Comp 411 (sections 41& 42) **Numerical Methods with Programming**

Instructor: Yahya M. Masalmah, PhD

SECOND EXAM
(Take-home)

Wednesday, April 14, 2021 at 11:00AM

Due: Wednesday, April 14, 2021 before 11:59PM

Do these problems on your own. Do not discuss them with anyone. You may use whatever text, notes, web resources you want. You should reference resources you used.

GOOD LUCK!

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SCORE:

Question	Grade
1	/30
2	/40
3	/30
Total	/100

Name:-…………

Problem I (30 points): (Matrix Properties and Ill-condition systems)

1. considered the following example 3x3 matrix:

$$A = \begin{bmatrix} 8 & 2 & -10 \\ -9 & 1 & 3 \\ 15 & -1 & 6 \end{bmatrix}$$

(a) Find the following norms of the matrix

i. $||A||_\infty =$

$= \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}| = \max(|8| + |2| + |-10|, |-9| + |1| + |3|, |15| + |-1| + |6|)$

$||A||_\infty = \max(20, 13, 22) = \boxed{22}$

$\boxed{||A||_\infty = 22}$

ii. $||A||_{Frob}$

iii. $||A||_2$

iv. Find the eigenvalues of the matrix A

ii) $\|A\|_{Frob} = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{3} |a_{ij}|} = \sqrt{8^2 + 2^2 + 10 + 9^2 + 1 + 3^2 + 15^2 + 1^2 + 6^2}$

$= \sqrt{64 + 4 + 100 + 81 + 1 + 9 + 225 + 1 + 36}$

$\|A\|_{Frob} = 22.8$

iii) $\|A\|_2 = \sqrt{\lambda_{max}(A^*A)}$, $A^*A = \begin{pmatrix} 8 & -9 & 15 \\ 2 & 1 & -1 \\ -10 & 3 & 6 \end{pmatrix}\begin{pmatrix} 8 & 2 & -10 \\ -9 & 1 & 3 \\ 15 & -1 & 6 \end{pmatrix}$

$= \begin{pmatrix} 370 & -8 & -17 \\ -8 & 6 & -23 \\ -17 & -23 & 145 \end{pmatrix}$

$A^*A = 1.986,\ 147.6,\ 371.3$

$\lambda_{max} = 371.3$

$\|A\|_2 = \sqrt{371.3} = 19.26$

$= 19.26$

iv) Find the eigenValues of the matrix A

$|A - \lambda I| = 0 \Rightarrow (8-\lambda)\left[(1-\lambda)(6-\lambda) - (3)(-1)\right]$

$-2\left[-9 \times (6-\lambda) - (3 \times 15)\right]$

$-10\left[(-9 \times -1) - 15(1-\lambda)\right] = 0$

$\Rightarrow [8-\lambda)\left[\lambda^2 - 7\lambda + 9\right] - 2\left[9\lambda - 99\right] - 10\left[15\lambda - 6\right] = 0$

$\Rightarrow (8\lambda^2 - 56\lambda + 72) - (\lambda^3 - 7\lambda^2 - 9\lambda) - (18\lambda - 198) - \left[150\lambda - 60\right] = 0$

$\Rightarrow -\lambda^3 + 15\lambda^2 - 215\lambda + 320 = 0$

$\Rightarrow \lambda^3 - 15\lambda^2 + 215\lambda - 320 = 0$

$\Rightarrow \lambda = 6.7 \pm 12.9i$

$\boxed{\lambda = 1.56}$

(b) What is the condition number of this matrix? Use any convenient matrix norms in this calculation.

