QR Decomposition of a Matrix
Numerical Linear Algebra Math 4330

Rafael Colunga: rafaelcolunga@dusty.tamiu.edu
Texas A&M International University
College of Arts and Sciences
Department of Mathematics and Physics
Dr. Runchang Lin

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1 Introduction

As its own name suggests, the QR decomposition of a matrix allows to split a given matrix, A, into two other matrices. Let's begin with the matrix A, which is a matrix of dimension mxn $(m \geq n)$, into a product of an mxn matrix Q with orthonormal columns and R is an upper triangular matrix. If A is nonsingular, then this factorization is unique.

For the purpose of this project, we will obtain the QR Factorization of a given matrix employing three different methods: Gram-Schmidt process, Givens rotation and Householder reflection. Furthermore, while each method allows us to reach the desired QR factorization, each one has its own set of benefits and drawbacks over each other, such as the numerical stability, the number of repetitions of the algorithm or the amount of operations per cycle of the algorithm.

2 Mathematical Tools

2.1 Gram–Schmidt Process

The Gram-Schmidt process is a method which takes a nonorthogonal set of linearly independent functions and constructs an orthogonal basis. Which then is applied to obtain the orthonormal set of vectors. Let the matrix A be decomposed in column vectors, such as $\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}$
Then, the algorithm proceeds as follows:

$$\begin{array}{ll}
\mathbf{u_1} = \mathbf{a_1} & \mathbf{q_1} = \frac{\mathbf{u_1}}{\|\mathbf{u_1}\|} \\
\mathbf{u_2} = \mathbf{a_2} - \langle \mathbf{q_1}, \mathbf{a_2} \rangle \mathbf{q_1} & \mathbf{q_2} = \frac{\mathbf{u_2}}{\|\mathbf{u_2}\|} \\
\mathbf{u_3} = \mathbf{a_3} - \langle \mathbf{q_1}, \mathbf{a_3} \rangle \mathbf{q_1} - \langle \mathbf{q_2}, \mathbf{a_3} \rangle \mathbf{q_2} & \mathbf{q_3} = \frac{\mathbf{u_3}}{\|\mathbf{u_3}\|} \\
\cdots & \cdots
\end{array}$$

$$Q = [\mathbf{q_1}, \mathbf{q_2}, ..., \mathbf{q_n}]$$

$$R = \begin{bmatrix} \mathrm{r}_{11} & \mathrm{r}_{12} & \mathrm{r}_{13}\dots & \mathrm{r}_{1n} \\ 0 & \mathrm{r}_{22} & \mathrm{r}_{23}\dots & \mathrm{r}_{2n} \\ 0 & 0 & \mathrm{r}_{33}\dots & \mathrm{r}_{3n} \\ 0 & 0 & 0 & \mathrm{r}_{4n} \end{bmatrix}$$

Where $r_{mn} = \langle \mathbf{q_m}, \mathbf{a_n} \rangle$

2.2 Givens Rotation

A Givens rotation defined as $R = \begin{bmatrix} \cos\theta & \text{-}\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

rotates $x \in \mathbf{R}^2$ by θ. Moreover, to set an element to zero, choose $\cos\theta$ and $\sin\theta$ such that:

$$\cos\theta = \frac{x_i}{\sqrt{x_i^2 + x_j^2}}$$

$$\sin\theta = \frac{-x_j}{\sqrt{x_i^2 + x_j^2}}$$

2.3 Householder Reflection

A Householder reflection is an alternative to a Givens rotation, which only zeros out one element per computation, which leads to a series of lengthy rotations. However, the Householder algorithm requires approximately two-thirds the flop count of the Givens algorithm because it zeros out all of the elements under a diagonal element a_{ii} in one multiplication. A Householder reflection is a transformation which takes a vector u and reflects it about a plane in $\mathbf{R^n}$. Such transformation has the following form:

$$H = I - \frac{2\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}}$$

3 QR Decompositions

3.1 Gram–Schmidt Process

Let $A = \begin{bmatrix} 3 & -5 & 1 & 2 \\ 1 & 1 & 1 & 4 \\ -1 & 5 & -2 & 3 \\ 3 & -7 & 8 & 2 \\ 5 & -4 & -3 & 7 \end{bmatrix}$

