ECE 495/595 Lecture Slides

Winter 2017

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Summary and Quick Links

These slides contain the following concepts:

Review of State Space Concepts

- State space is a method of describing the behavior of a system using a set of first-order differential equations.
- State space models consist of a state equation and an observation equation.
- In a linear state space model, the two equations are typically written as:

Linear State Equation

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU}$$

> State – $\mathbf{X}$
> Control signal – $\mathbf{U}$
> State matrix – $\mathbf{A}$
> Input matrix – $\mathbf{B}$

Linear Observation Equation

$$\mathbf{z} = \mathbf{CX} + \mathbf{DU}$$

> Observations – $\mathbf{z}$
> Measurement matrix – $\mathbf{C}$
> Feed-forward matrix – $\mathbf{D}$

Review of State Space Concepts

- When describing a non-linear system, the state space equations are written as:

Non-Linear State Equation

$$\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{U})$$

Non-Linear Observation Eqn.

$$\mathbf{z} = h(\mathbf{X}, \mathbf{U})$$

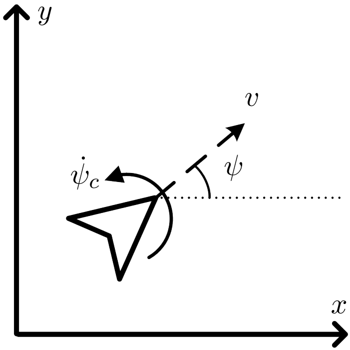
- Developing navigation algorithms for robots usually begins with representing the system in this form.

- For now, only the modeling of the state equation is considered. The observation equation will be revisited when discussing sensor fusion techniques.

Unicycle Navigation Model

- Describe how the vehicle's position (x, y) and heading (ψ) are affected by controlling the forward velocity (v) and yaw rate ($\dot{\psi}_c$)
- Express the system in non-linear state space form:

$$\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{U})$$



Unicycle Navigation Model

- First define the state vector:

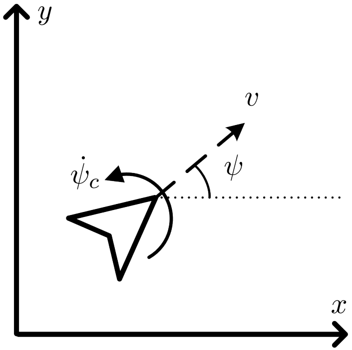
State Vector

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}$$

- ... and the control vector:

Control Vector

$$\mathbf{U} = \begin{bmatrix} v \\ \dot{\psi}_c \end{bmatrix}$$



Unicycle Navigation Model

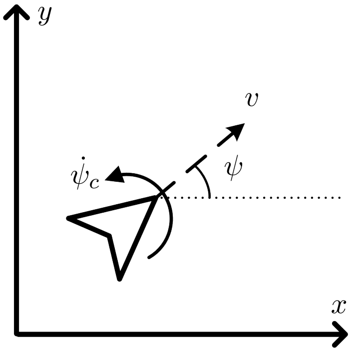
- Need to come up with equations for the derivatives of each state variable.

- Project v onto the x axis:

$$\dot{x} = v\cos\psi$$

- Project v onto the y axis:

$$\dot{y} = v\sin\psi$$



- The actual yaw rate is whatever the command is:

$$\dot{\psi} = \dot{\psi}_c$$

Unicycle Navigation Model

State Vector

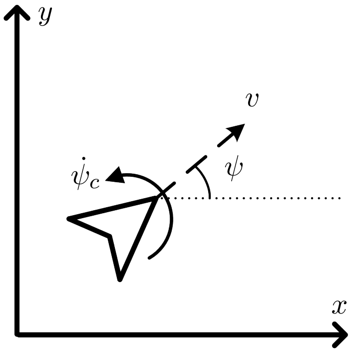
$$\mathbf{X} = \begin{bmatrix} x & y & \psi \end{bmatrix}^T$$

Control Vector

$$\mathbf{U} = \begin{bmatrix} v & \dot{\psi}_c \end{bmatrix}^T$$

State Function

$$f(\mathbf{X}, \mathbf{U}) = \begin{cases} \dot{x} & = & v \cos \psi \\ \dot{y} & = & v \sin \psi \\ \dot{\psi} & = & \dot{\psi}_c \end{cases}$$



Holonomic Navigation Model

- State vector remains the same:

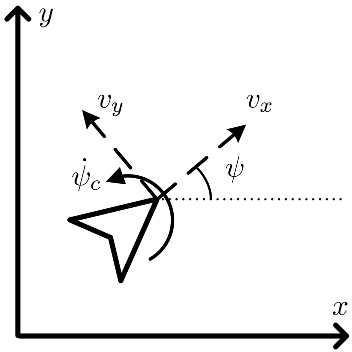
State Vector

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}$$

- The control vector has one more signal:

Control Vector

$$\mathbf{U} = \begin{bmatrix} v_x \\ v_y \\ \dot{\psi}_c \end{bmatrix}$$



- Holonomic vehicles have three degrees of freedom.

