# Type Theories with Computation Rules for the Univalence Axiom

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## 1 Preliminaries

```
module Prelims where

postulate Level : Set
postulate zro : Level
postulate suc : Level → Level
{-# BUILTIN LEVEL Level #-}
{-# BUILTIN LEVELZERO zro #-}
{-# BUILTIN LEVELSUC suc #-}
```

### 1.1 The Empty Type

```
data False : Set where
```

### 1.2 Conjunction

```
data _∧_ {i} (P Q : Set i) : Set i where
  _,_ : P → Q → P ∧ Q

π₁ : ∀ {i} {P Q : Set i} → P ∧ Q → P
π₁ (x , _) = x

π₂ : ∀ {i} {P Q : Set i} → P ∧ Q → Q
π₂ (_ , y) = y
```

### 1.3 Functions

```
infix 75 _∘_
_∘_ : ∀ {A B C : Set} → (B → C) → (A → B) → A → C
(g ∘ f) x = g (f x)
```

### 1.4 Equality

We use the inductively defined equality $=$ on every datatype.

```
infix 50 _≡_
data _≡_ {A : Set} (a : A) : A → Set where
  ref : a ≡ a

subst : ∀ {i} {A : Set} (P : A → Set i) {a} {b} → a ≡ b → P a → P b
subst P ref Pa = Pa

subst2 : ∀ {A B : Set} (P : A → B → Set) {a a' b b'} → a ≡ a' → b ≡ b' → P a b → P
subst2 P ref ref Pab = Pab

sym : ∀ {A : Set} {a b : A} → a ≡ b → b ≡ a
sym ref = ref

trans : ∀ {A : Set} {a b c : A} → a ≡ b → b ≡ c → a ≡ c
trans ref ref = ref

wd : ∀ {A B : Set} (f : A → B) {a a' : A} → a ≡ a' → f a ≡ f a'
wd _ ref = ref

wd2 : ∀ {A B C : Set} (f : A → B → C) {a a' : A} {b b' : B} → a ≡ a' → b ≡ b' → f a
wd2 _ ref ref = ref

module Equational-Reasoning (A : Set) where
  infix 2 ∵_
  ∵_ : ∀ (a : A) → a ≡ a
  ∵ _ = ref

  infix 1 _≡_[_]
  _≡_[_] : ∀ {a b : A} → a ≡ b → ∀ c → b ≡ c → a ≡ c
  δ ≡ c [ δ' ] = trans δ δ'

  infix 1 _≡_[[_]]
  _≡_[[_]] : ∀ {a b : A} → a ≡ b → ∀ c → c ≡ b → a ≡ c
  δ ≡ c [[ δ' ]] = trans δ (sym δ')
```

## 2 Datatypes

We introduce a universe **FinSet** of (names of) finite sets. There is an empty set $\emptyset$ : **FinSet**, and for every $A$ : **FinSet**, the type $A+1$ : **FinSet** has one more element:

$$A + 1 = \{\bot\} \uplus \{\uparrow a : a \in A\}$$

```
data FinSet : Set where
  ∅ : FinSet
  Lift : FinSet → FinSet

data El : FinSet → Set where
  ⊥ : ∀ {V} → El (Lift V)
  ↑ : ∀ {V} → El V → El (Lift V)

lift : ∀ {A} {B} → (El A → El B) → El (Lift A) → El (Lift B)
lift _ ⊥ = ⊥
lift f (↑ x) = ↑ (f x)
```

# 3 Grammars

```
module Grammar where

open import Prelims
```

Before we begin investigating the several theories we wish to consider, we present a general theory of syntax and capture-avoiding substitution.

A *grammar* consists of:

- a set of *expression kinds*;
- a subset of expression kinds, the *variable kinds*;
- a set of *constructors*, each with an associated *constructor kind* of the form

$$((A_{11}, \ldots, A_{1r_1})B_1, \ldots, (A_{m1}, \ldots, A_{mr_m})B_m)C \tag{1}$$

  where each $A_{ij}$ is a variable kind, and each $B_i$ and $C$ is an expression kind.

- a function assigning, to each variable kind $K$, an expression kind, the *parent* of $K$.

A constructor $c$ of kind (1) is a constructor that takes $m$ arguments of kind $B_1, \ldots, B_m$, and binds $r_i$ variables in its $i$th argument of kind $A_{ij}$, producing an expression of kind $C$. We write this expression as

$$c([x_{11}, \ldots, x_{1r_1}]E_1, \ldots, [x_{m1}, \ldots, x_{mr_m}]E_m) \ . \tag{2}$$

The subexpressions of the form $[x_{i1}, \ldots, x_{ir_i}]E_i$ shall be called *abstractions*, and the pieces of syntax of the form $(A_{i1}, \ldots, A_{ij})B_i$ that occur in constructor kinds shall be called *abstraction kinds*.

```
record Taxonomy : Set₁ where
  field
```

```
  VarKind : Set
  NonVarKind : Set

data ExpressionKind : Set where
  varKind : VarKind → ExpressionKind
  nonVarKind : NonVarKind → ExpressionKind

data KindClass : Set where
  -Expression  : KindClass
  -Abstraction : KindClass
  -Constructor : ExpressionKind → KindClass

data Kind : KindClass → Set where
  base : ExpressionKind → Kind -Expression
  out  : ExpressionKind → Kind -Abstraction
  Π    : VarKind → Kind -Abstraction → Kind -Abstraction
  out₂ : ∀ {K} → Kind (-Constructor K)
  Π₂   : ∀ {K} → Kind -Abstraction → Kind (-Constructor K) → Kind (-Constructor K)
```

An *alphabet* $V = \{V_E\}_E$ consists of a set $V_E$ of *variables* of kind $E$ for each expression kind $E$. The *expressions* of kind $E$ over the alphabet $V$ are defined inductively by:

- Every variable of kind $E$ is an expression of kind $E$.
- If $c$ is a constructor of kind (1), each $E_i$ is an expression of kind $B_i$, and each $x_{ij}$ is a variable of kind $A_{ij}$, then (2) is an expression of kind $C$.

Each $x_{ij}$ is bound within $E_i$ in the expression (2). We identify expressions up to $\alpha$-conversion.

```
data Alphabet : Set where
  ∅ : Alphabet
  _,_ : Alphabet → VarKind → Alphabet

data Var : Alphabet → VarKind → Set where
  x₀ : ∀ {V} {K} → Var (V , K) K
  ↑ : ∀ {V} {K} {L} → Var V L → Var (V , K) L

extend : Alphabet → VarKind → FinSet → Alphabet
extend A K ∅ = A
extend A K (Lift F) = extend A K F , K

embed : ∀ {A} {K} {F} → El F → Var (extend A K F) K
embed ⊥ = x₀
embed (↑ x) = ↑ (embed x)
```

```
record ToGrammar (T : Taxonomy) : Set₁ where
  open Taxonomy T
  field
    Constructor    : ∀ {K : ExpressionKind} → Kind (-Constructor K) → Set
    parent         : VarKind → ExpressionKind

  data Subexpression (V : Alphabet) : ∀ C → Kind C → Set where
    var : ∀ {K} → Var V K → Subexpression V -Expression (base (varKind K))
    app : ∀ {K} {C : Kind (-Constructor K)} → Constructor C → Subexpression V (-Constru
    out : ∀ {K} → Subexpression V -Expression (base K) → Subexpression V -Abstraction
    Λ   : ∀ {K} {A} → Subexpression (V , K) -Abstraction A → Subexpression V -Abstracti
    out₂ : ∀ {K} → Subexpression V (-Constructor K) out₂
    app₂ : ∀ {K} {A} {C} → Subexpression V -Abstraction A → Subexpression V (-Construct

  var-inj : ∀ {V} {K} {x y : Var V K} → var x ≡ var y → x ≡ y
  var-inj ref = ref

  Expression : Alphabet → ExpressionKind → Set
  Expression V K = Subexpression V -Expression (base K)
```

