COMPUTER SCIENCE E-20, SPRING 2014
Homework Problems
Induction I, Strong Induction

Due Thursday, February 12, 2015 before 9PM EST. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/17757

1. Prove that for all nonnegative integers n

$$\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$$

Hint: the following identity may be useful

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Solution:

- Proof: for all nonnegative integers n

$$\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$$

- The Induction hypothesis P(n), is:

$$0^3 + 1^3 + 2^3 \ldots + n^3 = (0 + 1 + 2 + \ldots + n)^2$$

$$0^3 + 1^3 + 2^3 \ldots + n^3 = (n(n+1)/2)^2$$

- Base Case (n = 0) : 0 == 0
- Base Case (n = 1) : 1 == 1
- Inductive Step : Assume P(n) is true ∀ n ≥ 0, and prove P(n+1)

$$0^3 + 1^3 + 2^3 \ldots n^3 + (n+1)^3 = (0 + 1 + 2 + \ldots + n + n + 1)^2$$

$$0^3 + 1^3 + 2^3 \ldots n^3 + (n+1)^3 = ((n+1)(n+2)/2)^2$$

from Inductive Step:

$$0^3 + 1^3 + 2^3 \ldots n^3 = (n(n+1)/2)^2$$

Therefore

$$(n(n+1)/2)^2 + (n+1)^3 = ((n+1)(n+2)/2)^2$$

$$(n+1)^2/2^2\left(n^2+4(n+1)\right) = ((n+1)(n+2)/2)^2$$
$$(n+1)^2/2^2\left(n^2+4n+4\right) = ((n+1)(n+2)/2)^2$$
$$(n+1)^2/2^2\left(n+2)^2\right) = ((n+1)(n+2)/2)^2$$
$$\left(n+1)^2(n+2)^2/2^2\right) = ((n+1)(n+2)/2)^2$$
$$(n+1)(n+2)/2)^2) = ((n+1)(n+2)/2)^2$$

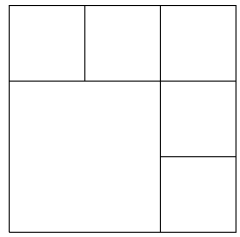
This proves P(n+1), completing the proof by induction.

2. Consider the sequence $a_1 = 1, a_2 = 3, ..., a_n = a_{n-1} + a_{n-2}$. Using strong induction prove that $a_n \leq \left(\frac{7}{4}\right)^n$ for all positive integers n.

   Solution:

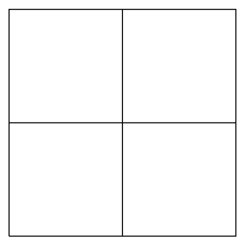
   - Proof: By strong induction on n that for all positive integers n, $a_n \leq \left(\frac{7}{4}\right)^n$.
   - The Induction hypothesis P(n), is: $a_n \leq \left(\frac{7}{4}\right)^n \; \forall$ n ≥ 1
   - Base Case (n = 1) : $a_1 \leq \left(\frac{7}{4}\right)^1 = 1 \leq \left(\frac{7}{4}\right) = 1 \leq 1.75$
   - Base Case (n = 2) : $a_2 \leq \left(\frac{7}{4}\right)^2 = 3 \leq \left(\frac{7}{4}\right)^2 = 3 \leq 1.75*1.75 = 3 \leq 3.06$
   - Inductive Step : Assume P(n) is true $\forall$ n ≥ 1, and prove
     P(n+1) : $a_{n+1} \leq \left(\frac{7}{4}\right)^{n+1}$
     substituting (n+1) in $a_n = a_{n-1} + a_{n-2}$
     $a_{n+1} = a_{n+1-1} + a_{n+1-2} \leq \left(\frac{7}{4}\right)^{n+1}$
     $= a_n + a_{n-1} \leq \left(\frac{7}{4}\right)^{n+1}$
     $= \left(\frac{7}{4}\right)^n + \left(\frac{7}{4}\right)^{n-1} \leq \left(\frac{7}{4}\right)^{n+1}$
     $= \left(\frac{7}{4}\right)^{n-1} * \left(\frac{7}{4}+1\right) \leq \left(\frac{7}{4}\right)^{n+1}$
     $= \left(\frac{7}{4}\right)^{n-1} * \left(\frac{7}{4}+1\right) \leq \left(\frac{7}{4}\right)^n * \left(\frac{7}{4}\right)$
     $= \left(\frac{7}{4}\right)^{n-1} * \left(\frac{7}{4}+1\right) \leq \left(\frac{7}{4}\right)^{n-1} * \left(\frac{7}{4}\right)^2$
     $= \left(\frac{11}{4}\right) \leq \left(\frac{49}{16}\right)$
     This proves P(n+1), completing the proof by induction.

