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Project 2: Fourier Analysis

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MATH 4340 – Numerical Analysis I

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Contents

Introduction

Fourier Analysis is based on the premise that a more complicated function can be represented as a sum of simple trigonometric functions, since these are periodic. Recall a periodic function *f(t)* is one for which *f(t) = f(t + T)*, where *T* a constant called the "period". Fourier analysis is extensively used in fields such as physics and engineering, where periodic phenomena mostly occur, such as acoustic waves. Moreover, it has an extensive array of applications in areas such as communications, astronomy, optics, etc. In the case of mathematics, in numerical analysis and modeling.

Mathematical Tools

The Fourier decomposition of a given function *f(x)*, defined over [-*L, L*], is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

As we see, there are three unknowns, a_0, a_n, b_n. To compute their values, it goes as follows:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx\,, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx\,, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

Problem

Consider a function whose definition on [-1,1] is defined as:

$$f(x) = 5x, -1 < x \leq 0 \ \wedge \ -5x + 2, 0 < x \leq 1$$

Solution:

1) Find a Fourier Series expansion of *f(x)* on [-1,1]. Provide the formulae for all coefficients. Provide details of calculation.

Now, since it's a piecewise defined function, to compute the coefficients, it's as follows:

$$a_0 = \frac{1}{2}\int_{-1}^{0} 5x\, dx + \frac{1}{2}\int_{0}^{1} -5x + 2\, dx \quad , a_n = \int_{-1}^{0} 5x \cos n\pi x\, dx + \int_{0}^{1} (-5x+2)\cos n\pi x\, dx \ ,$$

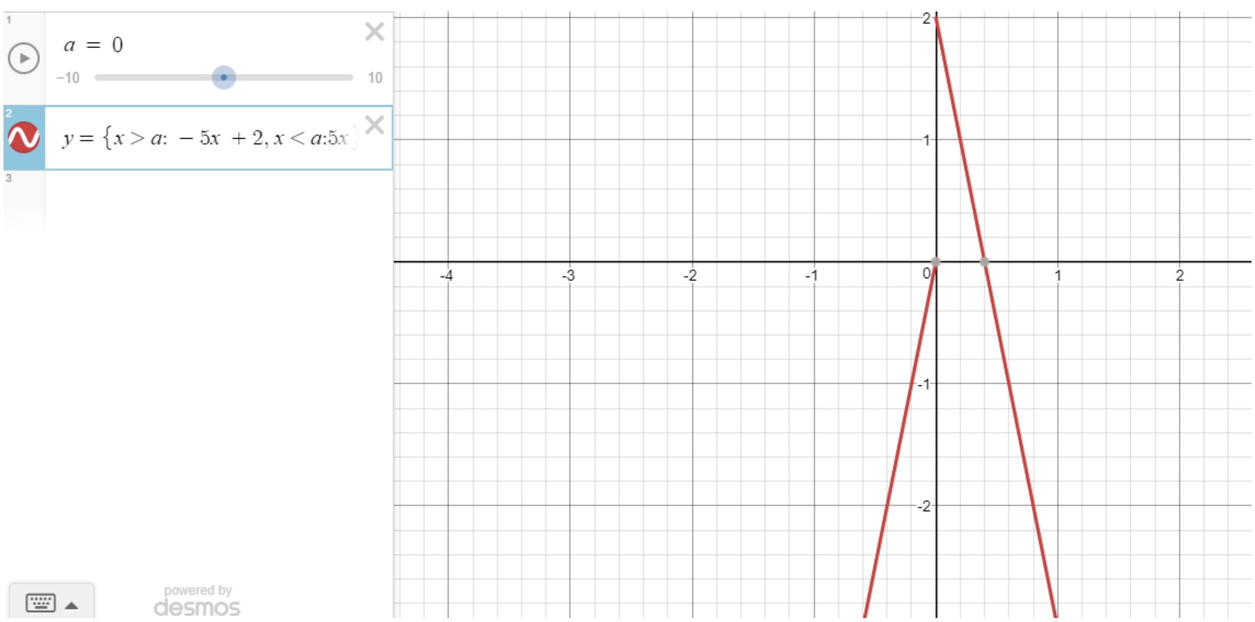
$$b_n = \int_{-1}^{0} 5x \sin n\pi x\, dx + \int_{0}^{1} (-5x+2)\sin n\pi x\, dx$$

