Type Theories with Computation Rules for the Univalence Axiom

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1 Preliminaries

```
module Prelims where

open import Relation.Binary public hiding (_⇒_)
import Relation.Binary.EqReasoning
open import Relation.Binary.PropositionalEquality public using (_≡_;refl;sym;trans;cong;

module EqReasoning {s₁ s₂} (S : Setoid s₁ s₂) where
  open Setoid S using (_≈_)
  open Relation.Binary.EqReasoning S public

  infixr 2 _≡⟨⟨_⟩⟩_
  _≡⟨⟨_⟩⟩_ : ∀ x {y z} → y ≈ x → y ≈ z → x ≈ z
  _ ≡⟨⟨ y≈x ⟩⟩ y≈z = Setoid.trans S (Setoid.sym S y≈x) y≈z

module ≡-Reasoning {a} {A : Set a} where
  open Relation.Binary.PropositionalEquality
  open ≡-Reasoning {a} {A} public

  infixr 2 _≡⟨⟨_⟩⟩_
  _≡⟨⟨_⟩⟩_ : ∀ (x : A) {y z} → y ≡ x → y ≡ z → x ≡ z
  _ ≡⟨⟨ y≡x ⟩⟩ y≡z = trans (sym y≡x) y≡z
--TODO Add this to standard library
```

2 Grammars

```
module Grammar where

open import Function
open import Data.Empty
open import Data.Product
```

```
open import Data.Nat public
open import Data.Fin public using (Fin;zero;suc)
open import Prelims
```

Before we begin investigating the several theories we wish to consider, we present a general theory of syntax and capture-avoiding substitution.

A *taxononmy* consists of:

- a set of *expression kinds*;
- a subset of expression kinds, called the *variable kinds*. We refer to the other expession kinds as *non-variable kinds*.

A *grammar* over a taxonomy consists of:

- a set of *constructors*, each with an associated *constructor kind* of the form

$$((A_{11}, \ldots, A_{1r_1})B_1, \ldots, (A_{m1}, \ldots, A_{mr_m})B_m)C \tag{1}$$

  where each A_{ij} is a variable kind, and each B_i and C is an expression kind.

- a function assigning, to each variable kind K, an expression kind, the *parent* of K.

A constructor c of kind (1) is a constructor that takes m arguments of kind B_1, …, B_m, and binds r_i variables in its ith argument of kind A_{ij}, producing an expression of kind C. We write this expression as

$$c([x_{11}, \ldots, x_{1r_1}]E_1, \ldots, [x_{m1}, \ldots, x_{mr_m}]E_m) \ . \tag{2}$$

The subexpressions of the form $[x_{i1}, \ldots, x_{ir_i}]E_i$ shall be called *abstractions*, and the pieces of syntax of the form $(A_{i1}, \ldots, A_{ij})B_i$ that occur in constructor kinds shall be called *abstraction kinds*.

We formalise this as follows. First, we construct the sets of expression kinds, constructor kinds and abstraction kinds over a taxonomy:

```
record Taxonomy : Set₁ where
  field
    VarKind : Set
    NonVarKind : Set

  data ExpressionKind : Set where
    varKind : VarKind → ExpressionKind
    nonVarKind : NonVarKind → ExpressionKind

  data KindClass : Set where
    -Expression  : KindClass
    -Abstraction : KindClass
```

```
  -Constructor : ExpressionKind → KindClass

data Kind : KindClass → Set where
  base : ExpressionKind → Kind -Expression
  out  : ExpressionKind → Kind -Abstraction
  Π    : VarKind → Kind -Abstraction → Kind -Abstraction
  out₂ : ∀ {K} → Kind (-Constructor K)
  Π₂   : ∀ {K} → Kind -Abstraction → Kind (-Constructor K) → Kind (-Constructor K)
```

An *alphabet* A consists of a finite set of *variables*, to each of which is assigned a variable kind K. Let $\emptyset$ be the empty alphabet, and (A, K) be the result of extending the alphabet A with one fresh variable x_0 of kind K. We write $\mathsf{Var}\ A\ K$ for the set of all variables in A of kind K.

```
data Alphabet : Set where
  ∅ : Alphabet
  _,_ : Alphabet → VarKind → Alphabet

data Var : Alphabet → VarKind → Set where
  x₀ : ∀ {V} {K} → Var (V , K) K
  ↑ : ∀ {V} {K} {L} → Var V L → Var (V , K) L
```

We can now define a grammar over a taxonomy:

```
record ToGrammar : Set₁ where
  field
    Constructor    : ∀ {K} → Kind (-Constructor K) → Set
    parent         : VarKind → ExpressionKind
```

The *expressions* of kind E over the alphabet V are defined inductively by:

- Every variable of kind E is an expression of kind E.
- If c is a constructor of kind (1), each E_i is an expression of kind B_i, and each x_{ij} is a variable of kind A_{ij}, then (2) is an expression of kind C.

Each x_{ij} is bound within E_i in the expression (2). We identify expressions up to α-conversion.

```
data Subexpression : Alphabet → ∀ C → Kind C → Set
Expression : Alphabet → ExpressionKind → Set
Body : Alphabet → ∀ {K} → Kind (-Constructor K) → Set
Abstraction : Alphabet → Kind -Abstraction → Set

Expression V K = Subexpression V -Expression (base K)
Body V {K} C = Subexpression V (-Constructor K) C

alpha : Alphabet → Kind -Abstraction → Alphabet
```

```
alpha V (out _) = V
alpha V (Π K A) = alpha (V , K) A

beta : Kind -Abstraction → ExpressionKind
beta (out K) = K
beta (Π _ A) = beta A

Abstraction V A = Expression (alpha V A) (beta A)

data Subexpression where
  var : ∀ {V} {K} → Var V K → Expression V (varKind K)
  app : ∀ {V} {K} {C} → Constructor C → Body V {K} C → Expression V K
  out₂ : ∀ {V} {K} → Body V {K} out₂
  app₂ : ∀ {V} {K} {A} {C} → Abstraction V A → Body V {K} C → Body V (Π₂ A C)

var-inj : ∀ {V} {K} {x y : Var V K} → var x ≡ var y → x ≡ y
var-inj refl = refl
```

2.1 Families of Operations

We now wish to define the operations of *replacement* (replacing one variable with another) and *substitution* of expressions for variables. To this end, we define the following.

A *family of operations* consists of the following data:

- Given alphabets U and V, a set of *operations* $\sigma : U \to V$.
- Given an operation $\sigma : U \to V$ and a variable x in U of kind K, an expression $\sigma(x)$ over V of kind K, the result of *applying* σ to x.
- For every alphabet V, an operation $\mathsf{id}_V : V \to V$, the *identity* operation.
- For any operations $\rho : U \to V$ and $\sigma : V \to W$, an operation $\sigma \circ \rho : U \to W$, the *composite* of σ and ρ
- For every alphabet V and variable kind K, an operation $\uparrow : V \to (V, K)$, the *successor* operation.
- For every operation $\sigma : U \to V$, an operation $(\sigma, K) : (U, K) \to (V, K)$, the result of *lifting* σ. We write $(\sigma, K_1, K_2, \ldots, K_n)$ for $((\cdots(\sigma, K_1), K_2), \cdots), K_n)$.

such that

1. $\uparrow(x) \equiv x$
2. $\mathsf{id}_V(x) \equiv x$
3. $(\sigma \circ \rho)(x) \equiv \sigma[\rho(x)]$
4. Given $\sigma : U \to V$ and $x \in U$, we have $(\sigma, K)(x) \equiv \sigma(x)$

5. $(\sigma, K)(x_0) \equiv x_0$

where, given an operation $\sigma : U \to V$ and expression E over U, the expression $\sigma[E]$ over V is defined by

$$\sigma[x] \overset{=}{\text{def}} \sigma(x) \sigma[c([x_{11}, \ldots, x_{1r_1}]E_1, \ldots, [x_{n1}, \ldots, x_{nr_n}]E_n)] \quad \overset{=}{\text{def}} c([x_{11}, \ldots, x_{1r_1}](\sigma, K_{11}, \ldots, K_{1r_1})[E_1], \ldots, [$$

where K_{ij} is the kind of x_{ij}.

We say two operations $\rho, \sigma : U \to V$ are *equivalent*, $\rho \sim \sigma$, iff $\rho(x) \equiv \sigma(x)$ for all x. Note that this is equivalent to $\rho[E] \equiv \sigma[E]$ for all E.

```
record PreOpFamily : Set₂ where
  field
    Op : Alphabet → Alphabet → Set
    apV : ∀ {U} {V} {K} → Op U V → Var U K → Expression V (varKind K)
    up : ∀ {V} {K} → Op V (V , K)
    apV-up : ∀ {V} {K} {L} {x : Var V K} → apV (up {K = L}) x ≡ var (↑ x)
    idOp : ∀ V → Op V V
    apV-idOp : ∀ {V} {K} (x : Var V K) → apV (idOp V) x ≡ var x

  _∼op_ : ∀ {U} {V} → Op U V → Op U V → Set
  _∼op_ {U} {V} ρ σ = ∀ {K} (x : Var U K) → apV ρ x ≡ apV σ x

  ∼-refl : ∀ {U} {V} {σ : Op U V} → σ ∼op σ
  ∼-refl _ = refl

  ∼-sym : ∀ {U} {V} {σ τ : Op U V} → σ ∼op τ → τ ∼op σ
  ∼-sym σ-is-τ x = sym (σ-is-τ x)

  ∼-trans : ∀ {U} {V} {ρ σ τ : Op U V} → ρ ∼op σ → σ ∼op τ → ρ ∼op τ
  ∼-trans ρ-is-σ σ-is-τ x = trans (ρ-is-σ x) (σ-is-τ x)

  OP : Alphabet → Alphabet →  Setoid _ _
  OP U V = record {
    Carrier = Op U V ;
    _≈_ = _∼op_ ;
    isEquivalence = record {
      refl = ∼-refl ;
      sym = ∼-sym ;
      trans = ∼-trans } }

  record IsLiftFamily : Set₁ where
    field
      liftOp : ∀ {U} {V} K → Op U V → Op (U , K) (V , K)
      liftOp-cong : ∀ {V} {W} {K} {ρ σ : Op V W} → ρ ∼op σ → liftOp K ρ ∼op liftOp
```

Given an operation $\sigma : U \to V$ and an abstraction kind $(x_1 : A_1, \ldots, x_n : A_n)B$, define the *repeated lifting* σ^A to be $((\cdots(\sigma, A_1), A_2), \cdots), A_n)$.

```
    liftOp' : ∀ {U} {V} A → Op U V → Op (alpha U A) (alpha V A)
    liftOp' (out _) σ = σ
    liftOp' (Π K A) σ = liftOp' A (liftOp K σ)
--TODO Refactor to deal with sequences of kinds instead of abstraction kinds?

    liftOp'-cong : ∀ {U} {V} A {ρ σ : Op U V} → ρ ∼op σ → liftOp' A ρ ∼op liftOp'
    liftOp'-cong (out _) ρ-is-σ = ρ-is-σ
    liftOp'-cong (Π _ A) ρ-is-σ = liftOp'-cong A (liftOp-cong ρ-is-σ)

    ap : ∀ {U} {V} {C} {K} → Op U V → Subexpression U C K → Subexpression V C K
    ap ρ (var x) = apV ρ x
    ap ρ (app c EE) = app c (ap ρ EE)
    ap _ out₂ = out₂
    ap ρ (app₂ {A = A} E EE) = app₂ (ap (liftOp' A ρ) E) (ap ρ EE)

    ap-congl : ∀ {U} {V} {C} {K} {ρ σ : Op U V} (E : Subexpression U C K) →
      ρ ∼op σ → ap ρ E ≡ ap σ E
    ap-congl (var x) ρ-is-σ = ρ-is-σ x
    ap-congl (app c E) ρ-is-σ = cong (app c) (ap-congl E ρ-is-σ)
    ap-congl out₂ _ = refl
    ap-congl (app₂ {A = A} E F) ρ-is-σ = cong₂ app₂ (ap-congl E (liftOp'-cong A ρ-is-

    ap-cong : ∀ {U} {V} {C} {K} {ρ σ : Op U V} {M N : Subexpression U C K} →
      ρ ∼op σ → M ≡ N → ap ρ M ≡ ap σ N
    ap-cong {ρ = ρ} {σ} {M} {N} ρ∼σ M≡N = let open ≡-Reasoning in
      begin
        ap ρ M
      ≡⟨ ap-congl M ρ∼σ ⟩
        ap σ M
      ≡⟨ cong (ap σ) M≡N ⟩
        ap σ N
        □

  record LiftFamily : Set₂ where
    field
      preOpFamily : PreOpFamily
      isLiftFamily : PreOpFamily.IsLiftFamily preOpFamily
    open PreOpFamily preOpFamily public
    open IsLiftFamily isLiftFamily public
```