Given alphabets $U$, $V$, and a function $\rho$ that maps every variable in $U$ of kind $K$ to a variable in $V$ of kind $K$, we denote by $E\{\rho\}$ the result of *replacing* every variable $x$ in $E$ with $\rho(x)$.

```
  record PreOpFamily : Set₂ where
    field
      Op : Alphabet → Alphabet → Set
      apV : ∀ {U} {V} {K} → Op U V → Var U K → Expression V (varKind K)
      liftOp : ∀ {U} {V} {K} → Op U V → Op (U , K) (V , K)
      comp : ∀ {U} {V} {W} → Op V W → Op U V → Op U W

    _∼op_ : ∀ {U} {V} → Op U V → Op U V → Set
    _∼op_ {U} {V} ρ σ = ∀ {K} (x : Var U K) → apV ρ x ≡ apV σ x

    ap : ∀ {U} {V} {C} {K} → Op U V → Subexpression U C K → Subexpression V C K
    ap ρ (var x) = apV ρ x
    ap ρ (app c EE) = app c (ap ρ EE)
    ap ρ (out E) = out (ap ρ E)
    ap ρ (Λ E) = Λ (ap (liftOp ρ) E)
    ap _ out₂ = out₂
    ap ρ (app₂ E EE) = app₂ (ap ρ E) (ap ρ EE)

  record IsOpFamily (opfamily : PreOpFamily) : Set₂ where
    open PreOpFamily opfamily
    field
      liftOp-wd : ∀ {V} {W} {K} {ρ σ : Op V W} → ρ ∼op σ →
```

```
      liftOp {K = K} ρ ∼op liftOp σ
    apV-comp : ∀ {U} {V} {W} {K} {σ : Op V W} {ρ : Op U V} {x : Var U K} →
      apV (comp σ ρ) x ≡ ap σ (apV ρ x)
    liftOp-comp : ∀ {U} {V} {W} {K} {σ : Op V W} {ρ : Op U V} →
      liftOp {K = K} (comp σ ρ) ∼op comp (liftOp σ) (liftOp ρ)

  ap-wd : ∀ {U} {V} {C} {K} {ρ σ : Op U V} {E : Subexpression U C K} →
    ρ ∼op σ → ap ρ E ≡ ap σ E
  ap-wd {E = var x} ρ-is-σ = ρ-is-σ x
  ap-wd {E = app c EE} ρ-is-σ = wd (app c) (ap-wd {E = EE} ρ-is-σ)
  ap-wd {E = out E} ρ-is-σ = wd out (ap-wd {E = E} ρ-is-σ)
  ap-wd {E = Λ {K} E} ρ-is-σ = wd Λ (ap-wd {E = E} (liftOp-wd {K = K} ρ-is-σ))
  ap-wd {E = out₂} _ = ref
  ap-wd {E = app₂ E F} ρ-is-σ = wd2 app₂ (ap-wd {E = E} ρ-is-σ) (ap-wd {E = F} ρ-is-σ)

  ap-comp : ∀ {U} {V} {W} {C} {K} {E : Subexpression U C K} {F : Op V W} {G : Op U V} →
  ap-comp {E = var x} = apV-comp
  ap-comp {E = app c EE} = wd (app c) (ap-comp {E = EE})
  ap-comp {E = out E} = wd out (ap-comp {E = E})
  ap-comp {U} {V} {W} {E = Λ E} {σ} {ρ} = wd Λ (let open Equational-Reasoning _ in
    ∵ ap (liftOp (comp σ ρ)) E
    ≡ ap (comp (liftOp σ) (liftOp ρ)) E [ ap-wd {E = E} (liftOp-comp {σ = σ} {ρ = ρ})
    ≡ ap (liftOp σ) (ap (liftOp ρ) E)   [ ap-comp {E = E} ])
  ap-comp {E = out₂} = ref
  ap-comp {E = app₂ E F} = wd2 app₂ (ap-comp {E = E}) (ap-comp {E = F})

record OpFamily : Set₂ where
  field
    preOpFamily : PreOpFamily
    isOpFamily  : IsOpFamily preOpFamily
  open PreOpFamily preOpFamily public
  open IsOpFamily isOpFamily public

Rep : Alphabet → Alphabet → Set
Rep U V = ∀ K → Var U K → Var V K

Rep↑ : ∀ {U} {V} {K} → Rep U V → Rep (U , K) (V , K)
Rep↑ _ _ x₀ = x₀
Rep↑ ρ K (↑ x) = ↑ (ρ K x)

infixl 75 _•R_
_•R_ : ∀ {U} {V} {W} → Rep V W → Rep U V → Rep U W
(ρ' •R ρ) K x = ρ' K (ρ K x)

pre-replacement : PreOpFamily
pre-replacement = record {
```

```
  Op = Rep;
  apV = λ ρ x → var (ρ _ x);
  liftOp = Rep↑;
  comp = _•R_ }

_∼R_ : ∀ {U} {V} → Rep U V → Rep U V → Set
_∼R_ = PreOpFamily._∼op_ pre-replacement

Rep↑-wd : ∀ {U} {V} {K} {ρ ρ' : Rep U V} → ρ ∼R ρ' → Rep↑ {K = K} ρ ∼R Rep↑ ρ'
Rep↑-wd ρ-is-ρ' x₀ = ref
Rep↑-wd ρ-is-ρ' (↑ x) = wd (var ∘ ↑) (var-inj (ρ-is-ρ' x))

Rep↑-comp : ∀ {U} {V} {W} {K} {ρ' : Rep V W} {ρ : Rep U V} → Rep↑ {K = K} (ρ' •R ρ) ∼
Rep↑-comp x₀ = ref
Rep↑-comp (↑ _) = ref

replacement : OpFamily
replacement = record {
  preOpFamily = pre-replacement;
  isOpFamily = record {
    liftOp-wd = Rep↑-wd;
    apV-comp = ref;
    liftOp-comp = Rep↑-comp } }

embedl : ∀ {A} {K} {F} → Rep A (extend A K F)
embedl {F = ∅} _ x = x
embedl {F = Lift F} K x = ↑ (embedl {F = F} K x)
```

The alphabets and replacements form a category.

```
idRep : ∀ V → Rep V V
idRep _ _ x = x

--We choose not to prove the category axioms, as they hold up to judgemental equality.
```

Given a replacement $\rho : U \to V$, we can 'lift´ this to a replacement $(\rho, K) : (U, K) \to (V, K)$. Under this operation, the mapping $(-, K)$ becomes an endofunctor on the category of alphabets and replacements.

```
Rep↑-id : ∀ {V} {K} → Rep↑ (idRep V) ∼R idRep (V , K)
Rep↑-id x₀ = ref
Rep↑-id (↑ _) = ref
```

Finally, we can define $E\langle\rho\rangle$, the result of replacing each variable $x$ in $E$ with $\rho(x)$. Under this operation, the mapping $\mathsf{Expression}\ -\ K$ becomes a functor from the category of alphabets and replacements to the category of sets.

```
infix 60 _⟨_⟩
_⟨_⟩ : ∀ {U} {V} {C} {K} → Subexpression U C K → Rep U V → Subexpression V C K
E ⟨ ρ ⟩ = OpFamily.ap replacement ρ E

rep-wd : ∀ {U} {V} {C} {K} {E : Subexpression U C K} {ρ ρ' : Rep U V} → ρ ∼R ρ' → E
rep-wd {U} {V} {C} {K} {E} {ρ} {ρ'} ρ-is-ρ' = OpFamily.ap-wd replacement {U} {V} {C} {I

rep-id : ∀ {V} {C} {K} {E : Subexpression V C K} → E ⟨ idRep V ⟩ ≡ E
rep-id {E = var _} = ref
rep-id {E = app c EE} = wd (app c) rep-id
rep-id {E = out E} = wd out rep-id
rep-id {V} {E = Λ {K} {A} E} = wd Λ (let open Equational-Reasoning (Subexpression (V ,
  ∵ E ⟨ Rep↑ (idRep V) ⟩
  ≡ E ⟨ idRep (V , K) ⟩         [ rep-wd {E = E} Rep↑-id ]
  ≡ E                           [ rep-id ])
rep-id {E = out₂} = ref
rep-id {E = app₂ E EE} = wd2 app₂ rep-id rep-id

rep-comp : ∀ {U} {V} {W} {C} {K} {E : Subexpression U C K} {ρ : Rep U V} {σ : Rep V W}
  E ⟨ σ •R ρ ⟩ ≡ E ⟨ ρ ⟩ ⟨ σ ⟩
rep-comp {E = var _} = ref
rep-comp {E = app c EE} = wd (app c) (rep-comp {E = EE})
rep-comp {E = out E} = wd out (rep-comp {E = E})
rep-comp {E = Λ E} {ρ} {σ} = wd Λ (let open Equational-Reasoning _ in
  ∵ E ⟨ Rep↑ (σ •R ρ) ⟩
  ≡ E ⟨ Rep↑ σ •R Rep↑ ρ ⟩ [ rep-wd {E = E} Rep↑-comp ]
  ≡ E ⟨ Rep↑ ρ ⟩ ⟨ Rep↑ σ ⟩ [ rep-comp {E = E} ])
rep-comp {E = out₂} = ref
rep-comp {E = app₂ E EE} = wd2 app₂ (rep-comp {E = E}) (rep-comp {E = EE})
```

This provides us with the canonical mapping from an expression over $V$ to an expression over $(V, K)$:

```
liftE : ∀ {V} {K} {L} → Expression V L → Expression (V , K) L
liftE E = E ⟨ (λ _ → ↑) ⟩
```

A *substitution* $\sigma$ from alphabet $U$ to alphabet $V$, $\sigma : U \Rightarrow V$, is a function $\sigma$ that maps every variable $x$ of kind $K$ in $U$ to an *expression* $\sigma(x)$ of kind $K$ over $V$. Then, given an expression $E$ of kind $K$ over $U$, we write $E[\sigma]$ for the result of substituting $\sigma(x)$ for $x$ for each variable in $E$, avoiding capture.

```
Sub : Alphabet → Alphabet → Set
Sub U V = ∀ K → Var U K → Expression V (varKind K)

_∼_ : ∀ {U} {V} → Sub U V → Sub U V → Set
σ ∼ τ = ∀ K x → σ K x ≡ τ K x
```

The *identity* substitution $\mathsf{id}_V : V \to V$ is defined as follows.

```
idSub : ∀ {V} → Sub V V
idSub _ = var
```

Given $\sigma : U \to V$ and an expression $E$ over $U$, we want to define the expression $E[\sigma]$ over $V$, the result of applying the substitution $\sigma$ to $M$. Only after this will we be able to define the composition of two substitutions. However, there is some work we need to do before we are able to do this.

We can define the composition of a substitution and a replacement as follows

```
infix 75 _•₁_
_•₁_ : ∀ {U} {V} {W} → Rep V W → Sub U V → Sub U W
(ρ •₁ σ) K x = (σ K x) 〈 ρ 〉

infix 75 _•₂_
_•₂_ : ∀ {U} {V} {W} → Sub V W → Rep U V → Sub U W
(σ •₂ ρ) K x = σ K (ρ K x)
```

Given a substitution $\sigma : U \Rightarrow V$, define a substitution $(\sigma, K) : (U, K) \Rightarrow (V, K)$ as follows.

```
Sub↑ : ∀ {U} {V} {K} → Sub U V → Sub (U , K) (V , K)
Sub↑ _ _ x₀ = var x₀
Sub↑ σ K (↑ x) = liftE (σ K x)

Sub↑-wd : ∀ {U} {V} {K} {σ σ' : Sub U V} → σ ∼ σ' → Sub↑ {K = K} σ ∼ Sub↑ σ'
Sub↑-wd {K = K} σ-is-σ' ._ x₀ = ref
Sub↑-wd σ-is-σ' L (↑ x) = wd liftE (σ-is-σ' L x)
```

**Lemma 1.** *The operations we have defined satisfy the following properties.*

1. $(\mathsf{id}_V, K) = \mathsf{id}_{(V,K)}$
2. $(\rho \bullet_1 \sigma, K) = (\rho, K) \bullet_1 (\sigma, K)$
3. $(\sigma \bullet_2 \rho, K) = (\sigma, K) \bullet_2 (\rho, K)$

```
Sub↑-id : ∀ {V} {K} → Sub↑ {V} {V} {K} idSub ∼ idSub
Sub↑-id {K = K} ._ x₀ = ref
Sub↑-id _ (↑ _) = ref

Sub↑-comp₁ : ∀ {U} {V} {W} {K} {ρ : Rep V W} {σ : Sub U V} → Sub↑ (ρ •₁ σ) ∼ Rep↑ ρ •
Sub↑-comp₁ {K = K} ._ x₀ = ref
Sub↑-comp₁ {U} {V} {W} {K} {ρ} {σ} L (↑ x) = let open Equational-Reasoning (Expression
  ∵ liftE ((σ L x) 〈 ρ 〉)
  ≡ (σ L x) 〈 (λ _ x → ↑ (ρ _ x)) 〉 [[ rep-comp {E = σ L x} ]]
  ≡ (liftE (σ L x)) 〈 Rep↑ ρ 〉     [ rep-comp {E = σ L x} ]

Sub↑-comp₂ : ∀ {U} {V} {W} {K} {σ : Sub V W} {ρ : Rep U V} → Sub↑ {K = K} (σ •₂ ρ) ∼ S
Sub↑-comp₂ {K = K} ._ x₀ = ref
Sub↑-comp₂ L (↑ x) = ref
```

