COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
States and Invariants, Directed Graphs, Graphs and Relations

Due Thursday, March 26, 2015 before 9PM EDT. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/20670

1. It's the first day of CS 20, and Professor Lewis is running a little late. To pass the time, the 57 students go around introducing themselves to one another, and sometimes two students shake hands.

   When class starts, everyone stops mingling and goes back to their seats. Assuming that each handshake involves exactly one hand from each of the two participants, use the invariant principle to show that the number of people who have shaken an odd number of hands is even.

   Proof:

   Let x = represents the number of people shaking hands y = represent the number of hands they each shake.

   Initial State : P(Start) = (0,0) = No people have shaken hands

   State Transition: P(x,y) = (2x) = (2y)+1

   This is because the invariant is 2 people shake hands will result in each of them increasing the number of hands they shake by 1

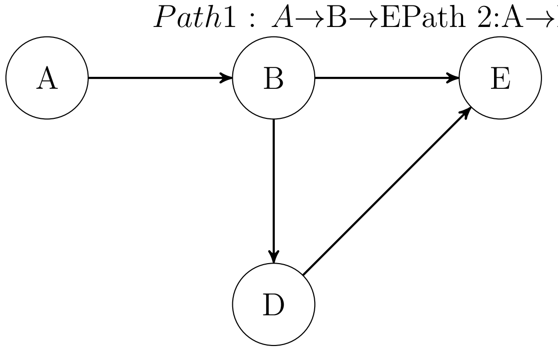
   This proves that number of people that shake odd numbers of hands is even

2. An arborescence is a directed graph with a root node u such that for every other vertex v in the graph, there is exactly one directed path from u to v.

   (a) Give an example of a DAG (directed acyclic graph) that is not an arborescence. Briefly explain.

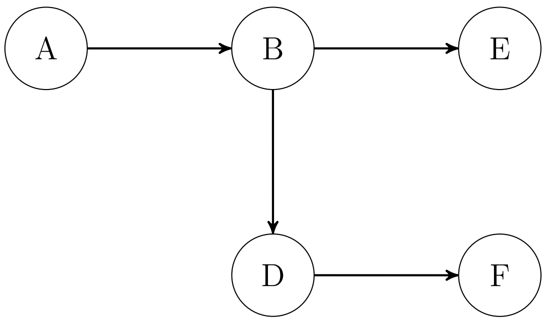
   The graph below shows a DAG such that there are 2 directed walks to node E

   $Path1:\ A$→B→EPath 2:A→B→D→E



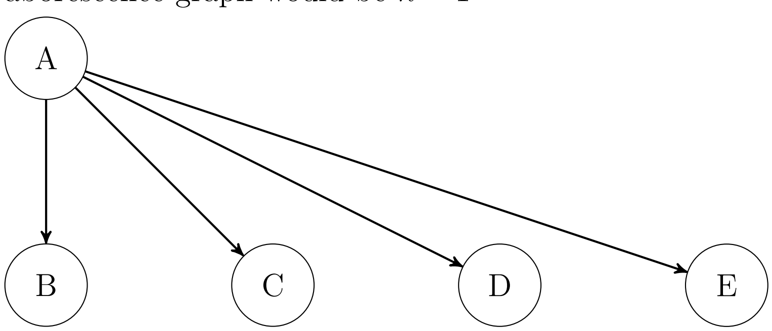
(b) What is the maximum in-degree of a vertex in an arborescence with n vertices? Prove your answer. Provide an example of an arborescence that achieves the maximum for $n = 5$.

Maximum in-degree of a vertex in an aborescence is 1. This is because an arborescence can have only one directed path from the root. This means starting from the root to the final vertex, the path is unique and directed. This implies that each node can only have exactly one parent. This proves that each vertex can have a maximum in-degree of 1.



(c) What is the maximum out-degree of a vertex in an arborescence with n vertices? Prove your answer. Provide an example of an arborescence that achieves the maximum for $n = 5$.

Maximum out-degree of a vertex in an aborescence is $n - 1$.This is because a node will only need out-degrees equal to the number of vertices it has an edge too. Consider a very simple aborescence graph such that all nodes are connected directly to the root. Then the only unique path from the root to the nodes is the edge from root to the node. Therefore the maximum out-degree of a vertex in an aborescence graph would be $n - 1$



3. For each of the following relations, indicate whether it is (1) transitive, (2) reflexive, (3), irreflexive, (4) symmetric, (5) antisymmetric, and/or (6) asymmetric.