In this case, we arrive to the following results:

$$a_0 = \frac{-3}{2}, a_n = \frac{10}{n^2\pi^2}(1-\cos n\pi), b_n = \frac{2}{n\pi}(1-\cos n\pi)$$

As for a_n and b_n, we have that when $n = 2k$, it gives us a value of 0, and when $n = 2k+1$ it gives an actual value. Therefore, doing the substitutions, we now have that the Fourier series is:

$$f(x) = \frac{-3}{2} + \sum_{k=0}^{\infty}\left(\frac{20}{(2k+1)^2\pi^2}\cos(2k+1)\pi x + \frac{4}{(2k+1)\pi}\sin(2k+1)\pi x\right)$$



Function created to graph the function on the given domain

MyFourierSers.m | FouriеProject.m | *y=f(x)

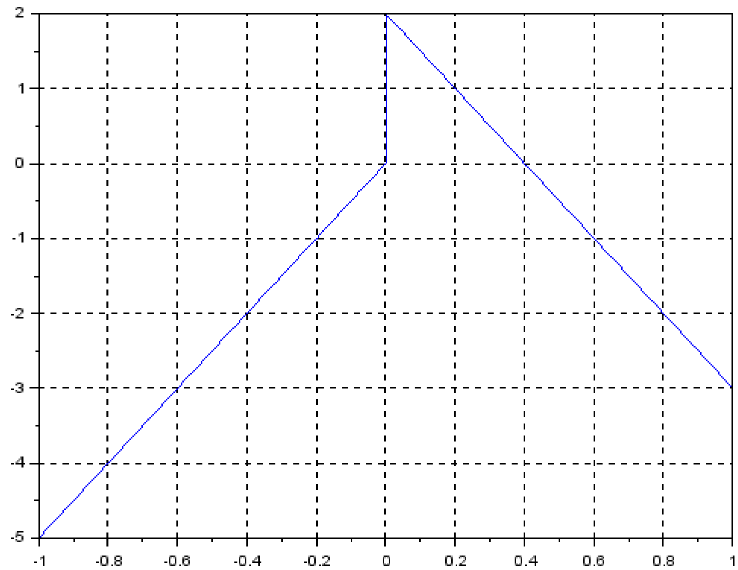
```
function y=f(t),

    y1 = (5*t).*(-1 < t & t <= 0);
    y2 = (-5*t + 2).*(0 < t & t <= 1);

    y = y1 + y2

endfunction
```

Code to graph the Fourier Series

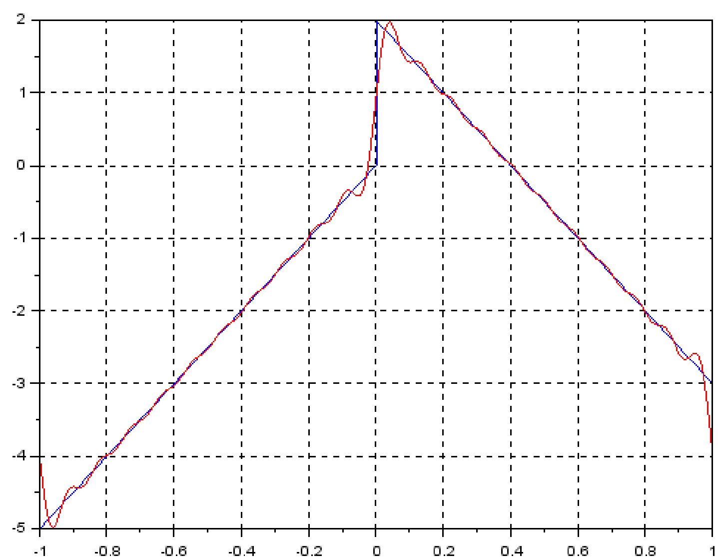
MyFourierSers.m | *FourieProject.m | y=f(x)

```
//%   This example shows Fourier analysis for a periodic piecewise function//
//%   defined over [-3,3] with period 2
//%   f(x) = 5x if -1<x<=0; -5x + 2 if 0<x<=1
x=[-1:0.0001:1];

f=x*0-3/2;
for k=0:10,
  f = f + 20/((2*k+1).^2*%pi^2)*cos((2*k + 1)*%pi*x) + 4/((2*k+1)*%pi)*sin((2*k + 1)*%pi*x);
end
plot(x,f,'r-',x-2,f,'r-',x-2,f,'r-',x+2,f,'r-',x+2,f,'r-')

for k=11:50,
  f = f + 20/((2*k+1).^2*%pi^2)*cos((2*k + 1)*%pi*x) + 4/((2*k+1)*%pi)*sin((2*k + 1)*%pi*x);
end
plot(x,f,'m-',x-2,f,'m-',x-2,f,'m-',x+2,f,'m-',x+2,f,'m-')

for k=51:250,
  f = f + 20/((2*k+1).^2*%pi^2)*cos((2*k + 1)*%pi*x) + 4/((2*k+1)*%pi)*sin((2*k + 1)*%pi*x);
end
plot(x,f,'g-',x-2,f,'g-',x-2,f,'g-',x+2,f,'g-',x+2,f,'g-')
```