We can now define the result of applying a substitution $\sigma$ to an expression $E$, which we denote $E[\sigma]$.

```
infix 60 _⟦_⟧
_⟦_⟧ : ∀ {U} {V} {C} {K} → Subexpression U C K → Sub U V → Subexpression V C K
var x ⟦ σ ⟧ = σ _ x
app c EE ⟦ σ ⟧ = app c (EE ⟦ σ ⟧)
out E ⟦ σ ⟧ = out (E ⟦ σ ⟧)
Λ E ⟦ σ ⟧ = Λ (E ⟦ Sub↑ σ ⟧)
out₂ ⟦ _ ⟧ = out₂
app₂ E EE ⟦ σ ⟧ = app₂ (E ⟦ σ ⟧) (EE ⟦ σ ⟧)

sub-wd : ∀ {U} {V} {C} {K} {E : Subexpression U C K} {σ σ' : Sub U V} → σ ∼ σ' → E ⟦
sub-wd {E = var x} σ-is-σ' = σ-is-σ' _ x
sub-wd {E = app c EE} σ-is-σ' = wd (app c) (sub-wd {E = EE} σ-is-σ')
sub-wd {E = out E} σ-is-σ' = wd out (sub-wd {E = E} σ-is-σ')
sub-wd {E = Λ E} σ-is-σ' = wd Λ (sub-wd {E = E} (Sub↑-wd σ-is-σ'))
sub-wd {E = out₂} _ = ref
sub-wd {E = app₂ E EE} σ-is-σ' = wd2 app₂ (sub-wd {E = E} σ-is-σ') (sub-wd {E = EE} σ-i
```

**Lemma 2.**

1. $M[\mathsf{id}_V] \equiv M$
2. $M[\rho \bullet_1 \sigma] \equiv M[\sigma]\langle\rho\rangle$
3. $M[\sigma \bullet_2 \rho] \equiv M\langle\rho\rangle[\sigma]$

```
sub-id : ∀ {V} {C} {K} {E : Subexpression V C K} → E ⟦ idSub ⟧ ≡ E
sub-id {E = var _} = ref
sub-id {E = app c EE} = wd (app c) sub-id
sub-id {E = out E} = wd out sub-id
sub-id {E = Λ E} = wd Λ (let open Equational-Reasoning _ in
  ∵ E ⟦ Sub↑ idSub ⟧
  ≡ E ⟦ idSub ⟧          [ sub-wd {E = E} Sub↑-id ]
  ≡ E                     [ sub-id ])
sub-id {E = out₂} = ref
sub-id {E = app₂ E EE} = wd2 app₂ sub-id sub-id

sub-comp₁ : ∀ {U} {V} {W} {C} {K} {E : Subexpression U C K} {ρ : Rep V W} {σ : Sub U V
    E ⟦ ρ •₁ σ ⟧ ≡ E ⟦ σ ⟧ 〈 ρ 〉
sub-comp₁ {E = var _} = ref
sub-comp₁ {E = app c EE} = wd (app c) (sub-comp₁ {E = EE})
sub-comp₁ {E = out₂} = ref
sub-comp₁ {E = app₂ A EE} = wd2 app₂ (sub-comp₁ {E = A}) (sub-comp₁ {E = EE})
sub-comp₁ {E = out E} = wd out (sub-comp₁ {E = E})
sub-comp₁ {E = Λ A} {ρ} {σ} =
  wd Λ (let open Equational-Reasoning _ in
```

```
  ∵ A ⟦ Sub↑ (ρ •₁ σ) ⟧
  ≡ A ⟦ Rep↑ ρ •₁ Sub↑ σ ⟧   [ sub-wd {E = A} Sub↑-comp₁ ]
  ≡ A ⟦ Sub↑ σ ⟧ 〈 Rep↑ ρ 〉  [ sub-comp₁ {E = A} ])

sub-comp₂ : ∀ {U} {V} {W} {C} {K} {E : Subexpression U C K} {σ : Sub V W} {ρ : Rep U V
sub-comp₂ {E = var _} = ref
sub-comp₂ {E = app c EE} = wd (app c) (sub-comp₂ {E = EE})
sub-comp₂ {E = out₂} = ref
sub-comp₂ {E = app₂ A EE} = wd2 app₂ (sub-comp₂ {E = A}) (sub-comp₂ {E = EE})
sub-comp₂ {E = out E} = wd out (sub-comp₂ {E = E})
sub-comp₂ {E = Λ A} {σ} {ρ} = wd Λ (let open Equational-Reasoning _ in
  ∵ A ⟦ Sub↑ (σ •₂ ρ) ⟧
  ≡ A ⟦ Sub↑ σ •₂ Rep↑ ρ ⟧ [ sub-wd {E = A} Sub↑-comp₂ ]
  ≡ A 〈 Rep↑ ρ 〉 ⟦ Sub↑ σ ⟧ [ sub-comp₂ {E = A} ])
```

We define the composition of two substitutions, as follows.

```
subid : ∀ {V} → Sub V V
subid {V} K x = var x

infix 75 _•_
_•_ : ∀ {U} {V} {W} → Sub V W → Sub U V → Sub U W
(σ • ρ) K x = ρ K x ⟦ σ ⟧
```

**Lemma 3.** *Let $\sigma : V \Rightarrow W$ and $\rho : U \Rightarrow V$.*

1. $(\sigma \bullet \rho, K) \sim (\sigma, K) \bullet (\rho, K)$
2. $E[\sigma \bullet \rho] \equiv E[\rho][\sigma]$

```
pre-substitution : PreOpFamily
pre-substitution = record {
  Op = Sub;
  apV = λ σ x → σ _ x;
  liftOp = Sub↑;
  comp = _•_ }

--TODO Remove this
sub-is-sub : ∀ {U} {V} {σ : Sub U V} {C} {K} {E : Subexpression U C K} →
             E ⟦ σ ⟧ ≡ PreOpFamily.ap pre-substitution σ E
sub-is-sub {E = var _} = ref
sub-is-sub {E = app c E} = wd (app c) (sub-is-sub {E = E})
sub-is-sub {E = out E} = wd out (sub-is-sub {E = E})
sub-is-sub {E = Λ E} = wd Λ (sub-is-sub {E = E})
sub-is-sub {E = out₂} = ref
sub-is-sub {E = app₂ E F} = wd2 app₂ (sub-is-sub {E = E}) (sub-is-sub {E = F})
```

```
Sub↑-comp : ∀ {U} {V} {W} {ρ : Sub U V} {σ : Sub V W} {K} →
  Sub↑ {K = K} (σ • ρ) ∼ Sub↑ σ • Sub↑ ρ
Sub↑-comp _ x₀ = ref
Sub↑-comp {W = W} {ρ = ρ} {σ = σ} {K = K} L (↑ x) =
  let open Equational-Reasoning (Expression (W , K) (varKind L)) in
    ∵ liftE ((ρ L x) ⟦ σ ⟧)
    ≡ ρ L x ⟦ (λ _ → ↑) •₁ σ ⟧  [[ sub-comp₁ {E = ρ L x} ]]
    ≡ (liftE (ρ L x)) ⟦ Sub↑ σ ⟧ [ sub-comp₂ {E = ρ L x} ]

substitution : OpFamily
substitution = record {
  preOpFamily = pre-substitution;
  isOpFamily = record {
    liftOp-wd = λ ρ-is-σ → Sub↑-wd (λ _ → ρ-is-σ) _;
    apV-comp = λ {U} {V} {W} {K} {σ} {ρ} {x} → sub-is-sub {E = ρ K x};
    liftOp-comp = Sub↑-comp _ } }

mutual
  sub-compA : ∀ {U} {V} {W} {K} {A : Subexpression U -Abstraction K} {σ : Sub V W} {ρ
    A ⟦ σ • ρ ⟧ ≡ A ⟦ ρ ⟧ ⟦ σ ⟧
  sub-compA {A = out E} = wd out (sub-comp {E = E})
  sub-compA {U} {V} {W} .{Π K L} {Λ {K} {L} A} {σ} {ρ} = wd Λ (let open Equational-Rea
    ∵ A ⟦ Sub↑ (σ • ρ) ⟧
    ≡ A ⟦ Sub↑ σ • Sub↑ ρ ⟧    [ sub-wd {E = A} Sub↑-comp ]
    ≡ A ⟦ Sub↑ ρ ⟧ ⟦ Sub↑ σ ⟧  [ sub-compA {A = A} ])

  sub-compB : ∀ {U} {V} {W} {K} {C : Kind (-Constructor K)} {EE : Subexpression U (-Co
    EE ⟦ σ • ρ ⟧ ≡ EE ⟦ ρ ⟧ ⟦ σ ⟧
  sub-compB {EE = out₂} = ref
  sub-compB {U} {V} {W} {K} {(Π₂ L C)} {app₂ A EE} = wd2 app₂ (sub-compA {A = A}) (sub-

  sub-comp : ∀ {U} {V} {W} {K} {E : Expression U K} {σ : Sub V W} {ρ : Sub U V} →
    E ⟦ σ • ρ ⟧ ≡ E ⟦ ρ ⟧ ⟦ σ ⟧
  sub-comp {E = var _} = ref
  sub-comp {U} {V} {W} {K} {app c EE} = wd (app c) (sub-compB {EE = EE})
```

**Lemma 4.** *The alphabets and substitutions form a category under this composition.*

```
assoc : ∀ {U V W X} {ρ : Sub W X} {σ : Sub V W} {τ : Sub U V} → ρ • (σ • τ) ∼ (ρ • σ)
assoc {τ = τ} K x = sym (sub-comp {E = τ K x})

sub-unitl : ∀ {U} {V} {σ : Sub U V} → idSub • σ ∼ σ
sub-unitl _ _ = sub-id

sub-unitr : ∀ {U} {V} {σ : Sub U V} → σ • idSub ∼ σ
sub-unitr _ _ = ref
```

Replacement is a special case of substitution:

**Lemma 5.** *Let $\rho$ be a replacement $U \to V$.*

1. *The replacement $(\rho, K)$ and the substitution $(\rho, K)$ are equal.*

2.
$$E\langle\rho\rangle \equiv E[\rho]$$

```
Rep↑-is-Sub↑ : ∀ {U} {V} {ρ : Rep U V} {K} → (λ L x → var (Rep↑ {K = K} ρ L x)) ∼ Su
Rep↑-is-Sub↑ K x₀ = ref
Rep↑-is-Sub↑ K₁ (↑ x) = ref

mutual
  rep-is-sub : ∀ {U} {V} {K} {E : Expression U K} {ρ : Rep U V} →
           E 〈 ρ 〉 ≡ E ⟦ (λ K x → var (ρ K x)) ⟧
  rep-is-sub {E = var _} = ref
  rep-is-sub {U} {V} {K} {app c EE} = wd (app c) (rep-is-subB {EE = EE})

  rep-is-subB : ∀ {U} {V} {K} {C : Kind (-Constructor K)} {EE : Subexpression U (-Cons
    EE 〈 ρ 〉 ≡ EE ⟦ (λ K x → var (ρ K x)) ⟧
  rep-is-subB {EE = out₂} = ref
  rep-is-subB {EE = app₂ A EE} = wd2 app₂ (rep-is-subA {A = A}) (rep-is-subB {EE = EE})

  rep-is-subA : ∀ {U} {V} {K} {A : Subexpression U -Abstraction K} {ρ : Rep U V} →
    A 〈 ρ 〉 ≡ A ⟦ (λ K x → var (ρ K x)) ⟧
  rep-is-subA {A = out E} = wd out (rep-is-sub {E = E})
  rep-is-subA {U} {V} .{Π K L} {Λ {K} {L} A} {ρ} = wd Λ (let open Equational-Reasoning
    ∵ A 〈 Rep↑ ρ 〉
    ≡ A ⟦ (λ M x → var (Rep↑ ρ M x)) ⟧ [ rep-is-subA {A = A} ]
    ≡ A ⟦ Sub↑ (λ M x → var (ρ M x)) ⟧ [ sub-wd {E = A} Rep↑-is-Sub↑ ])
```

Let $E$ be an expression of kind $K$ over $V$. Then we write $[x_0 := E]$ for the following substitution $(V, K) \Rightarrow V$:

```
x₀:= : ∀ {V} {K} → Expression V (varKind K) → Sub (V , K) V
x₀:= E _ x₀ = E
x₀:= E K₁ (↑ x) = var x
```

**Lemma 6.** 1.
$$\rho \bullet_1 [x_0 := E] \sim [x_0 := E\langle\rho\rangle] \bullet_2 (\rho, K)$$

2.
$$\sigma \bullet [x_0 := E] \sim [x_0 := E[\sigma]] \bullet (\sigma, K)$$

```
comp₁-botsub : ∀ {U} {V} {K} {E : Expression U (varKind K)} {ρ : Rep U V} →
  ρ •₁ (x₀:= E) ∼ (x₀:= (E 〈 ρ 〉)) •₂ Rep↑ ρ
comp₁-botsub _ x₀ = ref
```

```
  comp₁-botsub _ (↑ _) = ref

  comp-botsub : ∀ {U} {V} {K} {E : Expression U (varKind K)} {σ : Sub U V} →
    σ • (x₀:= E) ∼ (x₀:= (E ⟦ σ ⟧)) • Sub↑ σ
  comp-botsub _ x₀ = ref
  comp-botsub {σ = σ} L (↑ x) = trans (sym sub-id) (sub-comp₂ {E = σ L x})
```

# 4 Contexts

A *context* has the form $x_1 : A_1, \ldots, x_n : A_n$ where, for each $i$:

- $x_i$ is a variable of kind $K_i$ distinct from $x_1, \ldots, x_{i-1}$;
- $A_i$ is an expression of some kind $L_i$;
- $L_i$ is a parent of $K_i$.

The *domain* of this context is the alphabet $\{x_1, \ldots, x_n\}$.

```
  data Context (K : VarKind) : Alphabet → Set where
    〈〉 : Context K ∅
    _,_ : ∀ {V} → Context K V → Expression V (parent K) → Context K (V , K)

  typeof : ∀ {V} {K} (x : Var V K) (Γ : Context K V) → Expression V (parent K)
  typeof x₀ (_ , A) = liftE A
  typeof (↑ x) (Γ , _) = liftE (typeof x Γ)

  data Context' (A : Alphabet) (K : VarKind) : FinSet → Set where
    〈〉 : Context' A K ∅
    _,_ : ∀ {F} → Context' A K F → Expression (extend A K F) (parent K) → Context' A K

  typeof' : ∀ {A} {K} {F} → El F → Context' A K F → Expression (extend A K F) (parent
  typeof' ⊥ (_ , A) = liftE A
  typeof' (↑ x) (Γ , _) = liftE (typeof' x Γ)

record Grammar : Set₁ where
  field
    taxonomy : Taxonomy
    toGrammar : ToGrammar taxonomy
  open Taxonomy taxonomy public
  open ToGrammar toGrammar public

module PL where

open import Prelims
open import Grammar
import Reduction
```

# 5 Propositional Logic

Fix sets of *proof variables* and *term variables*.

The syntax of the system is given by the following grammar.

| | | | |
|---|---|---|---|
| Proof | $\delta$ | $::=$ | $p \mid \delta\delta \mid \lambda p : \phi.\delta$ |
| Proposition | $f$ | $::=$ | $\bot \mid \phi \to \phi$ |
| Context | $\Gamma$ | $::=$ | $\langle\rangle \mid \Gamma, p : \phi$ |
| Judgement | $\mathcal{J}$ | $::=$ | $\Gamma \vdash \delta : \phi$ |

where $p$ ranges over proof variables and $x$ ranges over term variables. The variable $p$ is bound within $\delta$ in the proof $\lambda p : \phi.\delta$, and the variable $x$ is bound within $M$ in the term $\lambda x : A.M$. We identify proofs and terms up to $\alpha$-conversion.

```
data PLVarKind : Set where
  -Proof : PLVarKind

data PLNonVarKind : Set where
  -Prp   : PLNonVarKind

PLtaxonomy : Taxonomy
PLtaxonomy = record {
  VarKind = PLVarKind;
  NonVarKind = PLNonVarKind }

module PLgrammar where
  open Grammar.Taxonomy PLtaxonomy

  data PLCon : ∀ {K : ExpressionKind} → Kind (-Constructor K) → Set where
    app : PLCon (Π₂ (out (varKind -Proof)) (Π₂ (out (varKind -Proof)) (out₂ {K = varKind
    lam : PLCon (Π₂ (out (nonVarKind -Prp)) (Π₂ (Π -Proof (out (varKind -Proof))) (out₂ {
    bot : PLCon (out₂ {K = nonVarKind -Prp})
    imp : PLCon (Π₂ (out (nonVarKind -Prp)) (Π₂ (out (nonVarKind -Prp)) (out₂ {K = nonVar

  PLparent : VarKind → ExpressionKind
  PLparent -Proof = nonVarKind -Prp

open PLgrammar

Propositional-Logic : Grammar
Propositional-Logic = record {
  taxonomy = PLtaxonomy;
  toGrammar = record {
    Constructor = PLCon;
    parent = PLparent } }

open Grammar.Grammar Propositional-Logic
```

```
open Reduction Propositional-Logic

Prp : Set
Prp = Expression ∅ (nonVarKind -Prp)

⊥P : Prp
⊥P = app bot out₂

_⇒_ : ∀ {P} → Expression P (nonVarKind -Prp) → Expression P (nonVarKind -Prp) → Expre
φ ⇒ ψ = app imp (app₂ (out φ) (app₂ (out ψ) out₂))

Proof : Alphabet → Set
Proof P = Expression P (varKind -Proof)

appP : ∀ {P} → Expression P (varKind -Proof) → Expression P (varKind -Proof) → Express
appP δ ε = app app (app₂ (out δ) (app₂ (out ε) out₂))

ΛP : ∀ {P} → Expression P (nonVarKind -Prp) → Expression (P , -Proof) (varKind -Proof)
ΛP φ δ = app lam (app₂ (out φ) (app₂ (Λ (out δ)) out₂))

data β : Reduction where
  βI : ∀ {V} {φ} {δ} {ε} → β {V} app (app₂ (out (ΛP φ δ)) (app₂ (out ε) out₂)) (δ ⟦ x₀:=

β-respects-rep : respect-rep β
β-respects-rep {U} {V} {ρ = ρ} (βI .{U} {φ} {δ} {ε}) = subst (β app _)
  (let open Equational-Reasoning (Expression V (varKind -Proof)) in
  ∵ δ ⟨ Rep↑ ρ ⟩ ⟦ x₀:= (ε ⟨ ρ ⟩) ⟧
   ≡ δ ⟦ x₀:= (ε ⟨ ρ ⟩) •₂ Rep↑ ρ ⟧ [[ sub-comp₂ {E = δ} ]]
   ≡ δ ⟦ ρ •₁ x₀:= ε ⟧ [[ sub-wd {E = δ} comp₁-botsub ]]
   ≡ δ ⟦ x₀:= ε ⟧ ⟨ ρ ⟩ [ sub-comp₁ {E = δ} ])
  βI

β-creates-rep : create-rep β
β-creates-rep = record {
  created = λ {U} {V} {K} {C} {c} {EE} {F} {ρ} → created {U} {V} {K} {C} {c} {EE} {F} {ρ
  red-created = λ {U} {V} {K} {C} {c} {EE} {F} {ρ} → red-created {U} {V} {K} {C} {c} {EE
  rep-created = λ {U} {V} {K} {C} {c} {EE} {F} {ρ} → rep-created {U} {V} {K} {C} {c} {EE
  created : ∀ {U V : Alphabet} {K} {C} {c : PLCon C} {EE : Subexpression U (-Constructor
  created {c = app} {EE = app₂ (out (var _)) _} ()
  created {c = app} {EE = app₂ (out (app app _)) _} ()
  created {c = app} {EE = app₂ (out (app lam (app₂ (out φ) (app₂ (Λ (out δ)) out₂)))) (ap
  created {c = lam} ()
  created {c = bot} ()
  created {c = imp} ()
  red-created : ∀ {U} {V} {K} {C} {c : PLCon C} {EE : Subexpression U (-Constructor K) C
  red-created {c = app} {EE = app₂ (out (var _)) _} ()
```

```
  red-created {c = app} {EE = app₂ (out (app app _)) _} ()
  red-created {c = app} {EE = app₂ (out (app lam (app₂ (out φ) (app₂ (Λ (out δ)) out₂)))) 
  red-created {c = lam} ()
  red-created {c = bot} ()
  red-created {c = imp} ()
  rep-created : ∀ {U} {V} {K} {C} {c : PLCon C} {EE : Subexpression U (-Constructor K) C
  rep-created {c = app} {EE = app₂ (out (var _)) _} ()
  rep-created {c = app} {EE = app₂ (out (app app _)) _} ()
  rep-created {c = app} {EE = app₂ (out (app lam (app₂ (out φ) (app₂ (Λ (out δ)) out₂)))) 
    ∵ δ ⟦ x₀:= ε ⟧ ⟨ ρ ⟩
    ≡ δ ⟦ ρ •₁ x₀:= ε ⟧                    [[ sub-comp₁ {E = δ} ]]
    ≡ δ ⟦ x₀:= (ε ⟨ ρ ⟩) •₂ Rep↑ ρ ⟧     [ sub-wd {E = δ} comp₁-botsub ]
    ≡ δ ⟨ Rep↑ ρ ⟩ ⟦ x₀:= (ε ⟨ ρ ⟩) ⟧ [ sub-comp₂ {E = δ} ]
  rep-created {c = lam} ()
  rep-created {c = bot} ()
  rep-created {c = imp} ()
```