   (a) xRy for $x, y \in \mathbb{Z}$ if x divides y. Transitive, reflexive, symmetric

   (b) xRy for people x and y if x is older than y. Transitive, irreflexive, asymmetric

(c) XRY for sets X and Y if $X \subseteq Y$. Transitive, reflexive, antisymmetric

(d) xRy for binary trees x and y if x is a subgraph in y. Transitive,reflexive, antisymmetric

(e) xRy for boolean formulas x and y if $x \implies y$ reflexive, antisymmetric

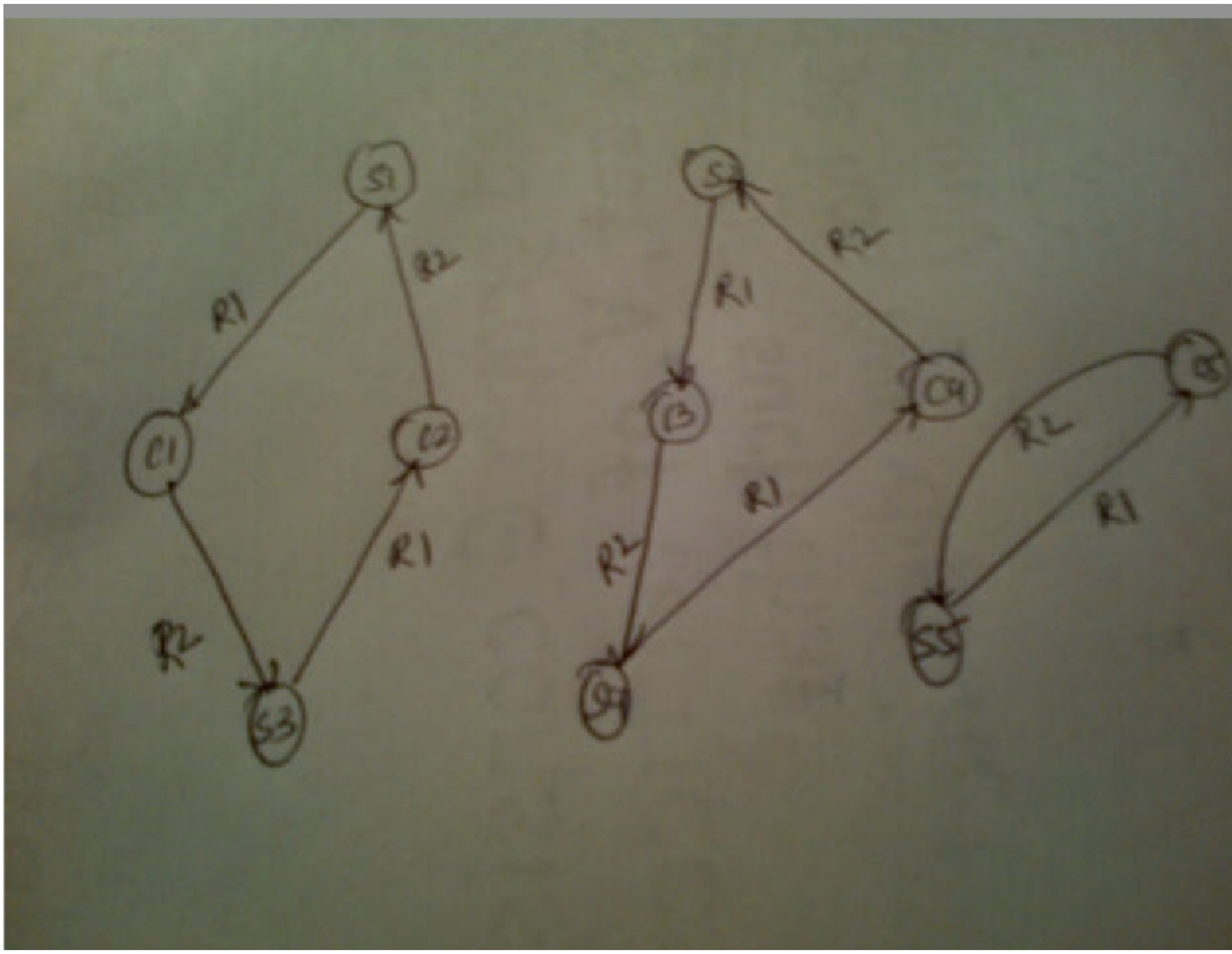
(f) xRy for $x, y \in \{\text{Rock, Paper, Scissors}\}$ if x beats y. irreflexive, asymmetric

4. CS 1234: Cat Photography is a new highly exclusive course that teaches computer science concentrators how to take cat pictures for the Internet. Enrollment is limited to 5 students and 5 cats. Let S be the set of 5 students in the course, and let C be the set of 5 cats. A relation $R_1 \subseteq S \times C$ is defined by students' preferences for cats: if sR_1c, then student s would be willing to work with cat c. Another relation $R_2 \subseteq C \times S$ is defined by cats' preferences for students: if cR_2s, then cat c would be willing to work with student s. A successful matching occurs when all students and cats can be paired in accordance with all parties' preferences.

   (a) Find relations R_1 and R_2 with a successful matching such that $|R_1| + |R_2|$ is minimum. Draw the corresponding directed graph. (Make sure to indicate a distinction between R_1 and R_2 on your graph.)

