COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Propositional Logic, Normal Forms, Logic Gates

Due Thursday, February 19, 2015 before 9PM EST. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/18601

1. (a) Define the propositions p="You obey the speed limit" and the q="You are going to a wedding". Write the following sentences as compound propositions using p and q:
   i. Failing to obey the speed limit implies that you are going to a wedding : $\neg p \to q$.
   ii. You drive below the speed limit only if you are going to a wedding: $\neg p \to q$
   iii. You do not obey the speed limit unless you are going to a wedding: $q \to \neg p$
   iv. You drive above the speed limit whenever you are going to a wedding: $q \to \neg p$

   (b) Define the propositions p="The home team wins," q="It is raining," r="There is an earthquake " Write the following sentences as compound propositions using p,q, and r:
   i. Either rain or an earthquake is sufficient for the home team to win.: $(q \vee r) \to p$
   ii. Rain and earthquake are necessary but not sufficient for the home team to win. : $\neg(q \wedge r) \to \neg p$
   iii. The home team wins only if it is not raining and there is no earthquake.: $(\ q \wedge \ r) \to p$
   iv. If it is raining the home team will win unless there is an earthquake: $if q and not r implies p : (q \wedge \neg r) \to p$

2. Using just the operators $\neg$ and $\wedge$ (and parentheses), write formulas involving p and q that are logically equivalent to:
   (a) $p \vee q$:$\neg(\neg p \wedge \neg q)$
   (b) $p \oplus q$: $\neg(\neg p \wedge \neg q) \wedge \neg(p \wedge q)$
   (c) $p \to q$: $\neg(p \wedge \neg q)$
   (d) $p \leftrightarrow q$: $\neg(p \wedge \neg q) \wedge \neg(q \wedge \neg p)$

3. The Boolean function $G(p, q, r)$ is defined in the truth table below. **Note: We are using Meyer's definition for conjunctive normal form and disjunctive normal form for this pset.**

p	q	r	G(p,q,r)
F	F	F	T
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	T
T	F	T	F
T	T	F	F
T	T	T	T

(a) Construct a proposition in disjunctive normal form for $G(p, q, r)$.
Solution:
To construct the Disjunctive Normal Form, find all the True lines in the Truth table, join these terms with AND and disjoin the lines with OR
$(\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$

(b) Using only the AND and NOT operators, construct the simplest proposition equivalent to the one in (a).
Solution:
$(\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$
$(\neg p \wedge \neg q) \wedge (\neg r \vee r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$
$(\neg p \wedge \neg q) \wedge (T) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$
$(\neg p \wedge \neg q) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$
$\neg p \wedge ((\neg q) \vee (q \wedge r)) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$
$\neg p \wedge ((\neg q \vee q) \wedge (\neg q \vee r)) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$
$\neg p \wedge ((T) \wedge (\neg q \vee r)) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$
$\neg p \wedge (\neg q \vee r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$
$(\neg p \wedge \neg q) \vee (\neg p \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$
$\neg q \wedge ((\neg p) \vee (p \wedge \neg r)) \vee (\neg p \wedge r) \vee (p \wedge q \wedge r)$
$\neg q \wedge ((\neg p \vee p) \wedge (\neg p \vee \neg r)) \vee (\neg p \wedge r) \vee (p \wedge q \wedge r)$
$\neg q \wedge (T \wedge (\neg p \vee \neg r)) \vee (\neg p \wedge r) \vee (p \wedge q \wedge r)$
$\neg q \wedge (\neg p \vee \neg r) \vee (\neg p \wedge r) \vee (p \wedge q \wedge r)$
$(\neg q \wedge \neg p) \vee (\neg q \wedge \neg r) \vee (\neg p \wedge r) \vee (p \wedge q \wedge r)$
$(\neg q \wedge \neg p) \vee (\neg q \wedge \neg r) \vee r \wedge (\neg p \vee (p \wedge q))$
$(\neg q \wedge \neg p) \vee (\neg q \wedge \neg r) \vee r \wedge (\neg p \vee p) \wedge (\neg p \vee q))$

$(\neg q \wedge \neg p) \vee (\neg q \wedge \neg r) \vee r \wedge ((T) \wedge (\neg p \vee q))$
$(\neg q \wedge \neg p) \vee (\neg q \wedge \neg r) \vee r \wedge (\neg p \vee q)$
$(\neg q \wedge (\neg p \vee \neg r) \vee (r \wedge \neg(p \wedge \neg q))$
$(\neg q \wedge \neg(p \wedge r) \vee (r \wedge \neg(p \wedge \neg q))$
$\neg(\neg(\neg q \wedge \neg(p \wedge r)) \wedge \neg(r \wedge \neg(p \wedge \neg q)))$

4. (a) Write $(p \wedge q) \oplus (p \vee r)$ in conjunctive normal form.

   To construct the Conjunctive Normal Form, find all the False lines in the Truth table, join these terms with OR and join the lines with AND
   $(p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

   (b) Prove that any propositional formula can be written in conjunctive normal form.
   Using a Truth Table, we can derive the Conjunctive Normal Form of any propositional formula as follows.
   Find all the False lines in the Truth table, represent each line by joining the variables with OR. Then finally join the OR representation of these lines with AND.
   This method can be applied to any truth table, which implies that any propositional formula can be written in conjunctive normal form.

5. While in a maze looking for lost treasure, Grace finds a stone door with three keyholes, one colored red, one blue, and one green. A sign next to the door reads:

   To reach the treasure behind this door:

   Use only two keys on the door, one must be colored blue.

   If blue keys you do not have, the other 2 keys will get you through.
   Luckily, Grace has a red key, a blue key, and a green key. Define r, b, g:

   r: Grace uses the red key

   b: Grace uses the blue key

   g: Grace uses the green key

   (a) Write a proposition that evaluates to True if Grace uses the right combination of keys and can get through the door.
   Solution:
   $(\neg r \wedge b \wedge g) \vee (r \wedge b \wedge \neg g) \vee (r \wedge \neg b \wedge g)$

   (b) Draw a logic circuit that implements the proposition in (a) with as few gates as possible from this list: NOT, OR, AND, XOR, NAND, XOR, NXOR.

Solution:
So In order to create the gates, lets begin with the truth table to list out all the possible values

Solution: