COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Pigeonhole, Proofs
SOLUTIONS

Due Thursday, February 5, 2015 before 9PM EST. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/17263

1. What is the minimum number of unique integers that you have to be pick from $\{1, 2, ..., 16\}$ to ensure that there are at least two integers whose sum is equal to 17?

   Solution: 9. Note that it is possible to choose 8 integers without the guarantee that at least two integers sum to 17: $\{1, 2, ...8\}$ or $\{9, 10, ..., 16\}$ suffice. Group the integers in $\{1, 2, ..., 16\}$ into 8 pairs that each sum to 17, i.e. $(1, 16), (2, 15), ..., (8, 9)$. This grouping *partitions* the original set; each element of the original set appears in exactly one pair. For each pair, we can choose at most one of the integers to be in the final set; if we choose both, those two integers will sum to 17. However, since there are 8 pairs, choosing 9 integers will force us to include two elements from the same pair. Thus, choosing 9 integers guarantees that at least two integers will sum to 17.

2. Every day a ketchup factory produces a whole number of gallons of ketchup. Show, using the pigeonhole principle, that within the next two months there will be a period of some number of consecutive days, in which the total production will fit into 50 gallon containers with nothing left over.

   Solution: Let S be the set of all days in the next two months, labelled according to the number of days in the future, e.g. $S = \{0, 1, ..., 60\}$. Let $T = \{0, 1, ..., 49\}$. Let g_i be the gallons of ketchup produced on Day i, and let s_i be the sum of all the gallons produced from Day 0 through Day i, i.e. $s_i = \sum_{j=0}^{i} g_j$. Next let $f : S \to T$ be given by $f : i \mapsto s_i \pmod{50}$. Remember that $s_i \pmod{50}$ means the remainder after s_i is divided by 50. By the pigeonhole principle, since $|S| > |T|$, there are at least two days that are mapped to the same element in T, call them a and b. We claim that the ketchup production from day $a + 1$ to day b will fit into 50 gallon containers with nothing left over.

   In math terms, our claim is that $(\sum_{i=a+1}^{b} g_i) \pmod{50} = 0$. We know that $s_a = \sum_{i=0}^{a} g_i$ and similarly that $s_b = \sum_{i=0}^{b} g_i$. Observe that $(\sum_{i=a+1}^{b} g_i) = (\sum_{i=0}^{b} g_i) - (\sum_{i=0}^{a} g_i)$, so $(\sum_{i=a+1}^{b} g_i) \pmod{50} = s_b - s_a$. We also know that $s_a = s_b$ from above. Thus, $(\sum_{i=a+1}^{b} g_i) \pmod{50} = 0$, and we are done.

3. Prove by contradiction that if $17n + 2$ is odd then n is odd.

   Solution: Suppose that $17n + 2$ is odd and n is not odd, so it's even. By the definition of even, there is some natural number k s.t. $n = 2k$. Thus $17(2k) + 2$ is odd. Factoring, we have $2(17k + 1)$ is odd. However, we see that $2(17k + 1)$ fits the definition of even, since $(17k + 1)$ is a natural number. Thus we have a contradiction, proving that n must be odd.