COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Functions and Relations, Countability, Recursive Definitions, Structural Induction

Due Thursday, March 12, 2015 before 9PM EDT. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/20343

1. (a) Give an example of an injective relation that is not a function (or even a partial function).

   Solution: E.g. $\{(1,1),(1,2)\}$

   (b) The *inverse* of a relation R on $X \times Y$ is the following relation R^{-1} on $Y \times X$:
   $$R^{-1}(y,x) \iff R(x,y)$$
   Fill in the blanks in the following table

R is...	iff R^{-1} is...
injective	a (possibly partial) function
surjective	total
bijective	bijective
total	surjective
a (possibly partial) function	injective

2. Prove or disprove the following statement: If f is a function from X to Y and g a function from Y to Z such that g and $g \circ f$ are surjective, then f is surjective.

   Solution: This statement is false. Consider $X = \{1\}, Y = \{1,2\}, Z = \{1\}$, and $f : x \mapsto x$ and $g : y \mapsto 1$. Then $g(1) = g(2) = 1$, which is the only element in Z, so g is surjective; also $g \circ f(1) = 1 \in Z$, so $g \circ f$ is surjective. However, f is not surjective, since $f(1) = 1$, and there is no element in X that f maps to $2 \in Y$.

3. Trees Let T be the set of rooted trees.

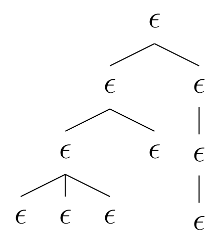
   - Base Case:
   $$\epsilon \in T$$
   This represents a node with no edges.

- Constructor Case: If $t_1, ..., t_n \in T$, then

$$(\epsilon, t_1, ..., t_n) \in T$$

  Here, ϵ represents the root node and t_i represent the subtrees of our new tree. We say that there is an *e*dge between ϵ and each of the root nodes of the subtrees.

- Example: $(\epsilon, (\epsilon, (\epsilon, \epsilon, \epsilon, \epsilon), \epsilon), (\epsilon, (\epsilon, \epsilon)))$ represents the following tree:



For all $n \geq 1$, prove that any tree with n nodes has $n - 1$ edges.

Solution: Base case: A tree with one node has no edges because self-loops are not allowed.
Inductive step: Assume a tree with n nodes has $n - 1$ edges. Consider a tree with $n+1$ nodes. If there is no vertex of degree 1, we could find a cycle by starting at an arbitrary vertex and following any path until returning to a previously visited vertex (since we can't get stuck), which contradicts the fact that trees have no cycles. Thus the tree must have a vertex of degree 1. Removing this vertex gives us a tree with n nodes, meaning by our inductive hypothesis it has $n - 1$ edges. So the original tree before removing the vertex has n edges since we only removed one edge with it. Thus we are done.

4. Show that the set of real numbers between 0 and 1 is uncountable. (Hint: Can you represent the numbers as strings using some alphabet?)

Solution: Any number can be represented as an infinite sequence of the digits from 0 to 9. So every real number between 0 and 1 corresponds to an element of $\{0, 1, \cdots, 9\}^\omega$, which is uncountable.