Original function, plotted

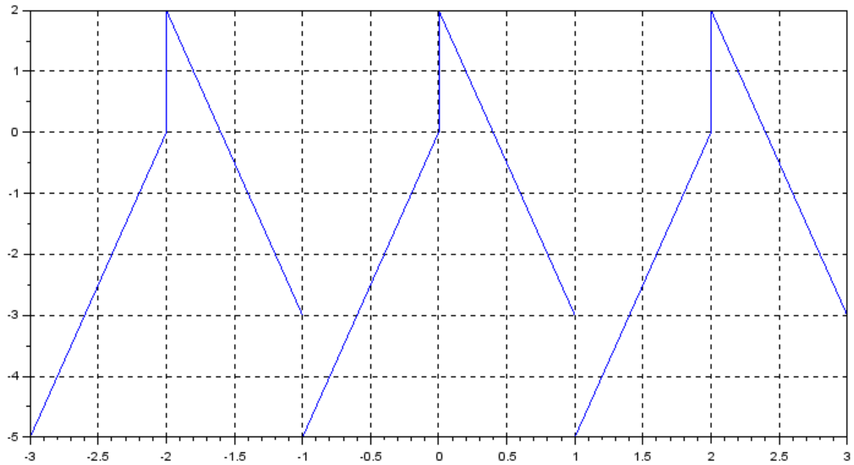


Original function plotted, with 11 terms of Fourier Series

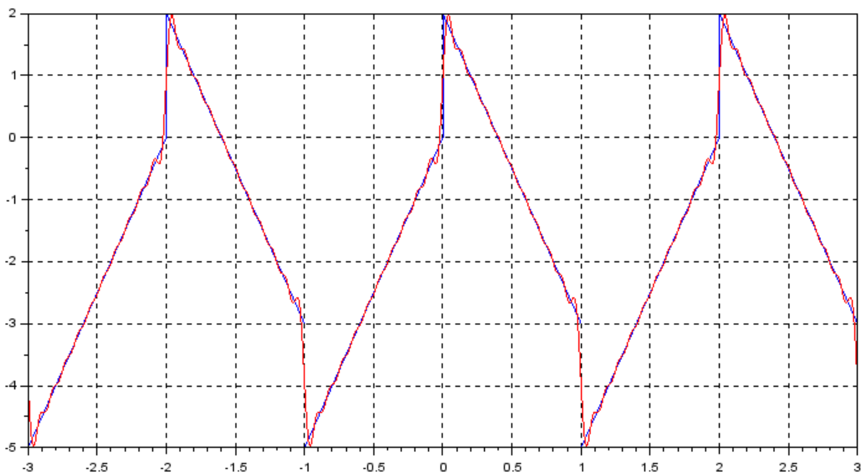


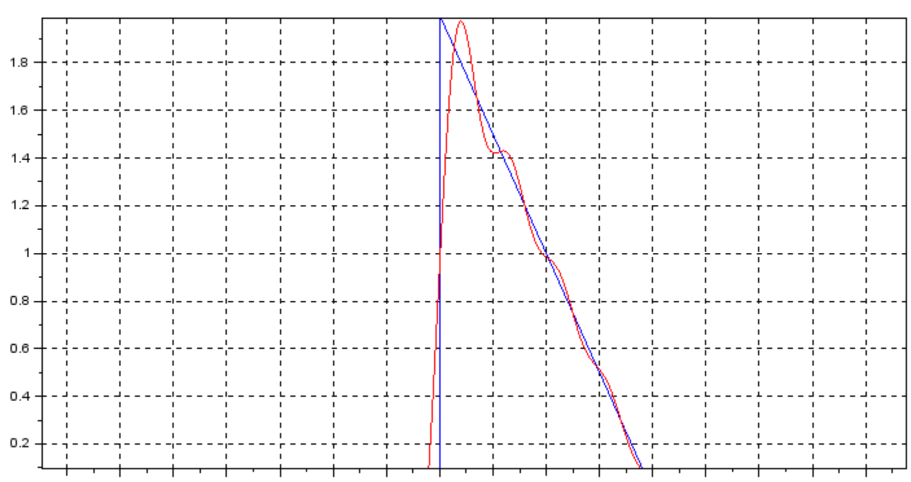
2) Plot the partial sum of the Fourier series with the first 11 terms (i.e. 1 constant term, 10 sine terms, and 10 cosine terms) on [−3, 3]. Plot *f(x)* on [−3, 3] in the same figure for comparison purpose.

Original function plotted on [-3,3]



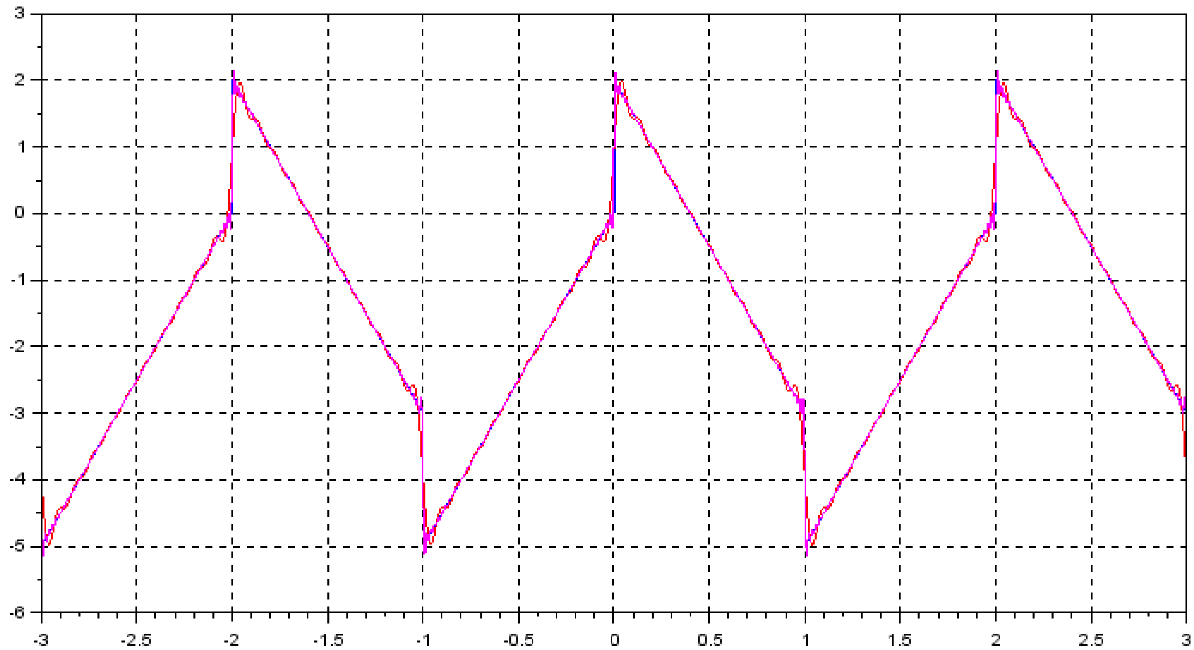
Function with 11 Fourier Series terms, and zoom



