Solution (End Semester Examination)

Q1. Observe that

$$U_n = \frac{X_1 + \cdots + X_n}{X_1^5 + \cdots + X_n^5} = \frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)}{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^5\right)} := \frac{A_n}{B_n}$$

By CLT, note that

$$A_n \xrightarrow{d} N(0, EX_1^2) \ \& \ B_n \xrightarrow{d} N(0, EX_1^{10}).$$

& by cont^n mapping theorem,

$$\frac{A_n}{B_n} \xrightarrow{d} \frac{A}{B}, \text{ where } A \sim N(0, EX_1^2)$$
$$\& \ B \sim N(0, EX_1^{10}).$$

3 marks

Moreover, since for any $C_1 \in \mathbb{R}$, & $C_2 \in \mathbb{R}$,

$$C_1 A_n + C_2 B_n \xrightarrow[\text{by CLT}]{d} N(0, \ldots), \text{ we have}$$

$$(A_n, B_n) \xrightarrow{d} (A, B) \sim BVN\Big(\underset{\mu_A}{0}, \underset{\mu_B}{0}, \underset{\sigma_A^2}{EX_1^2}, \underset{\sigma_B^2}{EX_1^{10}}, \underbrace{\frac{EX_1^6}{\sqrt{EX_1^2 \, EX_1^{10}}}}_{\rho_{A,B}}\Big).$$

3 marks.

Now, we need to derive the p.d.f. of

$U = \frac{A}{B}$. Let $V = B$.

$\Leftrightarrow A = UV, \quad B = V.$

$$J\left(\frac{A, B}{U, V}\right) = |v|.$$

Joint p.d.f. of (U, V) is

$$f_{U,V}(u,v) = \frac{|v|}{2\pi\sigma_A\sigma_B\sqrt{1-\rho_{A,B}^2}} e^{-\frac{v^2}{2(1-\rho_{A,B}^2)}\left(\frac{u^2}{\sigma_A^2} - 2\rho_{A,B}\frac{u}{\sigma_A\sigma_B} + \frac{1}{\sigma_B^2}\right)}.$$

Hence,

$$f_U(u) = \int_{-\infty}^{\infty} \frac{|v|}{2\pi \sigma_A \sigma_B \sqrt{1-\rho^2_{A,B}}} e^{-\frac{v^2}{2(1-\rho_{A,B}^2)}\left(\frac{u^2}{\sigma_A^2} - 2\rho_{A,B}\frac{u}{\sigma_A\sigma_B} + \frac{1}{\sigma_B^2}\right)} dv.$$

$$= 2\int_0^{\infty} \frac{v}{2\pi \sigma_A \sigma_B \sqrt{1-\rho^2_{A,B}}} e^{-\frac{v^2}{2(1-\rho_{A,B}^2)}\left(\frac{u^2}{\sigma_A^2} - 2\rho_{A,B}\frac{u}{\sigma_A\sigma_B} + \frac{1}{\sigma_B^2}\right)} dv$$

3 marks.

$$= \frac{\frac{\sigma_A}{\sigma_B}\sqrt{1-\rho_{A,B}^2}}{\pi\left(\left(u-\rho_{A,B}\frac{\sigma_A}{\sigma_B}\right)^2 + \frac{\sigma_A^2}{\sigma_B^2}\left(1-\rho^2_{A,B}\right)\right)},$$

where

$$\sigma_A^2 = EX_1^2, \quad \sigma_B^2 = EX_1^{10},$$

$$\rho_{A,B} = \frac{EX_1^6}{\sqrt{EX_1^2 EX_1^{10}}}.$$

Remark:- The last integration can be done in several ways.

80.

