COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Functions and Relations, Countability, Recursive Definitions, Structural Induction

Due Thursday, March 12, 2015 before 9PM EDT. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/20343

1. (a) Give an example of an injective relation that is not a function (or even a partial function). Set of Students

   (b) The *inverse* of a relation R on $X \times Y$ is the following relation R^{-1} on $Y \times X$:
   $$R^{-1}(y,x) \iff R(x,y)$$
   Fill in the blanks in the following table

R is...	iff R^{-1} is...
injective	injective
surjective	relation (set of ordered pairs)
bijective	bijection
total	relation (set of ordered pairs)
a (possibly partial) function	relation

2. Prove or disprove the following statement: If f is a function from X to Y and g a function from Y to Z such that g and $g \circ f$ are surjective, then f is surjective.

   Proof:
   Since g is surjective - every element of Z has at least one elements mapped from Y

   So $(\forall z \in Z)\ (\exists_{\geq 1} y \in Y)\ g(y) = z$

   Since $g \circ f$ is surjective - every element in f has at least one element mapped from g.

   That is to say
   every element in Y has at least one mapping from X and
   every element in Z has at least one mapping from Y

   The fact that every element in Y has at least one mapping from X proves that f is surjective

3. Trees Let T be the set of rooted trees.

- Base Case:

$$\epsilon \in T$$

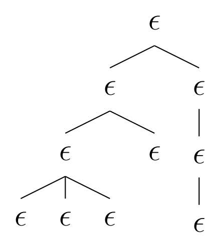
  This represents a node with no edges.

- Constructor Case: If $t_1, ..., t_n \in T$, then

$$(\epsilon, t_1, ..., t_n) \in T$$

  Here, ϵ represents the root node and t_i represent the subtrees of our new tree. We say that there is an $e$dge between ϵ and each of the root nodes of the subtrees.

- Example: $(\epsilon, (\epsilon, (\epsilon, \epsilon, \epsilon, \epsilon), \epsilon), (\epsilon, (\epsilon, \epsilon)))$ represents the following tree:



For all $n \geq 1$, prove that any tree with n nodes has $n - 1$ edges.

Proof:For all $n \geq 1$, prove that any tree with n nodes has $n - 1$ edges.

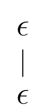
Structural Definition for tree construction

Base rule : Tree with only one node

Constructor rule (c): The root node of Tree T is connected to each of its subtree $(t_1...t_n)$ root through an edge. Note each subtree $(t_1...t_n)$ represents a tree and should have been constructed via the base and/or construction rule.

Base case - there is only one node ($n = 1$) and this tree has no edges.

Construction case -If a Tree (T) has only one subtree and this subtree is a tree created via base construction rule i.e has only one node (and zero edges).This represents a tree like below with 2 nodes and 1 edges.



Since any tree can only be created by using the Base and the Constructor rule, and we have shown above that both the Base and Constructor case satisfies the $n - 1$ edges count for a tree with n node.Any tree with n node has $n - 1$ edges.

4. Show that the set of real numbers between 0 and 1 is uncountable. (Hint: Can you represent the numbers as strings using some alphabet?)

   Proof:

   Suppose the set of real numbers between 0 and 1 is countable and let $\{0, 1\}^\omega$ represent the set of all infinite binary strings i.e. strings representing infinite decimals (fractions)

   This means that we can enumerate all the values in the set $\{0, 1\}^\omega$

   So let $s_0, s_1, s_2 ... \in \{0, 1\}^\omega$ - represents the infinite binary strings that belong to this set.

   $s_0 = 0, 0, 0, 0, 0, 1, ...$
   $s_1 = 0, 1, 0, 0, 0, 1, ...$
   $s_2 = 0, 0, 1, 0, 0, 1, ...$ ...

   Consider a string s_m created by flipping the n^{th} digit of the s_n string listed above. Thus $s_m = 1, 0, 0, 0, 1, 0, ...$

   This represents a new binary string not in the list s_0, s_1, s_2 , as it was constructed by flipping the n^{th} digit bit of s_n binary string.

   This contradicts the assumption of being able to enumerate (and thus count) all values in the set $\{0, 1\}^\omega$.

   Therefore the set of real numbers between 0 and 1 is uncountable.