Let F, G and H be three families of operations. For all U, V, W, let $\circ$ be a function

$$\circ : FVW \times GUV \to HUW$$

Lemma 1. *If $\circ$ respects lifting, then it respects repeated lifting.*

```
  liftOp-liftOp' : ∀ F G H
```

```
    (circ : ∀ {U} {V} {W} → LiftFamily.Op F V W → LiftFamily.Op G U V → LiftFamily.O
    (∀ {U V W K σ ρ} → LiftFamily._∼op_ H (LiftFamily.liftOp H K (circ {U} {V} {W} σ
    ∀ {U V W} A {σ ρ} → LiftFamily._∼op_ H (LiftFamily.liftOp' H A (circ {U} {V} {W}
  liftOp-liftOp' _ _ H circ hyp (out _) = LiftFamily.∼-refl H
  liftOp-liftOp' F G H circ hyp {U} {V} {W} (Π K A) {σ} {ρ} = let open EqReasoning (Li
    begin
      LiftFamily.liftOp' H A (LiftFamily.liftOp H K (circ σ ρ))
    ≈⟨ LiftFamily.liftOp'-cong H A hyp ⟩
      LiftFamily.liftOp' H A (circ (LiftFamily.liftOp F K σ) (LiftFamily.liftOp G K ρ))
    ≈⟨ liftOp-liftOp' F G H circ hyp A ⟩
      circ (LiftFamily.liftOp' F A (LiftFamily.liftOp F K σ)) (LiftFamily.liftOp' G A
      □

  record IsOpFamily (F : LiftFamily) : Set₂ where
    open LiftFamily F public
    field
        liftOp-x₀ : ∀ {U} {V} {K} {σ : Op U V} → apV (liftOp K σ) x₀ ≡ var x₀
        liftOp-↑ : ∀ {U} {V} {K} {L} {σ : Op U V} (x : Var U L) →
          apV (liftOp K σ) (↑ x) ≡ ap up (apV σ x)
        comp : ∀ {U} {V} {W} → Op V W → Op U V → Op U W
        apV-comp : ∀ {U} {V} {W} {K} {σ : Op V W} {ρ : Op U V} {x : Var U K} →
          apV (comp σ ρ) x ≡ ap σ (apV ρ x)
        liftOp-comp : ∀ {U} {V} {W} {K} {σ : Op V W} {ρ : Op U V} →
          liftOp K (comp σ ρ) ∼op comp (liftOp K σ) (liftOp K ρ)
```

The following results about operationsare easy to prove.

Lemma 2. *1.* $(\sigma, K) \circ \uparrow \sim \uparrow \circ \sigma$

2. $(\mathsf{id}_V, K) \sim \mathsf{id}_{V,K}$

3. $\mathsf{id}_V[E] \equiv E$

4. $(\sigma \circ \rho)[E] \equiv \sigma[\rho[E]]$

```
    liftOp-up : ∀ {U} {V} {K} {σ : Op U V} → comp (liftOp K σ) up ∼op comp up σ
    liftOp-up {U} {V} {K} {σ} {L} x =
        let open ≡-Reasoning {A = Expression (V , K) (varKind L)} in
          begin
            apV (comp (liftOp K σ) up) x
          ≡⟨ apV-comp ⟩
            ap (liftOp K σ) (apV up x)
          ≡⟨ cong (ap (liftOp K σ)) apV-up ⟩
            apV (liftOp K σ) (↑ x)
          ≡⟨ liftOp-↑ x ⟩
            ap up (apV σ x)
          ≡⟨⟨ apV-comp ⟩⟩
            apV (comp up σ) x
```

□

```
liftOp-idOp : ∀ {V} {K} → liftOp K (idOp V) ∼op idOp (V , K)
liftOp-idOp {V} {K} x₀ = let open ≡-Reasoning in
    begin
      apV (liftOp K (idOp V)) x₀
    ≡⟨ liftOp-x₀ ⟩
      var x₀
    ≡⟨⟨ apV-idOp x₀ ⟩⟩
      apV (idOp (V , K)) x₀
      □
liftOp-idOp {V} {K} {L} (↑ x) = let open ≡-Reasoning in
    begin
      apV (liftOp K (idOp V)) (↑ x)
    ≡⟨ liftOp-↑ x ⟩
      ap up (apV (idOp V) x)
    ≡⟨ cong (ap up) (apV-idOp x) ⟩
      ap up (var x)
    ≡⟨ apV-up ⟩
      var (↑ x)
    ≡⟨⟨ apV-idOp (↑ x) ⟩⟩
      (apV (idOp (V , K)) (↑ x)
      □)

liftOp'-idOp : ∀ {V} A → liftOp' A (idOp V) ∼op idOp (alpha V A)
liftOp'-idOp (out _) = ∼-refl
liftOp'-idOp {V} (Π K A) = let open EqReasoning (OP (alpha (V , K) A) (alpha (V , 
    begin
      liftOp' A (liftOp K (idOp V))
    ≈⟨ liftOp'-cong A liftOp-idOp ⟩
      liftOp' A (idOp (V , K))
    ≈⟨ liftOp'-idOp A ⟩
      idOp (alpha (V , K) A)
      □

ap-idOp : ∀ {V} {C} {K} {E : Subexpression V C K} → ap (idOp V) E ≡ E
ap-idOp {E = var x} = apV-idOp x
ap-idOp {E = app c EE} = cong (app c) ap-idOp
ap-idOp {E = out₂} = refl
ap-idOp {E = app₂ {A = A} E F} = cong₂ app₂ (trans (ap-congl E (liftOp'-idOp A)) ap

liftOp'-comp : ∀ {U} {V} {W} A {σ : Op U V} {τ : Op V W} → liftOp' A (comp τ σ) ∼
liftOp'-comp A = liftOp-liftOp' F F F comp liftOp-comp A

ap-comp : ∀ {U} {V} {W} {C} {K} (E : Subexpression U C K) {σ : Op V W} {ρ : Op U V
ap-comp (var x) = apV-comp
```

```
ap-comp (app c E) = cong (app c) (ap-comp E)
ap-comp out₂ = refl
ap-comp (app₂ {A = A} E F) = cong₂ app₂ (trans (ap-congl E (liftOp'-comp A)) (ap-co

comp-cong : ∀ {U} {V} {W} {σ σ' : Op V W} {ρ ρ' : Op U V} → σ ∼op σ' → ρ ∼op ρ'
comp-cong {σ = σ} {σ'} {ρ} {ρ'} σ∼σ' ρ∼ρ' x = let open ≡-Reasoning in
    begin
      apV (comp σ ρ) x
    ≡⟨ apV-comp ⟩
      ap σ (apV ρ x)
    ≡⟨ ap-cong σ∼σ' (ρ∼ρ' x) ⟩
      ap σ' (apV ρ' x)
    ≡⟨⟨ apV-comp ⟩⟩
      apV (comp σ' ρ') x
      □
```

The alphabets and operations up to equivalence form a category, which we denote **Op**. The action of application associates, with every operator family, a functor $\mathbf{Op} \to \mathbf{Set}$, which maps an alphabet U to the set of expressions over U, and every operation σ to the function $\sigma[-]$. This functor is faithful and injective on objects, and so **Op** can be seen as a subcategory of **Set**.

```
assoc : ∀ {U} {V} {W} {X} {τ : Op W X} {σ : Op V W} {ρ : Op U V} → comp τ (comp σ
assoc {U} {V} {W} {X} {τ} {σ} {ρ} {K} x = let open ≡-Reasoning {A = Expression X (
    begin
      apV (comp τ (comp σ ρ)) x
    ≡⟨ apV-comp ⟩
      ap τ (apV (comp σ ρ) x)
    ≡⟨ cong (ap τ) apV-comp ⟩
      ap τ (ap σ (apV ρ x))
    ≡⟨⟨ ap-comp (apV ρ x) ⟩⟩
      ap (comp τ σ) (apV ρ x)
    ≡⟨⟨ apV-comp ⟩⟩
      apV (comp (comp τ σ) ρ) x
      □

unitl : ∀ {U} {V} {σ : Op U V} → comp (idOp V) σ ∼op σ
unitl {U} {V} {σ} {K} x = let open ≡-Reasoning {A = Expression V (varKind K)} in
    begin
      apV (comp (idOp V) σ) x
    ≡⟨ apV-comp ⟩
      ap (idOp V) (apV σ x)
    ≡⟨ ap-idOp ⟩
      apV σ x
      □
```

```
unitr : ∀ {U} {V} {σ : Op U V} → comp σ (idOp U) ∼op σ
unitr {U} {V} {σ} {K} x = let open ≡-Reasoning {A = Expression V (varKind K)} in
    begin
      apV (comp σ (idOp U)) x
    ≡⟨ apV-comp ⟩
      ap σ (apV (idOp U) x)
    ≡⟨ cong (ap σ) (apV-idOp x) ⟩
      apV σ x
      □
```

```
record OpFamily : Set₂ where
  field
    liftFamily : LiftFamily
    isOpFamily  : IsOpFamily liftFamily
  open IsOpFamily isOpFamily public
```

