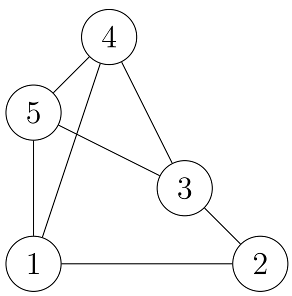
COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Undirected Graphs, Connectivity

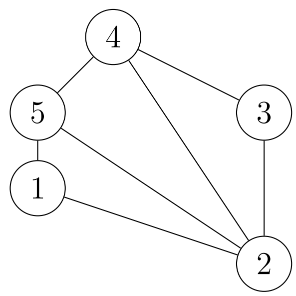
Due Thursday, April 2, 2015 before 9PM EDT. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/21389

1. For each of the following pairs of graphs, either define an isomorphism between them, or prove that there is none.

(a)

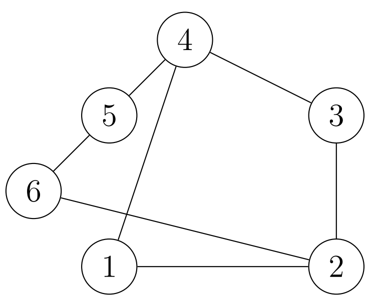


G_1

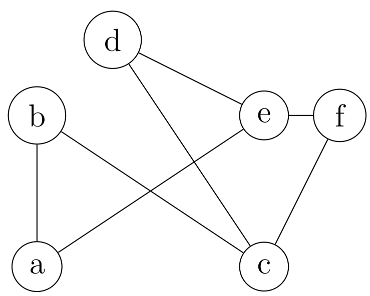


G_2

(b)



G_1



G_2

Solution:
(a) is not isomorphic as the degree of vertex in G1 is not equal to degree of vertex in G2. G1 has 4 vertices (5,4,3,1) with a degree of 3 while G2 has only 2 vertices (5,4) with degree of 3. G2's other vertices (1 and 3) have a degree of 2 , and its vertex 2 has a degree of 4.

(b) is isomorphic as the total number of edges and vertices of G1 and G2 are the same, and the degree of a vertex in G1 is equal to degree of vertex of G2.

2. (a) If G is a simple graph with n vertices and exactly k connected components, what is the maximum number of edges it can have?

   Solution: The maximum number of edges in to connect n vertices is = n , and then to connect the k components we will need (k-1) edges. So total edges will be = n + (k-1)

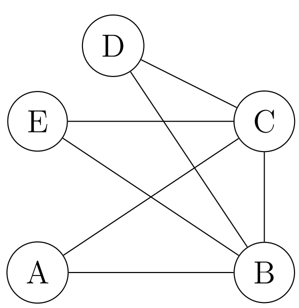
   (b) Let D be a 2-edge connected graph ($n \geq 3$). Prove that every edge is in a cycle.

   Proof: Let (u,v) be an edge in the Graph D. Let us remove this edge from the graph D. Since D is 2-edge connected, there must exist another path from v to u (i.e. no vertex in this walk is repeated more than once). This path from (v,u), added with the edge (u,v), results in a path from that starts at v and finishes at v, with no repeated vertex. Such a path is called a cycle.

   This proves that every edge in a 2-edge connected graph is in a cycle.

3. Consider a group of 5 people. Within the group Amy knows 3 other people, Bob knows 4 others, and Cal knows 4 others. How many people in the group does each of Dan and Eva know? List all the possible values, and prove that there are no others.

   Proof:
   We know that Bob and Cal know 4 people, while Amy know only 3 other. The graph below models the "knowing" relationship that we definitely know from above.



   Given that Amy knows 3 other people, Amy can know Dan or Eva, but not both. There is also a possibility that Dan and Eva know each other . With this information we can conclude the following

   Case 1 - If Dan and Eva know each other and
   (A) Amy knows Dan - then Dan knows 4 and Eva knows 3 people
   (B) Amy knows Eva - then Dan knows 3 and Evan knows 4 people

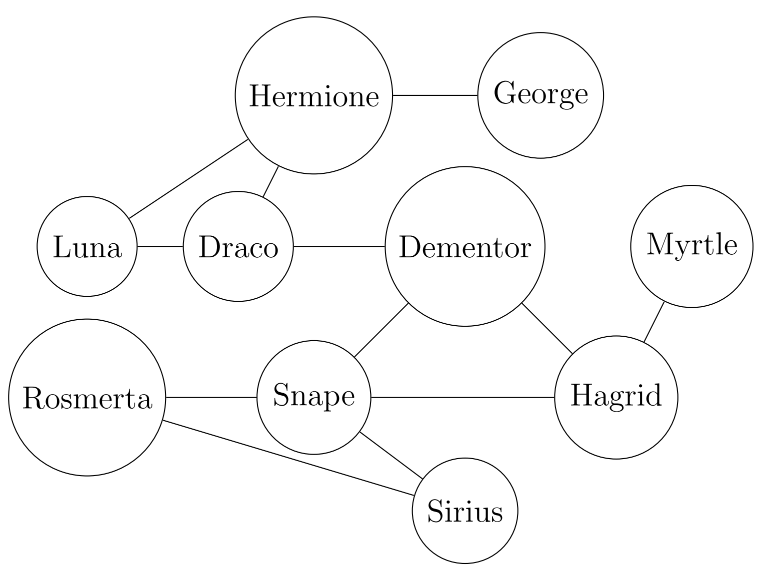
   Case 2 - If Dan and Eva Do not know each other and
   (C) Amy knows Dan - then Dan knows 3 and Eva knows 2 people

(D) Amy knows Eva - then Dan knows 2 and Evan knows 3 people

4. (a) Corinne is writing Harry Potter fan-fiction. To get a lot of readers, her stories inevitably end up with a large network of relationships. We can represent all of these relationships with simple graphs (no self-loops), where nodes are the characters and the edges represent that two characters have dated. To create more drama, Corinne makes sure that when characters enter new relationships, they will only be paired with other characters they can reach from this graph. Determine if the binary relation for who a student could end up dating is any of the following: transitive, reflexive, irreflexive, symmetric, antisymmetric, and/or asymmetric.

   Solution: The binary relation is
   irreflexive - As the graph allows no self loops no character can date themsleves.
   transitive - As an edge from A to B and an edge from B to C, means there is a path from A to B to C. So in this case A dates B and B dates C, implies that A can date C, as C can be reached from A in the graph.
   symmetric - If A dates B , then B can date A. This is because there is no information above that prevents a loop/cycle between 2 nodes in the graph. As long as B can reach A on the graph, B will be able to enter into a "new relatioship" with the same person.

   (b) The following is a graph showing relationships from Corinne's latest fanfic, *The Dementor's Kiss*.



   For this graph, please give the vertex connectivity, edge connectivity, and any articulation points or bridges that exist.

Solution:

(1) Vertex Connectivity : 1: Dementor, if removed will make Snape (or any of connected vertex in this graph) disconnected from Draco (or any of its connected vertex)

(2) Edge Connectivity : 1, because removing the edge between Hermione and George, will make George unreachable, or removing the edge between Hagrid and Myrtle will make Myrtle unreachable.

(3) Articularion Points :
Dementor - as removing this creates 2 connected components.
Hermione - as removing this will make George unreachable
Draco - as removing this will create 2 connected components
Snape - as removing this will create 2 connected components
Hagrid - as removing will make Myrtle unreachable.

(4) Bridges : 2 bridges : Edge (Hermione , George) and Edge (Hagrid, Myrtle)