SICP EXERCISE 1.13 (PROOF)

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- **Proof:** Let $P(n)$ be the mathematical statement

  Fib(n) is the closest integer to $\dfrac{\phi^n}{\sqrt{5}}$ where Fib(n) is the nth Fibonacci number: 0,1,1,2,3,5,8,13,21...

Base Case:

When $n = 0$ we have $\frac{\phi^0}{\sqrt{5}} = 0.4472135954999579$ and the nearest integer is 0. So $P(0)$ is correct.
When $n = 1$ we have $\frac{\phi^1}{\sqrt{5}} = 0.7236067977499789$ and the nearest integer is 1. So $P(1)$ is correct.

Induction hypothesis:

Assume that $P(k)$ and $P(k+1)$ is correct for some positive integer k.

Induction step:

We will now show that $P(k+2)$ is correct using the Fibonacci property $F_n + F_{n+1} = F_{n+2}$

$$P(k) + P(k+1)$$

$$= \frac{\phi^k}{\sqrt{5}} + \frac{\phi^{k+1}}{\sqrt{5}} = \frac{\phi^k}{\sqrt{5}} + \frac{\phi^k}{\sqrt{5}} \cdot \frac{(1+\sqrt{5})}{2}$$

$$= \frac{\phi^k}{\sqrt{5}} + \frac{\phi^k}{\sqrt{5}} \cdot \frac{1}{2} + \frac{\phi^k}{\sqrt{5}} \cdot \frac{(\sqrt{5})}{2}$$

$$= \frac{2}{2} \cdot \frac{\phi^k}{\sqrt{5}} + \frac{\phi^k}{\sqrt{5}} \cdot \frac{1}{2} + \frac{\phi^k}{\sqrt{5}} \cdot \frac{(\sqrt{5})}{2}$$

$$= \frac{3\phi^k + \phi^k\sqrt{5}}{2\sqrt{5}} = \frac{(3+\sqrt{5})\phi^k}{2\sqrt{5}}$$

$$= \left(\frac{3+\sqrt{5}}{2}\right) \cdot \frac{\phi^k}{\sqrt{5}} = \left(\frac{1+\sqrt{5}}{2}\right)^2 \cdot \frac{\phi^k}{\sqrt{5}} = \phi^2 \cdot \frac{\phi^k}{\sqrt{5}} = \frac{\phi^{k+2}}{\sqrt{5}}$$

$$= P(k+2)$$

How do we show that Fib(n) is the closest integer to $\frac{\phi^n}{\sqrt{5}}$?

Well, if Fib(n) is the closest integer to $\frac{\phi^n}{\sqrt{5}}$ then $\frac{\phi^n}{\sqrt{5}}$ must be within $\frac{1}{2}$ of Fib(n).

Basically, we need to prove that $|(Fib(n) - \frac{\phi^n}{\sqrt{5}}| \leq \frac{1}{2}$

Using Fib(n) $= \dfrac{\phi^n - \psi^n}{\sqrt{5}}^{**}$:

$|(Fib(n) - \frac{\phi^n}{\sqrt{5}}| \leq \frac{1}{2}$
$|\frac{\phi^n - \psi^n}{\sqrt{5}} - \frac{\phi^n}{\sqrt{5}}| \leq \frac{1}{2}$
$|\frac{-\psi^n}{\sqrt{5}}| \leq \frac{1}{2}$
$|\frac{\psi^n}{\sqrt{5}}| \leq \frac{1}{2}$
$|\psi^n| \leq \frac{\sqrt{5}}{2}$
$|\psi^n| \leq$ 1.1180339...

We know that for any x where $-1 \leq x \leq 1$ and n is an integer where $n \geq 0$, it will always be the case that $-1 \leq x^n \leq 1$.

Since $\psi = -0.6180339$, $-1 \leq \psi^n \leq 1$ for all positive integers n.

So we know $|\psi^n| \leq \frac{\sqrt{5}}{2}$ since $\frac{\sqrt{5}}{2} = 1.1180339...$

Hence by mathematical induction $P(n)$ is correct for all positive integers n. ■

** I have discovered a truly marvelous proof of this, which this margin is too narrow to contain.