2.2 Replacement

The operation family of *replacement* is defined as follows. A replacement $\rho : U \to V$ is a function that maps every variable in U to a variable in V of the same kind. Application, idOpentity and composition are simply function application, the idOpentity function and function composition. The successor is the canonical injection $V \to (V, K)$, and (σ, K) is the extension of σ that maps x_0 to x_0.

```
Rep : Alphabet → Alphabet → Set
Rep U V = ∀ K → Var U K → Var V K

Rep↑ : ∀ {U} {V} {K} → Rep U V → Rep (U , K) (V , K)
Rep↑ _ _ x₀ = x₀
Rep↑ ρ K (↑ x) = ↑ (ρ K x)

upRep : ∀ {V} {K} → Rep V (V , K)
upRep _ = ↑

idOpRep : ∀ V → Rep V V
idOpRep _ _ x = x

pre-replacement : PreOpFamily
pre-replacement = record {
  Op = Rep;
  apV = λ ρ x → var (ρ _ x);
  up = upRep;
  apV-up = refl;
  idOp = idOpRep;
  apV-idOp = λ _ → refl }
```

```
_∼R_ : ∀ {U} {V} → Rep U V → Rep U V → Set
_∼R_ = PreOpFamily._∼op_ pre-replacement

Rep↑-cong : ∀ {U} {V} {K} {ρ ρ' : Rep U V} → ρ ∼R ρ' → Rep↑ {K = K} ρ ∼R Rep↑ ρ'
Rep↑-cong ρ-is-ρ' x₀ = refl
Rep↑-cong ρ-is-ρ' (↑ x) = cong (var ∘ ↑) (var-inj (ρ-is-ρ' x))

proto-replacement : LiftFamily
proto-replacement = record {
  preOpFamily = pre-replacement;
  isLiftFamily = record {
    liftOp = λ _ → Rep↑;
    liftOp-cong = Rep↑-cong }}

infix 60 _⟨_⟩
_⟨_⟩ : ∀ {U} {V} {C} {K} → Subexpression U C K → Rep U V → Subexpression V C K
E ⟨ ρ ⟩ = LiftFamily.ap proto-replacement ρ E

infixl 75 _•R_
_•R_ : ∀ {U} {V} {W} → Rep V W → Rep U V → Rep U W
(ρ' •R ρ) K x = ρ' K (ρ K x)

Rep↑-comp : ∀ {U} {V} {W} {K} {ρ' : Rep V W} {ρ : Rep U V} → Rep↑ {K = K} (ρ' •R ρ)
Rep↑-comp x₀ = refl
Rep↑-comp (↑ _) = refl

replacement : OpFamily
replacement = record {
  liftFamily = proto-replacement;
  isOpFamily = record {
    liftOp-x₀ = refl;
    comp = _•R_;
    apV-comp = refl;
    liftOp-comp = Rep↑-comp;
    liftOp-↑ = λ _ → refl }
  }

rep-cong : ∀ {U} {V} {C} {K} {E : Subexpression U C K} {ρ ρ' : Rep U V} → ρ ∼R ρ' -
rep-cong {U} {V} {C} {K} {E} {ρ} {ρ'} ρ-is-ρ' = OpFamily.ap-congl replacement E ρ-is-

rep-idOp : ∀ {V} {C} {K} {E : Subexpression V C K} → E ⟨ idOpRep V ⟩ ≡ E
rep-idOp = OpFamily.ap-idOp replacement

rep-comp : ∀ {U} {V} {W} {C} {K} {E : Subexpression U C K} {ρ : Rep U V} {σ : Rep V W
  E ⟨ σ •R ρ ⟩ ≡ E ⟨ ρ ⟩ ⟨ σ ⟩
```

```
    rep-comp {U} {V} {W} {C} {K} {E} {ρ} {σ} = OpFamily.ap-comp replacement E

    Rep↑-idOp : ∀ {V} {K} → Rep↑ (idOpRep V) ∼R idOpRep (V , K)
    Rep↑-idOp = OpFamily.liftOp-idOp replacement
--TODO Inline many of these
```

This providOpes us with the canonical mapping from an expression over V to an expression over (V, K):

```
    liftE : ∀ {V} {K} {L} → Expression V L → Expression (V , K) L
    liftE E = E 〈 upRep 〉
--TOOD Inline this
```

2.3 Substitution

A *substitution* σ from alphabet U to alphabet V, $\sigma : U \Rightarrow V$, is a function σ that maps every variable x of kind K in U to an *expression* $\sigma(x)$ of kind K over V. We now aim to prov that the substitutions form a family of operations, with application and idOpentity being simply function application and idOpentity.

```
    Sub : Alphabet → Alphabet → Set
    Sub U V = ∀ K → Var U K → Expression V (varKind K)

    idOpSub : ∀ V → Sub V V
    idOpSub _ _ = var
```

The *successor* substitution $V \to (V, K)$ maps a variable x to itself.

```
    Sub↑ : ∀ {U} {V} {K} → Sub U V → Sub (U , K) (V , K)
    Sub↑ _ _ x0 = var x0
    Sub↑ σ K (↑ x) = (σ K x) 〈 upRep 〉

    pre-substitution : PreOpFamily
    pre-substitution = record {
      Op = Sub;
      apV = λ σ x → σ _ x;
      up = λ _ x → var (↑ x);
      apV-up = refl;
      idOp = λ _ _ → var;
      apV-idOp = λ _ → refl }

    _∼_ : ∀ {U} {V} → Sub U V → Sub U V → Set
    _∼_ = PreOpFamily._∼op_ pre-substitution

    Sub↑-cong : ∀ {U} {V} {K} {σ σ' : Sub U V} → σ ∼ σ' → Sub↑ {K = K} σ ∼ Sub↑ σ'
    Sub↑-cong {K = K} σ-is-σ' x0 = refl
    Sub↑-cong σ-is-σ' (↑ x) = cong (λ E → E 〈 upRep 〉) (σ-is-σ' x)
```

```
proto-substitution : LiftFamily
proto-substitution = record {
  preOpFamily = pre-substitution;
  isLiftFamily = record {
    liftOp = λ _ → Sub↑;
    liftOp-cong = Sub↑-cong }
  }
```

Then, given an expression E of kind K over U, we write $E[\sigma]$ for the application of σ to E, which is the result of substituting $\sigma(x)$ for x for each variable in E, avoidOping capture.

```
infix 60 _⟦_⟧
_⟦_⟧ : ∀ {U} {V} {C} {K} → Subexpression U C K → Sub U V → Subexpression V C K
E ⟦ σ ⟧ = LiftFamily.ap proto-substitution σ E
```

Composition is defined by $(\sigma \circ \rho)(x) \equiv \rho(x)[\sigma]$.

```
infix 75 _•_
_•_ : ∀ {U} {V} {W} → Sub V W → Sub U V → Sub U W
(σ • ρ) K x = ρ K x ⟦ σ ⟧

sub-cong : ∀ {U} {V} {C} {K} {E : Subexpression U C K} {σ σ' : Sub U V} → σ ∼ σ' →
sub-cong {E = E} = LiftFamily.ap-congl proto-substitution E
```

Most of the axioms of a family of operations are easy to verify.

```
infix 75 _•₁_
_•₁_ : ∀ {U} {V} {W} → Rep V W → Sub U V → Sub U W
(ρ •₁ σ) K x = (σ K x) 〈 ρ 〉

Sub↑-comp₁ : ∀ {U} {V} {W} {K} {ρ : Rep V W} {σ : Sub U V} → Sub↑ (ρ •₁ σ) ∼ Rep↑ ρ
Sub↑-comp₁ {K = K} x₀ = refl
Sub↑-comp₁ {U} {V} {W} {K} {ρ} {σ} {L} (↑ x) = let open ≡-Reasoning {A = Expression
  begin
    (σ L x) 〈 ρ 〉 〈 upRep 〉
  ≡⟨⟨ rep-comp {E = σ L x} ⟩⟩
    (σ L x) 〈 upRep •R ρ 〉
  ≡⟨⟩
    (σ L x) 〈 Rep↑ ρ •R upRep 〉
  ≡⟨ rep-comp {E = σ L x} ⟩
    (σ L x) 〈 upRep 〉 〈 Rep↑ ρ 〉
    □

liftOp'-comp₁ : ∀ {U} {V} {W} A {ρ : Rep V W} {σ : Sub U V} →
  LiftFamily.liftOp' proto-substitution A (ρ •₁ σ) ∼ OpFamily.liftOp' replacement A
```

```
  liftOp'-comp₁ = liftOp-liftOp' proto-replacement proto-substitution proto-substitutio

  sub-comp₁ : ∀ {U} {V} {W} {C} {K} {E : Subexpression U C K} {ρ : Rep V W} {σ : Sub U
    E ⟦ ρ •₁ σ ⟧ ≡ E ⟦ σ ⟧ ⟨ ρ ⟩
  sub-comp₁ {E = var _} = refl
  sub-comp₁ {E = app c EE} = cong (app c) (sub-comp₁ {E = EE})
  sub-comp₁ {E = out₂} = refl
  sub-comp₁ {E = app₂ {A = A} E F} {ρ} {σ} = cong₂ app₂
    (let open ≡-Reasoning {A = Expression (alpha _ A) (beta A)} in
    begin
      E ⟦ LiftFamily.liftOp' proto-substitution A (ρ •₁ σ) ⟧
    ≡⟨ LiftFamily.ap-congl proto-substitution E (liftOp'-comp₁ A) ⟩
      E ⟦ OpFamily.liftOp' replacement A ρ •₁ LiftFamily.liftOp' proto-substitution A σ
    ≡⟨ sub-comp₁ {E = E} ⟩
      E ⟦ LiftFamily.liftOp' proto-substitution A σ ⟧ ⟨ OpFamily.liftOp' replacement A ρ
      ∎)
    (sub-comp₁ {E = F})

  infix 75 _•₂_
  _•₂_ : ∀ {U} {V} {W} → Sub V W → Rep U V → Sub U W
  (σ •₂ ρ) K x = σ K (ρ K x)

  Sub↑-comp₂ : ∀ {U} {V} {W} {K} {σ : Sub V W} {ρ : Rep U V} → Sub↑ {K = K} (σ •₂ ρ) ∼
  Sub↑-comp₂ {K = K} x₀ = refl
  Sub↑-comp₂ (↑ x) = refl

  liftOp'-comp₂ : ∀ {U} {V} {W} A {σ : Sub V W} {ρ : Rep U V} → LiftFamily.liftOp' pro
  liftOp'-comp₂ = liftOp-liftOp' proto-substitution proto-replacement proto-substitutio

  sub-comp₂ : ∀ {U} {V} {W} {C} {K} {E : Subexpression U C K} {σ : Sub V W} {ρ : Rep U
  sub-comp₂ {E = var _} = refl
  sub-comp₂ {E = app c EE} = cong (app c) (sub-comp₂ {E = EE})
  sub-comp₂ {E = out₂} = refl
  sub-comp₂ {E = app₂ {A = A} E F} {σ} {ρ} = cong₂ app₂
    (let open ≡-Reasoning {A = Expression (alpha _ A) (beta A)} in
    begin
      E ⟦ LiftFamily.liftOp' proto-substitution A (σ •₂ ρ) ⟧
    ≡⟨ LiftFamily.ap-congl proto-substitution E (liftOp'-comp₂ A) ⟩
      E ⟦ LiftFamily.liftOp' proto-substitution A σ •₂ OpFamily.liftOp' replacement A ρ
    ≡⟨ sub-comp₂ {E = E} ⟩
      E ⟨ OpFamily.liftOp' replacement A ρ ⟩ ⟦ LiftFamily.liftOp' proto-substitution A σ
      ∎)
    (sub-comp₂ {E = F})
--TODO Common pattern with sub-comp₁

  Sub↑-comp : ∀ {U} {V} {W} {ρ : Sub U V} {σ : Sub V W} {K} →
```

```
    Sub↑ {K = K} (σ • ρ) ∼ Sub↑ σ • Sub↑ ρ
  Sub↑-comp x₀ = refl
  Sub↑-comp {W = W} {ρ = ρ} {σ = σ} {K = K} {L} (↑ x) =
    let open ≡-Reasoning {A = Expression (W , K) (varKind L)} in
    begin
      (ρ L x) [ σ ] 〈 upRep 〉
    ≡〈〈 sub-comp₁ {E = ρ L x} 〉〉
      ρ L x [ upRep •₁ σ ]
    ≡〈 sub-comp₂ {E = ρ L x} 〉
      (ρ L x) 〈 upRep 〉 [ Sub↑ σ ]
      □
```

Replacement is a special case of substitution:

Lemma 3. *Let ρ be a replacement $U \to V$.*

1. *The replacement (ρ, K) and the substitution (ρ, K) are equal.*
2. $$E\langle\rho\rangle \equiv E[\rho]$$

```
  Rep↑-is-Sub↑ : ∀ {U} {V} {ρ : Rep U V} {K} → (λ L x → var (Rep↑ {K = K} ρ L x)) ∼
  Rep↑-is-Sub↑ x₀ = refl
  Rep↑-is-Sub↑ (↑ _) = refl

  liftOp'-is-liftOp' : ∀ {U} {V} {ρ : Rep U V} {A} → (λ K x → var (OpFamily.liftOp' r
  liftOp'-is-liftOp' {ρ = ρ} {A = out _} = LiftFamily.∼-refl proto-substitution {σ = λ
  liftOp'-is-liftOp' {U} {V} {ρ} {Π K A} = LiftFamily.∼-trans proto-substitution
    (liftOp'-is-liftOp' {ρ = Rep↑ ρ} {A = A})
    (LiftFamily.liftOp'-cong proto-substitution A (Rep↑-is-Sub↑ {ρ = ρ} {K = K}) )

  rep-is-sub : ∀ {U} {V} {K} {C} {E : Subexpression U K C} {ρ : Rep U V} → E 〈 ρ 〉 ≡ E
  rep-is-sub {E = var _} = refl
  rep-is-sub {E = app c E} = cong (app c) (rep-is-sub {E = E})
  rep-is-sub {E = out₂} = refl
  rep-is-sub {E = app₂ {A = A} E F} {ρ} = cong₂ app₂
    (let open ≡-Reasoning {A = Expression (alpha _ A) (beta A)} in
    begin
      E 〈 OpFamily.liftOp' replacement A ρ 〉
    ≡〈 rep-is-sub {E = E} 〉
      E [ (λ K x → var (OpFamily.liftOp' replacement A ρ K x)) ]
    ≡〈 LiftFamily.ap-congl proto-substitution E (liftOp'-is-liftOp' {A = A}) 〉
      E [ LiftFamily.liftOp' proto-substitution A (λ K x → var (ρ K x)) ]
      □)
    (rep-is-sub {E = F})

  substitution : OpFamily
```

```
substitution = record {
  liftFamily = proto-substitution;
  isOpFamily = record {
    liftOp-x₀ = refl;
    comp = _•_;
    apV-comp = refl;
    liftOp-comp = Sub↑-comp;
    liftOp-↑ = λ {_} {_} {_} {_} {σ} x → rep-is-sub {E = σ _ x}
    }
  }

Sub↑-idOp : ∀ {V} {K} → Sub↑ {V} {V} {K} (idOpSub V) ∼ idOpSub (V , K)
Sub↑-idOp = OpFamily.liftOp-idOp substitution

sub-idOp : ∀ {V} {C} {K} {E : Subexpression V C K} → E ⟦ idOpSub V ⟧ ≡ E
sub-idOp = OpFamily.ap-idOp substitution

sub-comp : ∀ {U} {V} {W} {C} {K} {E : Subexpression U C K} {σ : Sub V W} {ρ : Sub U V
  E ⟦ σ • ρ ⟧ ≡ E ⟦ ρ ⟧ ⟦ σ ⟧
sub-comp {E = E} = OpFamily.ap-comp substitution E

assoc : ∀ {U V W X} {ρ : Sub W X} {σ : Sub V W} {τ : Sub U V} → ρ • (σ • τ) ∼ (ρ •
assoc {τ = τ} = OpFamily.assoc substitution {ρ = τ}

sub-unitl : ∀ {U} {V} {σ : Sub U V} → idOpSub V • σ ∼ σ
sub-unitl {σ = σ} = OpFamily.unitl substitution {σ = σ}

sub-unitr : ∀ {U} {V} {σ : Sub U V} → σ • idOpSub U ∼ σ
sub-unitr {σ = σ} = OpFamily.unitr substitution {σ = σ}
```

Let E be an expression of kind K over V. Then we write $[x_0 := E]$ for the following substitution $(V, K) \Rightarrow V$:

```
x₀:= : ∀ {V} {K} → Expression V (varKind K) → Sub (V , K) V
x₀:= E _ x₀ = E
x₀:= E K₁ (↑ x) = var x
```

Lemma 4. *1.*

$$\rho \bullet_1 [x_0 := E] \sim [x_0 := E\langle\rho\rangle] \bullet_2 (\rho, K)$$

2.

$$\sigma \bullet [x_0 := E] \sim [x_0 := E[\sigma]] \bullet (\sigma, K)$$

```
comp₁-botsub : ∀ {U} {V} {K} {E : Expression U (varKind K)} {ρ : Rep U V} →
  ρ •₁ (x₀:= E) ∼ (x₀:= (E 〈 ρ 〉)) •₂ Rep↑ ρ
comp₁-botsub x₀ = refl
comp₁-botsub (↑ _) = refl
```

```
comp-botsub : ∀ {U} {V} {K} {E : Expression U (varKind K)} {σ : Sub U V} →
  σ • (x₀:= E) ∼ (x₀:= (E ⟦ σ ⟧)) • Sub↑ σ
comp-botsub x₀ = refl
comp-botsub {σ = σ} {L} (↑ x) = trans (sym sub-idOp) (sub-comp₂ {E = σ L x})
```

2.4 Congruences

A *congruence* is a relation R on expressions such that:

1. if MRN, then M and N have the same kind;
2. if $M_i R N_i$ for all i, then $c[[\vec{x_1}]M_1, \ldots, [\vec{x_n}]M_n] R c[[\vec{x_1}]N_1, \ldots, [\vec{x_n}]N_n]$.

```
Relation : Set₁
Relation = ∀ {V} {C} {K} → Subexpression V C K → Subexpression V C K → Set

--TODO Abbreviations for Subexpression V (-Constructor... and Subexpression V -Abstracti
  record IsCongruence (R : Relation) : Set where
    field
      ICapp : ∀ {V} {K} {C} {c} {MM NN : Subexpression V (-Constructor K) C} → R MM NN
      ICout₂ : ∀ {V} {K} → R {V} { -Constructor K} {out₂} out₂ out₂
      ICappl : ∀ {V} {K} {A} {C} {M N : Abstraction V A} {PP : Body V {K} C} → R M N -
      ICappr : ∀ {V} {K} {A} {C} {M : Abstraction V A} {NN PP : Body V {K} C} → R NN P
```

2.5 Contexts

A *context* has the form $x_1 : A_1, \ldots, x_n : A_n$ where, for each i:

- x_i is a variable of kind K_i distinct from $x_1, \ldots, x_{i-1}$;
- A_i is an expression of some kind L_i;
- L_i is a parent of K_i.

The *domain* of this context is the alphabet $\{x_1, \ldots, x_n\}$.

We give ourselves the following operations. Given an alphabet A and finite set F, let extend A K F be the alphabet $A \uplus F$, where each element of F has kind K. Let embedr be the canonical injection $F \rightarrow$ extend A K F; thus, for all $x \in F$, we have embedr x is a variable of extend A K F of kind K.

```
extend : Alphabet → VarKind → ℕ → Alphabet
extend A K zero = A
extend A K (suc F) = extend A K F , K

embedr : ∀ {A} {K} {F} → Fin F → Var (extend A K F) K
embedr zero = x₀
embedr (suc x) = ↑ (embedr x)
```

Let embedl be the canonical injection $A \to$ extend A K F, which is a replacement.

```
embedl : ∀ {A} {K} {F} → Rep A (extend A K F)
embedl {F = zero} _ x = x
embedl {F = suc F} K x = ↑ (embedl {F = F} K x)

data Context (K : VarKind) : Alphabet → Set where
  〈〉 : Context K ∅
  _,_ : ∀ {V} → Context K V → Expression V (parent K) → Context K (V , K)

typeof : ∀ {V} {K} (x : Var V K) (Γ : Context K V) → Expression V (parent K)
typeof x₀ (_ , A) = A 〈 upRep 〉
typeof (↑ x) (Γ , _) = typeof x Γ 〈 upRep 〉

data Context' (A : Alphabet) (K : VarKind) : ℕ → Set where
  〈〉 : Context' A K zero
  _,_ : ∀ {F} → Context' A K F → Expression (extend A K F) (parent K) → Context' A

typeof' : ∀ {A} {K} {F} → Fin F → Context' A K F → Expression (extend A K F) (pare
typeof' zero (_ , A) = A 〈 upRep 〉
typeof' (suc x) (Γ , _) = typeof' x Γ 〈 upRep 〉

record Grammar : Set₁ where
  field
    taxonomy : Taxonomy
    toGrammar : Taxonomy.ToGrammar taxonomy
  open Taxonomy taxonomy public
  open ToGrammar toGrammar public

module PL where

open import Function
open import Data.Empty
open import Data.Product
open import Data.Nat
open import Data.Fin
open import Prelims
open import Grammar
import Reduction
```

3 Propositional Logic

Fix sets of *proof variables* and *term variables*.

The syntax of the system is given by the following grammar.

$$
\begin{array}{llll}
\text{Proof} & \delta & ::= & p \mid \delta\delta \mid \lambda p : \phi.\delta \\
\text{Proposition} & f & ::= & \bot \mid \phi \to \phi \\
\text{Context} & \Gamma & ::= & \langle\rangle \mid \Gamma, p : \phi \\
\text{Judgement} & \mathcal{J} & ::= & \Gamma \vdash \delta : \phi
\end{array}
$$

where p ranges over proof variables and x ranges over term variables. The variable p is bound within δ in the proof $\lambda p : \phi.\delta$, and the variable x is bound within M in the term $\lambda x : A.M$. We identify proofs and terms up to α-conversion.

```
data PLVarKind : Set where
  -Proof : PLVarKind

data PLNonVarKind : Set where
  -Prp   : PLNonVarKind

PLtaxonomy : Taxonomy
PLtaxonomy = record {
  VarKind = PLVarKind;
  NonVarKind = PLNonVarKind }

module PLgrammar where
  open Grammar.Taxonomy PLtaxonomy

  data PLCon : ∀ {K : ExpressionKind} → Kind (-Constructor K) → Set where
    app : PLCon (Π₂ (out (varKind -Proof)) (Π₂ (out (varKind -Proof)) (out₂ {K = varKind
    lam : PLCon (Π₂ (out (nonVarKind -Prp)) (Π₂ (Π -Proof (out (varKind -Proof))) (out₂ {
    bot : PLCon (out₂ {K = nonVarKind -Prp})
    imp : PLCon (Π₂ (out (nonVarKind -Prp)) (Π₂ (out (nonVarKind -Prp)) (out₂ {K = nonVar

  PLparent : VarKind → ExpressionKind
  PLparent -Proof = nonVarKind -Prp

open PLgrammar

Propositional-Logic : Grammar
Propositional-Logic = record {
  taxonomy = PLtaxonomy;
  toGrammar = record {
    Constructor = PLCon;
    parent = PLparent } }

open Grammar.Grammar Propositional-Logic

Prp : Set
Prp = Expression ∅ (nonVarKind -Prp)
```

```
⊥P : Prp
⊥P = app bot out₂

_⇛_ : ∀ {P} → Expression P (nonVarKind -Prp) → Expression P (nonVarKind -Prp) → Expre
φ ⇛ ψ = app imp (app₂ φ (app₂ ψ out₂))

Proof : Alphabet → Set
Proof P = Expression P (varKind -Proof)

appP : ∀ {P} → Expression P (varKind -Proof) → Expression P (varKind -Proof) → Express
appP δ ε = app app (app₂ δ (app₂ ε out₂))

ΛP : ∀ {P} → Expression P (nonVarKind -Prp) → Expression (P , -Proof) (varKind -Proof)
ΛP φ δ = app lam (app₂ φ (app₂ δ out₂))

data β : ∀ {V} {K} {C : Kind (-Constructor K)} → Constructor C → Subexpression V (-Cons
  βI : ∀ {V} {φ} {δ} {ε} → β {V} app (app₂ (ΛP φ δ) (app₂ ε out₂)) (δ ⟦ x₀:= ε ⟧)

open Reduction Propositional-Logic β

β-respects-rep : Respects-Creates.respects' replacement
β-respects-rep {U} {V} {σ = ρ} (βI .{U} {φ} {δ} {ε}) = subst (β app _)
  (let open ≡-Reasoning {A = Expression V (varKind -Proof)} in
  begin
    δ 〈 Rep↑ ρ 〉 ⟦ x₀:= (ε 〈 ρ 〉) ⟧
  ≡⟨⟨ sub-comp₂ {E = δ} ⟩⟩
    δ ⟦ x₀:= (ε 〈 ρ 〉) •₂ Rep↑ ρ ⟧
  ≡⟨⟨ sub-cong {E = δ} comp₁-botsub ⟩⟩
    δ ⟦ ρ •₁ x₀:= ε ⟧
  ≡⟨ sub-comp₁ {E = δ} ⟩
    δ ⟦ x₀:= ε ⟧ 〈 ρ 〉
    ∎)
  βI

β-creates-rep : Respects-Creates.creates' replacement
β-creates-rep {c = app} (app₂ (var _) _) ()
β-creates-rep {c = app} (app₂ (app app _) _) ()
β-creates-rep {c = app} (app₂ (app lam (app₂ A (app₂ δ out₂))) (app₂ ε out₂)) {σ = σ} βI
  created = δ ⟦ x₀:= ε ⟧ ;
  red-created = βI ;
  ap-created = let open ≡-Reasoning {A = Expression _ (varKind -Proof)} in
    begin
      δ ⟦ x₀:= ε ⟧ 〈 σ 〉
    ≡⟨⟨ sub-comp₁ {E = δ} ⟩⟩
      δ ⟦ σ •₁ x₀:= ε ⟧
    ≡⟨ sub-cong {E = δ} comp₁-botsub ⟩
```

```
      δ [ x₀:= (ε ⟨ σ ⟩) •₂ Rep↑ σ ]
    ≡⟨ sub-comp₂ {E = δ} ⟩
      δ ⟨ Rep↑ σ ⟩ [ x₀:= (ε ⟨ σ ⟩) ]
      □ }
β-creates-rep {c = lam} _ ()
β-creates-rep {c = bot} _ ()
β-creates-rep {c = imp} _ ()
```

