COMPUTER SCIENCE E-20, SPRING 2015
Homework Problems
Induction I, Strong Induction

Due Thursday, February 12, 2015 before 9PM EST. Upload a PDF of your answers at https://canvas.harvard.edu/courses/1815/assignments/17757

1. Prove that for all nonnegative integers n

$$\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$$

Hint: the following identity may be useful

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Solution: We will proceed by induction. As a base case, note that $\sum_{i=0}^{0} i^3 = 0$, and $\left(\sum_{i=0}^{0} i\right)^2 = 0$, so $\sum_{i=0}^{n} i^3 = (\sum_{i=0}^{n} i)^2$ for $n = 0$.

Induction Hypothesis: Assume $P(n)$ is true for some $n \geq 0$:

$$P(n) : \sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$$

Show $P(n+1)$:

$$P(n+1) : \sum_{i=0}^{n+1} i^3 = \left(\sum_{i=0}^{n+1} i\right)^2$$

In the next step we will perform algebraic manipulations on the LFS to make it equal to the RHS. In the process, as is the case with all induction proofs, we will use the Induction Hypothesis (IH).

$$
\begin{aligned}
\sum_{i=0}^{n+1} i^3 &= (n+1)^3 + \sum_{i=0}^{n} i^3 \\
&= (n+1)^3 + \left(\sum_{i=0}^{n} i\right)^2 && \text{by the IH} \\
&= (n+1)^3 + \left[\frac{n(n+1)}{2}\right]^2 \\
&= \frac{\left(n^2 + 4(n+1)\right)(n+1)^2}{4} \\
&= \left[\frac{(n+1)(n+2)}{2}\right]^2 \\
&= \left(\sum_{i=0}^{n+1} i\right)^2
\end{aligned}
$$

2. Consider the sequence $a_1 = 1, a_2 = 3, ..., a_n = a_{n-1} + a_{n-2}$. Using strong induction prove that $a_n \leq \left(\frac{7}{4}\right)^n$ for all positive integers n.

   Solution: Let $P(n) = a_n \leq (\frac{7}{4})^n$ for positive integer n. We will proceed by strong induction.

   Base Cases: Since $a_1 = 1$ and $1 \leq \frac{7}{4}$, $P(1)$ is true.

   Since $a_2 = 3$ and $3 \leq (\frac{7}{4})^2$, $P(2)$ is true.

   Inductive Step: Suppose $P(m)$ holds for all positve integers $m \leq n$.

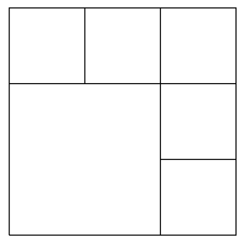
   We are given $a_{n+1} = a_n + a_{n-1}$.

   Since $n \leq n$ and $n-1 \leq n$, the induction hypothesis guarentees $a_n \leq (\frac{7}{4})^n$ and $a_{n-1} \leq (\frac{7}{4})^{n-1}$. As a result,

$$
\begin{aligned}
a_{n+1} &= a_n + a_{n-1} \\
&\leq (\frac{7}{4})^n + (\frac{7}{4})^{n-1} \\
&\leq (\frac{7}{4})^{n-1}(\frac{7}{4} + 1) \\
&\leq (\frac{7}{4})^{n-1}(\frac{7}{4})^2 \\
&\leq (\frac{7}{4})^{n+1}
\end{aligned}
$$

   which proves $P(n+1)$. It follows by strong induction that $P(n)$ is true for all positive integers n. ■

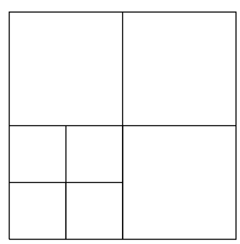
3. Prove using strong induction that any square can be subdivided into n smaller squares, where $n > 5$. For example, the large square below has been subdivided into 6 squares.
Hint: first show that any square subdivided into k squares can easily be subdivided into $k + 3$ squares, then think how many base cases you need show are true (it is not just the case of n=6).



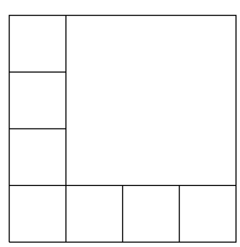
Solution: Let $P(n)$ = any square can be subdivided into n smaller squares, where $n > 5$. We will proceed by strong induction.

Base cases: The example above proves that $P(6)$ is true.

$P(7)$ is true since the following division is possible:



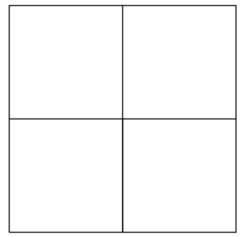
$P(8)$ is true since the following division is possible:



Inductive step: We have that $P(6)$, $P(7)$, and $P(8)$ are true. Suppose $P(m)$ holds for $8 < m \leq n$.

Consider $P(n - 2)$. Since $n - 2 \leq n$, the induction hypothesis guarentees that any square can be subdivided into $n-2$ smaller squares, where $n-2 > 5$ (since $8 < m \leq n$).

