ECE 495/595 Lecture Slides

Winter 2017

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Summary and Quick Links

These slides contain the following concepts:

- Statistics review (Slide 3)
- State space linearization (Slide 9)
- Extended Kalman filter algorithm (Slide 15)

Statistics

- The **mean** of a signal x is the average value over a number of samples N:

$$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$$

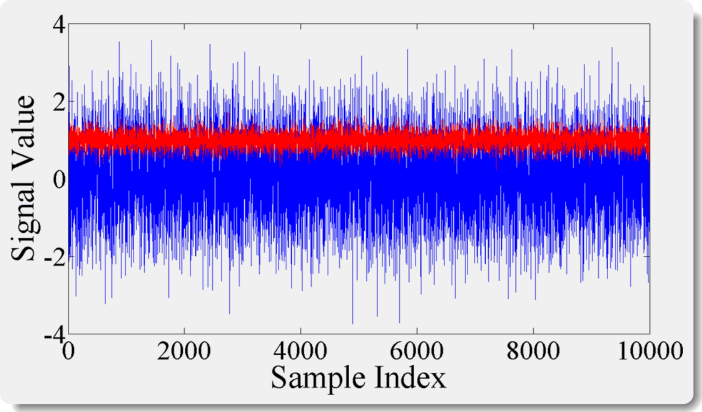
- The **variance** is the average squared deviation from the mean:

$$\sigma^2 = \frac{1}{N}\sum_{i+1}^{N} (x_i - \mu)^2$$

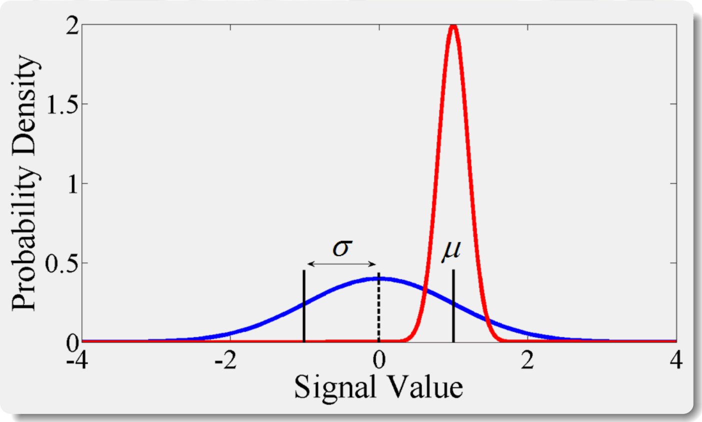
- The **standard deviation** is the square root of variance:

$$\sigma = \sqrt{\sigma^2}$$

Two Random Signals



Gaussian Distribution Functions



Statistics

- The notation $\frac{1}{N}\sum_{i=1}^{N}$ can be replaced by defining the **expected value** of a signal x:

$$E[x] \triangleq \frac{1}{N}\sum_{i=1}^{N} x_i$$

- Using the expected value notation, the mean and variance can be defined by:

$$\mu = E[x]$$

$$\sigma^2 = E[(x-\mu)^2]$$

Statistics

- In the multi-variate case, the mean is a vector that contains the average value of each individual variable:

$$\boldsymbol{\mu} = E[\mathbf{X}_i]$$

- The **covariance** matrix of the vector is adapted from the single variable case:

$$\boldsymbol{\Sigma} = E\left[\left(\mathbf{X}_i - \boldsymbol{\mu}\right)\left(\mathbf{X}_i - \boldsymbol{\mu}\right)^T\right]$$

- Diagonal elements are the variances of the individual variables.

- Off-diagonal elements are the cross-covariance between the different variables.

Statistics

- To see how covariance is a matrix, consider the two-variable case:

$$\begin{aligned}
\mathbf{\Sigma} &= E\left[\left(\begin{bmatrix} x_{1_i} - \mu_1 \\ x_{2_i} - \mu_2 \end{bmatrix} \begin{bmatrix} x_{1_i} - \mu_1 & x_{2_i} - \mu_2 \end{bmatrix}\right)\right] \\
&= E\left[\left(\begin{bmatrix} (x_{1_i} - \mu_1)^2 & (x_{1_i} - \mu_1)(x_{2_i} - \mu_2) \\ (x_{1_i} - \mu_1)(x_{2_i} - \mu_2) & (x_{2_i} - \mu_2)^2 \end{bmatrix}\right)\right]
\end{aligned}$$

- Expressing this in terms of single-variable standard deviations:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

State Space Linearization

- For use in the EKF algorithm, state space equations need to be linearized:

State Equation

$$\overbrace{\mathbf{X}_{k+1} = f(\mathbf{X}_k,\ \mathbf{U}_k)}^{\text{Non-linear}} \quad \Rightarrow \quad \overbrace{\mathbf{X}_{k+1} = \mathbf{A}_k\mathbf{X}_k + \mathbf{B}_k\mathbf{U}_k}^{\text{Linear}}$$

Observation Equation

$$\overbrace{\mathbf{Z}_k = h(\mathbf{X}_k)}^{\text{Non-linear}} \quad \Rightarrow \quad \overbrace{\mathbf{Z}_k = \mathbf{C}_k\mathbf{X}_k}^{\text{Linear}}$$

- Only possible if f and h are differentiable.