The rules of deduction of the system are as follows.

$$\frac{\Gamma \text{ valid}}{\Gamma \vdash p : \phi} \ (p : \phi \in \Gamma)$$

$$\frac{\Gamma \vdash \delta : \phi \to \psi}{\Gamma \vdash \delta\epsilon : \psi} \ \Gamma \vdash \epsilon : \phi$$

$$\frac{\Gamma, p : \phi \vdash \delta : \psi}{\Gamma \vdash \lambda p : \phi . \delta : \phi \to \psi}$$

```
PContext : ℕ → Set
PContext P = Context' ∅ -Proof P

Palphabet : ℕ → Alphabet
Palphabet P = extend ∅ -Proof P

Palphabet-faithful : ∀ {P} {Q} {ρ σ : Rep (Palphabet P) (Palphabet Q)} → (∀ x → ρ -Pro
Palphabet-faithful {zero} _ ()
Palphabet-faithful {suc _} ρ-is-σ x₀ = cong var (ρ-is-σ zero)
Palphabet-faithful {suc _} {Q} {ρ} {σ} ρ-is-σ (↑ x) = Palphabet-faithful {Q = Q} {ρ = ρ

infix 10 _⊢_::_
data _⊢_::_ : ∀ {P} → PContext P → Proof (Palphabet P) → Expression (Palphabet P) (non
  var : ∀ {P} {Γ : PContext P} {p : Fin P} → Γ ⊢ var (embedr p) :: typeof' p Γ
  app : ∀ {P} {Γ : PContext P} {δ} {ε} {φ} {ψ} → Γ ⊢ δ :: φ ⇛ ψ → Γ ⊢ ε :: φ → Γ ⊢ app
  Λ : ∀ {P} {Γ : PContext P} {φ} {δ} {ψ} → (_,_ {K = -Proof} Γ φ) ⊢ δ :: liftE ψ → Γ ⊢
```

A *replacement* ρ from a context Γ to a context Δ, $\rho : \Gamma \to \Delta$, is a replacement on the syntax such that, for every $x : \phi$ in Γ, we have $\rho(x) : \phi \in \Delta$.

```
toRep : ∀ {P} {Q} → (Fin P → Fin Q) → Rep (Palphabet P) (Palphabet Q)
toRep {zero} f K ()
toRep {suc P} f .-Proof x₀ = embedr (f zero)
toRep {suc P} {Q} f K (↑ x) = toRep {P} {Q} (f ∘ suc) K x

toRep-embedr : ∀ {P} {Q} {f : Fin P → Fin Q} {x : Fin P} → toRep f -Proof (embedr x) ≡
toRep-embedr {zero} {_} {_} {()}
toRep-embedr {suc _} {_} {_} {zero} = refl
```

```
toRep-embedr {suc P} {Q} {f} {suc x} = toRep-embedr {P} {Q} {f ∘ suc} {x}

toRep-comp : ∀ {P} {Q} {R} {g : Fin Q → Fin R} {f : Fin P → Fin Q} → toRep g •R toRep
toRep-comp {zero} ()
toRep-comp {suc _} {g = g} x₀ = cong var (toRep-embedr {f = g})
toRep-comp {suc _} {g = g} {f = f} (↑ x) = toRep-comp {g = g} {f = f ∘ suc} x

_::_⇒R_ : ∀ {P} {Q} → (Fin P → Fin Q) → PContext P → PContext Q → Set
ρ :: Γ ⇒R Δ = ∀ x → typeof' (ρ x) Δ ≡ (typeof' x Γ) ⟨ toRep ρ ⟩

toRep-↑ : ∀ {P} → toRep {P} {suc P} suc ∼R (λ _ → ↑)
toRep-↑ {zero} = λ ()
toRep-↑ {suc P} = Palphabet-faithful {suc P} {suc (suc P)} {toRep {suc P} {suc (suc P)} s

toRep-lift : ∀ {P} {Q} {f : Fin P → Fin Q} → toRep (lift (suc zero) f) ∼R Rep↑ (toRep
toRep-lift x₀ = refl
toRep-lift {zero} (↑ ())
toRep-lift {suc _} (↑ x₀) = refl
toRep-lift {suc P} {Q} {f} (↑ (↑ x)) = trans
  (sym (toRep-comp {g = suc} {f = f ∘ suc} x))
  (toRep-↑ {Q} (toRep (f ∘ suc) _ x))

↑-typed : ∀ {P} {Γ : PContext P} {φ : Expression (Palphabet P) (nonVarKind -Prp)} →
  suc :: Γ ⇒R (Γ , φ)
↑-typed {P} {Γ} {φ} x = rep-cong {E = typeof' x Γ} (λ x → sym (toRep-↑ {P} x))

Rep↑-typed : ∀ {P} {Q} {ρ} {Γ : PContext P} {Δ : PContext Q} {φ : Expression (Palphabet P
  lift 1 ρ :: (Γ , φ) ⇒R (Δ , φ ⟨ toRep ρ ⟩)
Rep↑-typed {P} {Q = Q} {ρ = ρ} {φ = φ} ρ::Γ→Δ zero =
  let open ≡-Reasoning {A = Expression (Palphabet Q , -Proof) (nonVarKind -Prp)} in
  begin
    liftE (φ ⟨ toRep ρ ⟩)
  ≡⟨⟨ rep-comp {E = φ} ⟩⟩
    φ ⟨ upRep •R toRep ρ ⟩
  ≡⟨⟨ rep-cong {E = φ} (OpFamily.liftOp-up replacement {σ = toRep ρ}) ⟩⟩
    φ ⟨ Rep↑ (toRep ρ) •R upRep ⟩
  ≡⟨⟨ rep-cong {E = φ} (OpFamily.comp-cong replacement {σ = toRep (lift 1 ρ)} toRep-lift
    φ ⟨ toRep (lift 1 ρ) •R upRep ⟩
  ≡⟨ rep-comp {E = φ} ⟩
    (liftE φ) ⟨ toRep (lift 1 ρ) ⟩
    □
Rep↑-typed {Q = Q} {ρ = ρ} {Γ = Γ} {Δ = Δ} ρ::Γ→Δ (suc x) = let open ≡-Reasoning {A = Ex
  begin
    liftE (typeof' (ρ x) Δ)
  ≡⟨ cong liftE (ρ::Γ→Δ x) ⟩
    liftE ((typeof' x Γ) ⟨ toRep ρ ⟩)
```

```
  ≡⟨⟨ rep-comp {E = typeof' x Γ} ⟩⟩
    (typeof' x Γ) ⟨ (λ K x → ↑ (toRep ρ K x)) ⟩
  ≡⟨⟨ rep-cong {E = typeof' x Γ} (λ x → toRep-↑ {Q} (toRep ρ _ x)) ⟩⟩
    (typeof' x Γ) ⟨ toRep {Q} suc •R toRep ρ ⟩
  ≡⟨ rep-cong {E = typeof' x Γ} (toRep-comp {g = suc} {f = ρ}) ⟩
    (typeof' x Γ) ⟨ toRep (lift 1 ρ) •R (λ _ → ↑) ⟩
  ≡⟨ rep-comp {E = typeof' x Γ} ⟩
    (liftE (typeof' x Γ)) ⟨ toRep (lift 1 ρ) ⟩
    □
```

The replacements between contexts are closed under composition.

```
•R-typed : ∀ {P} {Q} {R} {σ : Fin Q → Fin R} {ρ : Fin P → Fin Q} {Γ} {Δ} {Θ} → ρ :: Γ =
  (σ ∘ ρ) :: Γ ⇒R Θ
•R-typed {R = R} {σ} {ρ} {Γ} {Δ} {Θ} ρ::Γ→Δ σ::Δ→Θ x = let open ≡-Reasoning {A = Express
  begin
    typeof' (σ (ρ x)) Θ
  ≡⟨ σ::Δ→Θ (ρ x) ⟩
    (typeof' (ρ x) Δ) ⟨ toRep σ ⟩
  ≡⟨ cong (λ x₁ → x₁ ⟨ toRep σ ⟩) (ρ::Γ→Δ x) ⟩
    typeof' x Γ ⟨ toRep ρ ⟩ ⟨ toRep σ ⟩
  ≡⟨⟨ rep-comp {E = typeof' x Γ} ⟩⟩
    typeof' x Γ ⟨ toRep σ •R toRep ρ ⟩
  ≡⟨ rep-cong {E = typeof' x Γ} (toRep-comp {g = σ} {f = ρ}) ⟩
    typeof' x Γ ⟨ toRep (σ ∘ ρ) ⟩
    □
```

