. Observe that

4 marks

$$\mathbb{P} \circ X^{-1}(A) = \mathbb{P}[\omega \in \Omega : X(\omega) \in A]$$

$$= \mathbb{P}[\omega \in \Omega : X(\omega) \leq \ln 2]$$

$$= \mathbb{P}[\omega \in \Omega : -\ln(1-\omega) \leq \ln 2]$$

$$= \mathbb{P}\left[\omega \in \Omega : \ln\left(\frac{1}{1-\omega}\right) \leq \ln 2\right].$$

$$= \mathbb{P}\left[\omega \in \Omega : \frac{1}{1-\omega} \leq 2\right] \text{ since } \ln \uparrow$$

$$= \mathbb{P}\left[\omega \in \Omega : 1-\omega \leq \frac{1}{2}\right].$$

$$= \mathbb{P}\left[\omega \in \Omega : \omega \geq \frac{1}{2}\right].$$

2 marks

Now, $\Omega = (0,1)$, $\mathbb{P}((a,b)) = b-a$

So, $\mathbb{P}\left[\omega \in (0,1) : \omega \in \left[\frac{1}{2}, 1\right)\right]$

$$= 1 - \frac{1}{2} = \frac{1}{2}. \text{ (Answer)}.$$

2. 3 marks.

X: RV. $M_X(t) = e^{\frac{t^2}{2}} \Rightarrow X \sim N(0,1)$.

Now, $Y(\omega) = \Phi(X(\omega))$ $\forall \omega \in \Omega$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy.$$

So, $Y \sim \text{Unif}(0,1)$. [Proof is NOT required; taught in class].

3 marks

Now, $\sum_{n \geq 0}^{\infty} \{E[Y^n]\}^2 = \sum_{n \geq 0}^{\infty} \frac{1}{(n+1)^2} < \infty$.

So, it is true.

3. ~~Take~~

3 marks

Consider any $p \in (0, K)$. & $Y(\omega) = \{X(\omega)\}^p \; \forall \omega \in \Omega$.

so, $\frac{K}{p} > 1$.

Consider $g(y) = y^{\frac{K}{p}}$, where $g : (0,\infty) \to (0,\infty)$.

Observe that $g(y)$ is a convex f^n (Proof is not required; proof is straight forward).

Therefore, by Jenson's inequality, we have

3 marks

$$g(E(Y)) \leq E\, g(Y).$$

$$\Rightarrow \{EY\}^{\frac{K}{p}} \leq E\left(Y^{\frac{K}{p}}\right)$$

$$\Rightarrow \{EX^p\}^{\frac{K}{p}} \leq E\left(X^{p \times \frac{K}{p}}\right) = EX^K$$

$$\Rightarrow \{EX^p\}^{\frac{1}{p}} \leq \{EX^K\}^{\frac{1}{K}}. \text{ since } K \in [0,\infty).$$

Since $p \in (0, K)$, and it is true for all $p \in (0,K)$ the above implies that $(EX^K)^{\frac{1}{K}}$ is an increasing f^n in $K \in [0,\infty)$.

4.

(i) Yes, X & Y are identically distributed since $F_X(a) = F_Y(a)\ \forall a \in \mathbb{R}$, where $F_X(\cdot)$ & $F_Y(\cdot)$ are the CDF of X & Y, respectively.

1.5 marks (if it is NOT shown).

(ii) Since $F_X(a) = F_Y(a)\ \forall a \in \mathbb{R}$,

$$E(X^n) = E(Y^n)\ \forall n \in \mathbb{N}.$$

Answer: Yes.

Alternative argument:- Direct integration & show that $EX^n = EY^n\ \forall n \in \mathbb{N}$.

(iii) Since $F_X(a) = F_Y(a)\ \forall a \in \mathbb{R}$,

so ~~it is enough to derive E~~ $\forall m \in \mathbb{N}$.

$$E(X^{2m}) = E(Y^{2m})$$

Now, $E(Y^{2m}) = E(X^{2m})$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2m} e^{-\frac{x^2}{2}} dx$$

After a few steps $= (2m-1)(2m-3) \times \cdots \times 5 \times 3 \times 1.$

Alternative argument:- Directly compute

$$E(Y^{2m}) = \int_{-\infty}^{\infty} y^{2m} (\cdots)\, dy.$$

5.

(i) Observe that

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \begin{cases} 0 & \text{if } x < -1 \\ \int_{-1}^{x} f_X(y)\,dy & \text{if } -1 \leq x \leq 0 \\ \int_{-1}^{0} f_X(y)\,dy + \int_{0}^{x} f_X(y)\,dy & \text{if } 0 < x \leq \frac{3}{2} \\ 1 & \text{if } x > \frac{3}{2} \end{cases}$$

CDF of X

$$= \begin{cases} 0 & \text{if } x < -1 \\ \frac{x+1}{2} & \text{if } -1 \leq x \leq 0 \\ \frac{1}{2} + \frac{x}{3} & \text{if } 0 < x \leq \frac{3}{2} \\ 1 & \text{if } x > \frac{3}{2}. \end{cases} \quad \text{(**)}$$

Further, observe that

$$F_Y(y) = \mathbb{P}[X^4 \leq y] = \mathbb{P}[-y^{\frac{1}{4}} \leq X \leq y^{\frac{1}{4}}]$$

CDF of Y

$$= F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}}). \quad \text{(*)}$$

Using (*) & (**), we have

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}}) & \text{if } 0 \leq y \leq 1 \\ F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}}) & \text{if } 1 < y \leq \frac{81}{16} \\ 1 & \text{if } y > \frac{81}{16} \end{cases}$$

$$= \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{2} + \frac{1}{3} y^{\frac{1}{4}} - \frac{-y^{\frac{1}{4}} + 1}{2} & \text{if } 0 \le y \le 1 \\ \frac{1}{2} + \frac{1}{3} y^{\frac{1}{4}} & \text{if } 1 < y \le \frac{81}{16} \\ \cancel{0}\, 1 & \text{if } y > \frac{81}{16} \end{cases}$$

$$= \begin{cases} 0 & \text{, if } y < 0 \\ \frac{5}{6} y^{\frac{1}{4}} & \text{if } 0 \le y \le 1 \\ \frac{1}{2} + \frac{1}{3} y^{\frac{1}{4}} & \text{if } 1 < y \le \frac{81}{16} \\ 1 & \text{, if } y > \frac{81}{16}. \end{cases}$$

~~Upto this~~

(ii) Hence, the p.d.f. of Y is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{5}{24} y^{-\frac{3}{4}} & \text{if } 0 \le y \le 1 \\ \frac{1}{12} y^{-\frac{3}{4}} & \text{if } 1 < y \le \frac{81}{16} \\ 0 & \text{if } y > \frac{81}{16} \end{cases}$$