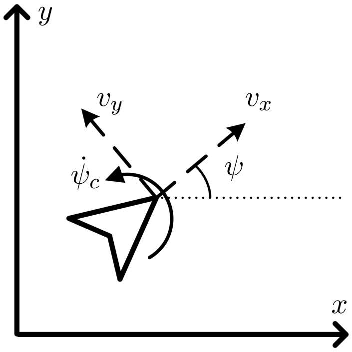
Holonomic Navigation Model

- Derive expressions for the derivatives of state variables.
- Project v_x and v_y onto the x axis:

$$\dot{x} = v_x \cos\psi - v_y \sin\psi$$

- Project v_x and v_y onto the y axis:

$$\dot{y} = v_x \sin\psi + v_y \cos\psi$$



- The actual yaw rate is whatever the command is:

$$\dot{\psi} = \dot{\psi}_c$$

Holonomic Navigation Model

State Vector

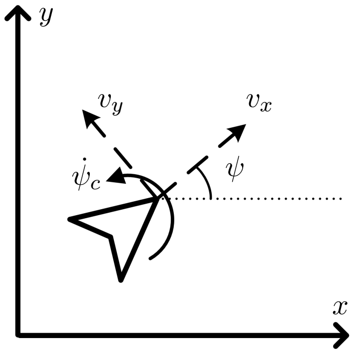
$$\mathbf{X} = \begin{bmatrix} x & y & \psi \end{bmatrix}^T$$

Control Vector

$$\mathbf{U} = \begin{bmatrix} v_x & v_y & \dot{\psi}_c \end{bmatrix}^T$$

State Function

$$f(\mathbf{X}, \mathbf{U}) = \left\{ \begin{array}{lcl} \dot{x} & = & v_x \cos\psi - v_y \sin\psi \\ \dot{y} & = & v_x \sin\psi + v_y \cos\psi \\ \dot{\psi} & = & \dot{\psi}_c \end{array} \right.$$



Bicycle Navigation Model

- The kinematic constraints of a car can be modeled simplistically by a bicycle navigation model.
- The state vector remains the same as the other vehicles, but the control vector becomes the combination of speed v and steering wheel angle α_s:

State Vector

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}$$

Control Vector

$$\mathbf{U} = \begin{bmatrix} v \\ \alpha_s \end{bmatrix}$$

Bicycle Navigation Model

- Recall the relationship between steering wheel angle, yaw rate and speed:

$$\alpha_s = \gamma \tan^{-1}\left(\frac{L\dot{\psi}}{v}\right)$$

- Rearranging this, the yaw rate $\dot{\psi}$ can be represented in terms of the control variables v and α_s, with the steering ratio γ and wheelbase L of the vehicle as constant parameters:

$$\dot{\psi} = \frac{v}{L}\tan\left(\frac{\alpha_s}{\gamma}\right)$$

Bicycle Navigation Model

State Vector

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ \psi \end{bmatrix}$$

Control Vector

$$\mathbf{U} = \begin{bmatrix} v \\ \alpha_s \end{bmatrix}$$

State Function

$$f(\mathbf{X}, \mathbf{U}) = \begin{cases} \dot{x} & = & v \cos \psi \\ \dot{y} & = & v \sin \psi \\ \dot{\psi} & = & \dfrac{v}{L} \tan \left(\dfrac{\alpha_s}{\gamma} \right) \end{cases}$$

State Space Discretization

- In order to implement a state space controller or observer in a computer, it must first be *discretized*.
- Discretization involves approximating the time derivative of the continuous state space equation.
- There are many techniques to discretize a system, each with different implications when analyzed in frequency domain.

State Space Discretization

- The simplest derivative approximation is the *Forward Euler* method:

Forward Euler Derivative Approximation

$$\frac{dg(t)}{dt} \approx \frac{g(t+\Delta t) - g(t)}{\Delta t}$$

- Applying the conventional discrete equation notation:

$$g(t) \triangleq g_k, \quad g(t+n\Delta t) \triangleq g_{k+n}, \quad \Delta t \triangleq T_s$$

$$\left.\frac{dg}{dt}\right|_k = \frac{g_{k+1} - g_k}{T_s} \Rightarrow g_{k+1} = g_k + T_s \left.\frac{dg}{dt}\right|_k$$

State Space Discretization

Equivalent state equations

$$\underbrace{\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{U})}_{\text{Continuous}} \quad \Rightarrow \quad \underbrace{\mathbf{X}_{k+1} = f(\mathbf{X}_k, \mathbf{U}_k)}_{\text{Discrete}}$$

- Apply forward Euler approximation:

$$\left.\frac{d\mathbf{X}}{dt}\right|_k \approx \frac{\mathbf{X}_{k+1} - \mathbf{X}_k}{T_s} \;\Rightarrow\; \mathbf{X}_{k+1} = \mathbf{X}_k + T_s \left.\frac{d\mathbf{X}}{dt}\right|_k$$

- where $d\mathbf{X}/dt$ can be replaced by the continuous state equation:

$$\mathbf{X}_{k+1} = \mathbf{X}_k + T_s f(\mathbf{X}_k,\ \mathbf{U}_k)$$

Unicycle Navigation Model

- Apply the forward Euler approximation:

$$\overbrace{\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{U})}^{\text{Continuous}} \quad \Rightarrow \quad \overbrace{\mathbf{X}_{k+1} = f(\mathbf{X}_k, \mathbf{U}_k)}^{\text{Discrete}}$$

$$\begin{cases} \dot{x} & = & v\cos\psi \\ \dot{y} & = & v\sin\psi \\ \dot{\psi} & = & \dot{\psi}_c \end{cases} \quad \Rightarrow \quad \begin{cases} x_{k+1} & = & x_k + T_s v_k \cos\psi_k \\ y_{k+1} & = & y_k + T_s v_k \sin\psi_k \\ \psi_{k+1} & = & \psi_k + T_s \dot{\psi}_{c|k} \end{cases}$$