State Space Linearization

- $\mathbf{A}_k$ and $\mathbf{B}_k$ are derived from a first-order Taylor Series expansion of f.
- Perturb nominal value of $\mathbf{X}_{k+1}$:

$$\bar{\mathbf{X}}_{k+1} = \mathbf{X}_{k+1} + \delta\mathbf{X}_{k+1}$$

- Expand series around nominal value:

$$f(\bar{\mathbf{X}}_k,\ \bar{\mathbf{U}}_k) \approx f(\mathbf{X}_k,\ \mathbf{U}_k) + \left.\frac{\partial f}{\partial \mathbf{X}}\right|_{\mathbf{X}_k,\ \mathbf{U}_k} \left(\bar{\mathbf{X}}_k - \mathbf{X}_k\right) + \left.\frac{\partial f}{\partial \mathbf{U}}\right|_{\mathbf{X}_k,\ \mathbf{U}_k} \left(\bar{\mathbf{U}}_k - \mathbf{U}_k\right)$$

State Space Linearization

- By inspection:

$$\delta \mathbf{X}_{k+1} = \underbrace{\left.\frac{\partial f}{\partial \mathbf{X}}\right|_{\mathbf{X}_k,\ \mathbf{U}_k}}_{\mathbf{A}_k} \delta \mathbf{X}_k + \underbrace{\left.\frac{\partial f}{\partial \mathbf{U}}\right|_{\mathbf{X}_k,\ \mathbf{U}_k}}_{\mathbf{B}_k} \delta \mathbf{U}_k$$

- Therefore, the $\mathbf{A}_k$ matrix is the partial derivative of the state equation w.r.t. the state vector, evaluated at the current time k.
- $\mathbf{B}_k$ is the partial derivative w.r.t. the input vector, evaluated at time k.

State Space Linearization

- Applying a similar procedure to the observation equation:

$$\mathbf{C}_k = \left.\frac{\partial h}{\partial \mathbf{X}}\right|_{\mathbf{X}_k}$$

- The $\mathbf{C}_k$ matrix is the partial derivative of the observation equation w.r.t. the state vector, evaluated at time k.

State Space Linearization

- Consider an arbitrary state vector:

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$$

- Each state variable has an equation for its derivative:

$$\dot{x}_i = f_i(\mathbf{X}, \mathbf{U}), \quad i = 1, 2, \ldots, n$$

- These individual functions combine to form the entire state space model function f:

$$\dot{\mathbf{X}} = f(\mathbf{X},\ \mathbf{U}) = \begin{cases} \dot{x}_1 & = & f_1(\mathbf{X},\ \mathbf{U}) \\ \dot{x}_2 & = & f_2(\mathbf{X},\ \mathbf{U}) \\ & \vdots & \\ \dot{x}_n & = & f_n(\mathbf{X},\ \mathbf{U}) \end{cases}$$

State Space Linearization

- The partial derivative of f w.r.t. the state vector $\mathbf{X}$ is defined as:

$$\frac{\partial f}{\partial \mathbf{X}} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

- This partial derivative matrix is commonly referred to as a **Jacobian** matrix.

Extended Kalman Filter

- A Kalman filter observer operates on a linearized state space model of the system being observed.
- The filter generates an estimate $\hat{\mathbf{X}}$ of the state vector, along with an estimated covariance $\mathbf{P}$ of that estimate.
- The 'extended' part of the name refers to the consideration of the time-varying nature of the linearized state space matrices $\mathbf{A}$ and $\mathbf{C}$.

Extended Kalman Filter

- Accounts for uncertainty in the state and noise in the measurements by adding random noise to the model.
- It is assumed that the random noise is zero-mean and obeys a Gaussian distribution.
- Uncertainty is quantified by specifying the covariance of the random noise variables.

Extended Kalman Filter

- Corrupt discrete, linearized state model with a process noise vector $\mathbf{w}$:

$$\hat{\mathbf{X}}_{k+1} = \mathbf{A}_k \hat{\mathbf{X}}_k + \mathbf{w}_k$$

- Corrupt measurement function with a measurement noise vector $\mathbf{v}$:

$$\mathbf{y}_k = \mathbf{C}_k \hat{\mathbf{X}}_k + \mathbf{v}_k$$

- User must specify the process noise covariance matrix $\mathbf{Q}$ and the measurement noise covariance matrix $\mathbf{R}$

$$\mathbf{Q}_k = E\left[(\mathbf{w}_k - \overline{\mathbf{w}})(\mathbf{w}_k - \overline{\mathbf{w}})^T\right]$$
$$\mathbf{R}_k = E\left[(\mathbf{v}_k - \overline{\mathbf{v}})(\mathbf{v}_k - \overline{\mathbf{v}})^T\right]$$

Extended Kalman Filter

- The state of a Kalman filter whose underlying state vector is n elements is defined by two quantites:

State Estimate

$\hat{\mathbf{X}}_{k|k}$ – $n \times 1$ vector

Estimate Error Covariance

$\mathbf{P}_{k|k}$ – $n \times n$ matrix

- The subscript notation $a|b$ is read as "value at time a, based on information available at time b".
- Each iteration of the EKF will update each of these quantities.

