



ECE 495/595 Lecture Slides

Winter 2017

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Summary and Quick Links

These slides contain the following concepts:

- ▷ Review of state space concepts (Slide [3](#))
- ▷ Unicycle navigation model (Slide [5](#))
- ▷ Holonomic navigation model (Slide [9](#))
- ▷ Bicycle navigation model (Slide [12](#))
- ▷ Discrete state space models (Slide [15](#))

Review of State Space Concepts

- ▷ State space is a method of describing the behavior of a system using a set of first-order differential equations.
- ▷ State space models consist of a state equation and an observation equation.
- ▷ In a linear state space model, the two equations are typically written as:

Linear State Equation

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U}$$

- > State – \mathbf{X}
- > Control signal – \mathbf{U}
- > State matrix – \mathbf{A}
- > Input matrix – \mathbf{B}

Linear Observation Equation

$$\mathbf{z} = \mathbf{C}\mathbf{X} + \mathbf{D}\mathbf{U}$$

- > Observations – \mathbf{z}
- > Measurement matrix – \mathbf{C}
- > Feed-forward matrix – \mathbf{D}

Review of State Space Concepts

- ▷ When describing a non-linear system, the state space equations are written as:

Non-Linear State Equation

$$\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{U})$$

Non-Linear Observation Eqn.

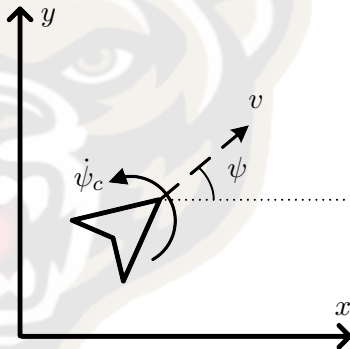
$$\mathbf{z} = h(\mathbf{X}, \mathbf{U})$$

- ▷ Developing navigation algorithms for robots usually begins with representing the system in this form.
- ▷ For now, only the modeling of the state equation is considered. The observation equation will be revisited when discussing sensor fusion techniques.

Unicycle Navigation Model

- ▷ Describe how the vehicle's position (x, y) and heading (ψ) are affected by controlling the forward velocity (v) and yaw rate $(\dot{\psi}_c)$
- ▷ Express the system in non-linear state space form:

$$\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{U})$$



Unicycle Navigation Model

- ▷ First define the state vector:

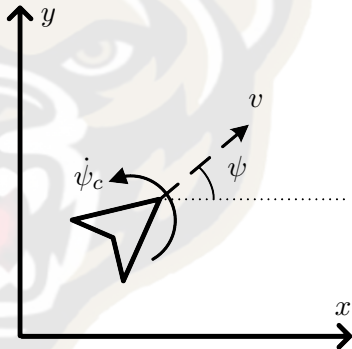
State Vector

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}$$

- ▷ ... and the control vector:

Control Vector

$$\mathbf{U} = \begin{bmatrix} v \\ \dot{\psi}_c \end{bmatrix}$$



Unicycle Navigation Model

- ▷ Need to come up with equations for the derivatives of each state variable.

- ▷ Project v onto the x axis:

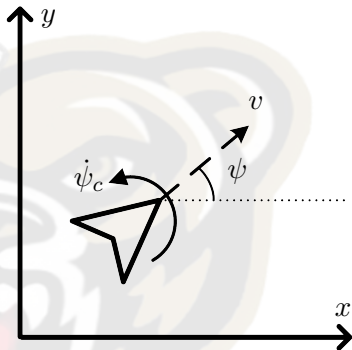
$$\dot{x} = v \cos \psi$$

- ▷ Project v onto the y axis:

$$\dot{y} = v \sin \psi$$

- ▷ The actual yaw rate is whatever the command is:

$$\dot{\psi} = \dot{\psi}_c$$



Unicycle Navigation Model

State Vector

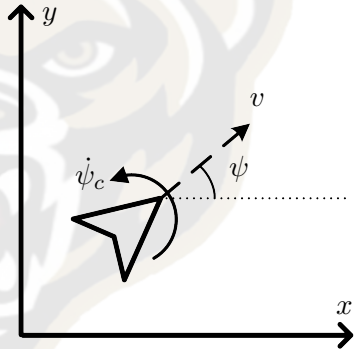
$$\mathbf{X} = [x \quad y \quad \psi]^T$$

Control Vector

$$\mathbf{U} = [v \quad \dot{\psi}_c]^T$$

State Function

$$f(\mathbf{X}, \mathbf{U}) = \begin{cases} \dot{x} &= v \cos \psi \\ \dot{y} &= v \sin \psi \\ \dot{\psi} &= \dot{\psi}_c \end{cases}$$



Holonomic Navigation Model

- ▷ State vector remains the same:

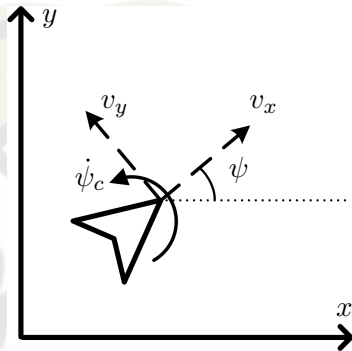
State Vector

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}$$

- ▷ The control vector has one more signal:

Control Vector

$$\mathbf{U} = \begin{bmatrix} v_x \\ v_y \\ \dot{\psi}_c \end{bmatrix}$$



- ▷ Holonomic vehicles have three degrees of freedom.