The rules of deduction of the system are as follows.

$$\frac{\Gamma \text{ valid}}{\Gamma \vdash p : \phi} \ (p : \phi \in \Gamma)$$

$$\frac{\Gamma \vdash \delta : \phi \to \psi}{\Gamma \vdash \delta\epsilon : \psi \quad \Gamma \vdash \epsilon : \phi}$$

$$\frac{\Gamma, p : \phi \vdash \delta : \psi}{\Gamma \vdash \lambda p : \phi . \delta : \phi \to \psi}$$

```
PContext : FinSet → Set
PContext P = Context' ∅ -Proof P

Palphabet : FinSet → Alphabet
Palphabet P = extend ∅ -Proof P

Palphabet-faithful : ∀ {P} {Q} {ρ σ : Rep (Palphabet P) (Palphabet Q)} → (∀ x → ρ -Proo
Palphabet-faithful {∅} _ ()
Palphabet-faithful {Lift _} ρ-is-σ x₀ = wd var (ρ-is-σ ⊥)
Palphabet-faithful {Lift _} {Q} {ρ} {σ} ρ-is-σ (↑ x) = Palphabet-faithful {Q = Q} {ρ = ρ

infix 10 _⊢_::_
data _⊢_::_ : ∀ {P} → PContext P → Proof (Palphabet P) → Expression (Palphabet P) (non
  var : ∀ {P} {Γ : PContext P} {p : El P} → Γ ⊢ var (embed p) :: typeof' p Γ
  app : ∀ {P} {Γ : PContext P} {δ} {ε} {φ} {ψ} → Γ ⊢ δ :: φ ⇒ ψ → Γ ⊢ ε :: φ → Γ ⊢ app
  Λ : ∀ {P} {Γ : PContext P} {φ} {δ} {ψ} → (_,_ {K = -Proof} Γ φ) ⊢ δ :: liftE ψ → Γ ⊢
```

A *replacement* $\rho$ from a context $\Gamma$ to a context $\Delta$, $\rho : \Gamma \to \Delta$, is a replacement on the syntax such that, for every $x : \phi$ in $\Gamma$, we have $\rho(x) : \phi \in \Delta$.

```
toRep : ∀ {P} {Q} → (El P → El Q) → Rep (Palphabet P) (Palphabet Q)
toRep {∅} f K ()
toRep {Lift P} f .-Proof x₀ = embed (f ⊥)
toRep {Lift P} {Q} f K (↑ x) = toRep {P} {Q} (f ∘ ↑) K x

toRep-embed : ∀ {P} {Q} {f : El P → El Q} {x : El P} → toRep f -Proof (embed x) ≡ embe
toRep-embed {∅} {_} {_} {()}
toRep-embed {Lift _} {_} {_} {⊥} = ref
toRep-embed {Lift P} {Q} {f} {↑ x} = toRep-embed {P} {Q} {f ∘ ↑} {x}

toRep-comp : ∀ {P} {Q} {R} {g : El Q → El R} {f : El P → El Q} → toRep g •R toRep f ∼
toRep-comp {∅} ()
toRep-comp {Lift _} {g = g} x₀ = wd var (toRep-embed {f = g})
toRep-comp {Lift _} {g = g} {f = f} (↑ x) = toRep-comp {g = g} {f = f ∘ ↑} x

_::_⇒R_ : ∀ {P} {Q} → (El P → El Q) → PContext P → PContext Q → Set
ρ :: Γ ⇒R Δ = ∀ x → typeof' (ρ x) Δ ≡ (typeof' x Γ) ⟨ toRep ρ ⟩

toRep-↑ : ∀ {P} → toRep {P} {Lift P} ↑ ∼R (λ _ → ↑)
toRep-↑ {∅} = λ ()
toRep-↑ {Lift P} = Palphabet-faithful {Lift P} {Lift (Lift P)} {toRep {Lift P} {Lift (Li

toRep-lift : ∀ {P} {Q} {f : El P → El Q} → toRep (lift f) ∼R Rep↑ (toRep f)
toRep-lift x₀ = ref
toRep-lift {∅} (↑ ())
toRep-lift {Lift _} (↑ x₀) = ref
toRep-lift {Lift P} {Q} {f} (↑ (↑ x)) = trans
  (sym (toRep-comp {g = ↑} {f = f ∘ ↑} x))
  (toRep-↑ {Q} (toRep (f ∘ ↑) _ x))

↑-typed : ∀ {P} {Γ : PContext P} {φ : Expression (Palphabet P) (nonVarKind -Prp)} →
  ↑ :: Γ ⇒R (Γ , φ)
↑-typed {P} {Γ} {φ} x = rep-wd {E = typeof' x Γ} (λ x → sym (toRep-↑ {P} x))

Rep↑-typed : ∀ {P} {Q} {ρ} {Γ : PContext P} {Δ : PContext Q} {φ : Expression (Palphabet P
  lift ρ :: (Γ , φ) ⇒R (Δ , φ ⟨ toRep ρ ⟩)
Rep↑-typed {P} {Q = Q} {ρ = ρ} {φ = φ} ρ::Γ→Δ ⊥ = let open Equational-Reasoning (Express
  ∵ φ ⟨ toRep ρ ⟩ ⟨ (λ _ → ↑) ⟩
  ≡ φ ⟨ (λ K x → ↑ (toRep ρ _ x)) ⟩      [[ rep-comp {E = φ} ]]
  ≡ φ ⟨ toRep (lift ρ) •R (λ _ → ↑) ⟩  [ rep-wd {E = φ} (λ x → trans (sym (toRep-↑ {Q}
  ≡ φ ⟨ (λ _ → ↑) ⟩ ⟨ toRep (lift ρ) ⟩ [ rep-comp {E = φ} ]
Rep↑-typed {Q = Q} {ρ = ρ} {Γ = Γ} {Δ = Δ} ρ::Γ→Δ (↑ x) = let open Equational-Reasoning
  ∵ liftE (typeof' (ρ x) Δ)
  ≡ liftE ((typeof' x Γ) ⟨ toRep ρ ⟩)        [ wd liftE (ρ::Γ→Δ x) ]
  ≡ (typeof' x Γ) ⟨ (λ K x → ↑ (toRep ρ K x)) ⟩ [[ rep-comp {E = typeof' x Γ} ]]
  ≡ (typeof' x Γ) ⟨ toRep {Q} ↑ •R toRep ρ ⟩                           [[ rep-wd {E = ty
```

```
  ≡ (typeof' x Γ) 〈 toRep (lift ρ) •R (λ _ → ↑) 〉 [ rep-wd {E = typeof' x Γ} (toRep-comp
  ≡ (liftE (typeof' x Γ)) 〈 toRep (lift ρ) 〉 [ rep-comp {E = typeof' x Γ} ]
```

The replacements between contexts are closed under composition.

```
•R-typed : ∀ {P} {Q} {R} {σ : El Q → El R} {ρ : El P → El Q} {Γ} {Δ} {Θ} → ρ :: Γ ⇒R Δ
  σ ∘ ρ :: Γ ⇒R Θ
•R-typed {R = R} {σ} {ρ} {Γ} {Δ} {Θ} ρ::Γ→Δ σ::Δ→Θ x = let open Equational-Reasoning (Exp
  ∵ typeof' (σ (ρ x)) Θ
  ≡ (typeof' (ρ x) Δ) 〈 toRep σ 〉     [ σ::Δ→Θ (ρ x) ]
  ≡ typeof' x Γ 〈 toRep ρ 〉 〈 toRep σ 〉          [ wd (λ x₁ → x₁ 〈 toRep σ 〉) (ρ::Γ→Δ x
  ≡ typeof' x Γ 〈 toRep σ •R toRep ρ 〉    [[ rep-comp {E = typeof' x Γ} ]]
  ≡ typeof' x Γ 〈 toRep (σ ∘ ρ) 〉         [ rep-wd {E = typeof' x Γ} (toRep-comp {g = σ}
```