Weakening Lemma

```
Weakening : ∀ {P} {Q} {Γ : PContext P} {Δ : PContext Q} {ρ} {δ} {φ} → Γ ⊢ δ :: φ → ρ ::
Weakening {P} {Q} {Γ} {Δ} {ρ} (var {p = p}) ρ::Γ→Δ = subst₂ (λ x y → Δ ⊢ var x :: y)
  (sym (toRep-embedr {f = ρ} {x = p}))
  (ρ::Γ→Δ p)
  (var {p = ρ p})
Weakening (app Γ⊢δ::φ→ψ Γ⊢ε::φ) ρ::Γ→Δ = app (Weakening Γ⊢δ::φ→ψ ρ::Γ→Δ) (Weakening Γ⊢ε::φ
Weakening .{P} {Q} .{Γ} {Δ} {ρ} (Λ {P} {Γ} {φ} {δ} {ψ} Γ,φ⊢δ::ψ) ρ::Γ→Δ = Λ
  (subst (λ P → (Δ , φ ⟨ toRep ρ ⟩) ⊢ δ ⟨ Rep↑ (toRep ρ) ⟩ :: P)
  (let open ≡-Reasoning {A = Expression (Palphabet Q , -Proof) (nonVarKind -Prp)} in
  begin
    liftE ψ ⟨ Rep↑ (toRep ρ) ⟩
  ≡⟨⟨ rep-comp {E = ψ} ⟩⟩
    ψ ⟨ (λ _ x → ↑ (toRep ρ _ x)) ⟩
  ≡⟨ rep-comp {E = ψ} ⟩
    liftE (ψ ⟨ toRep ρ ⟩)
    □)
  (subst₂ (λ x y → (Δ , φ ⟨ toRep ρ ⟩) ⊢ x :: y)
    (rep-cong {E = δ} (toRep-lift {f = ρ}))
```

```
    (rep-cong {E = liftE ψ} (toRep-lift {f = ρ}))
    (Weakening {suc P} {suc Q} {Γ , φ} {Δ , φ ⟨ toRep ρ ⟩} {lift 1 ρ} {δ} {liftE ψ}
      Γ,φ⊢δ::ψ
      claim))) where
  claim : ∀ (x : Fin (suc P)) → typeof' (lift 1 ρ x) (Δ , φ ⟨ toRep ρ ⟩) ≡ typeof' x (Γ
  claim zero = let open ≡-Reasoning {A = Expression (Palphabet (suc Q)) (nonVarKind -Prp
    begin
      liftE (φ ⟨ toRep ρ ⟩)
    ≡⟨⟨ rep-comp {E = φ} ⟩⟩
      φ ⟨ (λ _ → ↑) •R toRep ρ ⟩
    ≡⟨ rep-comp {E = φ} ⟩
      liftE φ ⟨ Rep↑ (toRep ρ) ⟩
    ≡⟨⟨ rep-cong {E = liftE φ} (toRep-lift {f = ρ}) ⟩⟩
      liftE φ ⟨ toRep (lift 1 ρ) ⟩
      □
  claim (suc x) = let open ≡-Reasoning {A = Expression (Palphabet (suc Q)) (nonVarKind -
    begin
      liftE (typeof' (ρ x) Δ)
    ≡⟨ cong liftE (ρ::Γ→Δ x) ⟩
      liftE (typeof' x Γ ⟨ toRep ρ ⟩)
    ≡⟨⟨ rep-comp {E = typeof' x Γ} ⟩⟩
      typeof' x Γ ⟨ (λ _ → ↑) •R toRep ρ ⟩
    ≡⟨ rep-comp {E = typeof' x Γ} ⟩
      liftE (typeof' x Γ) ⟨ Rep↑ (toRep ρ) ⟩
    ≡⟨⟨ rep-cong {E = liftE (typeof' x Γ)} (toRep-lift {f = ρ}) ⟩⟩
      liftE (typeof' x Γ) ⟨ toRep (lift 1 ρ) ⟩
      □
```

A *substitution* σ from a context Γ to a context Δ, $\sigma : \Gamma \to \Delta$, is a substitution σ on the syntax such that, for every $x : \phi$ in Γ, we have $\Delta \vdash \sigma(x) : \phi$.

```
_::_⇒_ : ∀ {P} {Q} → Sub (Palphabet P) (Palphabet Q) → PContext P → PContext Q → Set
σ :: Γ ⇒ Δ = ∀ x → Δ ⊢ σ _ (embedr x) :: typeof' x Γ ⟦ σ ⟧

Sub↑-typed : ∀ {P} {Q} {σ} {Γ : PContext P} {Δ : PContext Q} {φ : Expression (Palphabet P
Sub↑-typed {P} {Q} {σ} {Γ} {Δ} {φ} σ::Γ→Δ zero = subst (λ p → (Δ , φ ⟦ σ ⟧) ⊢ var x₀ :: p
  (let open ≡-Reasoning {A = Expression (Palphabet Q , -Proof) (nonVarKind -Prp)} in
  begin
    liftE (φ ⟦ σ ⟧)
  ≡⟨⟨ sub-comp₁ {E = φ} ⟩⟩
    φ ⟦ (λ _ → ↑) •₁ σ ⟧
  ≡⟨ sub-comp₂ {E = φ} ⟩
    liftE φ ⟦ Sub↑ σ ⟧
    □)
  (var {p = zero})
Sub↑-typed {Q = Q} {σ = σ} {Γ = Γ} {Δ = Δ} {φ = φ} σ::Γ→Δ (suc x) =
```

```
  subst
  (λ P → (Δ , φ ⟦ σ ⟧) ⊢ Sub↑ σ -Proof (↑ (embedr x)) ∷ P)
  (let open ≡-Reasoning {A = Expression (Palphabet Q , -Proof) (nonVarKind -Prp)} in
  begin
    liftE (typeof' x Γ ⟦ σ ⟧)
  ≡⟨⟨ sub-comp₁ {E = typeof' x Γ} ⟩⟩
    typeof' x Γ ⟦ (λ _ → ↑) •₁ σ ⟧
  ≡⟨ sub-comp₂ {E = typeof' x Γ} ⟩
    liftE (typeof' x Γ) ⟦ Sub↑ σ ⟧
    ∎)
  (subst₂ (λ x y → (Δ , φ ⟦ σ ⟧) ⊢ x ∷ y)
    (rep-cong {E = σ -Proof (embedr x)} (toRep-↑ {Q}))
    (rep-cong {E = typeof' x Γ ⟦ σ ⟧} (toRep-↑ {Q}))
    (Weakening (σ∷Γ→Δ x) (↑-typed {φ = φ ⟦ σ ⟧})))

botsub-typed : ∀ {P} {Γ : PContext P} {φ : Expression (Palphabet P) (nonVarKind -Prp)} {δ
  Γ ⊢ δ ∷ φ → x₀:= δ ∷ (Γ , φ) ⇒ Γ
botsub-typed {P} {Γ} {φ} {δ} Γ⊢δ∷φ zero = subst (λ P₁ → Γ ⊢ δ ∷ P₁)
  (let open ≡-Reasoning {A = Expression (Palphabet P) (nonVarKind -Prp)} in
  begin
    φ
  ≡⟨⟨ sub-idOp ⟩⟩
    φ ⟦ idOpSub _ ⟧
  ≡⟨ sub-comp₂ {E = φ} ⟩
    liftE φ ⟦ x₀:= δ ⟧
    ∎)
  Γ⊢δ∷φ
botsub-typed {P} {Γ} {φ} {δ} _ (suc x) = subst (λ P₁ → Γ ⊢ var (embedr x) ∷ P₁)
  (let open ≡-Reasoning {A = Expression (Palphabet P) (nonVarKind -Prp)} in
  begin
    typeof' x Γ
  ≡⟨⟨ sub-idOp ⟩⟩
    typeof' x Γ ⟦ idOpSub _ ⟧
  ≡⟨ sub-comp₂ {E = typeof' x Γ} ⟩
    liftE (typeof' x Γ) ⟦ x₀:= δ ⟧
    ∎)
  var
```

Substitution Lemma

```
Substitution : ∀ {P} {Q} {Γ : PContext P} {Δ : PContext Q} {δ} {φ} {σ} → Γ ⊢ δ ∷ φ → σ
Substitution var σ∷Γ→Δ = σ∷Γ→Δ _
Substitution (app Γ⊢δ∷φ→ψ Γ⊢ε∷φ) σ∷Γ→Δ = app (Substitution Γ⊢δ∷φ→ψ σ∷Γ→Δ) (Substitut
Substitution {Q = Q} {Δ = Δ} {σ = σ} (Λ {P} {Γ} {φ} {δ} {ψ} Γ,φ⊢δ∷ψ) σ∷Γ→Δ = Λ
  (subst (λ p → (Δ , φ ⟦ σ ⟧) ⊢ δ ⟦ Sub↑ σ ⟧ ∷ p)
  (let open ≡-Reasoning {A = Expression (Palphabet Q , -Proof) (nonVarKind -Prp)} in
```

```
  begin
    liftE ψ [ Sub↑ σ ]
  ≡⟨⟨ sub-comp₂ {E = ψ} ⟩⟩
    ψ [ Sub↑ σ •₂ (λ _ → ↑) ]
  ≡⟨ sub-comp₁ {E = ψ} ⟩
    liftE (ψ [ σ ])
    □)
  (Substitution Γ,φ⊢δ::ψ (Sub↑-typed σ::Γ→Δ)))
```

Subject Reduction

```
prop-triv-red : ∀ {P} {φ ψ : Expression (Palphabet P) (nonVarKind -Prp)} → φ ⇒ ψ → ⊥
prop-triv-red {_} {app bot out₂} (redex ())
prop-triv-red {P} {app bot out₂} (app ())
prop-triv-red {P} {app imp (app₂ _ (app₂ _ out₂))} (redex ())
prop-triv-red {P} {app imp (app₂ φ (app₂ ψ out₂))} (app (appl φ→φ')) = prop-triv-red {P}
prop-triv-red {P} {app imp (app₂ φ (app₂ ψ out₂))} (app (appr (appl ψ→ψ'))) = prop-triv-
prop-triv-red {P} {app imp (app₂ _ (app₂ _ out₂))} (app (appr (appr ())))

SR : ∀ {P} {Γ : PContext P} {δ ε : Proof (Palphabet P)} {φ} → Γ ⊢ δ :: φ → δ ⇒ ε → Γ ⊢
SR var ()
SR (app {ε = ε} (Λ {P} {Γ} {φ} {δ} {ψ} Γ,φ⊢δ::ψ) Γ⊢ε::φ) (redex βI) =
  subst (λ P₁ → Γ ⊢ δ [ x₀:= ε ] :: P₁)
  (let open ≡-Reasoning {A = Expression (Palphabet P) (nonVarKind -Prp)} in
  begin
    liftE ψ [ x₀:= ε ]
  ≡⟨⟨ sub-comp₂ {E = ψ} ⟩⟩
    ψ [ idOpSub _ ]
  ≡⟨ sub-idOp ⟩
    ψ
    □)
  (Substitution Γ,φ⊢δ::ψ (botsub-typed Γ⊢ε::φ))
SR (app Γ⊢δ::φ→ψ Γ⊢ε::φ) (app (appl δ→δ')) = app (SR Γ⊢δ::φ→ψ δ→δ') Γ⊢ε::φ
SR (app Γ⊢δ::φ→ψ Γ⊢ε::φ) (app (appr (appl ε→ε'))) = app Γ⊢δ::φ→ψ (SR Γ⊢ε::φ ε→ε')
SR (app Γ⊢δ::φ→ψ Γ⊢ε::φ) (app (appr (appr ())))
SR (Λ _) (redex ())
SR (Λ {P = P} {φ = φ} {δ = δ} {ψ = ψ} Γ⊢δ::φ) (app (appl {N = φ'} δ→ε)) = ⊥-elim (prop-t
SR (Λ Γ⊢δ::φ) (app (appr (appl δ→ε))) = Λ (SR Γ⊢δ::φ δ→ε)
SR (Λ _) (app (appr (appr ())))
```

