COMPUTER SCIENCE 20, SPRING 2015

Module #5 (Induction Review)

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Executive Summary

1. Ordinary Induction
   - A *predicate* is the statement you are trying to prove.
   - Let $P(x)$ be a predicate and m, n nonnegative integers. If $P(m)$ is true and $P(n) \Rightarrow P(n+1)$ for all $n \geq m$, then $P(n)$ is true for all $n \geq m$.
   - To write a proof by induction, first you need to identify the proposition to be proven. Then prove the base case, prove the inductive step (how you can get from the proposition holding for n to the proposition holding for $n+1$), and the conclusion. Be sure you identify properly the thing being inducted upon.
2. Strong Induction
   - In simple terms, strong induction is similar to ordinary induction, except you use $P(0)$, ..., $P(n)$ instead of just $P(n)$ in order to prove $P(n+1)$.
   - Formally, the difference between ordinary and strong induction is:
     Let $P(x)$ be a predicate and m, n nonnegative integers.
     - Ordinary Induction: If $P(m)$ is true and $P(n) \Rightarrow P(n+1)$ for all $n \geq m$, then $P(n)$ is true for all $n \geq m$.
     - Strong Induction: If $P(m)$ is true and $P(m), P(m+1), \ldots, P(n)$ together $\Rightarrow P(n+1)$ for all $n \geq m$, then $P(n)$ is true for all $n \geq m$.
   - Strong induction proofs begin with the identification of the proposition to be proven. The next step is to identify and verify the base case. Note that with strong induction there are often multiple base cases. Next comes the inductive step, where you show that the proposition's truth for $0 \ldots n$ entails its truth for $n+1$. Be sure to properly identify the proposition being inducted on.
   - Note that sometimes you will need to break your inductive step into multiple cases.

Check-in questions

1. Let P(n) mean "Postage of n cents can be made with only 4-cent and 7-cent stamps". For which of the following values of n is P(n) false?

   (a) n=14

   (b) n=15

   (c) n=16

   (d) n=17

   (e) n=18

2. Let P(n) mean "Postage of n cents can be made with only 4-cent and 7-cent stamps". Suppose that we want to prove by strong induction that P(n) is true for all $n > 17$. Which of the following base cases do we need to show?

   (a) P(0), P(1), P(2), P(3)

   (b) P(14), P(15), P(16), P(17)

   (c) P(17), P(18), P(19), P(20)

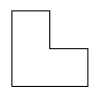
   (d) P(18), P(19), P(20), P(21)

In-class Problems

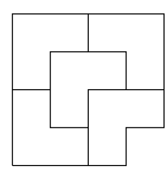
1. Using induction, prove that for all positive integers n :

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

2. A tromino is an arrangement of three squares like the one shown below. It can be flipped and rotated in any direction.



   (a) Prove by induction that every $2^n \times 2^n$ grid with one corner removed can be covered with trominos. Example:



   (b) Explain how you can use the result from the previous part to prove that:

$$2^{2n} - 1 \text{ is divisible by 3, for all } n > 0$$

3. Prove using strong induction that for all $n \in \mathbb{N}$ such that $n \geq 2$, n is divisible by a prime. *(Hint: In this problem, we consider all divisors, not just proper divisors, so a number is considered as being divisible by itself. Split the inductive step into cases based on whether $n+1$ is prime. What does it mean for a number to be composite?)*

4. You are putting together a jigsaw puzzle with $n \geq 1$ pieces. At each step you join together two matching pieces and produce a new (composite) piece. Prove by induction that no matter in which order you join the pieces, it takes $n-1$ steps to complete the puzzle.