In this case, each columns can be described as an individual vector, $\mathbf{a_1}, \mathbf{a_2}, ...$

(1) Now, let $r_{11} = \|\mathbf{a_1}\| = 6.7082$ and $\mathbf{q_1} = \frac{\mathbf{a_1}}{r_{11}} = \begin{bmatrix} 0.4472 \\ 0.1490 \\ -0.1490 \\ 0.4472 \\ 0.7453 \end{bmatrix}$

(2) $r_{12} = \langle \mathbf{q_1}, \mathbf{a_2} \rangle = -8.9442$ and $\mathbf{u_2} = \mathbf{a_2} - r_{12}\mathbf{q_1}$.

Moreover, $r_{22} = \|\mathbf{u_2}\| = 6$ and $\mathbf{q_2} = \frac{\mathbf{u_2}}{r_{22}} = \begin{bmatrix} -0.1666 \\ 0.3888 \\ 0.6111 \\ -0.5 \\ 0.4444 \end{bmatrix}$

After repeating the same algorithm two more times, one for $\mathbf{a_3}$ and one for $\mathbf{a_4}$, we arrive to the following answers.

(3) $r_{13} = 2.2360, r_{23} = -6.3333, r_{33} = 5.8214, \mathbf{q_3} = \begin{bmatrix} -0.1813 \\ 0.5376 \\ 0.3785 \\ 0.6584 \\ -0.3181 \end{bmatrix}$

(4) $r_{14} = 7.1554, r_{24} = 5.1666, r_{34} = 2.0136, r_{44} = 0.2254, \mathbf{q_4} = \begin{bmatrix} 0.1163 \\ -0.7029 \\ 0.6520 \\ 0.2545 \\ 0.0484 \end{bmatrix}$

Now that we have all of the elements of the matrix R, and the columns of the matrix Q, we now have that $Q = [\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}, \mathbf{q_4}] = \begin{bmatrix} 0.4472 & -0.1666 & -0.1813 & 0.1163 \\ 0.1490 & 0.3888 & 0.5376 & -0.7029 \\ -0.1490 & 0.6111 & 0.3785 & 0.6520 \\ 0.4472 & -0.5 & 0.6584 & 0.2545 \\ 0.7453 & 0.4444 & -0.3181 & 0.0484 \end{bmatrix}$

$$R = \begin{bmatrix} \mathrm{r}_{11} & \mathrm{r}_{12} & \mathrm{r}_{13}\ldots & \mathrm{r}_{1n} \\ 0 & \mathrm{r}_{22} & \mathrm{r}_{23}\ldots & \mathrm{r}_{2n} \\ 0 & 0 & \mathrm{r}_{33}\ldots & \mathrm{r}_{3n} \\ 0 & 0 & 0 & \mathrm{r}_{4n} \end{bmatrix} = \begin{bmatrix} 6.7082 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 6 & -6.3333 & 5.1666 \\ 0 & 0 & 5.8214 & 2.0136 \\ 0 & 0 & 0 & 0.2254 \end{bmatrix}$$

An example of the code used to generate the decomposition, is the following: $r_{11} = norm(a_1, 2)$, $q_1 = a_1/r_{11}$

3.2 Givens Rotation

Let $A = \begin{bmatrix} 3 & -5 & 1 & 2 \\ 1 & 1 & 1 & 4 \\ -1 & 5 & -2 & 3 \\ 3 & -7 & 8 & 2 \\ 5 & -4 & -3 & 7 \end{bmatrix}$

(1) Let $A^0 = A$ and use a^0_{41} to eliminate a^0_{51}. Let $r = \sqrt{(a^0_{41})^2 + (a^0_{51})^2} = 5.8309$, therefore, $cs(\cos) = \frac{a^0_{41}}{r} = 0.5144, sn(\sin) = \frac{a^0_{51}}{r} = 0.8574$. Now, the rotation matrix, let's call it $G145$ (1 is the number of the column, 4 is the element used and 5 is the one turned to zero).