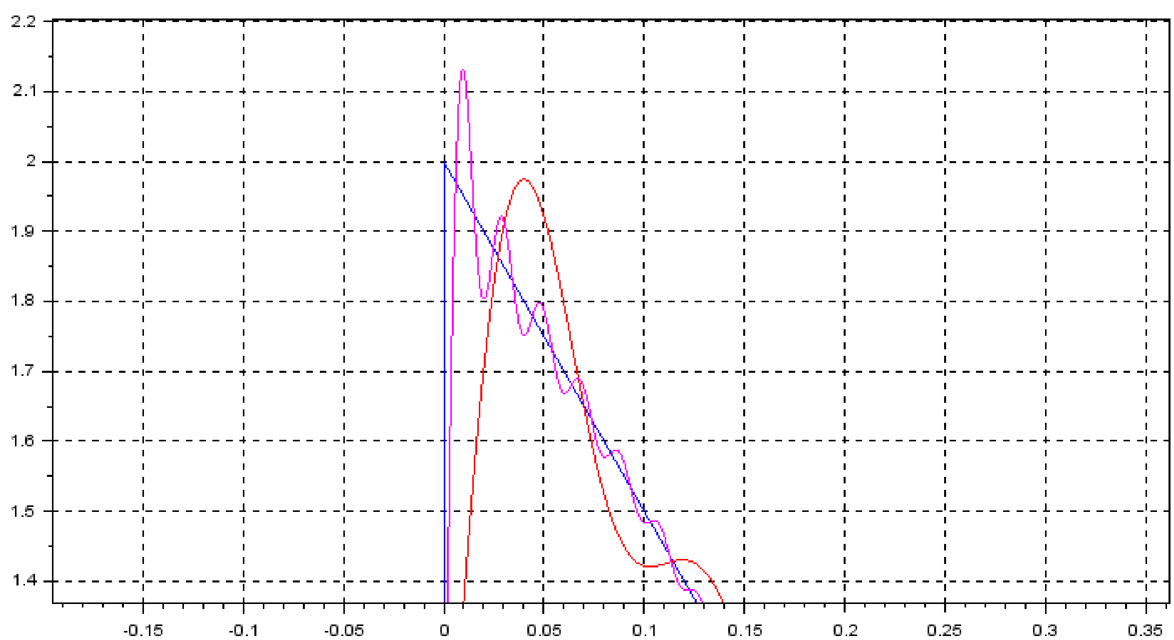


3) Repeat Step 2 with partial sums of the Fourier series with the first 51 terms and the first 251 terms, respectively.

