MTH 301 - ANALYSIS - I
IIT KANPUR
Instructor: Indranil Chowdhury
Odd Semester, 2025-26

Quiz 3	Max Marks: 25	6:30pm - 7:45pm

1. (a) Find an example of a space M and a *strict contraction* map $F : M \to M$ having no fixed point where M is *not complete*.

   (b) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a map defined for any $(x_1, x_2) \in \mathbb{R}^2$ by $F(x_1, x_2) = \frac{1}{4}(x_1 + x_2, x_1 + 2x_2)$. Show that F is a *strict contraction* and find the unique fixed point. [2+4]

2. (a) Let $g : \mathbb{R} \to \mathbb{R}$ be uniformly continuous, and define $f_n(x) := g\left(x - \frac{1}{n^2}\right)$. Show that f_n uniformly converges to g on $\mathbb{R}$.

   (b) Let $f : [0, 3] \to \mathbb{R}$ be a continuous function. Justify whether $\lim_{n \to \infty} \int_{\frac{1}{n}}^{2 + \frac{1}{n}} f(y) dy = \int_0^2 f(y) dy.$ [4+2]

3. (a) Let us consider $P(x) = \sum_{n=0}^{\infty} a_n x^n$ (whenever exists). If $P(5)$ exists, will $P(4)$ exist? Justify.

   (b) Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$

      i. Show that the series converges uniformly on every interval $[-r, r]$ with $0 \le r < 1$.

      ii. Justify whether the series also converges uniformly on the interval $[-1, 1]$. [1+3+2]

4. (a) Let $f : M \to N$ be continuous and $\{x_n\}$ be a Cauchy sequence in M. Will $\{f(x_n)\}$ be Cauchy? Justify.

   (b) Let $f : M \to N$ be uniformly continuous and $\{x_n\}$ be a Cauchy sequence in M. Will $\{f(x_n)\}$ be Cauchy? Justify.

   (c) Suppose $f : (0, 1) \to \mathbb{R}$ is continuous, and for each $n \in \mathbb{N}$, f is uniformly continuous on $[\frac{1}{n}, 1]$. Must f be uniformly continuous on $(0, 1)$? Justify your answer. [2 + 3 + 2]