Therefore, $G145 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & cs & sn \\ 0 & 0 & 0 & -sn & cs \end{bmatrix}$

$A^1 = G145 * A^0 = \begin{bmatrix} 3 & -5 & 1 & 2 \\ 1 & 1 & 1 & 4 \\ -1 & 5 & -2 & 3 \\ 5.8309 & -7.0314 & 1.5434 & 7.0314 \\ 0 & 3.9444 & -8.4034 & 1.8864 \end{bmatrix}$

(2) In A^1, use a^1_{31} to eliminate a^1_{41}. Following the same algorithm as the previous step, we now have that $r = 5.9160$, $cs = -0.1690$, $sn = 0.9856$. Now, the rotation matrix, now called G134 because we are in the first column using the third element to eliminate the fourth,

$G134 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & cs & sn & 0 \\ 0 & 0 & -sn & cs & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$A^2 = G134 * A^1 = \begin{bmatrix} 3 & -5 & 1 & 2 \\ 1 & 1 & 1 & 4 \\ 5.9160 & -7.7754 & 1.8593 & 6.4231 \\ 0 & -3.7395 & 1.7103 & -4.1453 \\ 0 & 3.9444 & -8.4034 & 1.8864 \end{bmatrix}$

(3) In this case, $r = 6, cs = 0.1666, sn = 0.9860$. Now let the rotation matrix be equal to

$$G123 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & cs & sn & 0 & 0 \\ 0 & -sn & cs & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = G123 * A^2 = \begin{bmatrix} 3 & -5 & 1 & 2 \\ 6 & -7.5 & 2 & 7 \\ 0 & -2.2819 & -0.6761 & -2.8735 \\ 0 & -3.7395 & 1.7103 & -4.1453 \\ 0 & 3.9444 & -8.4043 & 1.8864 \end{bmatrix}$$

(4) In this case, $r = 6.7082, cs = 0.4472, sn = 0.8944$. Now let the rotation matrix be equal to $G112 = \begin{bmatrix} cs & sn & 0 & 0 & 0 \\ -sn & cs & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$A^4 = G112 * A^3 = \begin{bmatrix} 6.7082 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 1.1180 & 0 & 1.3416 \\ 0 & -2.2819 & -0.6761 & -2.8735 \\ 0 & -3.7395 & 1.7103 & -4.1453 \\ 0 & 3.9444 & -8.4034 & 1.8864 \end{bmatrix}$$

(5) In this case, $r = 5.4345, cs = -0.6880, sn = 0.7257$. Now let the rotation matrix be equal to $G245 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & cs & sn \\ 0 & 0 & 0 & -sn & cs \end{bmatrix}$

$$A^5 = G245 * A^4 = \begin{bmatrix} 6.7082 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 1.1180 & 0 & 1.3416 \\ 0 & -2.2819 & -0.6761 & -2.8735 \\ 0 & 5.4353 & -7.2751 & 4.2210 \\ 0 & 0 & 4.5403 & 1.7104 \end{bmatrix}$$

(6) In this case, $r = 5.8949, cs = -0.3870, sn = 0.9220$. Now let the rotation matrix be equal to $G234 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & cs & sn & 0 \\ 0 & 0 & -sn & cs & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$A^6 = G234 * A^5 = \begin{bmatrix} 6.7082 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 1.1180 & 0 & 1.3416 \\ 0 & 5.8949 & -6.4462 & 5.0043 \\ 0 & 0 & 3.4396 & 1.0155 \\ 0 & 0 & 4.5403 & 1.7104 \end{bmatrix}$$

