COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Basic Probability, Conditional Probability, Bayes Theorem

Due Thursday, April 23, 2015 before 9PM EDT. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/22764

1. Sam is an avid Tinder user and a huge fan of knitting. For a given Tinder account, Sam will swipe right with probability $\frac{1}{3}$ if the person says nothing about knitting, and will swipe right with probability $\frac{3}{4}$ if the person expresses enthusiasm for knitting. The probability that any Tinder user will swipe right when looking at Sam's profile is $\frac{1}{2}$. Independently, the probability that any Tinder user will express enthusiasm for knitting is $\frac{1}{100}$. A date occurs if Sam swipes right for a person and that person also swipes right for Sam. What is the probability that a date occurs? (Assume for simplicity that Sam only considers one person.)

   Solution:

   Let A = event that Sam swipes right for a given user account

   Let B = event that a Tinder user swipes rights on Sam

   Let K = event that a Tinder user express enthusiasm for knitting

   $Pr(B) = \frac{1}{2}$ = probability that user swipes right on Sam's account

   $Pr(A|K) = \frac{3}{4}$ is probability that sam swipes right given user expresses enthusiasm for knitting

   $Pr(A|\neg K) = \frac{1}{4}$ = probability that sam swipes right given user does not say anything about knitting

   $Pr(K) = \frac{1}{100}$ = probability that user will express enthusiasm for knitting

   $Pr(\neg K) = \frac{99}{100}$ = probability that user will express enthusiasm for knitting

   To find : The probability that a date occurs.

   This is defined as the probability of both Sam and user swiping right.i.e $Pr(A \cap B)$.
   Since the event A and B are independent events (as probability of user swiping right $Pr(B)$ is not impacted by the probability of Sam swiping right $Pr(A)$ and vice-versa. $Pr(A \cap B) = Pr(A) * Pr(B)$

   $Pr(A)$ = probability that Sam swipes right on a Tinder user
   $= (Pr(A|K) * Pr(k) + Pr(A|\neg K) * Pr(\neg k))$
   $= ((\frac{3}{4} * \frac{1}{100}) + (\frac{1}{4} * \frac{99}{100})) = 0.3375$

$P(A \cap B) = Pr(A) * Pr(B) = 0.3375 * 0.5 = 0.16875$

2. For any competition between Harvard and Yale, the probability that Harvard wins is $\frac{95}{100}$. Suppose Harvard and Yale have 10 consecutive competitions.

   (a) What is the probability that Harvard wins exactly 8 competitions?

   Exactly 8 wins from 10 games can be chosen in $\binom{10}{8}$ ways. Since they play 10 competitions - and we are looking for exactly 8 wins, the other 2 games are loses.
   Thus probability $= \binom{10}{8} * (\frac{95}{100})^8 * (\frac{5}{100})^2$

   (b) What is the probability that Harvard wins at least 8 competitions?

   At least 8 wins implies - there can be 8 or 9 or 10 wins
   $P(8) = \binom{10}{8} * (\frac{95}{100})^8 * (\frac{5}{100})^2$
   $P(9) = \binom{10}{9} * (\frac{95}{100})^9 * (\frac{5}{100})$
   $P(10) = \binom{10}{10} * (\frac{95}{100})^{10}$

   So total probability of atleast 8 wins =

   $P(8) + P(9) + P(10) =$

   $\binom{10}{8} * (\frac{95}{100})^8 * (\frac{5}{100})^2 +$

   $\binom{10}{9} * (\frac{95}{100})^9 * (\frac{5}{100}) +$

   $\binom{10}{10} * (\frac{95}{100})^{10}$

   (c) What is the probability that Harvard wins at least 8 competitions in a row?

