COMPUTER SCIENCE E-20, SPRING 2014
In-class Problems - Group 5 (Rachna Sha)
4.2

1. Let S be the sequence a_1, a_2, a_3, ... where $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Use strong induction to prove that $a_n < 2^n$ $\forall$ n $\geq$ 4.

   Solution:

   - Proof: By strong induction on n that $a_n < 2^n$ $\forall$ n $\geq$ 4 .
   - The Induction hypothesis P(n), is: $a_n < 2^n$ $\forall$ n $\geq$ 4
   - Base Case (n = 4) : $a_4 < 2^4$.
     $a_4 = a_3 + a_2 + a_1 < 16$.
     $3 + 2 + 1 = 16$
     $6 < 16$.
   - Base Case (n = 5) : $a_5 < 2^5$.
     $a_5 = a_4 + a_3 + a_2 + a_1 < 32$.
     $16 + 3 + 2 + 1 = 32$
     $22 < 32$.
   - Inductive Step : Assume P(n) is true $\forall$ n $\geq$ 4, and prove
     P(n+1) : $a_{n+1} < 2^{n+1}$
     substituting (n+1) in

     $$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

     $$a_{n+1} = a_n + a_{n-1} + a_{n-2} < 2^{n+1}$$

     From the inductive hypothesis P(n) :

     $$a_n < 2^n, a_{n-1} < 2^{n-1}, a_{n-2} < 2^{n-2}$$

     $$(< (2^n + 2^{n-1} + 2^{n-2})) < 2^{n+1}$$

     $$(< 2^{n-2}(2^2 + 2^1 + 1)) < 2^{n-2}(2^3)$$

     $$(< 7) < 8$$

     This proves P(n+1), completing the proof by induction.