SICP EXERCISE 1.13 (PROOF)

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- **Proof:** Let $P(n)$ be the mathematical statement

  Fib(n) is the closest integer to $\dfrac{\phi^n}{\sqrt{5}}$ where Fib(n) is the nth Fibonacci number: 0,1,1,2,3,5,8,13,21...

Base Case:
When $n = 0$ we have $\frac{\phi^0}{\sqrt{5}} = 0.4472135954999579$ and the nearest integer is 0. So $P(0)$ is correct.
When $n = 1$ we have $\frac{\phi^1}{\sqrt{5}} = 0.7236067977499789$ and the nearest integer is 1. So $P(1)$ is correct.

Induction hypothesis:
Assume that $P(k)$ and $P(k+1)$ is correct for some positive integer k.

Induction step:
We will now show that $P(k+2)$ is correct using the Fibonacci property $F_n + F_{n+1} = F_{n+2}$

$$P(k) + P(k+1)$$

$$= \frac{p^k}{\sqrt{5}} + \frac{p^{k+1}}{\sqrt{5}} = \frac{p^k}{\sqrt{5}} + \frac{p^k}{\sqrt{5}} \cdot \frac{(1+\sqrt{5})}{2}$$

$$= \frac{p^k}{\sqrt{5}} + \frac{p^k}{\sqrt{5}} \cdot \frac{1}{2} + \frac{p^k}{\sqrt{5}} \cdot \frac{(\sqrt{5})}{2}$$

$$= \frac{2}{2} \cdot \frac{p^k}{\sqrt{5}} + \frac{p^k}{\sqrt{5}} \cdot \frac{1}{2} + \frac{p^k}{\sqrt{5}} \cdot \frac{(\sqrt{5})}{2}$$

$$= \frac{3p^k + p^k\sqrt{5}}{2\sqrt{5}} = \frac{(3+\sqrt{5})p^k}{2\sqrt{5}}$$

$$= \left(\frac{3+\sqrt{5}}{2}\right) \cdot \frac{p^k}{\sqrt{5}} = \left(\frac{1+\sqrt{5}}{2}\right)^2 \cdot \frac{p^k}{\sqrt{5}} = p^2 \cdot \frac{p^k}{\sqrt{5}} = \frac{p^{k+2}}{\sqrt{5}}$$

$$= P(k+2)$$

Hence by mathematical induction $P(n)$ is correct for all positive integers n. ■