COMPUTER SCIENCE 20, SPRING 2015

Module #4 (Strong Induction)

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Executive Summary

1. Strong Induction

   - In simple terms, strong induction is similar to ordinary induction, except you use $P(0)$, ..., $P(n)$ instead of just $P(n)$ in order to prove $P(n+1)$.
   - Formally, the difference between ordinary and strong induction is: Let $P(x)$ be a predicate and m, n nonnegative integers.
     - Ordinary Induction: If $P(m)$ is true and $P(n) \Rightarrow P(n+1)$ for all $n \geq m$, then $P(n)$ is true for all $n \geq m$.
     - Strong Induction: If $P(m)$ is true and $P(m), P(m+1), \ldots, P(n)$ together $\Rightarrow P(n+1)$ for all $n \geq m$, then $P(n)$ is true for all $n \geq m$.
   - Strong induction proofs begin with the identification of the proposition to be proven. The next step is to identify and verify the base case. Note that with strong induction there are often multiple base cases. Next comes the inductive step, where you show that the proposition's truth for $0 \ldots n$ entails its truth for $n+1$. Be sure to properly identify the proposition being inducted on.
   - Note that sometimes you will need to break your inductive step into multiple cases.

Check-in questions

1. In strong induction not all bases cases have to be true:

   (a) True

   (b) False

In-class Problems

1. The game of "Take Away" is played with a pile of 9 coins. Two teams take turns removing coins from the pile. At each turn, the team whose turn it is can choose to remove either one or two coins from the pile. The team that removes the last coin wins.

   (a) Does the game have a winning strategy? Note: for the game to have a winning strategy one of the two teams – either the team that removes coins first or the team that removes coins second – must always be able to win.

   (b) Generalize the game to a pile of any positive integer n coins and show that the game always has a wining strategy (not necessarily for the same team).

2. Let S be the sequence $a_1, a_2, a_3, \ldots$ where $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Use strong induction to prove that $a_n < 2^n$ for any positive integer n.

3. Prove using strong induction that for all $n \in \mathbb{N}$ such that $n \geq 2$, n is divisible by a prime. *(Hint: In this problem, we consider all divisors, not just proper divisors, so a number is considered as being divisible by itself. Split the inductive step into cases based on whether $n+1$ is prime. What does it mean for a number to be composite?)*

4. **Challenge:** Consider the sequence $a_1 = 2, a_2 = 5, a_3 = 13, \cdots, a_n = 5a_{n-1} - 6a_{n-2}$. Prove by strong induction that $a_n = 2^{n-1} + 3^{n-1}$ for all $n > 2$.