Weakening Lemma

```
Weakening : ∀ {P} {Q} {Γ : PContext P} {Δ : PContext Q} {ρ} {δ} {φ} → Γ ⊢ δ :: φ → ρ :: 
Weakening {P} {Q} {Γ} {Δ} {ρ} (var {p = p}) ρ::Γ→Δ = subst2 (λ x y → Δ ⊢ var x :: y)
  (sym (toRep-embed {f = ρ} {x = p}))
  (ρ::Γ→Δ p)
  (var {p = ρ p})
Weakening (app Γ⊢δ::φ→ψ Γ⊢ε::φ) ρ::Γ→Δ = app (Weakening Γ⊢δ::φ→ψ ρ::Γ→Δ) (Weakening Γ⊢ε::φ
Weakening .{P} {Q} .{Γ} {Δ} {ρ} (Λ {P} {Γ} {φ} {δ} {ψ} Γ,φ⊢δ::ψ) ρ::Γ→Δ = Λ
  (subst (λ P → (Δ , φ 〈 toRep ρ 〉) ⊢ δ 〈 Rep↑ (toRep ρ) 〉 :: P)
  (let open Equational-Reasoning (Expression (Palphabet Q , -Proof) (nonVarKind -Prp)) in
  ∵ liftE ψ 〈 Rep↑ (toRep ρ) 〉
  ≡ ψ 〈 (λ _ x → ↑ (toRep ρ _ x)) 〉          [[ rep-comp {E = ψ} ]]
  ≡ liftE (ψ 〈 toRep ρ 〉)                    [ rep-comp {E = ψ} ] )
  (subst2 (λ x y → Δ , φ 〈 toRep ρ 〉 ⊢ x :: y)
    (rep-wd {E = δ} (toRep-lift {f = ρ}))
    (rep-wd {E = liftE ψ} (toRep-lift {f = ρ}))
    (Weakening {Lift P} {Lift Q} {Γ , φ} {Δ , φ 〈 toRep ρ 〉} {lift ρ} {δ} {liftE ψ}
      Γ,φ⊢δ::ψ
      claim))) where
  claim : ∀ (x : El (Lift P)) → typeof' (lift ρ x) (Δ , φ 〈 toRep ρ 〉) ≡ typeof' x (Γ ,
  claim ⊥ = let open Equational-Reasoning (Expression (Palphabet (Lift Q)) (nonVarKind -
    ∵ liftE (φ 〈 toRep ρ 〉)
    ≡ φ 〈 (λ _ → ↑) •R toRep ρ 〉        [[ rep-comp {E = φ} ]]
    ≡ liftE φ 〈 Rep↑ (toRep ρ) 〉        [ rep-comp {E = φ} ]
    ≡ liftE φ 〈 toRep (lift ρ) 〉        [[ rep-wd {E = liftE φ} (toRep-lift {f = ρ}) ]]
  claim (↑ x) = let open Equational-Reasoning (Expression (Palphabet (Lift Q)) (nonVarKin
    ∵ liftE (typeof' (ρ x) Δ)
    ≡ liftE (typeof' x Γ 〈 toRep ρ 〉)             [ wd liftE (ρ::Γ→Δ x) ]
    ≡ typeof' x Γ 〈 (λ _ → ↑) •R toRep ρ 〉       [[ rep-comp {E = typeof' x Γ} ]]
    ≡ liftE (typeof' x Γ) 〈 Rep↑ (toRep ρ) 〉     [ rep-comp {E = typeof' x Γ} ]
    ≡ liftE (typeof' x Γ) 〈 toRep (lift ρ) 〉     [[ rep-wd {E = liftE (typeof' x Γ)} (to
```

A *substitution* $\sigma$ from a context $\Gamma$ to a context $\Delta$, $\sigma : \Gamma \to \Delta$, is a substitution $\sigma$ on the syntax such that, for every $x : \phi$ in $\Gamma$, we have $\Delta \vdash \sigma(x) : \phi$.

```
_::_⇒_ : ∀ {P} {Q} → Sub (Palphabet P) (Palphabet Q) → PContext P → PContext Q → Set
σ :: Γ ⇒ Δ = ∀ x → Δ ⊢ σ _ (embed x) :: typeof' x Γ ⟦ σ ⟧

Sub↑-typed : ∀ {P} {Q} {σ} {Γ : PContext P} {Δ : PContext Q} {φ : Expression (Palphabet I
Sub↑-typed {P} {Q} {σ} {Γ} {Δ} {φ} σ::Γ→Δ ⊥ = subst (λ p → (Δ , φ ⟦ σ ⟧) ⊢ var x₀ :: p)
  (let open Equational-Reasoning (Expression (Palphabet Q , -Proof) (nonVarKind -Prp)) ir
  ∵ liftE (φ ⟦ σ ⟧)
  ≡ φ ⟦ (λ _ → ↑) •₁ σ ⟧        [[ sub-comp₁ {E = φ} ]]
  ≡ liftE φ ⟦ Sub↑ σ ⟧         [ sub-comp₂ {E = φ} ])
  var
Sub↑-typed {Q = Q} {σ = σ} {Γ = Γ} {Δ = Δ} {φ = φ} σ::Γ→Δ (↑ x) =
  subst
  (λ P → Δ , φ ⟦ σ ⟧ ⊢ Sub↑ σ -Proof (↑ (embed x)) :: P)
  (let open Equational-Reasoning (Expression (Palphabet Q , -Proof) (nonVarKind -Prp)) ir
  ∵ liftE (typeof' x Γ ⟦ σ ⟧)
  ≡ typeof' x Γ ⟦ (λ _ → ↑) •₁ σ ⟧       [[ sub-comp₁ {E = typeof' x Γ} ]]
  ≡ liftE (typeof' x Γ) ⟦ Sub↑ σ ⟧       [ sub-comp₂ {E = typeof' x Γ} ])
  (subst2 (λ x y → Δ , φ ⟦ σ ⟧ ⊢ x :: y)
    (rep-wd {E = σ -Proof (embed x)} (toRep-↑ {Q}))
    (rep-wd {E = typeof' x Γ ⟦ σ ⟧} (toRep-↑ {Q}))
    (Weakening (σ::Γ→Δ x) (↑-typed {φ = φ ⟦ σ ⟧})))

botsub-typed : ∀ {P} {Γ : PContext P} {φ : Expression (Palphabet P) (nonVarKind -Prp)} {
  Γ ⊢ δ :: φ → x₀:= δ :: (Γ , φ) ⇒ Γ
botsub-typed {P} {Γ} {φ} {δ} Γ⊢δ::φ ⊥ = subst (λ P₁ → Γ ⊢ δ :: P₁)
  (let open Equational-Reasoning (Expression (Palphabet P) (nonVarKind -Prp)) in
  ∵ φ
  ≡ φ ⟦ idSub ⟧                 [[ sub-id ]]
  ≡ liftE φ ⟦ x₀:= δ ⟧          [ sub-comp₂ {E = φ} ])
  Γ⊢δ::φ
botsub-typed {P} {Γ} {φ} {δ} _ (↑ x) = subst (λ P₁ → Γ ⊢ var (embed x) :: P₁)
  (let open Equational-Reasoning (Expression (Palphabet P) (nonVarKind -Prp)) in
  ∵ typeof' x Γ
  ≡ typeof' x Γ ⟦ idSub ⟧                [[ sub-id ]]
  ≡ liftE (typeof' x Γ) ⟦ x₀:= δ ⟧       [ sub-comp₂ {E = typeof' x Γ} ])
  var
```

Substitution Lemma

```
Substitution : ∀ {P} {Q} {Γ : PContext P} {Δ : PContext Q} {δ} {φ} {σ} → Γ ⊢ δ :: φ → σ
Substitution var σ::Γ→Δ = σ::Γ→Δ _
Substitution (app Γ⊢δ::φ→ψ Γ⊢ε::φ) σ::Γ→Δ = app (Substitution Γ⊢δ::φ→ψ σ::Γ→Δ) (Substitut
Substitution {Q = Q} {Δ = Δ} {σ = σ} (Λ {P} {Γ} {φ} {δ} {ψ} Γ,φ⊢δ::ψ) σ::Γ→Δ = Λ
  (subst (λ p → Δ , φ ⟦ σ ⟧ ⊢ δ ⟦ Sub↑ σ ⟧ :: p)
```

```
  (let open Equational-Reasoning (Expression (Palphabet Q , -Proof) (nonVarKind -Prp)) in
  ∵ liftE ψ ⟦ Sub↑ σ ⟧
  ≡ ψ ⟦ Sub↑ σ •₂ (λ _ → ↑) ⟧  [[ sub-comp₂ {E = ψ} ]]
  ≡ liftE (ψ ⟦ σ ⟧)            [ sub-comp₁ {E = ψ} ])
  (Substitution Γ,φ⊢δ::ψ (Sub↑-typed σ::Γ→Δ)))
```

Subject Reduction

```
prop-triv-red : ∀ {P} {φ ψ : Expression (Palphabet P) (nonVarKind -Prp)} → φ →⟨ β ⟩ ψ -
prop-triv-red {_} {app bot out₂} (redex ())
prop-triv-red {P} {app bot out₂} (app ())
prop-triv-red {P} {app imp (app₂ _ (app₂ _ out₂))} (redex ())
prop-triv-red {P} {app imp (app₂ (out φ) (app₂ ψ out₂))} (app (appl (out φ→φ'))) = prop-
prop-triv-red {P} {app imp (app₂ φ (app₂ (out ψ) out₂))} (app (appr (appl (out ψ→ψ'))))
prop-triv-red {P} {app imp (app₂ _ (app₂ (out _) out₂))} (app (appr (appr ())))

SR : ∀ {P} {Γ : PContext P} {δ ε : Proof (Palphabet P)} {φ} → Γ ⊢ δ :: φ → δ →⟨ β ⟩ ε -
SR var ()
SR (app {ε = ε} (Λ {P} {Γ} {φ} {δ} {ψ} Γ,φ⊢δ::ψ) Γ⊢ε::φ) (redex βI) =
  subst (λ P₁ → Γ ⊢ δ ⟦ x₀:= ε ⟧ :: P₁)
  (let open Equational-Reasoning (Expression (Palphabet P) (nonVarKind -Prp)) in
  ∵ liftE ψ ⟦ x₀:= ε ⟧
  ≡ ψ ⟦ idSub ⟧                  [[ sub-comp₂ {E = ψ} ]]
  ≡ ψ                            [ sub-id ])
  (Substitution Γ,φ⊢δ::ψ (botsub-typed Γ⊢ε::φ))
SR (app Γ⊢δ::φ→ψ Γ⊢ε::φ) (app (appl (out δ→δ'))) = app (SR Γ⊢δ::φ→ψ δ→δ') Γ⊢ε::φ
SR (app Γ⊢δ::φ→ψ Γ⊢ε::φ) (app (appr (appl (out ε→ε')))) = app Γ⊢δ::φ→ψ (SR Γ⊢ε::φ ε→ε')
SR (app Γ⊢δ::φ→ψ Γ⊢ε::φ) (app (appr (appr ())))
SR (Λ Γ⊢δ::φ) (redex ())
SR {P} (Λ Γ⊢δ::φ) (app (appl (out φ→φ'))) with prop-triv-red {P} φ→φ'
... | ()
SR (Λ Γ⊢δ::φ) (app (appr (appl (Λ (out δ→δ'))))) = Λ (SR Γ⊢δ::φ δ→δ')
SR (Λ Γ⊢δ::φ) (app (appr (appr ())))
```

