COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Propositional Logic, Normal Forms, Logic Gates

Due Thursday, February 19, 2015 before 9PM EST. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/18601

1. ***Note that there are many ways to express each solution.***

   (a) Define the propositions p="You obey the speed limit" and the q="You are going to a wedding". Write the following sentences as compound propositions using p and q:

      i. Failing to obey the speed limit implies that you are going to a wedding.
      Solution: $\neg p \to q$

      ii. You drive below the speed limit only if you are going to a wedding.
      Solution: $p \to q$

      iii. You do not obey the speed limit unless you are going to a wedding.
      Solution: $p \to q$

      iv. You drive above the speed limit whenever you are going to a wedding.
      Solution: $q \to \neg p$

   (a) Define the propositions p="The home team wins," q="It is raining," r="There is an earthquake " Write the following sentences as compound propositions using p,q, and r:

      i. Either rain or an earthquake is sufficient for the home team to win.
      Solution: $(q \vee r) \to p$

      ii. Rain and earthquake are necessary but not sufficient for the home team to win.
      Solution: $p \to (q \wedge r)$

      iii. The home team wins only if it is not raining and there is no earthquake.
      Solution: $p \to (\neg q \wedge \neg r)$

      iv. If it is raining the home team will win unless there is an earthquake.
      Solution: $(q \wedge \neg r) \to p$

2. Using just the operators $\neg$ and $\wedge$ (and parentheses), write formulas involving p and q that are logically equivalent to:

(a) $p \vee q$

Solution: $\neg(\neg p \wedge \neg q)$

(b) $p \oplus q$

Solution: $\neg(\neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q))$

(c) $p \rightarrow q$

Solution: $\neg(p \wedge \neg q)$

(d) $p \leftrightarrow q$

Solution: $\neg(\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q))$

3. The Boolean function $G(p, q, r)$ is defined in the truth table below. **Note: We are using Meyer's definition for conjunctive normal form and disjunctive normal form for this pset.**

p	q	r	G(p,q,r)
F	F	F	T
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	T
T	F	T	F
T	T	F	F
T	T	T	T

(a) Construct a proposition in disjunctive normal form for $G(p, q, r)$.

Solution: Constructing it from the Truth lines in the truth table, we get $(\neg p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r)$.

(b) Using only the AND and NOT operators, construct the simplest proposition equivalent to the one in (a).

Solution: $\neg(p \wedge \neg q \wedge r) \wedge \neg(q \wedge \neg r)$

4. (a) Write $(p \wedge q) \oplus (p \vee r)$ in conjunctive normal form.

Solution: $(p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

(b) Prove that any propositional formula can be written in conjunctive normal form.

Solution: For any truth table, there is a formula to convert the logical proposition into the conjunctive normal form. The formula is as follows:

i. Single out the rows where the logical proposition evaluates to false. If there are not such rows, the proposition is a tautology meaning the correct form can just be T.

ii. For each false row, we can join together every variable in that line of the truth table with OR, adding a NOT in front of variables that read "T" for that line of the truth table. For example, a truth table row with p=T, q=F, and r=F that resulted in the logical proposition evaluating to F would result in the clause $(\neg p \vee q \vee r)$.

iii. We can join the clauses described in step 2 with AND. With this method, every clause will have every variable present, variables within clauses will all be joined by OR, and all clauses will be joined by AND, satisfying the conjunctive normal form requirements.

iv. Because we are negating all variables that are true for the false rows in the truth table, no clauses in our formula will evaluate to true for the same combination of variables that led to a false outcome in the truth table. And then because we combine the clauses with ANDs, every single clause must evaluate to true, which implies that no original false combination of our original variables can evaluate to true in this formula.

v. Because this can be done for every truth table, all logical propositions can be written in conjunctive normal form.

5. While in a maze looking for lost treasure, Grace finds a stone door with three keyholes, one colored red, one blue, and one green. A sign next to the door reads:

   To reach the treasure behind this door:

   Use only two keys on the door, one must be colored blue.

   If blue keys you do not have, the other 2 keys will get you through.

   Luckily, Grace has a red key, a blue key, and a green key. Define r, b, g:

   r: Grace uses the red key

   b: Grace uses the blue key

   g: Grace uses the green key

   (a) Write a proposition that evaluates to True if Grace uses the right combination of keys and can get through the door.

   Solution: $(b \wedge g) \oplus (r \wedge (b \vee g))$

   (b) Draw a logic circuit that implements the proposition in (a) with as few gates as possible from this list: NOT, OR, AND, XOR, NAND, XOR, NXOR.

Solution: