Homework #3
Norms & Iterative Methods
Numerical Analysis I - Math 4340

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1 Problems

- **Problem 11.13: Do parts (a) and (b) using 1-, 2-, and ∞-norm.**
- **Problem 12.3: Solve the equation system for solutions of accuracy of 5 significant digits using Gauss-Seidel method**
  $10x_1 + 2x_2 - x_3 = 27$
  $-3x_1 - 6x_2 + 2x_3 = -61.5$
  $x_1 + x_2 + 5x_3 = -21.5$
- **Problem 12.4: Solve the equation system for solutions of accuracy of 5 significant digits using Jacobi's method**
  $10x_1 + 2x_2 - x_3 = 27$
  $-3x_1 - 6x_2 + 2x_3 = -61.5$
  $x_1 + x_2 + 5x_3 = -21.5$

2 Norms

2.1 Problem 11.13

(a) Determine the matrix inverse and condition number for the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(b) Repeat (a) but change a_{33} slightly to 9.1

- **Determine the matrix inverse and condition number for A**

  In this case, matrix A is singular since its determinant is 0. Therefore, we cannot compute the inverse and neither the condition numbers for each norm.

```
--> A = [1,2,3;4,5,6;7,8,9]
 A  =

   1.   2.   3.
   4.   5.   6.
   7.   8.   9.


--> inv(A)
Warning :
matrix is close to singular or badly scaled. rcond = 2.2028E-18
 ans  =

   3.153D+15  -6.305D+15   3.153D+15
  -6.305D+15   1.261D+16  -6.305D+15
   3.153D+15  -6.305D+15   3.153D+15
```

- **Repeat (a) but change a_{33} slightly to 9.1**

  Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9.1 \end{bmatrix}$

  Now, recall that to obtain the condition number of a matrix, it is computed as:
  $Cond[A] = \parallel A \parallel_x * \parallel A^{-1} \parallel_x$, where x is the norm ($x = 1, 2, \infty, ...$)

  First, we must compute the inverse, which in this case, let
  $B = inv(A) = \begin{bmatrix} 8.3333333333333 & -19.333333333333 & 10 \\ -18.666666666667 & 39.666666666667 & -20 \\ 10 & -20 & 10 \end{bmatrix}$

- **1- norm**
  Recall the 1-norm is computed as the largest sum of the absolute values per column. In this case, $\parallel A \parallel_1 = 18.1$, and $\parallel B \parallel_1 = 79$.
  Therefore, $Cond[A]_1 = 18.1 * 79 = 1429.9$

- **2- norm**
  Recall the 2-norm is computed as the square root of the largest eigenvalue produced from $A^T * A$. In this case, let $C = A^T * A = \begin{bmatrix} 66 & 78 & 90.7 \\ 78 & 93 & 108.8 \\ 90.7 & 108.8 & 127.81 \end{bmatrix}$

  Computing the eigenvalues as $det(C - \lambda I) = 0$ results in:
  $\lambda_1 = 0.00028866767$, $\lambda_2 = 1.0912041027091$, $\lambda_3 = 285.71850722962$
  In this case, $\sqrt{\lambda_3 = 285.71850722962} = 16.903209968217$. Therefore, $\parallel A \parallel_2 = 16.903209968217$

  Now, let $D = inv(C) = \begin{bmatrix} 543.22222222111 & -1122.4444444421 & 569.99999999883 \\ -1122.4444444422 & 2321.8888888841 & -1179.9999999976 \\ 569.99999999884 & -1179.9999999976 & 599.99999999878 \end{bmatrix}$

  Computing the eigenvalues as $det(D - \lambda I) = 0$ results in:
  $\lambda_1 = 3464.1911923139$, $\lambda_2 = 0.9164188418256$, $\lambda_3 = 0.0034999482872$
  In this case, $\sqrt{\lambda_1 = 3464.1911923139} = 58.857380100663$. Therefore, $\parallel A^{-1} \parallel_2 = 58.857380100663$

  Finally, $Cond[A]_2 = 16.903209968217 * 58.857380100663 = 994.87865402166$

- **∞- norm**
  Recall the ∞-norm is computed as the largest sum of the absolute values per row. In this case, $\parallel A \parallel_\infty = 24.1$, and $\parallel B \parallel_\infty = 78.333333333333$.
  Therefore, $Cond[A]_\infty = 24.1 * 78.333333333333 = 1887.8333333333$

3 Iterative Methods

3.1 Problem 12.3

Solve the equation system for solutions of accuracy of 5 significant digits.

- **Use Gauss-Seidel Method**

  In this case, we have $A = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$.

  Now, let $x = [0, 0, 0]'$ and $U = triu(A) - diag(diag(A)) = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

  Now, if we let $x = inv(A - U) * (b - U * x)$, and we keep iterating until the result is accurate to 5 significant digits, then we have:

  $x =$ 2.7 8.9 -6.62

  $x =$ 0.258 7.9143333333333 -5.9344666666667

  $x =$ 0.5236866666667 8.0100011111111 -6.0067375555556

  $x =$ 0.4973260222222 7.999091137037 -5.9992834318519

  $x =$ 0.5002534294074 8.0001121413457 -6.0000731141506

  $x =$ 0.4999702603158 7.9999904984586 -5.9999921517549

  $x =$ **0.5000026851328 8.0000012735153 -6.0000007917296**

  As we could see, it took 7 iterations with $x = [0, 0, 0]'$ as the initial value. However, if we now employ a function, defined as follows:

```
function [xr, ns] = GaussSeidel(AMatrix, BMatrix)
A = AMatrix;
b = BMatrix;

x = b;
U = triu(A)-diag(diag(A));

xr = x*0;
ns = 0;

while norm(x - xr, 2) > 10^(-6),
    xr = x
    x = inv(A - U)*(b -U*x)
    ns = ns + 1
end



end
```

Moreover, and if we let now $x = b$, then we have the following output:

[xr, ns] = GaussSeidel(A, b)
ns = 11 (number of iterations)

xr = **0.5000001421183 7.9999999090598 -6.0000000102356**

Rounding both results to an accuracy of 5 significant digits, we now have:
$x_1 = 0.50000$, $x_2 = 8.0000$, $x_3 = -6.0000$

3.2 Problem 12.4

Solve the equation system for solutions of accuracy of 5 significant digits.

- **Use Jacobi Method**

  In this case, we have $A = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix}$.

  Now, let $x = [0, 0, 0]'$, $D = diag(diag(A)) = \begin{bmatrix} 10 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

, $IVD = diag(diag(A).^{-1}) = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & -0.1666666666667 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$

Finally, $x = IVD * (b - (A - D) * x)$, and we keep iterating until the result is accurate to 5 significant digits, then we have:

x = 2.7 10.25 -4.3

x = 0.22 7.4666666666667 -6.89

x = 0.5176666666667 7.8433333333333 -5.8373333333333

x = 0.5476 8.0453888888889 -5.9722

x = 0.4937022222222 7.9854666666667 -6.0185977777778

x = 0.5010468888889 7.9969496296296 -5.9958337777778

x = 0.5010266962963 8.0008652962963 -5.9995993037037

x = 0.4998670103704 7.9996202172839 -6.0003783985185

x = 0.5000381166914 7.9999403619753 -5.9998974455309

x = 0.5000221830519 8.0000151264774 -5.9999956957333

x = **0.4999974051312 7.9999903432296 -6.0000074619058**

As we could see, it took 11 iterations with $x = [0, 0, 0]'$ as the initial value. However, if we now employ a function, defined as follows:

```
function [xr, ns] = Jacobi(Amatrix, BMatrix)
A = Amatrix;
b = BMatrix;

x = b;
D = diag(diag(A));
IVD = diag(diag(A).^(-1));

xr = x * 0;
ns = 0;

while norm(x - xr,2) > 10^(-6),
    xr = x
    x = IVD*(b - (A-D)*x)
    ns = ns + 1
end


end
```

Moreover, and if we let now $x = b$, then we have the following output:

[xr, ns] = Jacobi(A, b)
ns = 16 (number of iterations)

xr = **0.5000005413866 7.9999990547645 -5.999999912233**

Rounding both results to an accuracy of 5 significant digits, we now have:
$x_1 = 0.50000$, $x_2 = 8.0000$, $x_3 = -6.0000$