Computer Science E-20, 2015 Final Exam

There are 8 problems on this exam. All problems count equally. Closed book; closed notes. No electronic devices.

There are three blank pages at the end of the exam for scratch work. Anything written on them will NOT be graded.

Name:

	Points	
Q1		/15
Q2		/15
Q3		/15
Q4		/15
Q5		/15
Q6		/15
Q7		/15
Q8		/15
Total		/120

1. Using the Pigeonhole Principle, show that in any group 105 people, at least three of them have birthdays within the same 7-day period. (For example, someone born on May 1, 1991 and someone born on May 7, 1995 have birthdays within the same 7-day period).

   Solution: Each birthday is 7 pigeons, because each birthday is part of exactly seven 7-day periods. There are 366 possible birthdays when we account for leap years. 105 people $\times$ 7 pigeons = 735 pigeons. There are 366 7-day periods (pigeonholes), and $735 > 2 \times 366$. By the PGP there are at least one 7-day period with at least three people's birthdays.

2. Using induction, prove DeMorgan's Law for multiple sets:

$$\overline{(A_1 \cup A_2 \cup \cdots \cup A_n)} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

You may use the two-set case of DeMorgan's Law in your proof, i.e. that

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}.$$

Solution: Base case $P(n = 2)$: $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$ True (as given)
Assume $P(n)$: $\overline{(A_1 \cup A_2 \cup \cdots \cup A_n)} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$.
Show $P(n+1)$: $\overline{(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1})} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} \cap \overline{A_{n+1}}$.

$$\begin{aligned}\overline{(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1})} &= \overline{(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_n)} \cap \overline{A_{n+1}} \\ &= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} \cap \overline{A_{n+1}}\end{aligned}$$

3. Let the domain of discourse be all **positive natural numbers** and define the predicates $E(x), O(x)$ and $D(x, y)$ as follows:

- $E(x)$: x is even
- $O(x)$: x is odd
- $D(x, y)$: x divides y

(a) Express the following sentences using quantificational logic.

i. Every number is either even or odd.

$$\forall x.E(x) \vee O(x)$$

$$\forall x.E(x) \oplus O(x)$$

ii. A number is even if and only if it is divisible by 2.

$$\forall x.E(x) \iff D(2, x)$$

iii. No odd number is divisible by an even number.

$$\forall x, y.(O(x) \wedge E(y)) \rightarrow \neg D(y, x)$$

(b) Label the following expressions as either (T)rue of (F)alse.

i. $\exists x \forall y.D(x, y)$
True, $x = 1$

ii. $\forall y \exists x.D(x, y)$
True, $x = y$

iii. $\exists y \forall x.(E(x) \wedge D(y, x))$
False, part of the statement claims that all numbers are even

(c) Write the predicate $P(x)$ that means that x is a prime number.
$P(x) : \forall y.(y \neq x \wedge y \neq 1) \rightarrow \neg D(y, x)$

4. Suppose S is a set and $\prec$ is a binary relation on S.

- $\prec$ is *well-founded* if for any $A \subseteq S$ there is some "minimal" $a \in A$ such that for all $b \in A$ it is not the case that $b \prec a$.
- $\prec$ is *total* if for any $a, b \in S$, if $a \neq b$ then either $a \prec b$ or $b \prec a$.

(a) Assume that S is a finite set of natural numbers. Is the relation *less than* "<" a well-founded relation? How about the relation *less than or equal* "$\leq$"? Explain briefly with two sentences or less.

Solution: Less is well-founded because there is always a smallest natural number. Less than or equal is not well-founded, because every natural number is related to itself.

(b) Prove that any well-founded, total relation on S is irreflexive and transitive. Remember, *irreflexive* means that no element bears the relation to itself.

Solution: Irreflexive (Proof by contradiction)

Proof by contradiction. Assume it is reflexive, therefore there is some $a : a \prec a$, consider the set $A = \{a\}$. Since $a : a \prec a$ violates the condition "for all $b \in A$ it is not the case that $b \prec a$" when $a = b$ – contradiction.

