COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Quantificational Logic I and II

Due Thursday, February 26, 2015 before 9PM EST. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/19314

1. Let the domain of discourse be all houses. Let $P(x, y)$ be the proposition "there is a path connecting x and y." Write quantificational formulas for each of the following statements:

   (a) Every house has at least one path connecting it to another house. $(\forall x \exists y)(P(x, y) \wedge (x \neq y))$

   (b) There is at least one house that is connected to three other houses. $(\exists x \exists a \exists b \exists c)(P(x, a) \wedge P(x, b) \wedge P(x, c) \wedge (x \neq a) \wedge (x \neq b) \wedge (x \neq c) \wedge (a \neq b) \wedge (a \neq c) \wedge (b \neq c))$

   (c) At least one house is isolated (it is not connected by paths to any other house). $(\exists x \forall y)(\neg(P(x, y)) \wedge x \neq y)$

   (d) If a house is connected to two other houses, then there isn't a path between the other two houses. $(\exists a \exists b \exists c)((P(a, b) \wedge P(a.c)) \Rightarrow \neg(P(b, c)))$

2. For each of the logical formulas, indicate whether or not it is true when the domain of discourse is (i) the natural (non-negative) numbers, (ii) the integers, (iii) the rationals, and (iv) the real numbers.

   (a) $\exists x.(x^3 = -8)$
   $(i) False$
   $(ii) True$
   $(iii) True$
   $(iv) True$

   (b) $\forall x \neq 0. \exists y.(xy = 1)$
   $(i) False$
   $(ii) False$
   $(iii) True$
   $(iv) True$

   (c) $\exists x. \forall y.(xy = 0)$
   $(i) False$
   $(ii) True$
   $(iii) True$
   $(iv) True$

(d) $\forall x.\exists y.(\sqrt{x} = y)$
$(i) False$
$(ii) False$
$(iii) False$
$(iv) True$

(e) $\forall x \forall y.\exists z.(x - y = z)$
$(i) False$
$(ii) True$
$(iii) True$
$(iv) True$

3. Determine whether each of the following statements is true or false. If false, write its logical negation (distributing the $\neg$ across any expressions as necessary). Note that the domain of discourse is all integers.

(a) $\forall x.\forall y.((y > x) \Rightarrow (x = 0))$:
$False.$
$\neg(\forall x.\forall y.((y > x) \Rightarrow (x = 0)))$
$\exists x.\exists y.\neg((y > x) \Rightarrow (x = 0))$
$\exists x.\exists y.((y > x) \wedge \neg(x = 0))$

(b) $\exists x.\forall y.((y > x) \Rightarrow (x = 0))$:
$False$
$\neg(\exists x.\forall y.((y > x) \Rightarrow (x = 0)))$
$\forall x.\exists y.\neg((y > x) \Rightarrow (x = 0)))$
$\forall x.\exists y.((y > x) \wedge \neg(x = 0))$

(c) $\forall x.\exists y.((y > x) \Rightarrow (x = 0))$
$False$
$\neg(\forall x.\exists y.((y > x) \Rightarrow (x = 0))$
$\exists x.\forall y.\neg((y > x) \Rightarrow (x = 0))$
$\exists x.\forall y.((y > x) \wedge \neg(x = 0))$

(d) $\forall x.((\forall y.(y > x)) \Rightarrow (x = 0))$
$False$
$\neg(\forall x.((\forall y.(y > x)) \Rightarrow (x = 0)))$
$\exists x.\neg((\forall y.(y > x)) \Rightarrow (x = 0))$
$\exists x.\neg(\neg(\forall y.(y > x)) \vee (x = 0))$
$\exists x.((\forall y.(y > x)) \wedge \neg(x = 0))$

4. Your younger sibling believes in the positive integers and understands that addition is commutative, but has not learned about zero or negative integers. You want to introduce new addition facts like:

$$3 + 0 = 3 \text{ (an example of an additive identity)}$$

and

$$5 + (-5) = 0 \text{ (an example of an additive inverse)}$$

Using quantifiers, create two axioms about the integers under addition that specify precisely the properties of zero and negation. Each will include one "exists" and one "for all," but the order of the quantifiers is crucial.

In particular, Axiom I should define an "additive identity" in the set of all integers. Axiom II should define the notion of an "additive inverse." (If you don't know what these terms mean, feel free to look them up on Wikipedia.)

Solution:

$Let P(x, y) := x + y = z$, be a function that represents the addition of two integers x and y whose sum is z.

Axiom I : Additive Identity can be written as $\exists y.\forall x.(P(x, y) = z \wedge (x = z))$

Axiom II : Additive Inverse can be written as $\forall x.\exists y.(P(x, y) = z \wedge (z = 0))$