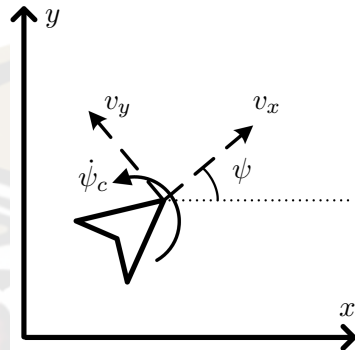
Holonomic Navigation Model

- ▷ Derive expressions for the derivatives of state variables.
- ▷ Project v_x and v_y onto the x axis:

$$\dot{x} = v_x \cos \psi - v_y \sin \psi$$

- ▷ Project v_x and v_y onto the y axis:

$$\dot{y} = v_x \sin \psi + v_y \cos \psi$$



- ▷ The actual yaw rate is whatever the command is:

$$\dot{\psi} = \dot{\psi}_c$$

Holonomic Navigation Model

State Vector

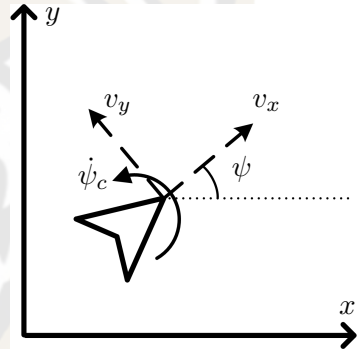
$$\mathbf{X} = [x \quad y \quad \psi]^T$$

Control Vector

$$\mathbf{U} = [v_x \quad v_y \quad \dot{\psi}_c]^T$$

State Function

$$f(\mathbf{X}, \mathbf{U}) = \begin{cases} \dot{x} &= v_x \cos \psi - v_y \sin \psi \\ \dot{y} &= v_x \sin \psi + v_y \cos \psi \\ \dot{\psi} &= \dot{\psi}_c \end{cases}$$



Bicycle Navigation Model

- ▷ The kinematic constraints of a car can be modeled simplistically by a bicycle navigation model.
- ▷ The state vector remains the same as the other vehicles, but the control vector becomes the combination of speed v and steering wheel angle α_s :

State Vector

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}$$

Control Vector

$$\mathbf{U} = \begin{bmatrix} v \\ \alpha_s \end{bmatrix}$$

Bicycle Navigation Model

- ▷ Recall the relationship between steering wheel angle, yaw rate and speed:

$$\alpha_s = \gamma \tan^{-1} \left(\frac{L\dot{\psi}}{v} \right)$$

- ▷ Rearranging this, the yaw rate $\dot{\psi}$ can be represented in terms of the control variables v and α_s , with the steering ratio γ and wheelbase L of the vehicle as constant parameters:

$$\dot{\psi} = \frac{v}{L} \tan \left(\frac{\alpha_s}{\gamma} \right)$$

Bicycle Navigation Model

State Vector

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}$$

Control Vector

$$\mathbf{U} = \begin{bmatrix} v \\ \alpha_s \end{bmatrix}$$

State Function

$$f(\mathbf{X}, \mathbf{U}) = \begin{cases} \dot{x} &= v \cos \psi \\ \dot{y} &= v \sin \psi \\ \dot{\psi} &= \frac{v}{L} \tan \left(\frac{\alpha_s}{\gamma} \right) \end{cases}$$

State Space Discretization

- ▷ In order to implement a state space controller or observer in a computer, it must first be **discretized**.
- ▷ Discretization involves approximating the time derivative of the continuous state space equation.
- ▷ There are many techniques to discretize a system, each with different implications when analyzed in frequency domain.

State Space Discretization

- ▷ The simplest derivative approximation is the **Forward Euler** method:

Forward Euler Derivative Approximation

$$\frac{dg(t)}{dt} \approx \frac{g(t + \Delta t) - g(t)}{\Delta t}$$

- ▷ Applying the conventional discrete equation notation:

$$g(t) \triangleq g_k, \quad g(t + n\Delta t) \triangleq g_{k+n}, \quad \Delta t \triangleq T_s$$

$$\left. \frac{dg}{dt} \right|_k = \frac{g_{k+1} - g_k}{T_s} \Rightarrow g_{k+1} = g_k + T_s \left. \frac{dg}{dt} \right|_k$$

State Space Discretization

Equivalent state equations

$$\underbrace{\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{U})}_{\text{Continuous}} \Rightarrow \underbrace{\mathbf{X}_{k+1} = f(\mathbf{X}_k, \mathbf{U}_k)}_{\text{Discrete}}$$

- ▷ Apply forward Euler approximation:

$$\left. \frac{d\mathbf{X}}{dt} \right|_k \approx \frac{\mathbf{X}_{k+1} - \mathbf{X}_k}{T_s} \Rightarrow \mathbf{X}_{k+1} = \mathbf{X}_k + T_s \left. \frac{d\mathbf{X}}{dt} \right|_k$$

- ▷ where $d\mathbf{X}/dt$ can be replaced by the continuous state equation:

$$\mathbf{X}_{k+1} = \mathbf{X}_k + T_s f(\mathbf{X}_k, \mathbf{U}_k)$$

Unicycle Navigation Model

▷ Apply the forward Euler approximation:

$$\begin{array}{ccc}
 \overbrace{\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{U})}^{\text{Continuous}} & \Rightarrow & \overbrace{\mathbf{X}_{k+1} = f(\mathbf{X}_k, \mathbf{U}_k)}^{\text{Discrete}} \\
 \left\{ \begin{array}{l} \dot{x} = v \cos \psi \\ \dot{y} = v \sin \psi \\ \dot{\psi} = \dot{\psi}_c \end{array} \right. & \Rightarrow & \left\{ \begin{array}{l} x_{k+1} = x_k + T_s v_k \cos \psi_k \\ y_{k+1} = y_k + T_s v_k \sin \psi_k \\ \psi_{k+1} = \psi_k + T_s \dot{\psi}_{c|k} \end{array} \right.
 \end{array}$$