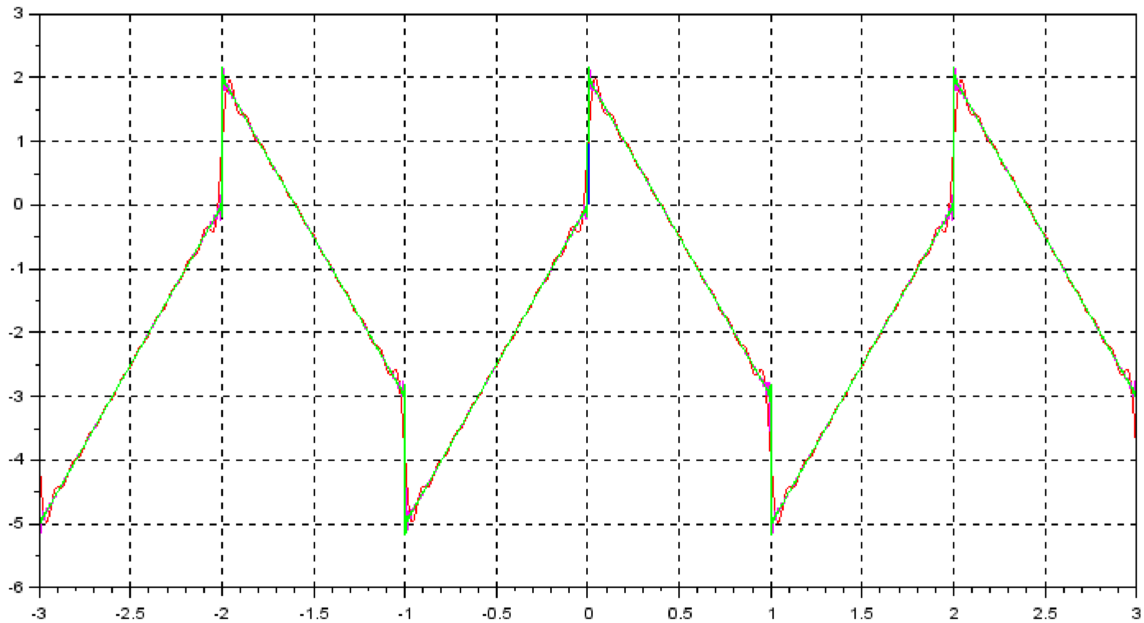
Function with 51 Fourier Series terms

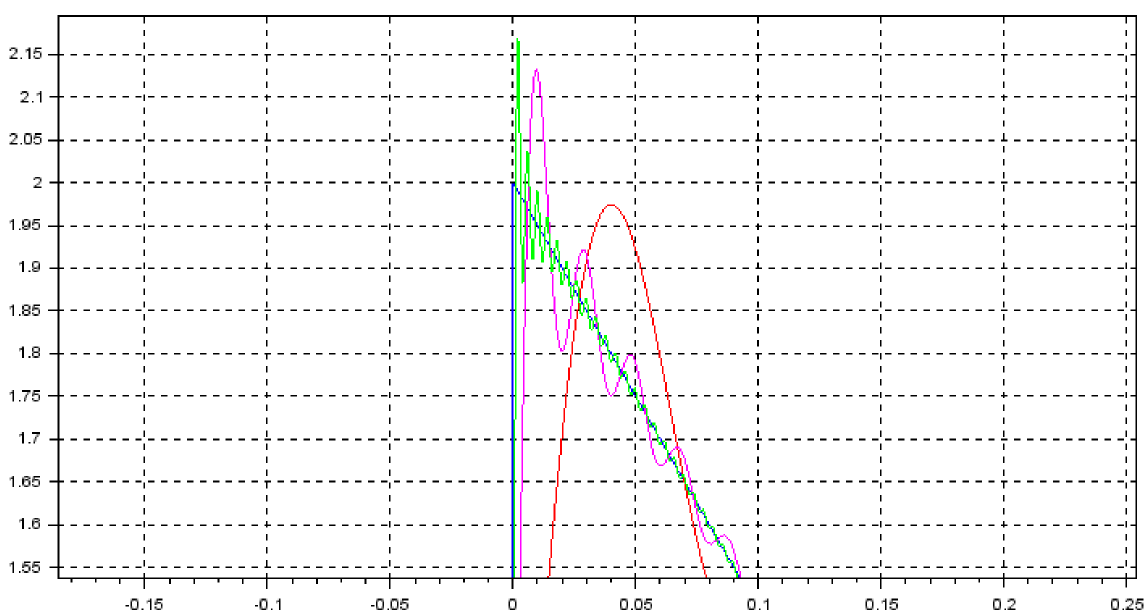