   Solution: For a successful match both R_1 and R_2 need to be met. Which is to say, that for each unique student there exist at least one cat such that $|R_1|$ relation is satisfied, and for each unique cat there is at least one student such that $|R_2|$ relation is satisfied.

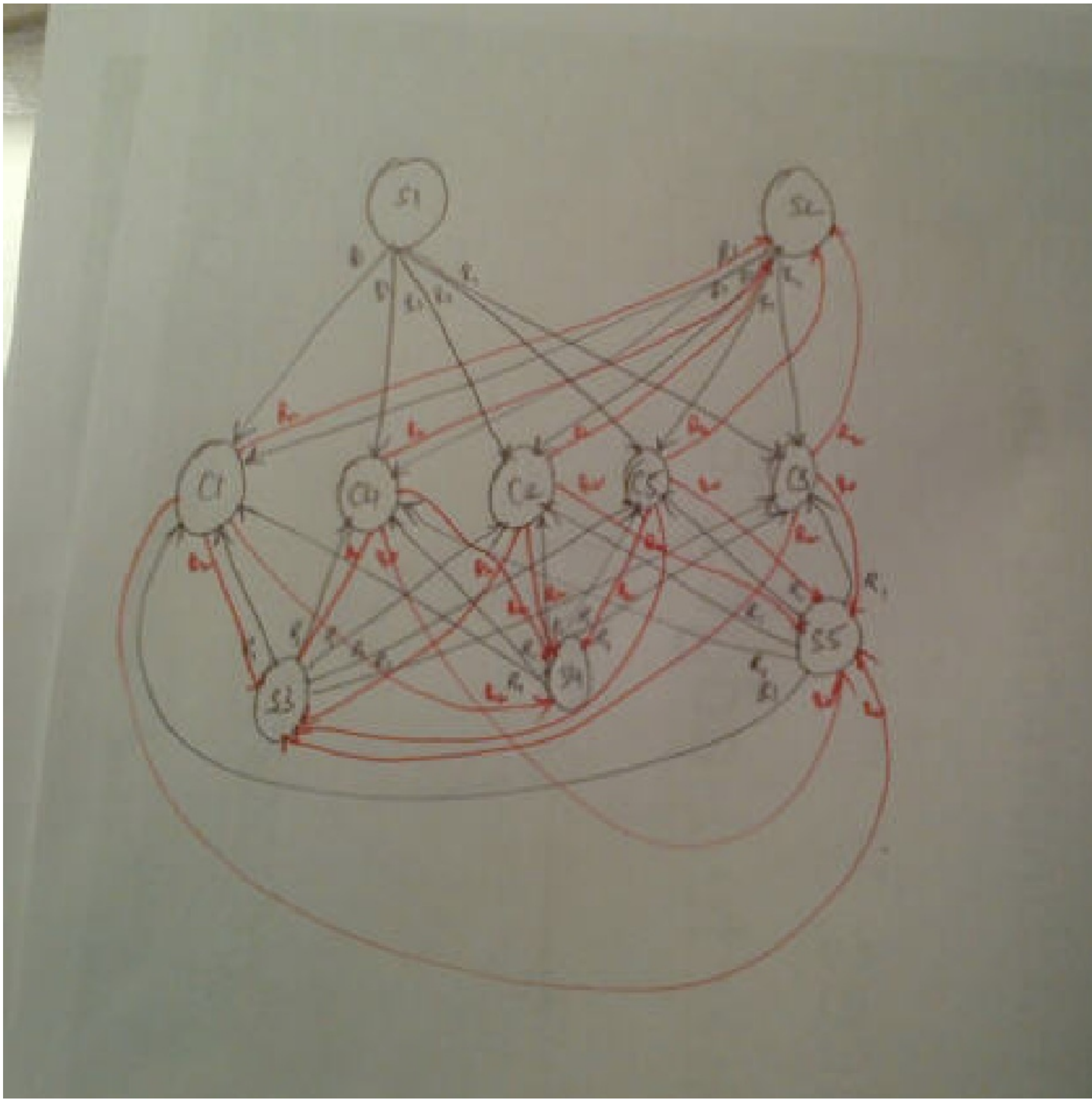
   The cardinality of such a set can be minimum if we pair each student with one cat such that $|R_1|$ holds and if we pair each cat with one student such that $|R_2|$ holds



(b) Find relations R_1 and R_2 with *no* successful matching such that $|R_1| + |R_2|$ is *maximum*. Draw the corresponding directed graph.

Solution: IF there exists at least one student that cannot be paired with a cat or at least one cat that cannot be paired with a student, the relations will result in no successful matching.

In order for this set to have maximum cardinality: If all students except $s5$ have a R1 with all cats and all cats have a R2 with all students the cardinality of this relation will be maximum


S1
S2
C1
C4
C2
C5
C3
S3
S4
S5