(7) In this case, $r = 6, cs = 0.1863, sn = 0.9824$. Now let the rotation matrix be equal to

$$G223 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & cs & sn & 0 & 0 \\ 0 & -sn & cs & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^7 = G223 * A^6 = \begin{bmatrix} 6.7082 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 6 & -6.3333 & 5.1666 \\ 0 & 0 & -1.2011 & -0.3856 \\ 0 & 0 & 3.4396 & 1.0155 \\ 0 & 0 & 4.5403 & 1.7104 \end{bmatrix}$$

(8) In this case, $r = 5.6961, cs = 0.6038, sn = 0.7970$. Now let the rotation matrix be equal to $G345 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & cs & sn \\ 0 & 0 & 0 & -sn & cs \end{bmatrix}$

$$A^8 = G345 * A^7 = \begin{bmatrix} 6.7820 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 6 & -6.3333 & 5.1666 \\ 0 & 0 & -1.2011 & -0.3856 \\ 0 & 0 & 5.6961 & 1.9765 \\ 0 & 0 & 0 & 0.2233 \end{bmatrix}$$

(9) In this case, $r = 5.8214, cs = -0.2063, sn = 0.9784$. Now let the rotation matrix be equal to $G334 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & cs & sn & 0 \\ 0 & 0 & -sn & cs & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$A^9 = G334 * A^8 = \begin{bmatrix} 6.7082 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 6 & -6.3333 & 5.1666 \\ 0 & 0 & 5.8214 & 2.0136 \\ 0 & 0 & 0 & -0.0305 \\ 0 & 0 & 0 & 0.2233 \end{bmatrix}$$

(10) In this case, $r = 0.2254, cs = -0.1353, sn = 0.9908$. Now let the rotation matrix be equal to $G445 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & cs & sn \\ 0 & 0 & 0 & -sn & cs \end{bmatrix}$

$A^{10} = G445 * A^9 \begin{bmatrix} 6.7082 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 6 & -6.3333 & 5.1666 \\ 0 & 0 & 5.8214 & 2.0136 \\ 0 & 0 & 0 & 0.2254 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now, we can conclude the fact that $R = A^{10}$. For Q, it is equal to the multiplication of the transpose of each rotation matrix.

That is, $Q = G145' * G134' * G123' * G112' * G245' * G234' * G223' * G345' * G334' * G445' =$

$$\begin{bmatrix} 0.4472 & -0.1666 & -0.1813 & 0.1163 & 0.8519 \\ 0.1490 & 0.3888 & 0.5376 & -0.7029 & 0.2082 \\ -0.1490 & 0.6111 & 0.3785 & 0.6520 & 0.1893 \\ 0.4472 & -0.5 & 0.6584 & 0.2545 & -0.2271 \\ 0.7453 & 0.4444 & -0.3181 & 0.0484 & -0.3786 \end{bmatrix}$$

An example of the code used to generate the decomposition, is the following:

$RTM = [cs, sn; -sn, cs]$, $G134 = eye(4)$, $G134(3:4, 3:4) = RTM$

3.3 Householder Reflection

(1) Let $A_0 = A = \begin{bmatrix} 3 & -5 & 1 & 2 \\ 1 & 1 & 1 & 4 \\ -1 & 5 & -2 & 3 \\ 3 & -7 & 8 & 2 \\ 5 & -4 & -3 & 7 \end{bmatrix}$

Therefore, $\mathbf{v_1} = \mathbf{a_1} - sign(a_{11}) * \|\mathbf{a_1}\| * \mathbf{e_1}$, and H1 is now defined as:

$$H_1 = I - \frac{2\mathbf{v_1}\mathbf{v_1}^T}{\mathbf{v_1}^T\mathbf{v_1}}$$

$$A_1 = H_1 * A_0 = \begin{bmatrix} 6.7082 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 2.0636 & 0.6666 & 2.6097 \\ 0 & 3.9363 & -1.6666 & 4.3902 \\ 0 & -3.8090 & 7 & -2.1708 \\ 0 & 1.3183 & -4.6666 & 0.0486 \end{bmatrix}$$

(2) If we observe the newly created matrix, we can obtain a new matrix ranging from the second to the last row, and from the second to the last column. That will be the matrix we will use to continue applying the Householder algorithm.