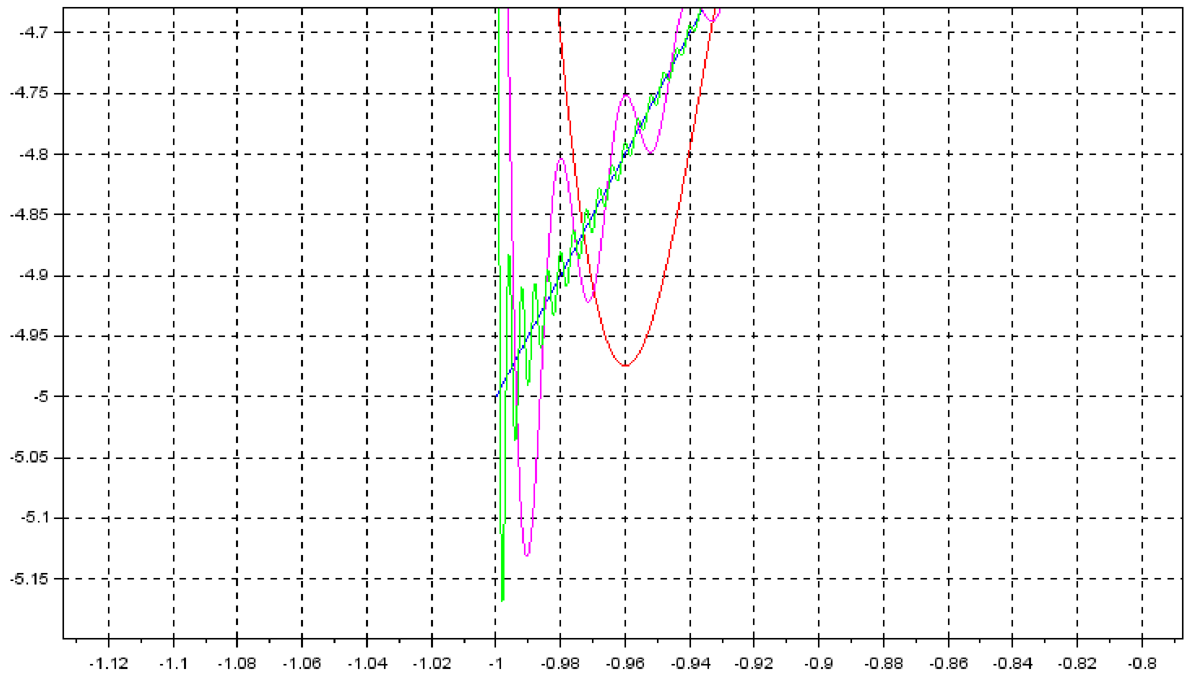
Zoom



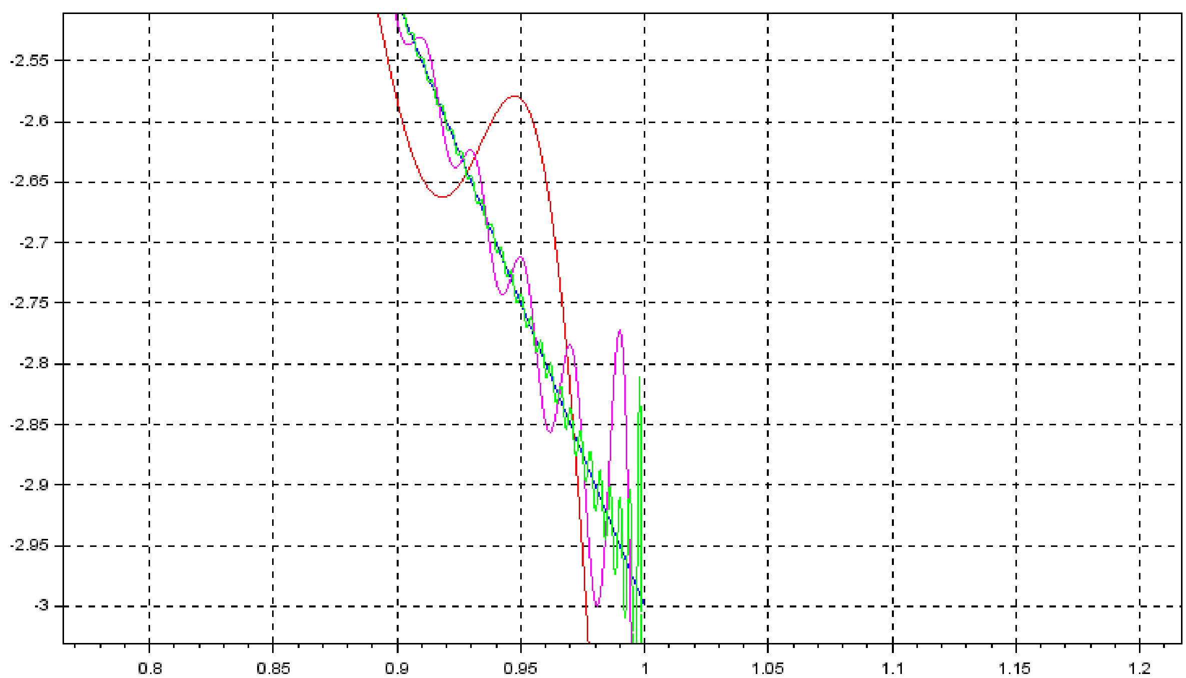
Function with 251 Fourier Series terms, and zooms