We define the sets of *computable* proofs $C_\Gamma(\phi)$ for each context Γ and proposition ϕ as follows:

$$
\begin{aligned}
C_\Gamma(\bot) &= \{\delta \mid \Gamma \vdash \delta : \bot, \delta \in SN\} \\
C_\Gamma(\phi \to \psi) &= \{\delta \mid \Gamma : \delta : \phi \to \psi, \forall \epsilon \in C_\Gamma(\phi). \delta\epsilon \in C_\Gamma(\psi)\}
\end{aligned}
$$

```
C : ∀ {P} → PContext P → Prp → Proof (Palphabet P) → Set
C Γ (app bot out₂) δ = (Γ ⊢ δ :: ⊥P 〈 (λ _ ()) 〉 ) × SN δ
C Γ (app imp (app₂ φ (app₂ ψ out₂))) δ = (Γ ⊢ δ :: (φ ⇛ ψ) 〈 (λ _ ()) 〉) ×
  (∀ Q {Δ : PContext Q} ρ ε → ρ :: Γ ⇒R Δ → C Δ φ ε → C Δ ψ (appP (δ 〈 toRep ρ 〉) ε))

C-typed : ∀ {P} {Γ : PContext P} {φ} {δ} → C Γ φ δ → Γ ⊢ δ :: φ 〈 (λ _ ()) 〉
C-typed {φ = app bot out₂} = proj₁
C-typed {Γ = Γ} {φ = app imp (app₂ φ (app₂ ψ out₂))} {δ = δ} = λ x → subst (λ P → Γ ⊢ δ
  (cong₂ _⇛_ (rep-cong {E = φ} (λ ())) (rep-cong {E = ψ} (λ ())))
  (proj₁ x)

C-rep : ∀ {P} {Q} {Γ : PContext P} {Δ : PContext Q} {φ} {δ} {ρ} → C Γ φ δ → ρ :: Γ ⇒R Δ
C-rep {φ = app bot out₂} (Γ⊢δ::x₀ , SNδ) ρ::Γ→Δ = (Weakening Γ⊢δ::x₀ ρ::Γ→Δ) , SNap β-creat
C-rep {P} {Q} {Γ} {Δ} {app imp (app₂ φ (app₂ ψ out₂))} {δ} {ρ} (Γ⊢δ::φ⇒ψ , Cδ) ρ::Γ→Δ = (
  (λ x → Δ ⊢ δ 〈 toRep ρ 〉 :: x)
  (cong₂ _⇛_
  (let open ≡-Reasoning {A = Expression (Palphabet Q) (nonVarKind -Prp)} in
    begin
      (φ 〈 _ 〉) 〈 toRep ρ 〉
    ≡⟨⟨ rep-comp {E = φ} ⟩⟩
      φ 〈 _ 〉
    ≡⟨ rep-cong {E = φ} (λ ()) ⟩
      φ 〈 _ 〉
      ∎)
--TODO Refactor common pattern
  (let open ≡-Reasoning {A = Expression (Palphabet Q) (nonVarKind -Prp)} in
    begin
      ψ 〈 _ 〉 〈 toRep ρ 〉
    ≡⟨⟨ rep-comp {E = ψ} ⟩⟩
      ψ 〈 _ 〉
    ≡⟨ rep-cong {E = ψ} (λ ()) ⟩
      ψ 〈 _ 〉
      ∎))
  (Weakening Γ⊢δ::φ⇒ψ ρ::Γ→Δ)) ,
  (λ R σ ε σ::Δ→Θ ε∈Cφ → subst (C _ ψ) (cong (λ x → appP x ε)
    (trans (sym (rep-cong {E = δ} (toRep-comp {g = σ} {f = ρ}))) (rep-comp {E = δ})))
    (Cδ R (σ ∘ ρ) ε (•R-typed {σ = σ} {ρ = ρ} ρ::Γ→Δ σ::Δ→Θ) ε∈Cφ))

C-red : ∀ {P} {Γ : PContext P} {φ} {δ} {ε} → C Γ φ δ → δ ⇒ ε → C Γ φ ε
C-red {φ = app bot out₂} (Γ⊢δ::x₀ , SNδ) δ→ε = (SR Γ⊢δ::x₀ δ→ε) , (SNred SNδ (osr-red δ—
C-red {Γ = Γ} {φ = app imp (app₂ φ (app₂ ψ out₂))} {δ = δ} (Γ⊢δ::φ⇒ψ , Cδ) δ→δ' = (SR (s
  (cong₂ _⇛_ (rep-cong {E = φ} (λ ())) (rep-cong {E = ψ} (λ ())))
  Γ⊢δ::φ⇒ψ) δ→δ') ,
  (λ Q ρ ε ρ::Γ→Δ ε∈Cφ → C-red {φ = ψ} (Cδ Q ρ ε ρ::Γ→Δ ε∈Cφ) (app (appl (Respects-Crea
```

The *neutral terms* are those that begin with a variable.

```
data Neutral {P} : Proof P → Set where
  varNeutral : ∀ x → Neutral (var x)
  appNeutral : ∀ δ ε → Neutral δ → Neutral (appP δ ε)
```

Lemma 5. *If δ is neutral and $\delta \to_\beta \epsilon$ then ϵ is neutral.*

```
neutral-red : ∀ {P} {δ ε : Proof P} → Neutral δ → δ ⇒ ε → Neutral ε
neutral-red (varNeutral _) ()
neutral-red (appNeutral .(app lam (app₂ _ (app₂ _ out₂))) _ ()) (redex βI)
neutral-red (appNeutral _ ε neutralδ) (app (appl δ→δ')) = appNeutral _ ε (neutral-red ne
neutral-red (appNeutral δ _ neutralδ) (app (appr (appl ε→ε'))) = appNeutral δ _ neutralδ
neutral-red (appNeutral _ _ _) (app (appr (appr ())))

neutral-rep : ∀ {P} {Q} {δ : Proof P} {ρ : Rep P Q} → Neutral δ → Neutral (δ 〈 ρ 〉)
neutral-rep {ρ = ρ} (varNeutral x) = varNeutral (ρ -Proof x)
neutral-rep {ρ = ρ} (appNeutral δ ε neutralδ) = appNeutral (δ 〈 ρ 〉) (ε 〈 ρ 〉) (neutral-re
```

Lemma 6. *Let $\Gamma \vdash \delta : \phi$. If δ is neutral and, for all ϵ such that $\delta \to_\beta \epsilon$, we have $\epsilon \in C_\Gamma(\phi)$, then $\delta \in C_\Gamma(\phi)$.*

```
NeutralC-lm : ∀ {P} {δ ε : Proof P} {X : Proof P → Set} →
  Neutral δ →
  (∀ δ' → δ ⇒ δ' → X (appP δ' ε)) →
  (∀ ε' → ε ⇒ ε' → X (appP δ ε')) →
  ∀ χ → appP δ ε ⇒ χ → X χ
NeutralC-lm () _ _ ._ (redex βI)
NeutralC-lm _ hyp1 _ .(app app (app₂ _ (app₂ _ out₂))) (app (appl δ→δ')) = hyp1 _ δ→δ'
NeutralC-lm _ _ hyp2 .(app app (app₂ _ (app₂ _ out₂))) (app (appr (appl ε→ε'))) = hyp2 _
NeutralC-lm _ _ _ .(app app (app₂ _ (app₂ _ _))) (app (appr (appr ())))

mutual
  NeutralC : ∀ {P} {Γ : PContext P} {δ : Proof (Palphabet P)} {φ : Prp} →
    Γ ⊢ δ :: φ 〈 (λ _ ()) 〉 → Neutral δ →
    (∀ ε → δ ⇒ ε → C Γ φ ε) →
    C Γ φ δ
  NeutralC {P} {Γ} {δ} {app bot out₂} Γ⊢δ::x₀ Neutralδ hyp = Γ⊢δ::x₀ , SNI δ (λ ε δ→ε → p
  NeutralC {P} {Γ} {δ} {app imp (app₂ φ (app₂ ψ out₂))} Γ⊢δ::φ→ψ neutralδ hyp = (subst (λ
    (λ Q ρ ε ρ::Γ→Δ ε∈Cφ → claim ε (CsubSN {φ = φ} {δ = ε} ε∈Cφ) ρ::Γ→Δ ε∈Cφ) where
    claim : ∀ {Q} {Δ} {ρ : Fin P → Fin Q} ε → SN ε → ρ :: Γ ⇒R Δ → C Δ φ ε → C Δ ψ (
    claim {Q} {Δ} {ρ} ε (SNI .ε SNε) ρ::Γ→Δ ε∈Cφ = NeutralC {Q} {Δ} {appP (δ 〈 toRep ρ 〉)
      (app (subst (λ P₁ → Δ ⊢ δ 〈 toRep ρ 〉 :: P₁)
      (cong₂ _⇒_
      (let open ≡-Reasoning {A = Expression (Palphabet Q) (nonVarKind -Prp)} in
        begin
          φ 〈 _ 〉 〈 toRep ρ 〉
        ≡〈〈 rep-comp {E = φ} 〉〉
          φ 〈 _ 〉
```

```
        ≡⟨⟨ rep-cong {E = φ} (λ ()) ⟩⟩
          φ ⟨ _ ⟩
          □)
      (  (let open ≡-Reasoning {A = Expression (Palphabet Q) (nonVarKind -Prp)} in
        begin
          ψ ⟨ _ ⟩ ⟨ toRep ρ ⟩
        ≡⟨⟨ rep-comp {E = ψ} ⟩⟩
          ψ ⟨ _ ⟩
        ≡⟨⟨ rep-cong {E = ψ} (λ ()) ⟩⟩
          ψ ⟨ _ ⟩
          □)
        ))
      (Weakening Γ⊢δ::φ→ψ ρ::Γ→Δ))
      (C-typed {Q} {Δ} {φ} {ε} ε∈Cφ))
      (appNeutral (δ ⟨ toRep ρ ⟩) ε (neutral-rep neutralδ))
      (NeutralC-lm {X = C Δ ψ} (neutral-rep neutralδ)
      (λ δ' δ⟨ρ⟩→δ' →
        let δ-creation = create-osr β-creates-rep δ δ⟨ρ⟩→δ' in
        let δ0 : Proof (Palphabet P)
            δ0 = Respects-Creates.creation.created δ-creation in
        let δ⇒δ0 : δ ⇒ δ0
            δ⇒δ0 = Respects-Creates.creation.red-created δ-creation in
        let δ0⟨ρ⟩≡δ' : δ0 ⟨ toRep ρ ⟩ ≡ δ'
            δ0⟨ρ⟩≡δ' = Respects-Creates.creation.ap-created δ-creation in
        let δ0∈C[φ⇒ψ] : C Γ (φ ⇛ ψ) δ0
            δ0∈C[φ⇒ψ] = hyp δ0 δ⇒δ0
        in let δ'∈C[φ⇒ψ] : C Δ (φ ⇛ ψ) δ'
              δ'∈C[φ⇒ψ] = subst (C Δ (φ ⇛ ψ)) δ0⟨ρ⟩≡δ' (C-rep {φ = φ ⇛ ψ} δ0∈C[φ⇒ψ]
        in subst (C Δ ψ) (cong (λ x → appP x ε) δ0⟨ρ⟩≡δ') (proj2 δ0∈C[φ⇒ψ] Q ρ ε ρ::Γ→Δ
      (λ ε' ε→ε' → claim ε' (SNε ε' ε→ε') ρ::Γ→Δ (C-red {φ = φ} ε∈Cφ ε→ε')))
```

Lemma 7.

$$C_\Gamma(\phi) \subseteq SN$$

```
CsubSN : ∀ {P} {Γ : PContext P} {φ} {δ} → C Γ φ δ → SN δ
CsubSN {P} {Γ} {app bot out2} P1 = proj2 P1
CsubSN {P} {Γ} {app imp (app2 φ (app2 ψ out2))} {δ} P1 =
  let φ' : Expression (Palphabet P) (nonVarKind -Prp)
      φ' = φ ⟨ (λ _ ()) ⟩ in
  let Γ' : PContext (suc P)
      Γ' = Γ , φ' in
  SNap' {replacement} {Palphabet P} {Palphabet P , -Proof} {E = δ} {σ = upRep} β-respe
    (SNsubbodyl (SNsubexp (CsubSN {Γ = Γ'} {φ = ψ}
    (subst (C Γ' ψ) (cong (λ x → appP x (var x0)) (rep-cong {E = δ} (toRep-↑ {P = P}))
    (proj2 P1 (suc P) suc (var x0) (λ x → sym (rep-cong {E = typeof' x Γ} (toRep-↑ {P
    (NeutralC {φ = φ}
```

```
        (subst (λ x → Γ' ⊢ var x₀ :: x)
          (trans (sym (rep-comp {E = φ})) (rep-cong {E = φ} (λ ())))
          (var {p = zero}))
        (varNeutral x₀)
        (λ _ ()))))))
```

```
module PHOPL where

open import Prelims
open import Grammar
import Reduction
```

4 Predicative Higher-Order Propositional Logic

Fix sets of *proof variables* and *term variables*.

