COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Recurrences

Due Thursday, April 30, 2015 before 9PM EDT. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/23409

1. A hacker replaces your MergeSort routine with a new recursive sorting algorithm, SlowSort. SlowSort runs in three phases. In the first phase, the first 3/4 of the list (rounding up) is sorted recursively. In the second phase, the final 3/4 of the list is sorted recursively. Finally, in the third phase, the first 3/4 of the list is sorted recursively again. SlowSort sorts correctly, but it is slow. Assume that at each step, we require an additional number of processing steps proportional to the size of the list, for overhead. Write a recurrence describing SlowSort's running time.

   Challenge, worth 0 points: Prove that SlowSort sorts correctly.
   Solution For all three phases, 3/4 of the list needs to be sorted recursively, which tells us that our recurrence will have $3T(3n/4)$ in the equation. At each step, there is also an additional number of processing steps proportional to the size of the list, which means we also need to add kn to the equation, where k is a some constant. Therefore, the recurrence relation $T(n) = 3T(3n/4) + kn$.

2. Find functions fitting the following recurrences, and prove your answers by induction.

   (a) $f(1) = 5, f(n) = f(n-1) + 2$
   (b) $g(0) = 0, g(n) = g(n-1) + 2n + 1$

   Solution

   (a) We propose that $f(n) = 2n + 3$.
   Base Case: $n = 1$:
   $$2(1) + 3 = 5 = f(1)$$
   The base case checks out. Inductive Step: Assume that $f(n) = 2n+3$. We hope to show that $f(n+1) = 2(n+1) + 3$. Now:
   $$f(n+1) = f(n) + 2$$
   $$= 2n + 3 + 2$$
   $$= 2(n+1) + 3$$
   As desired. Hence, we have shown that $f(n) = 2n + 3$ fits the recurrence by induction.

(b) We propose that $g(n) = n(n+2)$.
Base Case: $n = 0:$

$$0(0+2) = 0 = g(0)$$

The base case checks out. Inductive Step: Assume that $g(n) = n(n+2)$. We hope to show that $g(n+1) = (n+1)(n+3)$. Now:

$$g(n+1) = g(n) + 2(n+1) + 1$$

$$= n(n+2) + 2n + 2 + 1$$
$$= n^2 + 2n + 2n + 3$$
$$= n^2 + 4n + 3$$
$$= (n+3)(n+1)$$

As desired. Hence, we have shown that $g(n) = n(n+1)$ fits the recurrence by induction.