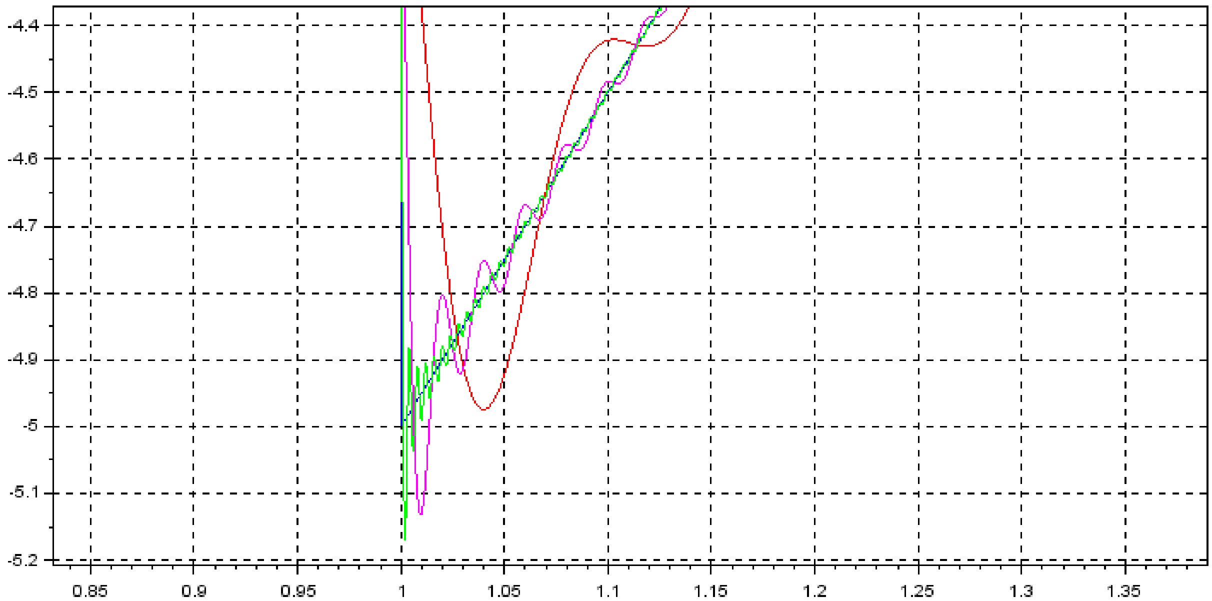




-4.7
-4.75
-4.8
-4.85
-4.9
-4.95
-5
-5.05
-5.1
-5.15
-1.12 -1.1 -1.08 -1.06 -1.04 -1.02 -1 -0.98 -0.96 -0.94 -0.92 -0.9 -0.88 -0.86 -0.84 -0.82 -0.8



-2.55
-2.6
-2.65
-2.7
-2.75
-2.8
-2.85
-2.9
-2.95
-3
0.8 0.85 0.9 0.95 1 1.05 1.1 1.15 1.2



4) Explain your observation.

At first sight, it seems that the first 11 terms of the Fourier series are relatively exact, however, one we zoom in to see the details, it's not so accurate. Now, if we increase the number of terms from 11 to 51, both at first glance and zooming in we see it's more accurate than the first try. Finally, if we increase the number of terms up to 251, we see it's even more accurate than the previous expansions with fewer terms. Moreover, we see that as the terms increase, the wavelength decreases along with the amplitude, therefore, creating a more precise approximation.

Conclusion

Periodic functions commonly occur in fields such as physics and engineering, which have the property that they could be represented as *f(t) = f(t + T)*, where *T* is a constant known as the "period". Fourier series are expressed in the form of:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Moreover, it's important to say that the more terms involved in the sum, the more precise the function representation is, since the wavelength and the amplitude both decrease, while increasing accuracy.