The syntax of the system is given by the following grammar.

Proof	δ	::=	$p \mid \delta\delta \mid \lambda p : \phi.\delta$
Term	M, ϕ	::=	$x \mid \bot \mid MM \mid \lambda x : A.M \mid \phi \to \phi$
Type	A	::=	$\Omega \mid A \to A$
Term Context	Γ	::=	$\langle\rangle \mid \Gamma, x : A$
Proof Context	Δ	::=	$\langle\rangle \mid \Delta, p : \phi$
Judgement	$\mathcal{J}$	::=	Γ valid $\mid \Gamma \vdash M : A \mid \Gamma, \Delta$ valid $\mid \Gamma, \Delta \vdash \delta : \phi$

where p ranges over proof variables and x ranges over term variables. The variable p is bound within δ in the proof $\lambda p : \phi.\delta$, and the variable x is bound within M in the term $\lambda x : A.M$. We identify proofs and terms up to α-conversion.

In the implementation, we write $\mathbf{Term}\,(V)$ for the set of all terms with free variables a subset of V, where $V : \mathbf{FinSet}$.

```
data PHOPLVarKind : Set where
  -Proof : PHOPLVarKind
  -Term : PHOPLVarKind

data PHOPLNonVarKind : Set where
  -Type : PHOPLNonVarKind

PHOPLTaxonomy : Taxonomy
PHOPLTaxonomy = record {
  VarKind = PHOPLVarKind;
  NonVarKind = PHOPLNonVarKind }

module PHOPLGrammar where
  open Taxonomy PHOPLTaxonomy
```

```
  data PHOPLcon : ∀ {K : ExpressionKind} → Kind (-Constructor K) → Set where
    -appProof : PHOPLcon (Π₂ (out (varKind -Proof)) (Π₂ (out (varKind -Proof)) (out₂ {K =
    -lamProof : PHOPLcon (Π₂ (out (varKind -Term)) (Π₂ (Π -Proof (out (varKind -Proof)))
    -bot : PHOPLcon (out₂ {K = varKind -Term})
    -imp : PHOPLcon (Π₂ (out (varKind -Term)) (Π₂ (out (varKind -Term)) (out₂ {K = varKin
    -appTerm : PHOPLcon (Π₂ (out (varKind -Term)) (Π₂ (out (varKind -Term)) (out₂ {K = va
    -lamTerm : PHOPLcon (Π₂ (out (nonVarKind -Type)) (Π₂ (Π -Term (out (varKind -Term)))
    -Omega : PHOPLcon (out₂ {K = nonVarKind -Type})
    -func  : PHOPLcon (Π₂ (out (nonVarKind -Type)) (Π₂ (out (nonVarKind -Type)) (out₂ {K

  PHOPLparent : PHOPLVarKind → ExpressionKind
  PHOPLparent -Proof = varKind -Term
  PHOPLparent -Term = nonVarKind -Type

  PHOPL : Grammar
  PHOPL = record {
    taxonomy = PHOPLTaxonomy;
    toGrammar = record {
      Constructor = PHOPLcon;
      parent = PHOPLparent } }

module PHOPL where
  open PHOPLGrammar using (PHOPLcon;-appProof;-lamProof;-bot;-imp;-appTerm;-lamTerm;-Ome
  open Grammar.Grammar PHOPLGrammar.PHOPL

  Type : Set
  Type = Expression ∅ (nonVarKind -Type)

  liftType : ∀ {V} → Type → Expression V (nonVarKind -Type)
  liftType (app -Omega out₂) = app -Omega out₂
  liftType (app -func (app₂ A (app₂ B out₂))) = app -func (app₂ (liftType A) (app₂ (liftT

  Ω : Type
  Ω = app -Omega out₂

  infix 75 _⇛_
  _⇛_ : Type → Type → Type
  φ ⇛ ψ = app -func (app₂ φ (app₂ ψ out₂))

  lowerType : ∀ {V} → Expression V (nonVarKind -Type) → Type
  lowerType (app -Omega out₂) = Ω
  lowerType (app -func (app₂ φ (app₂ ψ out₂))) = lowerType φ ⇛ lowerType ψ

{-  infix 80 _,_
  data TContext : Alphabet → Set where
    〈〉 : TContext ∅
```

```
  _,_ : ∀ {V} → TContext V → Type → TContext (V , -Term) -}

TContext : Alphabet → Set
TContext = Context -Term

Term : Alphabet → Set
Term V = Expression V (varKind -Term)

⊥ : ∀ {V} → Term V
⊥ = app -bot out₂

appTerm : ∀ {V} → Term V → Term V → Term V
appTerm M N = app -appTerm (app₂ M (app₂ N out₂))

ΛTerm : ∀ {V} → Type → Term (V , -Term) → Term V
ΛTerm A M = app -lamTerm (app₂ (liftType A) (app₂ M out₂))

_⊃_ : ∀ {V} → Term V → Term V → Term V
φ ⊃ ψ = app -imp (app₂ φ (app₂ ψ out₂))

PAlphabet : ℕ → Alphabet → Alphabet
PAlphabet zero A = A
PAlphabet (suc P) A = PAlphabet P A , -Proof

liftVar : ∀ {A} {K} P → Var A K → Var (PAlphabet P A) K
liftVar zero x = x
liftVar (suc P) x = ↑ (liftVar P x)

liftVar' : ∀ {A} P → Fin P → Var (PAlphabet P A) -Proof
liftVar' (suc P) zero = x₀
liftVar' (suc P) (suc x) = ↑ (liftVar' P x)

liftExp : ∀ {V} {K} P → Expression V K → Expression (PAlphabet P V) K
liftExp P E = E 〈 (λ _ → liftVar P) 〉

data PContext' (V : Alphabet) : ℕ → Set where
  〈〉 : PContext' V zero
  _,_ : ∀ {P} → PContext' V P → Term V → PContext' V (suc P)

PContext : Alphabet → ℕ → Set
PContext V = Context' V -Proof

P〈〉 : ∀ {V} → PContext V zero
P〈〉 = 〈〉

_P,_ : ∀ {V} {P} → PContext V P → Term V → PContext V (suc P)
```

```
_P,_ {V} {P} Δ φ = Δ , φ 〈 embedl {V} { -Proof} {P} 〉

Proof : Alphabet → ℕ → Set
Proof V P = Expression (PAlphabet P V) (varKind -Proof)

varP : ∀ {V} {P} → Fin P → Proof V P
varP {P = P} x = var (liftVar' P x)

appP : ∀ {V} {P} → Proof V P → Proof V P → Proof V P
appP δ ε = app -appProof (app₂ δ (app₂ ε out₂))

ΛP : ∀ {V} {P} → Term V → Proof V (suc P) → Proof V P
ΛP {P = P} φ δ = app -lamProof (app₂ (liftExp P φ) (app₂ δ out₂))

-- typeof' : ∀ {V} → Var V -Term → TContext V → Type
-- typeof' x₀ (_ , A) = A
-- typeof' (↑ x) (Γ , _) = typeof' x Γ

propof : ∀ {V} {P} → Fin P → PContext' V P → Term V
propof zero (_ , φ) = φ
propof (suc x) (Γ , _) = propof x Γ

data β : ∀ {V} {K} {C} → Constructor C → Subexpression V (-Constructor K) C → Expres
  βI : ∀ {V} A (M : Term (V , -Term)) N → β -appTerm (app₂ (ΛTerm A M) (app₂ N out₂))
open Reduction PHOPLGrammar.PHOPL β
```

The rules of deduction of the system are as follows.

$$\frac{}{\langle\rangle \text{ valid}} \qquad \frac{\Gamma \text{ valid}}{\Gamma, x : A \text{ valid}} \qquad \frac{\Gamma \vdash \phi : \Omega}{\Gamma, p : \phi \text{ valid}}$$

$$\frac{\Gamma \text{ valid}}{\Gamma \vdash x : A} \; (x : A \in \Gamma) \qquad \frac{\Gamma \text{ valid}}{\Gamma \vdash p : \phi} \; (p : \phi \in \Gamma)$$

$$\frac{\Gamma \text{ valid}}{\Gamma \vdash \bot : \Omega} \qquad \frac{\Gamma \vdash \phi : \Omega \quad \Gamma \vdash \psi : \Omega}{\Gamma \vdash \phi \to \psi : \Omega}$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad \frac{\Gamma \vdash \delta : \phi \to \psi \quad \Gamma \vdash \epsilon : \phi}{\Gamma \vdash \delta\epsilon : \psi}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \to B} \qquad \frac{\Gamma, p : \phi \vdash \delta : \psi}{\Gamma \vdash \lambda p : \phi.\delta : \phi \to \psi}$$

$$\frac{\Gamma \vdash \delta : \phi \quad \Gamma \vdash \psi : \Omega}{\Gamma \vdash \delta : \psi} \; (\phi \simeq \phi)$$

```
infix 10 _⊢_:_
data _⊢_:_ : ∀ {V} → TContext V → Term V → Expression V (nonVarKind -Type) → Set₁ w
  var : ∀ {V} {Γ : TContext V} {x} → Γ ⊢ var x : typeof x Γ
  ⊥R : ∀ {V} {Γ : TContext V} → Γ ⊢ ⊥ : Ω ⟨ (λ _ ()) ⟩
  imp : ∀ {V} {Γ : TContext V} {φ} {ψ} → Γ ⊢ φ : Ω ⟨ (λ _ ()) ⟩ → Γ ⊢ ψ : Ω ⟨ (λ _ ())
  app : ∀ {V} {Γ : TContext V} {M} {N} {A} {B} → Γ ⊢ M : app -func (app₂ A (app₂ B out
  Λ : ∀ {V} {Γ : TContext V} {A} {M} {B} → Γ , A ⊢ M : liftE B → Γ ⊢ app -lamTerm (ap

data Pvalid : ∀ {V} {P} → TContext V → PContext' V P → Set₁ where
  ⟨⟩ : ∀ {V} {Γ : TContext V} → Pvalid Γ ⟨⟩
  _,_ : ∀ {V} {P} {Γ : TContext V} {Δ : PContext' V P} {φ : Term V} → Pvalid Γ Δ → Γ

infix 10 _,,_⊢_::_
data _,,_⊢_::_ : ∀ {V} {P} → TContext V → PContext' V P → Proof V P → Term V → Set₁
  var : ∀ {V} {P} {Γ : TContext V} {Δ : PContext' V P} {p} → Pvalid Γ Δ → Γ ,, Δ ⊢ v
  app : ∀ {V} {P} {Γ : TContext V} {Δ : PContext' V P} {δ} {ε} {φ} {ψ} → Γ ,, Δ ⊢ δ ::
  Λ : ∀ {V} {P} {Γ : TContext V} {Δ : PContext' V P} {φ} {δ} {ψ} → Γ ,, Δ , φ ⊢ δ :: ψ
  convR : ∀ {V} {P} {Γ : TContext V} {Δ : PContext' V P} {δ} {φ} {ψ} → Γ ,, Δ ⊢ δ :: φ
```