A Type Theory with Native Homotopy Universes

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We present a type theory called $\lambda \simeq_2$ with an extensional equality relation; that is, the universe of types is closed by reflection into it of the logical relation defined by induction on the structure of types.

The type system has three universes:

- The universe **Prop** of *propositions*. An object of **Prop** is called a *proposition*, and the objects of a proposition are called *proofs*. There is a proposition $\mathbf{1}$, which has a unique proof $*$.
- The universe **Set** of *sets*.
- The universe **Groupoid** of *groupoids*.

The system has been designed in such a way that it should be straightforward to extend the system with three, four, ... dimensions.

For each universe U, we have an associated relation of equality between U-types $\simeq$, and between objects of U-equal types $\sim$. The associated rules of deduction are:

$$\frac{A : U \quad B : U}{A \simeq B : U} \qquad \frac{a : A \quad e : A \simeq B \quad b : B}{a \sim_e b : U^-}$$

where U^- is the universe one dimension below U. Thus:

- Given two propositions ϕ and ψ, we have the proposition $\phi \simeq \psi$ which denotes the proposition 'ϕ if and only if ψ'. If $\delta : \phi$, $\epsilon : \psi$ and $\chi : \phi \simeq \psi$, then $\delta \sim_\chi \epsilon = \mathbf{1}$. (*Cf* In homotopy type theory, any two objects of a proposition are equal.)
- Given two sets A and B, we have the set $A \simeq B$, which denotes the set of all bijections between A and B. Given $a : A$, $f : A \simeq B$ and $b : B$, we have the proposition $a \sim_f b :$ **Prop**, which denotes that a is mapped to b by the bijection f.
- Given two groupoids G and H, we have the groupoid $G \simeq H$, which denotes the groupoid of all groupoid isomorphisms between G and H. Given $g : G$, $\phi : G \simeq H$ and $h : H$, we have the set $g \sim_\phi h :$ **Set**, which can be thought of as the set of all paths between $\phi(g)$ and h in H.

The introduction and elimination rules for $\simeq$ ensure that $A \simeq B$ is the type of equivalences between A and B.

$$\frac{A : U}{1_A : A \simeq A} \qquad \frac{a : A}{r_a : a \sim_{1_A} a} \qquad \frac{e : A \simeq B \quad a : A}{e^+(a) : B} \qquad \frac{e : A \simeq B \quad b : B}{e^-(b) : A}$$

$$\frac{a : A \quad b : B \quad e : A \simeq B}{e^=(a,b) : (a \sim_{1_A} e^-(b)) \simeq (e^+(a) \sim_{1_B} b)} \qquad \frac{\begin{array}{ll} \Gamma, x : A & \vdash b : B \\ \Gamma, y : B & \vdash a : A \\ \Gamma, x : A, y : B & \vdash e : (x \sim_{1_A} a) \simeq (b \sim_{1_B} y) \end{array}}{\Gamma \vdash \mathsf{univ}([x : A]b, [y : B]a, [x : A, y : B]e) : A \simeq B}$$

Each universe is itself an object of the next universe; thus $\mathbf{1} : \mathbf{Prop} : \mathbf{Set} : \mathbf{Groupoid}$. We also have the following definitional equalities: $\phi \sim_{1_{\mathbf{Prop}}} \psi \stackrel{\text{def}}{=} \phi \simeq \psi$, $A \sim_{1_{\mathbf{Set}}} B \stackrel{\text{def}}{=} A \simeq B$. As well as the normal operation of substitution, we have an operation of *path substitution*:

$$\frac{\Gamma, x : A \vdash b : B \quad \Gamma \vdash e : a \sim_{1_A} a'}{\Gamma \vdash b[x//e] : b[x/a] =_{B[x//e]} b[x/a']}$$

The system $\lambda \simeq_2$ enjoys the following properties. Univalence holds definitionally; that is, an equality between types $A \simeq B$ is exactly (definitionally) the type of equivalences between A and B. Also, transport respects reflexivity and composition definitionally.

This type theory has been formalised in Agda, using the method of the system `Kipling` from McBride [5]. The method ensures that, if s and t are definitionally equal expressions in $\lambda \simeq_2$, then $[\![s]\!]$ and $[\![t]\!]$ are definitionally equal objects in Agda. We interpret each context with a groupoid in Agda; that is, we define a type `data Cx : Set₁` of contexts, and functions

```
[_]C : Cx → Set₁
EQC : ∀ Γ → [ Γ ]C → [ Γ ]C → Set
EQC₂ : ∀ {Γ} {a₁ a₂ b₁ b₂ : [ Γ ]C} →
   EQC Γ a₁ a₂ → EQC Γ b₁ b₂ → EQC Γ a₁ b₁ → EQC Γ a₂ b₂ → Set
```

The formalisation is available online at https://github.com/radams78/Equality2.

Related Work An earlier version of this system was presented in [2]. In this talk, we also give semantics to this system in Agda's type theory extended with a native type of groupoids, and show how the syntax and semantics are formalised in Agda.

Our system is closely related to the system PHOML (Predicative Higher-Order Minimal Logic) presented in [1]. The system $\lambda \simeq_2$ can be seen as an extension of PHOML with groupoids, and with a univalent equality for both sets and groupoids.

Cubical type theory [3, 4] has a very similar motivation to this work, and also offers a type theory with univalence and a computational interpretation. One difference with our system is that transport across the identity path is identity, and transport across $p \bullet q$ is the composition of transport across p with transport across q, up to definitional equality. In cubical type theory, these equations only hold up to propositional equality.

References

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