Type Theory with Native Homotopy Universes

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We present an implementation of a type system with explicit stratification of types into three homotopy levels, together with a model where the types are interpreted by propositions, setoids and groupoids. We work in the Agda proof assistant.

The type theory has an equality type $a \simeq_A b$ defined by recursion on the type A, in each case one level below A; thus, the equalities in a groupoid form a setoid, and the equalities in a setoid form a proposition. Univalence becomes the definition of equality between two types.

Our approach thus follows a "bottom-up" methodology. It includes a formalization of the setoid model of Martin-Löf type theory, and definitions of all nine cases of what it is to be a fibration of groupoids (setoids, propositions) indexed by a groupoid (setoid, proposition).

It is done in a modular way that should allow this work to generalize to a formalization of type theory with $(n+2)$-levels and an n-groupoid model.

The formalization is available at `https://github.com/radams78/Equality`