Coal-Mining Disasters

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$$x_j \sim \begin{cases} \text{Poisson}(\lambda_1) & j = 1, \ldots, \theta \\ \text{Poisson}(\lambda_2) & j = \theta+1, \ldots, 112 \end{cases}$$

$$\lambda_i \mid \alpha \sim \text{Gamma}(3, \alpha) \qquad i = 1, 2$$

$$\alpha \sim \text{Gamma}(10, 10)$$

$$\theta \sim \text{Uniform}(1, 2, \ldots, 111)$$

Likelihood

$$p(x_1, \ldots, x_{112} \mid \lambda_1, \lambda_2, \theta) = \prod_{j=1}^{\theta} \frac{\lambda_1^{x_j} e^{-\lambda_1}}{x_j!} \prod_{j=\theta+1}^{112} \frac{\lambda_2^{x_j} e^{-\lambda_2}}{x_j!}$$

Prior Dist.

$$p(\lambda_1 \mid \alpha) = \frac{\alpha^3 \lambda_1^{3-1} e^{-\alpha \lambda_1}}{\Gamma(3)}$$

$$p(\lambda_2 \mid \alpha) = \frac{\alpha^3 \lambda_2^{3-1} e^{-\alpha \lambda_2}}{\Gamma(3)}$$

$$p(\alpha) = \frac{10^{10} \alpha^{10-1}}{\Gamma(10)} e^{-10\alpha}$$

$$p(\theta) = \frac{1}{111} I\{\theta \in \{1, 2, \ldots, 111\}\}$$

$$p(\lambda_1, \lambda_2, \theta, \alpha \mid x_1, \ldots, x_{112}) \propto p(x_1, \ldots, x_{112} \mid \lambda_1, \lambda_2, \theta, \alpha)\, p(\lambda_1, \lambda_2, \theta, \alpha)$$

$$= p(x_1, \ldots, x_{112} \mid \lambda_1, \lambda_2, \theta)\, p(\lambda_1 \mid \alpha)\, p(\lambda_2 \mid \alpha)\, p(\theta)\, p(\alpha)$$

$$\propto \prod_{j=1}^{\theta} \lambda_1^{x_j} e^{-\lambda_1} \prod_{j=\theta+1}^{112} \lambda_2^{x_j} e^{-\lambda_2}\; \alpha^3 \lambda_1^2 e^{-\alpha\lambda_1}\; \alpha^3 \lambda_2^2 e^{-\alpha\lambda_2}\; I\{\theta \in \{1, \ldots, 111\}\}\; \alpha^9 e^{-10\alpha}$$

$$= \lambda_1^{2+\sum_{j=1}^{\theta} x_j} \lambda_2^{2+\sum_{j=\theta+1}^{112} x_j} e^{-\theta\lambda_1} e^{-(112-\theta)\lambda_2} \alpha^{15} e^{-(10+\lambda_1+\lambda_2)\alpha} I\{\theta \in \{1, 2, \ldots, 111\}\}$$

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"everything else", i.e. $\lambda_2, \theta, \alpha, x_1, \ldots, x_{112}$

$$p(\lambda_1 \mid \cdot) \propto p(\lambda_1, \lambda_2, \theta, \alpha \mid x_1, \ldots, x_{112})$$

[i.e. full conditionals are proportional to the joint posterior distribution.]

$$\propto \lambda_1^{2+\sum_{j=1}^{\theta} x_j} e^{-(\theta+\alpha)\lambda_1}$$

Kernel of a gamma distribution

$$\Rightarrow \lambda_1 \mid \cdot \sim \text{Gamma}\left(3 + \sum_{j=1}^{\theta} x_j, \ \theta + \alpha\right)$$

$$p(\lambda_2 \mid \cdot) \propto \lambda_2^{2+\sum_{j=\theta+1}^{112} x_j} e^{-(112-\theta+\alpha)\lambda_2}$$

$$\Rightarrow \lambda_2 \mid \cdot \sim \text{Gamma}\left(3 + \sum_{j=\theta+1}^{112} x_j, \ 112 - \theta + \alpha\right)$$

$$p(\alpha \mid \cdot) \propto \alpha^{15} e^{-(10+\lambda_1+\lambda_2)\alpha}$$

$$\Rightarrow \alpha \mid \cdot \sim \text{Gamma}(16, \ 10 + \lambda_1 + \lambda_2)$$

$$p(\theta \mid \cdot) \propto \lambda_1^{\sum_{j=1}^{\theta} x_j} \lambda_2^{\sum_{j=\theta+1}^{112} x_j} e^{(\lambda_2 - \lambda_1)\theta} \, I\{\theta \in \{1, 2, \ldots, 111\}\}$$

Discrete distribution on $\{1, 2, \ldots, 111\}$