(c) Is the system ill-condition? Explain your answer.

b) the Condition of Matrix

$= \|A\| \, \|A^{-1}\|$

$\|A\| \, \|A^{-1}\| = 24.2$ [we took the infinity norm]

c) Since Condition norm < 100

So it is not ill Conditioned

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Problem II (40points) : Direct Methods

You are to solve the following set of equations directly using the LU decomposition (Crouts method): (You need to show all your steps)

$$[A]\{x\} = \begin{bmatrix} 4 & -1 & 1 \\ 8 & 3 & -1 \\ 3 & 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 10 \\ 9 \end{Bmatrix} = \{b\}$$

a. Find L and U with $u_{ii}=1$ (you need to derive the equations)

b. Use the obtained L and U to find the determinant of A.

c. Solve the system for x

d. Use the obtained L and U to find the inverse of A

e. Find the infinity norm of A and its inverse

f. Find the condition number. Is the system ill-condtion? Justify your answer.

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 & 1 \\ 8 & 3 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$ $\begin{bmatrix} 100 \\ 2 \\ \end{bmatrix}$

using Gauss elimination method

$R_2 \leftarrow R_2 - 2R_1 \quad (\therefore \lambda_{21} = 2)$

$A = \begin{bmatrix} 4 & -1 & 1 \\ 8 & 3 & -1 \\ 3 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 4 & -1 & 1 \\ 0 & 5 & -3 \\ 3 & 1 & 1 \end{bmatrix} \xrightarrow[\lambda_{31} = 0.75]{R_3 \to R_3 - 0.75 R_1} \begin{bmatrix} 4 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 1.75 & 0.25 \end{bmatrix}$

$(\lambda_{31} = 0.35)$

$R_3 \to R_3 - (0.35) R_2$

$U = \begin{bmatrix} 4 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 1.3 \end{bmatrix}$

L is Just made up of the multiplier we used in ecuasion elimination with 1° an the diagonal.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0.75 & 0.35 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -1 & 1 \\ 8 & 3 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0.75 & 0.35 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 1.3 \end{bmatrix} = LU$$

b) $A = LU$

$$\det(A) = \det(LU) = \det(L)\ \det(U)$$
$$= 1.(4 \times 5 \times 1.3)$$

determinant of upper & lower triangle matrix is product of diagonal entry.

$$\det(A) = 20 \times \frac{13}{10} = 26 \Rightarrow \boxed{\det(A) = 26}$$

c) $Ax=b \Rightarrow LUx\ b$

Now let $Ux=y \Rightarrow Ly=b$

$$Ly=b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0.75 & 0.35 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 9 \end{bmatrix}$$

$\Rightarrow y_1 = 6,\ 2y_1 + y_2 = 10,\ 0.75y_1 + 0.35y_2 + y_3 = 9$

$\Rightarrow y_2 = 10 - 2y_1 = 10 - 12 = -2 \Rightarrow \boxed{y_2 = -2}$

$\Rightarrow \boxed{y_3 = 5.2}$

$$\text{Now } Ux = y \Rightarrow \begin{bmatrix} 4 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 1.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 5.2 \end{bmatrix}$$

$\Rightarrow 4x_1 - x_2 + x_3 = 6$

$5x_2 - 3x_3 = -2$

$1.3x_3 = 5.2$

$x_3 = \dfrac{5.2}{1.3} = 4$; $\boxed{x_3 = 4}$

$5x_2 = -2 + 3x_3 = -2 + 12 = 10$

$\boxed{x_2 = 2}$

$4x_1 = 6 + x_2 - x_3$

$4x_1 = 6 + 2 - 4 = 4$

$x = 1$

Hence solution of above system is

$(x_1, x_2, x_3) = (1, 2, 4)$

d) $A = LU \Rightarrow A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$

$$A^{-1} = \begin{bmatrix} 4 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 1.3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0.75 & 0.35 & 1 \end{bmatrix}^{-1}$$

$$U^{-1} = \begin{bmatrix} 1/4 & \frac{1}{20} & \frac{-1}{13} \\ 0 & \frac{1}{5} & \frac{6}{13} \\ 0 & 0 & \frac{10}{13} \end{bmatrix} \qquad L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{1}{20} & \frac{-7}{20} & 1 \end{bmatrix}$$

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{20} & \frac{-1}{13} \\ 0 & \frac{1}{5} & \frac{6}{13} \\ 0 & 0 & \frac{10}{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{1}{20} & \frac{-7}{20} & 1 \end{bmatrix} = \begin{bmatrix} 2/13 & \frac{1}{13} & \frac{-1}{13} \\ \frac{-11}{26} & \frac{1}{26} & \frac{6}{13} \\ \frac{-1}{26} & \frac{-7}{26} & \frac{10}{13} \end{bmatrix}$$

(7.1)

e) $\|A\|_\infty = \max\limits_{1 \le i \le n} \sum\limits_{j=1}^{n}$

$= \max\{4+1+1, 8+3+1, 3+1+1\}$

$= \max\{6, 12, 5\}$

$\therefore \|A\|_\infty = 12$

$\therefore$ Infinity norm of A is 12

Now $\|A^{-1}\|_\infty = \max\left\{\frac{4+2+2}{26}, \frac{11+1+12}{26}, \frac{1+7+20}{26}\right\}$

$= \max\left\{\frac{8}{26}, \frac{24}{26}, \frac{28}{26}\right\}$

$= \frac{28}{26}$

$\|A^{-1}\|_\infty = \frac{14}{13}$

$\therefore$ Infinity norm of A inverse of A is $\frac{14}{13}$

F) Consider the system $[A]\{x\} = \{b\}$

That is $\begin{bmatrix} 4 & -1 & 1 \\ 8 & 3 & -1 \\ 3 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 6 \\ 10 \\ 9 \end{Bmatrix}$

$$A = \begin{bmatrix} 4 & -1 & 1 \\ 8 & 3 & -1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$|A| = 4(3+1) + 1(8+3) + 1(8-9)$$
$$= 16 + 11 - 1$$
$$= 26$$

$$\text{adj } A = \begin{bmatrix} 4 & -11 & -1 \\ +2 & 1 & -7 \\ -2 & +12 & 20 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & -2 \\ -11 & 1 & 12 \\ -1 & -7 & 20 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj } A$$

$$= \frac{1}{26} \begin{bmatrix} 4 & 2 & -2 \\ -11 & 1 & 12 \\ -1 & -7 & 20 \end{bmatrix}$$

$$\|A\| = \max\left[\frac{8}{26}, \frac{24}{26}, \frac{28}{26}\right\} = \frac{28}{26} = 1.077$$

Conditional number $K(A) = \|A\| \, \|A^{-1}\|$

$$= 12 \times 1.077$$

$$\approx 13$$

So Condition number of A is $K(A) = 13$
since $K(A) = 13$ is larger than! the system is $\boxed{\text{ill - Conditioned}}$

Problem III (30points): Iterative methods

You are to solve the following set of equations iteratively by the Gauss-Seidel method:

$$[A]\{x\}=\begin{bmatrix}4 & 0 & 1\\ 1 & 3 & 0\\ 0 & 1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}=\begin{Bmatrix}8\\ 6\\ 4\end{Bmatrix}=\{b\}$$

(a) Will the Gauss-Seidel method converge for this problem? Justify your answer.

(b) Rewrite the system in the form of $\mathbf{x}=\mathbf{Cx}+\mathbf{d}$, identify C, and d.

(c) Perform (only) two (2) full iterations numerically, and calculate the approximate error of each iteration.

(d) Solve the previous system using SOR with $\lambda = 1.5$ (Only Two iterations), and calculate the approximate error of each iteration. Comment on your results. Compare the result you obtained using SOR with those you obtained using Gauss-Seidel. Do you think different value of λ **will improve the results? Justify your answer.**

$$[A]\{x\}=\begin{bmatrix}4 & 0 & 1\\ 1 & 3 & 0\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix}=\begin{Bmatrix}8\\ 6\\ 4\end{Bmatrix}=\{b\}$$

$$\Rightarrow 4X_1 + 0X_2 + X_3 = 8$$
$$X_1 + 3X_2 + 0X_3 = 6$$
$$0X_1 + X_2 + 2X_3 = 4$$

$$(X_1)_{x+1} = \frac{1}{4}\left(8 - 0(X_2)_x - (X_3)_x\right)$$

$$(X_2)_{x+1} = \frac{1}{3}\left(6 - (X_1)_{x+1} - 0(X_3)_x\right)$$

$$(X_3)_{x+1} = \frac{1}{2}\left(4 - (X_2)_{x+1} - 0(X_1)_{x+1}\right)$$

initial gaus $(X_1, X_2, X_3) = (0,0,0)$

1st aproximation

$(X_1)_1 = \frac{1}{4}(8-0-0) = \frac{1}{4}(8) = 2$

$(X_2)_1 = \frac{1}{3}(6-2) = \frac{1}{3}(4) = 1.33$

$(X_3)_1 = \frac{1}{2}(4-1.3333) = \frac{1}{2}(2.66667) = 1.3333$

2nd Aproximation

$(X_1)_2 = \frac{1}{4}(8-1.3333) = \frac{1}{4}(6.66667) = 1.66667$

$(X_2)_2 = \frac{1}{3}(6-1.68056) = \frac{1}{3}(4.3333) = 1.4444$

$(X_3)_3 = \frac{1}{2}(4-1.44444) = \frac{1}{2}(2.5556) = 1.2778$

3rd Aproximation

$(X_1)_3 = \frac{1}{4}(8-1.27778) = \frac{1}{4}(6.72222) = 1.68056$

$(X_2)_3 = \frac{1}{3}(6-1.68056) = \frac{1}{3}(4.31944) = 1.43981$

$(X_3)_3 = \frac{1}{2}(4-1.43981) = \frac{1}{2}(2.56019) = 1.28009$

4th Aproximation

$(X_1)_4 = \frac{1}{4}(8-1.28009) = \frac{1}{4}(6.71991) = 1.67998$

$(X_2)_4 = \frac{1}{3}(6-1.67998) = \frac{1}{3}(4.32002) = 1.44001$

$(X_3)_4 = \frac{1}{2}(4-1.44001) = \frac{1}{2}(2.55999) = 1.28$

Solution by Gauss seidel method

$X_1 = 1.67998 \simeq 1.68$

$X_2 = 1.44001 \approx 1.44$

$X_3 = 1.28 \approx 1.28$

a) Yes Gauss-Seidel method Converges For this Problem as we get $X_1 \simeq 1.68$ $X_2 \approx 1.44$ $X_3 \simeq 1.28$ Hence we can say that Gauss-Seidel method Converges For this.

b) $X = CX + d \Rightarrow \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = C \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ We can write many diferent values For c & d. suppose if $\boxed{C = \frac{1}{2}}$ then

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} -8 \\ -6 \\ -4 \end{bmatrix}$$

$$d = \begin{bmatrix} 0.84 \\ 0.72 \\ 0.64 \end{bmatrix}$$

c) Error of 3rd and 4th iterations

$|1.6817 - 1.68061|, |1.66497 - 1.44497 - 1.444502|, |1.2775 - 1.27729|$

$\Rightarrow \boxed{0.0011, 0.000432, 0.00621}$ ∴ su required errors

d) $X_1 = 1$, $X_2 = 1$ $X_3 = 1$

$X_1 = 2 - 1/4 X_3$

$X_2 = 2 - 1/3 X_1$

$X_3 = 2 - 1/2 X_2$

First iteration $X_1 = 1.7500$ $X_2 = 1.6667$ $X_3 = 1.5000$

Second Iteration $X_1 = 1.5625$ $X_2 = 1.4444$ $X_3 = 1.6250$

third Iteration $X_1 = 1.6094$ $X_2 = 1.5185$ $X_3 = 1.4583$