An example of using linear algebra

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The presenting question is simple. Given an arbitrary charge distribution in vacuum, how do we calculate the resulting electric field E?

Define a vector $\mathbf{E}$ the electric field, a scalar ϕ the electric potential, and a scalar ρ the charge distribution, i.e. where the charge is. By definition, the electric field is the negative gradient of the electric potential,

$$-\nabla\phi = \mathbf{E}. \tag{1}$$

You must have seen a similar expression between the gravitational force and the gravitational potential from your high school. They are mathematically the same.

A physical law, called Maxwell's equations, says that in vacuum,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{2}$$

where ϵ_0 is just a physical constant that we do not need care too much. Substituting eq.[1] into eq.[2], we have

$$-\nabla^2\phi = \frac{\rho}{\epsilon_0}. \tag{3}$$

So far, I haven't told you what ∇ represents, but in 1-D space it is equivalent to the operation of computing the derivative, i.e. $\nabla \equiv \frac{d}{dx}$, so eq.[2] becomes a relatively more friendly form,

$$-\frac{d^2\phi}{dx^2} = \frac{\rho}{\epsilon_0}. \tag{4}$$

Now we have an ordinary differential equation, and what meaning does it have? It means that if you give me a function $\rho(x)$ that describes where charges are located in a 1-D space, I can solve the differential equation eq.[4] for $\phi(x)$, and tell you the electric field $\mathbf{E}(x)$ at any point in the 1-D space!

How do we solve this equation? Easy, you just need to integrate the equation twice to get an expression for ϕ! However, what if $\rho(x)$ is very nasty that you cannot integrate it by hand? One tricky way is to solve it NUMERICALLY, which almost always needs the aid of a computer. I will show you how.

Imagine a straight line of finite length L, and we can cut the line into pieces of equal lengths, say h. we now have $n = \frac{L}{h}$ pieces of line, and $N = n + 1$ nodes. Denote the position of nodes by x_0, x_1, ..., x_N, and the values of them are 0, h, ..., Nh. We can denote the electric potential $\phi(x)$ at the nodes as ϕ_0, ϕ_1, ..., ϕ_N, which are what we want to find. If you give me a charge distribution function $\rho(x)$, I can certainly calculate the charge at the nodes: $\rho(x_0)$, $\rho(x_1)$, ..., $\rho(x_N)$. Let us write them down as vectors for now, and we will use them later.

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \dots \\ x_N \end{pmatrix} = \begin{pmatrix} 0 \\ h \\ 2h \\ \dots \\ Nh \end{pmatrix}, \mathbf{\Phi} = \begin{pmatrix} \phi(x_0) \\ \phi(x_1) \\ \phi(x_2) \\ \dots \\ \phi(x_N) \end{pmatrix} = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \dots \\ \phi_N \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \rho(x_0) \\ \rho(x_1) \\ \rho(x_2) \\ \dots \\ \rho(x_N) \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \dots \\ \rho_N \end{pmatrix} \tag{5}$$

Recall the definition of a derivative:

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{y(x+\frac{h}{2}) - y(x-\frac{h}{2})}{h}, \tag{6}$$

and the second derivative:

$$\frac{d^2y}{dx^2} = \lim_{h\to 0} \frac{[y(x+h) - y(x)] - [y(x) - y(x-h)]}{h^2} \tag{7}$$

$$= \lim_{h\to 0} \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}. \tag{8}$$

In our discretised line, if we make the spacing h small enough, we can approximate the second derivative with

$$\frac{d^2\phi_i}{dx^2} \approx \frac{\phi(x_i+h) - 2\phi(x_i) + \phi(x_i-h)}{h^2} \tag{9}$$

$$\approx \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2}, \tag{10}$$

which is a discretised form of eq.[4].

Let me give you a picture of this. For the origin differential equation, you have a smooth curve and on every random point x_c on the curve, eq.[4] must be satisfied, i.e.

$$-\frac{d^2\phi}{dx^2}\bigg|_{x=x_c} = \frac{\rho(x_c)}{\epsilon_0}. \tag{11}$$

Now we have cut the line into pieces so we only concern the $(N+1)$ points. The functions that they must satisfy are

$$\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} = \frac{\rho_i}{\epsilon_0}, \quad i = 1, 2, ..., N-1. \tag{12}$$

The end points will have slightly different expressions, which I am expecting you to work out yourself. Now we have in total $(N+1)$ eq.[12]'s, and$(N+1)$ unknown ϕ_i's. The natural thought comes to mind is that we can assemble all these into a matrix form!

$$\mathbf{A}\mathbf{\Phi} = \mathbf{b}, \tag{13}$$

where $\mathbf{\Phi}$ and $\mathbf{b}$ are vectors in eq.[5], and $\mathbf{A}$ is an (N+1) by (N+1) matrix with all its diagonal elements -2, sub-diagonal elements 1, and 0 otherwise:

$$\mathbf{A} = \frac{\epsilon_0}{h^2} \begin{pmatrix} -2 & 1 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 1 & -2 \end{pmatrix}. \tag{14}$$

This matrix $\mathbf{A}$ is named the difference matrix, and is the favourite matrix of Gilbert Strang.

If you do the matrix multiplication to expand eq.[13], you will find that each row (other than the end rows) in the matrix equation is exactly

$$\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} = \frac{\rho_i}{\epsilon_0}, \quad i = 1, 2, ..., N-1. \tag{15}$$

Hence, solving eq.[13] gives a discretised solution to eq.[4]! This is usually the way to efficiently solve a more complex differential equation.

How do we solve eq.[13]? Learn linear algebra!