   At least 8 wins in a row = there can be 8 or 9 or 10 wins in a row Since the wins are in a row, we can treat them as a single unit and the other loses as separate units, when trying to find the ways to choose wins from 10 competitions
   So $P(8) = \binom{3}{1} * (\frac{95}{100})^8 * (\frac{5}{100})^2$
   So $P(9) = \binom{2}{1} * (\frac{95}{100})^9 * (\frac{5}{100})^1$
   So $P(10) = \binom{1}{1} * (\frac{95}{100})^{10}$
   So total probability of atleast 8 wins in a row =

   $P(8) + P(9) + P(10) =$

$\binom{3}{1} * (\frac{95}{100})^8 * (\frac{5}{100})^2 +$

$\binom{2}{1} * (\frac{95}{100})^9 * (\frac{5}{100})^1 +$

$\binom{1}{1} * (\frac{95}{100})^{10}$

(d) Suppose Harvard loses the first 9 competitions. What is the probability that Harvard wins the last competition?
$P(10W) = (\frac{95}{100})^1$

3. (Wisdom of Solomon) In ancient Jerusalem, true prophets tell the truth 90% of the time, while false prophets tell the truth half the time. Solomon's servant has brought him a group of three prophets, two true, one false. Solomon asks Prophet 1, "Is Prophet 2 a true prophet?"

(a) If the answer is "No," what is the conditional probability that Prophet 3 is a true prophet?

Solution:
Let $P1$ = Prophet 1, $P2$ = Prophet 2, $P3$ = Prophet 3
Let A = event that Prophet 3 is a true Prophet
Let B = event that Prophet 1 answers " No "

Need to find:$Pr(A|B)$ i.e $P3$ is a true Prophet given $P1$ responds No.
Using Bayes Formula :

$Pr(A|B) = \frac{Pr(A)*Pr(B|A)}{(Pr(A)*Pr(B|A))+(Pr(\neg A)*Pr(B|\neg A))}$
$Pr(A) = 2/3$ - as there are 2 true prophets out of total 3 prophets

$Pr(B|A)$: Given event A that $P3$ is a True prophet. Event B will happen if either $P1$ is a True Prophet and speaks the truth or $P1$ is a false Prophet and speaks a Lie
Given event A (i.e. $P3$ is a true prophet) the probability of a True Prophet - 1/2 and Probability of False Prophet - 1/2

So, $Pr(B|A) = Pr(P1)$ speaks Truth as a True Prophet + $Pr(P1)$ speaks a Lie as a False Prophet

$Pr(B|A) = (1/2 * 9/10) + (1/2 * 1/2)$

$Pr(\neg A)$ = Probability that $P3$ is a False Prophet = 1/3 - as there is 1 false prophet out of total 3 prophets

$Pr(B|\neg A)$: Given event $\neg A$ that is $P3$ is a false prophet. Event B will only happen if $P1$ is a True Prophet and speaks a Lie. $P1$ cannot be a false prophet as event $\neg A$ means that $P3$ is a false prophet.

So, $Pr(B|\neg A) = Pr(P1)$ speaks a Lie as a True Prophet

$Pr(B|\neg A) = (1 * 1/10)$

$Pr(B|\neg A) = (1/10)$

Now, plugging all these values in the Bayes Formula above :

$Pr(A|B) = \frac{Pr(A)*Pr(B|A)}{(Pr(A)*Pr(B|A))+(Pr(\neg A)*Pr(B|\neg A))}$

$Pr(A|B) = \frac{2/3*((1/2*9/10)+(1/2*1/2))}{(2/3*((1/2*9/10)+(1/2*1/2)))+(1/3*1/10)}$

$Pr(A|B) = \frac{14}{15}$

(b) If the answer is "Yes," what is the conditional probability that Prophet 2 is a true prophet?

Solution:
Let $P1$ = Prophet 1, $P2$ = Prophet 2
Let A = event that Prophet 2 is a true Prophet
Let B = event that Prophet 1 answers " Yes "

Need to find: $Pr(A|B)$ i.e $P2$ is a true Prophet give $P1$ responds Yes.
Using Bayes Formula :

$Pr(A|B) = \frac{Pr(A)*Pr(B|A)}{(Pr(A)*Pr(B|A))+(Pr(\neg A)*Pr(B|\neg A))}$

$Pr(A) = 2/3$ - as there are 2 true prophets out of total 3 prophets

$Pr(B|A)$: Here we need to find the conditional probability of $P1$ speaking truth and identifying correctly that $P2$ is a true Prophet. This can happen in two ways.