Notice that if we have a square divided into k smaller squares, we can produce a division of this same square into $k + 3$ squares by dividing one of the original k squares as follows:



(Note that we no longer count the square that is divided; instead we count the 4 smaller squares that the division creates. After this division, there are $k-1+4=k+3$ sub-squares.)

Since any square can be subdivided into $n-2$ squares, any square can also be subdivided into $(n-2)+3=n+1$ squares. This proves $P(n+1)$.

By strong induction, we can conclude $P(n)$ is true for all integers $n > 5$. ■

(The reason we needed three bases cases should now be apparent. The proof relies on being able to assume that a square may be divided into $n-3$ sub-squares. If we only show that $P(n)$ holds for $n=6$ and try and rely on induction to prove the result for $n=7$ or $n=8$, we have a problem: the result does not necessarily hold for $n=4$ or $n=5$.)

4. The Fibonacci numbers are defined by $F_0=0, F_1=1, F_n=F_{n-1}+F_{n-2}$. Prove using strong induction that for all $n \geq 0$, F_{3n} is even.

   Solution 1 - Induction: Let $P(n) = F_{3n}$ is even for $n \geq 0$. We will proceed by induction.

   Base Case: Since $F_{3*0}=0$ and 0 is even, $P(0)$ is true.

   Inductive Step: Suppose $P(n)$ where $n \geq 0$ is true.

   Consider $F_{3(n+1)}$:

$$\begin{aligned} F_{3(n+1)} &= F_{3(n+1)-1}+F_{3(n+1)-2} \\ &= F_{3n+2}+F_{3n+1} \\ &= (F_{3n+1}+F_{3n})+F_{3n+1} \\ &= F_{3n}+2(F_{3n+1}) \end{aligned}$$

   The induction hypothesis guarentees F_{3n} is even. We can therefore express F_{3n} as $2k$ for some integer k. Then,

$$\begin{aligned} F_{3(n+1)} &= 2k+2(F_{3n+1}) \\ &= 2(k+F_{3n+1}) \end{aligned}$$

   Since $k+F_{3n+1}$ is an integer, $F_{3(n+1)}$ must be even. This proves $P(n+1)$. It follows by induction that $P(n)$ is true for all integers $n \geq 0$. ■

Solution 2 - Strong Induction: Let $P(n) = F_n$ is even if $n\%3 = 0$ and odd if $n\%3 = 1$ or $n\%3 = 2$ for $n \geq 0$. We will proceed by strong induction.

Base Cases: Since $F_0 = 0$, 0 is even, and $0\%3 = 0$, $P(0)$ is true.

Since $F_1 = 1$, 1 is odd, and $1\%3 = 1$, $P(1)$ is true.

Inductive Step: Suppose $P(m)$ is true for all $0 \leq m \leq n$.

Consider $F_{n+1} = F_n + F_{n-1}$.

There are three possible values for $(n+1)\%3$:

- If $(n+1)\%3 = 0$, then $n\%3 = 2$ and $(n-1)\%3 = 1$.

  Since $n \leq n$ and $n - 1 \leq n$, the inductive hypothesis guarentees F_n and F_{n-1} are both odd.

  As a result, there exist integers k and l such that $F_n = 2k + 1$ and $F_{n-1} = 2l + 1$. Then

$$\begin{aligned} F_{n+1} &= (2k+1) + (2l+1) \\ &= 2k + 2l + 2 \\ &= 2(k + l + 1) \end{aligned}$$

  Since $k + l + 1$ is an integer, F_{n+1} is even. $P(n+1)$ is therefore true.

- If $(n+1)\%3 = 1$, then $n\%3 = 0$ and $(n-1)\%3 = 2$.

  Since $n \leq n$ and $n - 1 \leq n$, the inductive hypothesis guarentees F_n is even and F_{n-1} is odd.

  As a result, there exist integers k and l such that $F_n = 2k$ and $F_{n-1} = 2l + 1$. Then

$$\begin{aligned} F_{n+1} &= (2k) + (2l+1) \\ &= 2(k + l) + 1 \end{aligned}$$

  Since $k + l$ is an integer, F_{n+1} is odd. $P(n+1)$ is therefore true.

- If $(n+1)\%3 = 2$, then $n\%3 = 1$ and $(n-1)\%3 = 0$.

  Since $n \leq n$ and $n - 1 \leq n$, the inductive hypothesis guarentees F_n is odd and F_{n-1} is even.

  As a result, there exist integers k and l such that $F_n = 2k + 1$ and $F_{n-1} = 2l$. Then

$$\begin{aligned} F_{n+1} &= (2k+1) + (2l) \\ &= 2(k + l) + 1 \end{aligned}$$

  Since $k + l$ is an integer, F_{n+1} is odd. $P(n+1)$ is therefore true.

This proves $P(n + 1)$. It follows by induction that $P(n)$ is true for all integers $n \geq 0$.

Since F_n is even if $n\%3 = 0$, we can conclude F_{3m} is even for all integers $m \geq 0$. $\blacksquare$