3. Prove using strong induction that any square can be subdivided into n smaller squares, where $n > 5$. For example, the large square below has been subdivided into 6 squares.
   Hint: first show that any square subdivided into k squares can easily be subdivided into $k+3$ squares, then think how many base cases you need show are true (it is not just the case of n=6).

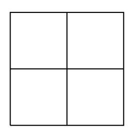


Solution:

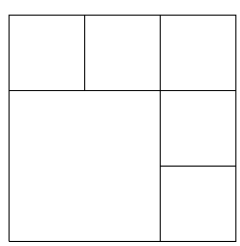
- Proof: By strong induction on n that any square can be subdivided into n smaller squares, where $n > 5$.
- The simplest way a square can be subdivided into smaller squares is by dividing it into 4 equal squares as shown in the picture below.

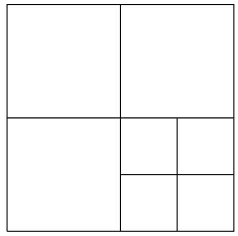
  Each of these smaller squares represent a square , and so can again be divided in the same way into 4 smaller squars, as shown below.

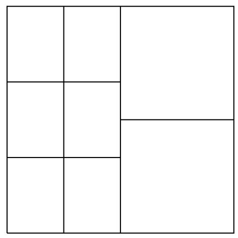
  This shows that any square subdivided into k squares can always be divided into $K + 3$ squares.
- Base Case (n = 6) :

- Base Case (n = 7) :



- Base Case (n = 8) :



- Inductive Step: Assume P(n) is true for all $n > 5$, and we now prove P(n+1):
  when $P(n + 1) = 6, 7, 8$, these form our base cases and have been proved above
  when $P(n+1) > 8$: From the inductive step we know that any square can be divided into $P(n)$ smaller squares. We have also shown above that a single square k can always be divided into $k + 3$ squares. Thus $P(n+1)$ can be divided into $P(n)+4$ smaller squares. This completes the prove by induction.

4. The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$. Prove using strong induction that for all $n \geq 0$, F_{3n} is even.

   Solution:

   - Proof: By strong induction on n that for all $n \geq 0$, F_{3n} is even.
   - The Induction hypothesis P(n), is: F_{3n} is even $\forall$ n $\geq$ 0
   - Base Case (n = 1) :$F_3 = F_2 + F_1 = F_1 + F_0 + F_1 = 1+0+1 = 2$.
   - Base Case (n = 2) : $F_6 = F_5 + F_4$
     $F_4 = F_3 + F_2 = 2+1 = 3$
     $F_6 = F_4 + F_3 + F_4 = 3 + 2 + 3 = 8$
   - Inductive Step : Assume P(n) is true $\forall$ n $\geq$ 0, and prove
     P(n+1) : $F_{3(n+1)}$ is even
     substituting (n+1) in $F_n = F_{n-1} + F_{n-2}$

$F_{3(n+1)} = F_{n+1-1} + F_{n+1-2}$
$F_{3(n+1)} = F_n + F_{n-1}$

From the inductive hypothesis P(n) :
F_n is even. So,$F_n = 2 * a(some integer a)$
F_{n-1} is even.So,$F_{n-1} = 2 * b(some integer b)$

$F_{3(n+1)} = 2 * a + 2 * b$
$F_{3(n+1)} = 2(a + b)$, which implies $F_{3(n+1)}$ is even.

This proves P(n+1), completing the proof by induction.