COMPUTER SCIENCE 20, SPRING 2014
Module #16 (States and Invariants)

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Readings from Meyer

- Meyer section 5.4.

Executive Summary

1. A state machine is a binary relation called a *transition relation* on a set of elements S called *states*.
   - A transition from state q to r is denoted $q \to r$.
   - A state machine has a start state $q_0 \in S$, which is denoted with an arrow pointed to that state's node.
   - A state machine may have final state(s), which are denoted with double-circles around the node(s) of the final state(s).
2. An *execution* of a state machine with states S is a plausible sequence of states (possibly an infinite one) with the property that
   - it starts with the start state $q_0 \in S$, and
   - for all consecutive states q and r, $q \to r$.
3. A state $s \in S$ is *reachable* if some execution of a state machine includes s.
4. Predicate P is a preserved invariant of a state machine if for all $q, r \in S$,

$$P(q) \wedge q \to r \implies P(r)$$

5. The Invariant Principle: If a preserved invariant of a state machine is true for the start state, then it is true for all reachable states.
6. A state machine is *deterministic* if for all states s in the machine, there is at most one transition out of s.
7. A state machine is *non-deterministic* if there exists a state s in the machine such that there is more than one transition out of s.

Check-in problem

1. The state machine for Tic-Tac-Toe that Professor Lewis describes in the mini-lecture is
   (a) deterministic
   (b) non-deterministic

Small group problems

1. Draw a state machine with states, labelled transitions, start state, and final state(s) that accepts (and only accepts) binary strings that begin and end with 1 (i.e. $1, 11, 101, 111, \ldots$).

2. Being the delapidated house that Eliot is, the card reader allowing Eliot students to pay for their laundry using CrimsonCash broke. Additionally, suppose their laundry machines only accept quarters. Each load of laundry costs \$1.25.
   (a) Design a state machine with transitions that models how a poor Eliot student must pay for their laundry using only quarters in order to start the laundry machine.
   (b) Use the Invariant Principle to show that each state in the state machine represents when some multiple of \$0.25 is in the laundry machine.

3. Using state machines, prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

4. **Bulgarian Solitaire:** This is a solitaire game. The game starts with 2 piles of 3 coins each. Now, repeat the following step:

   - Remove one coin from each existing pile and form a new pile.

   The order of the piles doesnt matter, so the state can be described as a sequence of positive integers in non-increasing order adding up to 6. For example, the first three moves are $(3, 3) \rightarrow (2, 2, 2)$ and $(2, 2, 2) \rightarrow (3, 1, 1, 1)$. On the next move, the last three piles disappear.

   (a) Trace the sequence of moves until it repeats and draw it on the board.
   (b) Draw the complete state space with six coins and varying initial piles.
   (c) What invariant does each state of the game hold? (No proof necessary; just state the invariant.)