$P1$ is a True Prophet and speaks the truth (OR) $P1$ is a False Prophet and speaks the truth.

Given event A (i.e. $P2$ is a true prophet) the probability of a True Prophet - 1/2 and Probability of False Prophet - 1/2

So, $Pr(B|A) = Pr(P1)$ speaks Truth as a True Prophet + $Pr(P1)$ speaks Truth as a False Prophet

$Pr(B|A) = (1/2 * 9/10) + (1/2 * 1/2)$

$Pr(\neg A)$ = Probability that $P2$ is a False Prophet = 1/3 - as there is 1 false prophet out of total 3 prophets

$Pr(B|\neg A)$: Here we need to find the conditional probability of $P1$ not speaking the Truth and incorrectly identifying $P2$ as a True Prophet.This can again happen in two ways.

$P1$ is a True Prophet and speaks a Lie (OR) $P1$ is a False Prophet and speaks a Lie. Given event $\neg A$ ($P2$ is a False prophet) the probability of a True Prophet - 1(= 2/2) and Probability of False Prophet - 0 , as event $\neg A$ means $P2$ is a False prophet.

So, $Pr(B|\neg A) = Pr(P1)$ speaks a Lie as a True Prophet + $Pr(P1)$ speaks a Lie as a False Prophet

$Pr(B|\neg A) = (1 * 1/10) + (0 * 1/2)$

$Pr(B|\neg A) = (1/10)$

Now, plugging all these values in the Bayes Formula above :

$Pr(A|B) = \frac{Pr(A)*Pr(B|A)}{(Pr(A)*Pr(B|A))+(Pr(\neg A)*Pr(B|\neg A))}$
$Pr(A|B) = \frac{2/3*((1/2*9/10)+(1/2*1/2))}{(2/3*((1/2*9/10)+(1/2*1/2)))+(1/3*1/10)}$
$Pr(A|B) = \frac{14}{15}$

4. There are three doors. Behind two of them are goats and behind one a 2015 Honda Civic LX Sedan. There is also Canadian-born MC, producer, actor, singer and sportscaster Monte "Monty Hall" Halparin who tonight, at the advanced age of 93, cannot recall what's behind what. And there you are, guessing a door, hoping against all odds to bring home this year's LX. Your family is watching from the studio audience. You sweat softly under the bright lights. Maybe you weren't born for television after all. You guess your door, and now all eyes are on Monte. The rules say he is supposed to reveal a goat behind one of the remaining doors, but he has no choice other than to make a guess himself. Luckily for him, his guess does reveal a goat. Is it in your best interest to switch doors? Explain.

   Solution:

   Let A = be the event of choosing the right door on switch Let B = event of choosing wrong door initially then $\neg B$ = event of choosing right door initially

Using Theorem of complete probability
$Pr(A) = Pr(A|B) * Pr(B) + Pr(A|\neg B) * Pr(\neg B)$

$Pr(A|B) = 1$: probability of choosing the right door when wrong door is choosen initially and when one of the wrong doors was opened by Monty

$Pr(\neg B) = 2/3$: probability of choosing the wrong door initially

$Pr(A|\neg B) = 0$: probability of choosing the right door when right door is choosen initially and when one of the wrong doors was opened by Monty

$Pr(\neg B) = 1/3$: probability of choosing the right door initially

Plugging in the values:

$Pr(A) = (1 * 2/3) + (0 * 1/3)$

$Pr(A) = (1 * 2/3)$

So from the above we can see that the chance on winning by switching is 2/3 is higher than being able to choose the right door initially (which is 1/3). So it is in the best interest to switch