Extended Kalman Filter

- At start-up, the Kalman filter must be initialized by:

Kalman Filter Initialization

- Initialize state estimate $\hat{\mathbf{X}}_0$
- Initialize covariance estimate $\mathbf{P}_0$
- Define process and measurement covariance $\mathbf{Q}$ and $\mathbf{R}$

- The initial error covariance $\mathbf{P}_0$ is typically set to an identity matrix.

- It is usually assumed that the state variables and sensor measurements are independent of each other, so the $\mathbf{Q}$ and $\mathbf{R}$ matrices are always diagonal.

Extended Kalman Filter

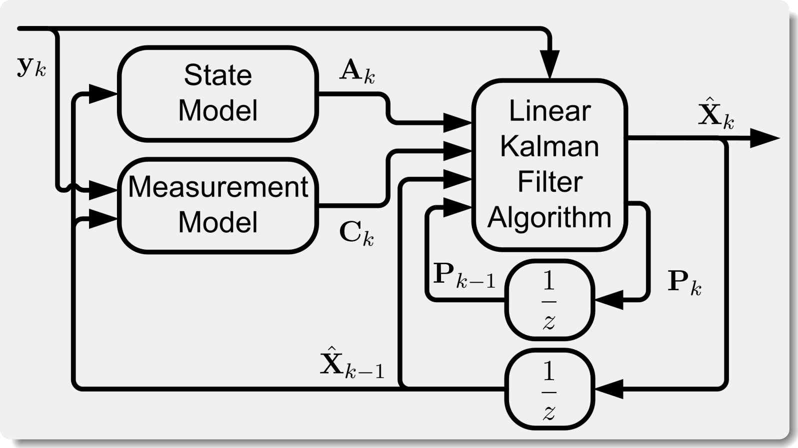
Prediction

- Update state Jacobian: $\mathbf{A}_k = (\partial f / \partial \mathbf{X})_{k-1|k-1}$
- Predict next state: $\hat{\mathbf{X}}_{k|k-1} = f(\hat{\mathbf{X}}_{k-1|k-1})$
- Predict next covariance: $\mathbf{P}_{k|k-1} = \mathbf{A}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{A}_{k-1}^T + \mathbf{Q}$

Update

- Compute residual covariance: $\mathbf{S} = \mathbf{C}_{k-1}\mathbf{P}_{k|k-1}\mathbf{C}_{k-1}^T + \mathbf{R}$
- Compute Kalman gain: $\mathbf{K} = \mathbf{P}_{k|k-1}\mathbf{C}_{k-1}^T\mathbf{S}^{-1}$
- Compute measurement error: $\epsilon = \mathbf{y}_k - h(\hat{\mathbf{X}}_{k|k-1})$
- Update state estimate: $\hat{\mathbf{X}}_{k|k} = \hat{\mathbf{X}}_{k|k-1} + \mathbf{K}\epsilon$
- Update covariance estimate: $\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}\mathbf{C}_{k-1})\,\mathbf{P}_{k|k-1}$

Extended Kalman Filter



Extended Kalman Filter

- Behavior and performance of a Kalman filter can be adjusted dramatically by tuning $\mathbf{Q}$ and $\mathbf{R}$.
- Consider a three-state system, where $\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$
- The $\mathbf{Q}$ matrix is the covariance of the state vector:

$$\mathbf{Q} = \begin{bmatrix} q_{x_1}^2 & 0 & 0 \\ 0 & q_{x_2}^2 & 0 \\ 0 & 0 & q_{x_3}^2 \end{bmatrix}$$

- A higher covariance in a particular state corresponds to a generally faster filter response in that variable.
- This is analogous to a change in the time constant of a low-pass filter.

Extended Kalman Filter

- If there are two measurements $\mathbf{Z} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T$

- The $\mathbf{R}$ matrix is the covariance of the measurement vector:

$$\mathbf{R} = \begin{bmatrix} r_{z_1}^2 & 0 \\ 0 & r_{z_2}^2 \end{bmatrix}$$

- A higher covariance in a particular measurement corresponds to higher noise in the sensor, which causes the filter to react less in response to a change in the measurement.

- Conversely, a lower r value indicates lower noise, thereby making the filter more responsive to that measurement.