We define the sets of *computable* proofs $C_\Gamma(\phi)$ for each context $\Gamma$ and proposition $\phi$ as follows:

$$C_\Gamma(\bot) = \{\delta \mid \Gamma \vdash \delta : \bot, \delta \in SN\}$$
$$C_\Gamma(\phi \to \psi) = \{\delta \mid \Gamma : \delta : \phi \to \psi, \forall \epsilon \in C_\Gamma(\phi). \delta\epsilon \in C_\Gamma(\psi)\}$$

```
C : ∀ {P} → PContext P → Prp → Proof (Palphabet P) → Set
C Γ (app bot out₂) δ = (Γ ⊢ δ :: ⊥P ⟨ (λ _ ()) ⟩ ) ∧ SN β δ
C Γ (app imp (app₂ (out φ) (app₂ (out ψ) out₂))) δ = (Γ ⊢ δ :: (φ ⇒ ψ) ⟨ (λ _ ()) ⟩) ∧
  (∀ Q {Δ : PContext Q} ρ ε → ρ :: Γ ⇒R Δ → C Δ φ ε → C Δ ψ (appP (δ ⟨ toRep ρ ⟩) ε))

C-typed : ∀ {P} {Γ : PContext P} {φ} {δ} → C Γ φ δ → Γ ⊢ δ :: φ ⟨ (λ _ ()) ⟩
```

```
C-typed {φ = app bot out₂} = π₁
C-typed {Γ = Γ} {φ = app imp (app₂ (out φ) (app₂ (out ψ) out₂))} {δ = δ} = λ x → subst (
  (wd2 _⇒_ (rep-wd {E = φ} (λ ())) (rep-wd {E = ψ} (λ ())))
  (π₁ x)

C-rep : ∀ {P} {Q} {Γ : PContext P} {Δ : PContext Q} {φ} {δ} {ρ} → C Γ φ δ → ρ :: Γ ⇒R Δ
C-rep {φ = app bot out₂} (Γ⊢δ::⊥ , SNδ) ρ::Γ→Δ = (Weakening Γ⊢δ::⊥ ρ::Γ→Δ) , SNrep β-crea
C-rep {P} {Q} {Γ} {Δ} {app imp (app₂ (out φ) (app₂ (out ψ) out₂))} {δ} {ρ} (Γ⊢δ::φ⇒ψ , Cδ
  (wd2 _⇒_
  (let open Equational-Reasoning (Expression (Palphabet Q) (nonVarKind -Prp)) in
    ∵ (φ ⟨ _ ⟩) ⟨ toRep ρ ⟩
    ≡ φ ⟨ _ ⟩              [[ rep-comp {E = φ} ]]
    ≡ φ ⟨ _ ⟩              [ rep-wd {E = φ} (λ ()) ])
--TODO Refactor common pattern
  (let open Equational-Reasoning (Expression (Palphabet Q) (nonVarKind -Prp)) in
    ∵ ψ ⟨ _ ⟩ ⟨ toRep ρ ⟩
    ≡ ψ ⟨ _ ⟩              [[ rep-comp {E = ψ} ]]
    ≡ ψ ⟨ _ ⟩              [ rep-wd {E = ψ} (λ ()) ]))
  (Weakening Γ⊢δ::φ⇒ψ ρ::Γ→Δ)) ,
  (λ R σ ε σ::Δ→Θ ε∈Cφ → subst (C _ ψ) (wd (λ x → appP x ε)
    (trans (sym (rep-wd {E = δ} (toRep-comp {g = σ} {f = ρ}))) (rep-comp {E = δ})))
    (Cδ R (σ ∘ ρ) ε (•R-typed {σ = σ} {ρ = ρ} ρ::Γ→Δ σ::Δ→Θ) ε∈Cφ))

C-red : ∀ {P} {Γ : PContext P} {φ} {δ} {ε} → C Γ φ δ → δ →⟨ β ⟩ ε → C Γ φ ε
C-red {φ = app bot out₂} (Γ⊢δ::⊥ , SNδ) δ→ε = (SR Γ⊢δ::⊥ δ→ε) , (SNred SNδ (osr-red δ→
C-red {Γ = Γ} {φ = app imp (app₂ (out φ) (app₂ (out ψ) out₂))} {δ = δ} (Γ⊢δ::φ⇒ψ , Cδ) δ-
  (wd2 _⇒_ (rep-wd {E = φ} (λ ())) (rep-wd {E = ψ} (λ ())))
  Γ⊢δ::φ⇒ψ) δ→δ') ,
  (λ Q ρ ε ρ::Γ→Δ ε∈Cφ → C-red {φ = ψ} (Cδ Q ρ ε ρ::Γ→Δ ε∈Cφ) (app (appl (out (reposr β
```

The *neutral terms* are those that begin with a variable.

```
data Neutral {P} : Proof P → Set where
  varNeutral : ∀ x → Neutral (var x)
  appNeutral : ∀ δ ε → Neutral δ → Neutral (appP δ ε)
```

**Lemma 7.** *If $\delta$ is neutral and $\delta \to_\beta \epsilon$ then $\epsilon$ is neutral.*

```
neutral-red : ∀ {P} {δ ε : Proof P} → Neutral δ → δ →⟨ β ⟩ ε → Neutral ε
neutral-red (varNeutral _) ()
neutral-red (appNeutral .(app lam (app₂ (out _) (app₂ (Λ (out _)) out₂))) _ ()) (redex βI
neutral-red (appNeutral _ ε neutralδ) (app (appl (out δ→δ'))) = appNeutral _ ε (neutral-
neutral-red (appNeutral δ _ neutralδ) (app (appr (appl (out ε→ε')))) = appNeutral δ _ ne
neutral-red (appNeutral _ _ _) (app (appr (appr ())))

neutral-rep : ∀ {P} {Q} {δ : Proof P} {ρ : Rep P Q} → Neutral δ → Neutral (δ ⟨ ρ ⟩)
neutral-rep {ρ = ρ} (varNeutral x) = varNeutral (ρ -Proof x)
neutral-rep {ρ = ρ} (appNeutral δ ε neutralδ) = appNeutral (δ ⟨ ρ ⟩) (ε ⟨ ρ ⟩) (neutral-re
```

**Lemma 8.** *Let* $\Gamma \vdash \delta : \phi$*. If* $\delta$ *is neutral and, for all* $\epsilon$ *such that* $\delta \rightarrow_\beta \epsilon$*, we have* $\epsilon \in C_\Gamma(\phi)$*, then* $\delta \in C_\Gamma(\phi)$*.*

```
NeutralC-lm : ∀ {P} {δ ε : Proof P} {X : Proof P → Set} →
  Neutral δ →
  (∀ δ' → δ →⟨ β ⟩ δ' → X (appP δ' ε)) →
  (∀ ε' → ε →⟨ β ⟩ ε' → X (appP δ ε')) →
  ∀ χ → appP δ ε →⟨ β ⟩ χ → X χ
NeutralC-lm () _ _ ._ (redex βI)
NeutralC-lm _ hyp1 _ .(app app (app2 (out _) (app2 (out _) out2))) (app (appl (out δ→δ')
NeutralC-lm _ _ hyp2 .(app app (app2 (out _) (app2 (out _) out2))) (app (appr (appl (out
NeutralC-lm _ _ _ .(app app (app2 (out _) (app2 (out _) _))) (app (appr (appr ())))

mutual
  NeutralC : ∀ {P} {Γ : PContext P} {δ : Proof (Palphabet P)} {φ : Prp} →
    Γ ⊢ δ :: φ ⟨ (λ _ ()) ⟩ → Neutral δ →
    (∀ ε → δ →⟨ β ⟩ ε → C Γ φ ε) →
    C Γ φ δ
  NeutralC {P} {Γ} {δ} {app bot out2} Γ⊢δ::⊥ Neutralδ hyp = Γ⊢δ::⊥ , SNI δ (λ ε δ→ε → π
  NeutralC {P} {Γ} {δ} {app imp (app2 (out φ) (app2 (out ψ) out2))} Γ⊢δ::φ→ψ neutralδ hyp
    (λ Q ρ ε ρ::Γ→Δ ε∈Cφ → claim ε (CsubSN {φ = φ} {δ = ε} ε∈Cφ) ρ::Γ→Δ ε∈Cφ) where
    claim : ∀ {Q} {Δ} {ρ : El P → El Q} ε → SN β ε → ρ :: Γ ⇒R Δ → C Δ φ ε → C Δ ψ (
    claim {Q} {Δ} {ρ} ε (SNI .ε SNε) ρ::Γ→Δ ε∈Cφ = NeutralC {Q} {Δ} {appP (δ ⟨ toRep ρ ⟩)
      (app (subst (λ P1 → Δ ⊢ δ ⟨ toRep ρ ⟩ :: P1)
      (wd2 _⇒_
      (let open Equational-Reasoning (Expression (Palphabet Q) (nonVarKind -Prp)) in
        ∵ φ ⟨ _ ⟩ ⟨ toRep ρ ⟩
        ≡ φ ⟨ _ ⟩        [[ rep-comp {E = φ} ]]
        ≡ φ ⟨ _ ⟩        [[ rep-wd {E = φ} (λ ()) ]])
      (  (let open Equational-Reasoning (Expression (Palphabet Q) (nonVarKind -Prp)) in
        ∵ ψ ⟨ _ ⟩ ⟨ toRep ρ ⟩
        ≡ ψ ⟨ _ ⟩        [[ rep-comp {E = ψ} ]]
        ≡ ψ ⟨ _ ⟩        [[ rep-wd {E = ψ} (λ ()) ]])
        ))
      (Weakening Γ⊢δ::φ→ψ ρ::Γ→Δ))
      (C-typed {Q} {Δ} {φ} {ε} ε∈Cφ))
      (appNeutral (δ ⟨ toRep ρ ⟩) ε (neutral-rep neutralδ))
      (NeutralC-lm {X = C Δ ψ} (neutral-rep neutralδ)
      (λ δ' δ⟨ρ⟩→δ' →
      let δ0 : Proof (Palphabet P)
          δ0 = create-reposr β-creates-rep {M = δ} {N = δ'} {ρ = toRep ρ} δ⟨ρ⟩→δ'
      in let δ→δ0 : δ →⟨ β ⟩ δ0
             δ→δ0 = red-create-reposr β-creates-rep δ⟨ρ⟩→δ'
      in let δ0⟨ρ⟩≡δ' : δ0 ⟨ toRep ρ ⟩ ≡ δ'
             δ0⟨ρ⟩≡δ' = rep-create-reposr β-creates-rep {M = δ} {N = δ'} {ρ = toRep ρ} δ(
      in let δ0∈C[φ⇒ψ] : C Γ (φ ⇒ ψ) δ0
```

```
        δ₀∈C[φ⇒ψ] = hyp δ₀ δ→δ₀
    in let δ'∈C[φ⇒ψ] : C Δ (φ ⇒ ψ) δ'
           δ'∈C[φ⇒ψ] = subst (C Δ (φ ⇒ ψ)) δ₀⟨ρ⟩≡δ' (C-rep {φ = φ ⇒ ψ} δ₀∈C[φ⇒ψ] ρ:
    in subst (C Δ ψ) (wd (λ x → appP x ε) δ₀⟨ρ⟩≡δ') (π₂ δ₀∈C[φ⇒ψ] Q ρ ε ρ::Γ→Δ ε∈Cφ)
    (λ ε' ε→ε' → claim ε' (SNε ε' ε→ε') ρ::Γ→Δ (C-red {φ = φ} ε∈Cφ ε→ε')))
```