Q 2:

Observe that

3 marks

$$D^2 = (X_1 - X_2)^2 + (Y_1 - Y_2)^2$$

$$\Rightarrow ED^2 = E(X_1 - X_2)^2 + E(Y_1 - Y_2)^2$$

$$= Var(X_1 - X_2) + Var(Y_1 - Y_2)$$ since $E(X_1 - X_2) = 0$ & $E(Y_1 - Y_2) = 0$.

$$= 2\,Var(X) + 2\,Var(Y)$$

since $X_1 \perp\!\!\!\perp X_2$ & $Y_1 \perp\!\!\!\perp Y_2$ (*)

1 mark

Since $f_{X,Y}(x,y) = \frac{1}{\pi} \mathbb{1}\{x^2 + y^2 < 1\}$,

Note that X & Y are symmetric R.V. around 0 on compact support ($\Rightarrow$ Expectation exists), and because of symmetry $EX = EY = 0$.

Hence (*) $ED^2 = 2EX^2 + 2EY^2 = 2E(X^2 + Y^2)$ (**)

~~Again by symmetry,~~ $\cancel{ED^2 = EX^2 = EY^2}$

Finally, Consider $X = R\cos\theta$ & $Y = R\sin\theta$.

4 marks

So, $$E(X^2 + Y^2) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r^2 \cdot r\,dr\,d\theta$$

$$= \frac{1}{2}.$$

Hence, $$ED^2 = 2 \times \frac{1}{2} = 1$$ Using (**)

Q. 3 :- Answer: Yes

Note that

3 marks. $\left\{ \begin{array}{l} E X_{(n-13;n)} = \theta \dfrac{n-(n-13)}{n+1} \\ \text{needs to show the derivation} = \theta \times \dfrac{13}{n+1} \longrightarrow 0 \text{ as } n \to \infty. \end{array} \right.$

Read \theta

& $\left\{ \begin{array}{l} Var\left(X_{(n-13;n)}\right) = \theta^2 . \dfrac{\{n-(n-13)\}\{13+1\}}{(n+1)^2(n+2)} \\ \text{need to show the derivation} \longrightarrow 0 \text{ as } n \to \infty. \end{array} \right.$

Hence $X_{(n-13;n)} \xrightarrow{P} 0$ as $n \to \infty$

as $E X_{(n-13;n)} \xrightarrow{\theta} \theta$ as $n \to \infty$

& $Var\left(X_{(n-13;n)}\right) \longrightarrow 0$ as $n \to \infty$.

Q4. Here $c = 0$. It can be done in different ways.

Justification :-

Let $S_n = \sum_{i=1}^{n} X_i$. So, $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{S_n}{n}$. Need to show $\bar{X}_n \xrightarrow{P} 0$ as $n \to \infty$.

2 marks.

Observe that

$$E\bar{X}_n = 0$$

$$\& \; Var(\bar{X}_n) = \Big[\underbrace{\sum_{i=1}^{n} Var(X_i)}_{A} + \underbrace{2\sum_{i<j} Cov(X_i, X_j)}_{B}\Big] \Big/ n^2$$

Since $Var(X_n) = 1 \; \forall \; n \in \mathbb{N}$,

$$A = n. \Rightarrow \frac{A}{n^2} \to 0 \text{ as } n \to \infty$$

~~No~~ Now, $|Cov(X_n, X_m)| < \delta(|n-m|)$, when $\delta(k) \to 0$ as $k \to \infty$.

$$\text{So, } \Big|2\sum_{i<j} Cov(X_i, X_j)\Big| \leq 2\sum_{i<j} \delta|n-m|$$

Read \delta li - jl

$$= 2\sum_{k=1}^{n-1} (n-k)\,\delta(k).$$

$$\leq 2\sum_{k=}^{n-1} n\,\delta(k).$$

5 marks.

Now, note that

OR

$$\frac{B}{n^2} \leq \frac{\sum_{k=1}^{n-1} n\,\delta(k)}{n^2} = \frac{1}{n}\sum_{k=1}^{n-1} \delta(k) \longrightarrow 0 \text{ as } n \to \infty$$

by cesaro-summability as $\delta(k) \to 0$ as $k \to \infty$.

Hence, $Var(\bar{X}_n) \longrightarrow 0$ as $n \to \infty$.

Hence, $\bar{X}_n \xrightarrow{P} 0$ as $n \to \infty$

since $E\bar{X}_n \to 0$ as $n \to \infty$ & $Var(\bar{X}_n) \to 0$ as $n \to \infty$