Let the previous matrix be defined as A_{14}; thus:
$\mathbf{v_2} = A_{14}(:,1) - sign(A_{l4}(1,1)) * \|A_{14}(:,1)\| * \mathbf{e_1}$

In this case, H_2 will be equal to a 5x5 matrix with a 1 in the 11 entry, and the rest of the elements in the first row and column will be zeros. As for the remaining entries, those will be filled with with the same mathematical expression as the original H_1 but instead of using $\mathbf{v_1}$, we will use $\mathbf{v_2}$

$$A_2 = H_2 * A_1 = \begin{bmatrix} 6.7082 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 6 & -6.3333 & 5.1666 \\ 0 & 0 & 5.3333 & 1.8333 \\ 0 & 0 & 0.2264 & 0.3034 \\ 0 & 0 & -2.3223 & -0.8077 \end{bmatrix}$$

(3) Applying the same process as before. We can obtain another matrix inside A_2. In this case, it is a 3x2 matrix. Let's define such matrix as $A_{24} = A_2(3:5, 3:4)$, which stands from the third to the fifth row and third to fourth column, respectively.

$\mathbf{v_3} = A_{24}(:,1) - sign(A_{24}(1,1)) * \|A_{24}(:,1)\| * \mathbf{e_1}$

$$A_3 = H_3 * A_2 = \begin{bmatrix} 6.7082 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 6 & -6.3333 & 5.1666 \\ 0 & 0 & 5.8214 & 2.0136 \\ 0 & 0 & 0 & 0.2197 \\ 0 & 0 & 0 & 0.0501 \end{bmatrix}$$

(4) Let $A_{34} = A_3(4:5, 4)$. Therefore, $\mathbf{v_4} = A_{34}(:,1) - sign(A_{34}(1,1)) * \|A_34(:,1)\| * \mathbf{e_1}$

$$A_4 = H_4 * A_3 = R = \begin{bmatrix} 6.7082 & -8.9442 & 2.2360 & 7.1554 \\ 0 & 6 & -6.3333 & 5.1666 \\ 0 & 0 & 5.8214 & 2.0136 \\ 0 & 0 & 0 & 0.2254 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q = H_1 * H_2 * H_3 * H_4 = \begin{bmatrix} 0.4472 & -0.1666 & -0.1813 & 0.1163 & 0.8519 \\ 0.1490 & 0.3888 & 0.5376 & -0.7029 & 0.2082 \\ -0.1490 & 0.6111 & 0.3785 & 0.6520 & 0.1893 \\ 0.4472 & -0.5 & 0.6584 & 0.2545 & -0.2271 \\ 0.7453 & 0.4444 & -0.3181 & 0.0484 & -0.3786 \end{bmatrix}$$

An example of the code used to generate the decomposition, is the following:
$v1 = A0(:,1) - sign(A0(1,1)) * norm(A0(:,1),2) * [1;0;0;0;0]$,
$H1 = eye(5) - 2 * v1 * v1'/(v1' * v1)$

4 Conclusion

Just as the computations showed, each process has its respective pros and cons. Of all of the algorithms, the lengthiest one in terms of number of computations, is the Givens rotation which had to be repeated 10 times. The fastest one overall was the Householder reflection; meanwhile, the Gram-Schmidt has the less numerical stability due to rounding errors of all of them, but it allows for a much cleaner computation of the Q and R matrices. Moreover, Householder reflections can turn out to be a bit messy when working step by step in platforms such as MATLAB, while it is still manageable, the user has to be really careful at the moment of inputting the matrices.

5 Bibliography

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