COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Trees, Coloring

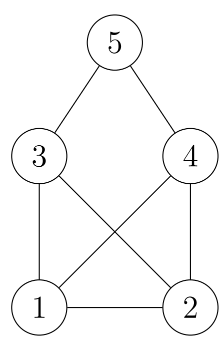
Due Thursday, April 9, 2015 before 9PM EDT. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/21777

1. (a) Prove that a graph is 2-colorable if and only if it is bipartite.
   Proof:
   If G is bipartite, then we can divide the vertices into two parts A and B such that all edges go between A and B. Since there are no edges within A and B,we can color all vertices in A with the one color,and those in B the other color.This proves G is two-colorable
   Conversely, suppose G is two-colorable.Since the graph is two-colorable there cannot be an edge connecting vertices of the same color, so the only edges possible will be those connecting the vertices of different color. Let A be the vertices of color 1 and B be the vertices of the other color. All edges will be go between A and B. This proves G is bipartite.

   (b) Prove that any tree is 2-colorable.
   Proof:
   We will prove this by showing that every tree is a bipartite graph. We know that in every tree there exists only a unique path from the root r to any vertex v in the tree. And that there can be no cycles in a tree. Let r be the root of the tree and let it be in the set of vertices V1. Now all vertices that have a even length path from r can be in this set V1 and those with odd length can be in another set V2. We now have a partition of all vertices v in the tree set such that every edge has one vertex in V1 and one in V2. This proves that every tree is a bipartite graph.
   And from (a) we know that all bipartite graphs are two-colorable. Hence every tree is two-colorable.

2. Determine, with justification, the chromatic numbers for the graphs below:

   (a)

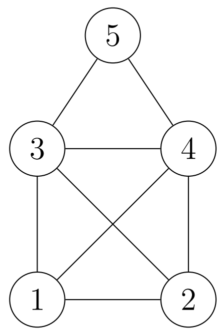


   Solution: We can start off by coloring node 5. We can color node 3 and 4 using a new color. Note the reason node 3 and 4 can be colored with same color is because they are not connected. This leaves node 1 and 2. Since 1 is connected to 4 and 2 and 2 is connected 3 and 1., we can color node 1 by say using the same color that was used to color node 5 and we need a new color for node 2. So to summarize

node 5 - uses color c1
node 3, 4 - uses color c2
node 1 - uses color c1
node 2 - uses color c3

So this graph has a chromatic number of 3.

(b)



Solution: the picture above shows a connected graph.We can start off by coloring node 5. We need to choose 2 diff colors for node 3 and 4 as they are connected to 5. Now since node 1 is connected to both node 4 and 3, we cant use those colors.But node 4 can use the color as node 5. Now the only node left is 2, and since node 2 is connected to nodes 1, 4, 3, its cant share any of those colors. Which means node 2 gets a new color. So to summarize

node 5 - uses color c1
node 3 - uses color c2
node 4 - uses color c3
node 1 - uses color c1
node 2 - uses color c4

This proves that the chromatic number of this graph is 4

(c) K_n, the complete graph on n vertices.
Solution : Complete graph is one in which every pair of vertices is connected by a unique edge. Since a vertex would be connected to every other vertex in the graph , in order to color this graph all n vertices should have a different color. So the Chromatic number of a complete graph with n vertices is n.

3. Let d be the maximum degree of any vertex in a graph G. Prove that the chromatic number of G is at most $d + 1$.

Proof :We can start coloring the graph G at any node.We now color its adjacent nodes (i.e neighbours). Since any vertex in this graph G can have at most d neighbours (i.e. $\leq d$) , a $d + 1$ colors will always be sufficient to color the graph. This proves that the chromatic number of the graph G with maximum degree d is at most $d + 1$