COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Expectation, Probability Review, Series

Due Thursday, April 30, 2015 before 9PM EDT. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/23127

1. Recall that the expectation of a random natural number X is the sum

$$\mathbb{E}[X] = \sum_{n=0}^{\infty} n\mathbb{P}[X = n]$$

(a) Show that $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.
Solution :

$\mathbb{E}[X] + \mathbb{E}[Y]$

Since
$\mathbb{E}[X] = \sum_{n=0}^{\infty} n\mathbb{P}[X = n]$

$\mathbb{E}[Y] = \sum_{n=0}^{\infty} n\mathbb{P}[Y = n]$

$\mathbb{E}[X] + \mathbb{E}[Y] =$
$\sum_{n=0}^{\infty} n\mathbb{P}[X = n] \; + \sum_{n=0}^{\infty} n\mathbb{P}[Y = n]$

$\sum_{n=0}^{\infty}(n\mathbb{P}[X = n] \; + n\mathbb{P}[Y = n])$

$\sum_{n=0}^{\infty} n(\mathbb{P}[X = n] \; + \mathbb{P}[Y = n])$

$= \mathbb{E}[X+Y]$

(b) Show that it is not always the case that $\mathbb{E}[X^2] = \mathbb{E}[X]^2$.
$\mathbb{E}[X^2]$ = represents the expectation of a square of the value. So the expectation is given by

$$\mathbb{E}[X^2] = \sum_{n=0}^{\infty} n\mathbb{P}[X = n^2]$$

$\mathbb{E}[X]^2$ = represents the square of the expectation itself.So,

$$\mathbb{E}[X]^2 = (\sum_{n=0}^{\infty} n\mathbb{P}[X = n])^2$$

-

And these are 2 very different things.

2. Let $n_0 = n$ and having chosen n_i, pick a number n_{i+1} between 0 and n_i (inclusive) at random.

   (a) Let ℓ be the least number such that $n_\ell = 0$. In terms of n, what is the expected value of ℓ? (Hint: let e_n be the expected value of ℓ when $n_0 = n$. Find an expression for e_n in terms of e_k for $0 \leq k \leq n$, and solve for e_n.)
   Given $n = n_0, EV(l) = \sum_{i=1}^{\ell} n_i$
   Using $n = n_k, EV(l) = \sum_{i=k}^{\ell} n_i$ After choosing n_0,

   n_1 can be choosen in $1/(n+1)$ ways
   n_2 can be choosen in $1/(n_1+1)$ ways
   n_k can be choosen in $1/(n_{k-1}+1)$ ways
   n_{l-1} can be choosen in $1/(n_{l-2}+1)$ ways
   n_l can be choosen in $1/(n_{l-1}+1)$ ways

   $\sum_{i=1}^{\ell} n_i =$

   $1/(n+1) + 1/(n_1+1) + .. + 1/(n_{l-2}+1) + 1/(n_{l-1}+1)$

   (b) What is the expected value of the sum $\sum_{i=1}^{\ell} n_i$?

3. (a) Taylor, a computer science concentrator, must complete 3 problem sets before doing laundry. Each problem set requires 2 days with probability $\frac{2}{3}$ and 3 days with probability $\frac{1}{3}$. What is the expected number of days Taylor delays laundry?

   (For example, if the first problem set requires 3 days and the second and third problem set each require 2 days, then Taylor delays laundry for 7 days.)

   Solution :

   The number of possible days to finish the 3 problem sets are

   Finish all in 2 days = (2,2,2) = total 6 days

   Finish one of the 3 days and rest in 2 days = (3,2,2), (2,3,2), (2,2,3) = total days = total 7 days

Finish 2 problem sets 3 days and 1 in in 2 days = (3,3,2), (3,2,3), (2,3,3) = total 8 days

Finish all in 3 days = (3,3,3) = 9 days
Therefore the possible values of number of days to finish 3 problem sets = (6,7,8,9)

Expected Value to delay laundry (and finish Problem Sets) =

$$\mathbb{E}[D] = \sum_{n=6}^{9} n\mathbb{P}[D = n]$$

=

$$(6 * \mathbb{P}[D = 6] + 7 * \mathbb{P}[D = 7] + 8 * \mathbb{P}[D = 8] + 9 * \mathbb{P}[D = 9])$$

=

$$(6 * 1/8 + 7 * 3/8 + 8 * 3/8 + 9 * 1/8)$$

= 7.5 days

(b) Blake, an English concentrator, must complete an essay before doing laundry. The length of the essay is equal to the sum of the numbers rolled on 2 fair, 6-sided dice. If Blake can write 1 page each day, what is the expected number of days Blake delays laundry?

