COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Growth Rates of Functions, Counting, Counting Subsets

Due Thursday, April 16, 2015 before 9PM EDT. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/22296

1. Fill in the following table—check off the box corresponding to functions f and g and relation ρ iff $f(x)$ is $\rho(g(x))$. For example, you would check off the lower-right box if you think that $\sqrt{x} \sim \log(x)$.

$f(x)$	$g(x)$	O	o	Ω	Θ	$\sim$
x	$x+100$	Y				Y
$(0.1)^x$	1	Y	Y			
2^x	3^x	Y	Y			
$\log(x)$	$\log(x^2)$				Y	
$x!$	2^x			Y		
$x!$	x^x	Y	Y			

2. Show that O is a transitive relation. That is, show that if h is $O(g)$ and g is $O(f)$, then h is $O(f)$.

   Proof :

   From the definition of O (in the lecture material), we know that

   if h is $O(g)$ then $\exists c, n_0$ such that $h(n) = c.g(n) \forall n \geq n_0$ (1)
   similarly if g is $O(f)$ then $\exists b, n_1$ such that $g(n) = b.f(n) \forall n \geq n_1$ (2)

   $h(n) = c.g(n) \forall n \geq n_0$ From (1)
   $g(n) = b.f(n) \forall n \geq n_1$ From (2)
   $h(n) = c.(b.f(n))\ \forall n \geq n_2$ where $n2 = max(n_1, n_0)$
   $h(n) = (b.c).f(n) = a.f(n)$ where a is the constant such that $a = b.c\ \forall n \geq n_2$

   Therefore $h(n) = a.f(n) \forall n \geq n_2$ - which is the definition of h is $O(f)$. This proves that if h is $O(g)$ and g is $O(f)$, then h is $O(f)$.

3. Recall that simple undirected graphs are graphs that contain at most one edge between any pair of vertices and no self-loops. Assume all graphs are simple and undirected in this question. For each of the following, provide a short (three sentences or less) explanation for your answer.

   (a) How many edges are there in the complete graph on n vertices?

   Solution: Total number of edges = n(n-1)/2. Each vertex has a degree of (n-1), if we sum this for all n vertices, this gives us a value of n(n-1)/2

(b) How many edges are there in the complete bipartite graph on $2n$ vertices? (Where each of the parts of the graph have n vertices)

Solution: Since this is a complete bipartite graph, the vertices can be divided into 2 sets of n each such that, there exist a unique edge between 2 vertices and that there is no edge among the vertices of the same set. Since each of the vertex in one set has a edge connecting to each of the vertex in the other set , the total number of edges $= n * n = n^2$

(c) How many isomorphisms are there from a complete graph on n vertices to itself?

Solution : Total isomorphism to itself will be the defined by the number of arrangements possible of the n vertices on the graph. The total number of permutations/arrangements of n is $n!$. Hence there are $n!$ isomorphisms from a complete graph to itself.

4. In this question, we will consider the security of various types of passwords. For each password constraint below, how many different passwords are possible?

(a) Passwords must consist of lowercase letters (a through z), must be at least 3 characters long, and no more than 8 characters long.

Solution: Set of Letters = L::= {a....z}
P_n = password of length n consisting of letters in L
$= (26)^n$
Set of password of at least length 3 and at most length 8
$= P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8$
$= |P_3| + |P_4| + |P_5| + |P_6| + |P_7| + |P_8|$
$= 26^3(1 + 26 + 26^2 + 26^3 + 26^4 + 26^5)$

(b) Passwords must consist of lowercase letters (a through z), must also contain exactly one exclamation point, and must be exactly 5 characters long.

Solution : Set of Letters = L::= {a....z}
Set of other symbols = S ::= {!}
P_5 = password of length 5 consisting of letters in L and one exclamation point
$= |S| * |L|^4$
$= 1 * (24)^4$
$= 456976$

(c) Same as part (a), except passwords cannot contain a letter more than twice.

Solution : Set of Letters = L::= {a....z}
P_n = password of length n consisting of letters in L with a letter repeated at most twice
P_3 = choose 2 distinct letters from L and the 3^{rd} letter can be any of the letters from L

$= (26!)/(26 - 2!) * 26$
$= (26!)/(24!) * 26$

P_4 = choose 2 distinct letters from L and the 3^{rd} and 4^{th} letters can be any of the letters from L
$= (26!)/(26 - 2!) * 26 * 26$
$= (26!)/(24!) * 26^2$

P_5 = choose 3 distinct letters from L and the 4^{th} and 5^{th}letters can be any of the letters from L
$= (26!)/(26 - 3!) * 26 * 26$
$= (26!)/(23!) * 26^2$

P_6 = choose 3 distinct letters from L and the $4^{th}, 5^{th}, 6^{th}$letters can be any of the letters from L
$= (26!)/(26 - 3!) * 26 * 26 * 26$
$= (26!)/(23!) * 26^3$

P_7 = choose 4 distinct letters from L and $5^{th}, 6^{th}, 7^{th}$letters can be any of the letters from L
$= (26!)/(26 - 4!) * 26 * 26 * 26$
$= (26!)/(22!) * 26^3$

P_8 = choose 4 distinct letters from L and $5^{th}, 6^{th}, 7^{th}, 8^{th}$letters can be any of the letters from L
$= (26!)/(26 - 4!) * 26 * 26 * 26 * 26$
$= (26!)/(22!) * 26^4$

Thus the set of password of at least length 3 and at most length 8 with a letter repeated at most twice will be

$= P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8$
$= |P_3| + |P_4| + |P_5| + |P_6| + |P_7| + |P_8|$
$= ((26!)/(24!) * 26) + ((26!)/(24!) * 26^2) + ((26!)/(23!) * 26^2) + ((26!)/(23!) * 26^3) + ((26!)/(22!) * 26^3) + ((26!)/(22!) * 26^4)$