Transitive (Proof by contradiction)

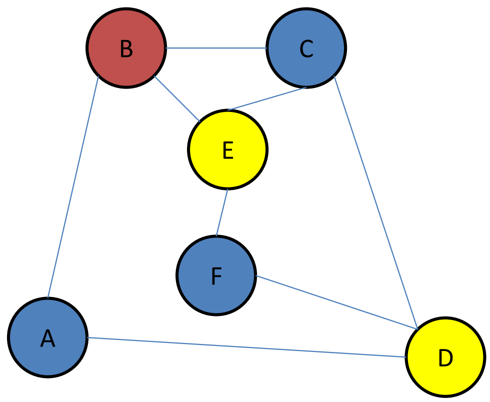
Assume it is not transitive, which means that there is a triple a, b, c such that: $a \prec b \wedge b \prec c \rightarrow a \not\prec c$. Equivalent to $a \prec b \wedge b \prec c \rightarrow c \prec a$ because totality. This means that there is no minimal element in the set a, b, c – contradiction.

5. (In-class) Show that the set of all finite sequences of integers is countably infinite.

   Solution: Assign the integers into sets according to their length: S_1 contains all integers of length 1, S_2 – of length 2, and so on. The size of each of these sets is finite and thus countable. The infinite union $S_1 \cup S_2 \cup ...$ is also countable, because it is indexed by a natural number. A countable union of countable sets is countable.

6. (In-class) Schedule the final exams for the 6 courses — A, B, C, D, E, and F — using the fewest number of different time slots. Courses that have students in common cannot be scheduled at the same time, and the following 8 pairs share at least one student:

(A, B), (A, D), (B, C), (B, E), (C, D), (C, E), (F, D), (E, F)



Solution:
One possible solution:
A, F, C
D, E
B
Note that because there is a cycle of length 3: (B,C), (C,E), (B,E) the minimum number of colors required to color the graph is 3, so there cannot be a solution with fewer colors than ours.

7. (In-class) Suppose that 40% of apartment listings on Craigslist are scams. 60% of scam listings do not include pictures of the apartment, whereas only 5% of legitimate listings do not include pictures. In your desperate search for summer housing, you find a new listing that does not include any pictures. What is the probability that it is a scam?

Solution: S = scam N = no pictures

$$\begin{aligned} P(S) &= 0.4 \\ P(N|S) &= 0.6 \\ P(N|\overline{S}) &= 0.05 \\ P(S|N) &=? \end{aligned}$$

$$\begin{aligned} P(S|N) &= \frac{P(N|S)P(S)}{P(N)} \\ &= \frac{P(N|S)P(S)}{P(N|S)P(S) + P(N|\overline{S})P(\overline{S})} \\ &= \frac{0.6 \times 0.4}{0.6 \times 0.4 + 0.05 \times 0.6} \\ &= \frac{8}{9} \end{aligned}$$

8. (In-class) Suppose there are two boxes, **each** of which contains 200 balls: 100 red balls numbered from 1 to 100, and 100 blue balls numbered from 1 to 100. We pick one ball at random from each box.

   (a) Given that at least one of the two balls picked is red, what is the probability that both are red?

   Solution: 1/3 Enumerate the number of ways one can select at least one red ball:
   $rr = 100 \times 100$
   $rb = 100 \times 100$
   $br = 100 \times 100$
   $P(rr|r) = \frac{100\times 100}{100\times 100+100\times 100+100\times 100} = \frac{1}{3}$

   (b) Given that at least one of the two balls is red and numbered 42, what is the probability that both balls are red?

   Solution:
   $r42.r = 1 \times 99$
   $r.r42 = 99 \times 1$
   $r42.r42 = 1 \times 1$
   $r42.b = 1 \times 100$
   $b.r42 = 100 \times 1$
   $P(rr|r42) = \frac{199}{199+200} = \frac{199}{399}$