(For example, if the rolls are 5 and 3, then Blake delays laundry for 8 days.)

Solution :

The number of possible days to finish the essay
Dice roll sum (i.e essay pages) = how these are possible are listed below
2 = (1,1)
3 = (1,2), (2,1)
4 = (1,3), (2,2), (3,1)
5 = (1,4), (2,3), (3,2), (4,1)
6 = (1,5), (2,4), (3,3), (4,2), (5,1)
7 = (1,6), (2,5), (3,4),(4,3),(5,2), (6,1)
8 = (2,6), (3,5), (4,4), (5,3), (6,2)
9 = (3,6), (4,5), (5,4), (6,3)
10 = (4,6), (5,5), (6,4)

11 = (6,5), (5,6)
12 = (6,6)

Expected Value to delay laundry =

$$\mathbb{E}[D] = \sum_{n=2}^{12} n\mathbb{P}[D = n]$$

$2*\mathbb{P}[D = 2]+3*\mathbb{P}[D = 3]+4*\mathbb{P}[D = 4]+5*\mathbb{P}[D = 5]+6*\mathbb{P}[D = 6]+7*\mathbb{P}[D = 7]+8*\mathbb{P}[D = 8]+9$

= (2*1/36 + 3* 2/26 + 4 * 3/36 + 5 * 4/36 +
6 *5/36 + 7 * 6/36 + 8* 5/36 + 9 * 4/36 +
10 *3/36 + 11 * 2/26 + 12 * 1/36)

= 7 days

4. Evariste Galois (a great mathematician who actually died in a duel at age 19) is fighting a duel to the death with Gaston. They fire alternately, with Evariste going first: EGEGEG... Evariste is a terrible shot, and has only one chance in five of killing Gaston with any given shot. Gaston is a somewhat better marksman, and has one chance in four of killing Evariste with any given shot. Either a shot kills the opponent, or it misses completely.

   (a) By summing a geometric series, determine the probability that Evariste wins the duel.
   Hint: Evariste wins if both duelists miss r times, and he then hits.
   Solution :

   P(E) = 1/5 - probability of making the shot
   P(G) = 1/4 - probability of making the shot
   P(E') = 4/5 - probability of missing the shot
   P(G') = 3/4 - probability of missing the shot

   For Evariste to win =

$$\sum_{n=1}^{\infty} \mathbb{P}[E']^{n-1} * P[G']^{n-1} * P[E$$

=

$$1/5 * (\sum_{i=1}^{(} P[E'])^{n-1} * P[G']^{n-1})$$

=

$$1/5 * (\sum_{n=1}^{(} 4/5)^{n-1} * (3/4)^{n-1}$$

Sum of the series :

$$\sum_{n=1}^{(} 4/5)^{n-1} * (3/4)^{n-1}$$

= (1/ 1-r). In this case r = (4/5 * 3/4) = 3/5

=

$$1/5 * (1/1 - (3/5))$$

= 1/2

(b) To check your answer, sum a series to calculate directly the probability that Gaston wins.

Solution :

For Gaston to win =

$$\sum_{n=1}^{\infty} \mathbb{P}[E']^{n-1} * P[G']^{n-1} * (P[E']) * P[G]$$

=

$$4/5 * 1/4(\sum_{n=1}^{\infty} \mathbb{P}[E']^{n-1} * P[G']^{n-1})$$

Sum of the series :

$$\sum_{n=1}^{(} 4/5)^{n-1} * (3/4)^{n-1}$$

= (1/ 1-r). In this case r = (4/5 * 3/4) = 3/5

=

$$4/5 * 1/4 * (1/1 - 3/5)$$

= 1/2