**Lemma 9.**

$$C_\Gamma(\phi) \subseteq SN$$

```
CsubSN : ∀ {P} {Γ : PContext P} {φ} {δ} → C Γ φ δ → SN β δ
CsubSN {P} {Γ} {app bot out₂} P₁ = π₂ P₁
CsubSN {P} {Γ} {app imp (app₂ (out φ) (app₂ (out ψ) out₂))} {δ} P₁ =
  let φ' : Expression (Palphabet P) (nonVarKind -Prp)
      φ' = φ 〈 (λ _ ()) 〉 in
  let Γ' : PContext (Lift P)
      Γ' = Γ , φ' in
  SNrep' {Palphabet P} {Palphabet P , -Proof} { varKind -Proof} {λ _ → ↑} β-respects-r
    (SNsubbodyl (SNsubexp (CsubSN {Γ = Γ'} {φ = ψ}
    (subst (C Γ' ψ) (wd (λ x → appP x (var x₀)) (rep-wd {E = δ} (toRep-↑ {P = P})))
    (π₂ P₁ (Lift P) ↑ (var x₀) (λ x → sym (rep-wd {E = typeof' x Γ} (toRep-↑ {P = P}))
    (NeutralC {φ = φ}
      (subst (λ x → Γ' ⊢ var x₀ :: x)
        (trans (sym (rep-comp {E = φ})) (rep-wd {E = φ} (λ ())))
        var)
      (varNeutral x₀)
      (λ _ ()))))))))
```

```
module PHOPL where
open import Prelims hiding (⊥)
open import Grammar
open import Reduction
```

# 6 Predicative Higher-Order Propositional Logic

Fix sets of *proof variables* and *term variables*.

The syntax of the system is given by the following grammar.

| | | | |
|---|---|---|---|
| Proof | $\delta$ | $::=$ | $p \mid \delta\delta \mid \lambda p : \phi.\delta$ |
| Term | $M, \phi$ | $::=$ | $x \mid \bot \mid MM \mid \lambda x : A.M \mid \phi \to \phi$ |
| Type | $A$ | $::=$ | $\Omega \mid A \to A$ |
| Term Context | $\Gamma$ | $::=$ | $\langle\rangle \mid \Gamma, x : A$ |
| Proof Context | $\Delta$ | $::=$ | $\langle\rangle \mid \Delta, p : \phi$ |
| Judgement | $\mathcal{J}$ | $::=$ | $\Gamma \text{ valid} \mid \Gamma \vdash M : A \mid \Gamma, \Delta \text{ valid} \mid \Gamma, \Delta \vdash \delta : \phi$ |

where $p$ ranges over proof variables and $x$ ranges over term variables. The variable $p$ is bound within $\delta$ in the proof $\lambda p : \phi.\delta$, and the variable $x$ is bound within $M$ in the term $\lambda x : A.M$. We identify proofs and terms up to $\alpha$-conversion.

In the implementation, we write **Term** $(V)$ for the set of all terms with free variables a subset of $V$, where $V$ : **FinSet**.

```
data PHOPLVarKind : Set where
  -Proof : PHOPLVarKind
  -Term : PHOPLVarKind

data PHOPLNonVarKind : Set where
  -Type : PHOPLNonVarKind

PHOPLTaxonomy : Taxonomy
PHOPLTaxonomy = record {
  VarKind = PHOPLVarKind;
  NonVarKind = PHOPLNonVarKind }

module PHOPLGrammar where
  open Taxonomy PHOPLTaxonomy

  data PHOPLcon : ∀ {K : ExpressionKind} → Kind (-Constructor K) → Set where
    -appProof : PHOPLcon (Π₂ (out (varKind -Proof)) (Π₂ (out (varKind -Proof)) (out₂ {K =
    -lamProof : PHOPLcon (Π₂ (out (varKind -Term)) (Π₂ (Π -Proof (out (varKind -Proof)))
    -bot : PHOPLcon (out₂ {K = varKind -Term})
    -imp : PHOPLcon (Π₂ (out (varKind -Term)) (Π₂ (out (varKind -Term)) (out₂ {K = varKin
    -appTerm : PHOPLcon (Π₂ (out (varKind -Term)) (Π₂ (out (varKind -Term)) (out₂ {K = va
    -lamTerm : PHOPLcon (Π₂ (out (nonVarKind -Type)) (Π₂ (Π -Term (out (varKind -Term)))
    -Omega : PHOPLcon (out₂ {K = nonVarKind -Type})
    -func  : PHOPLcon (Π₂ (out (nonVarKind -Type)) (Π₂ (out (nonVarKind -Type)) (out₂ {K

  PHOPLparent : PHOPLVarKind → ExpressionKind
  PHOPLparent -Proof = varKind -Term
  PHOPLparent -Term = nonVarKind -Type

  PHOPL : Grammar
  PHOPL = record {
    taxonomy = PHOPLTaxonomy;
    toGrammar = record {
      Constructor = PHOPLcon;
      parent = PHOPLparent } }

module PHOPL where
  open PHOPLGrammar using (PHOPLcon;-appProof;-lamProof;-bot;-imp;-appTerm;-lamTerm;-Ome
  open Grammar.Grammar PHOPLGrammar.PHOPL

  Type : Set
  Type = Expression ∅ (nonVarKind -Type)
```

```
  liftType : ∀ {V} → Type → Expression V (nonVarKind -Type)
  liftType (app -Omega out₂) = app -Omega out₂
  liftType (app -func (app₂ (out A) (app₂ (out B) out₂))) = app -func (app₂ (out (liftTyp

  Ω : Type
  Ω = app -Omega out₂

  infix 75 _⇒_
  _⇒_ : Type → Type → Type
  φ ⇒ ψ = app -func (app₂ (out φ) (app₂ (out ψ) out₂))

  lowerType : ∀ {V} → Expression V (nonVarKind -Type) → Type
  lowerType (app -Omega out₂) = Ω
  lowerType (app -func (app₂ (out φ) (app₂ (out ψ) out₂))) = lowerType φ ⇒ lowerType ψ

{-  infix 80 _,_
  data TContext : Alphabet → Set where
    〈〉 : TContext ∅
    _,_ : ∀ {V} → TContext V → Type → TContext (V , -Term) -}

  TContext : Alphabet → Set
  TContext = Context -Term

  Term : Alphabet → Set
  Term V = Expression V (varKind -Term)

  ⊥ : ∀ {V} → Term V
  ⊥ = app -bot out₂

  appTerm : ∀ {V} → Term V → Term V → Term V
  appTerm M N = app -appTerm (app₂ (out M) (app₂ (out N) out₂))

  ΛTerm : ∀ {V} → Type → Term (V , -Term) → Term V
  ΛTerm A M = app -lamTerm (app₂ (out (liftType A)) (app₂ (Λ (out M)) out₂))

  _⊃_ : ∀ {V} → Term V → Term V → Term V
  φ ⊃ ψ = app -imp (app₂ (out φ) (app₂ (out ψ) out₂))

  PAlphabet : FinSet → Alphabet → Alphabet
  PAlphabet ∅ A = A
  PAlphabet (Lift P) A = PAlphabet P A , -Proof

  liftVar : ∀ {A} {K} P → Var A K → Var (PAlphabet P A) K
  liftVar ∅ x = x
  liftVar (Lift P) x = ↑ (liftVar P x)
```

```
liftVar' : ∀ {A} P → El P → Var (PAlphabet P A) -Proof
liftVar' (Lift P) Prelims.⊥ = x₀
liftVar' (Lift P) (↑ x) = ↑ (liftVar' P x)

liftExp : ∀ {V} {K} P → Expression V K → Expression (PAlphabet P V) K
liftExp P E = E ⟨ (λ _ → liftVar P) ⟩

data PContext' (V : Alphabet) : FinSet → Set where
  ⟨⟩ : PContext' V ∅
  _,_ : ∀ {P} → PContext' V P → Term V → PContext' V (Lift P)

PContext : Alphabet → FinSet → Set
PContext V = Context' V -Proof

P⟨⟩ : ∀ {V} → PContext V ∅
P⟨⟩ = ⟨⟩

_P,_ : ∀ {V} {P} → PContext V P → Term V → PContext V (Lift P)
_P,_ {V} {P} Δ φ = Δ , φ ⟨ embedl {V} { -Proof} {P} ⟩

Proof : Alphabet → FinSet → Set
Proof V P = Expression (PAlphabet P V) (varKind -Proof)

varP : ∀ {V} {P} → El P → Proof V P
varP {P = P} x = var (liftVar' P x)

appP : ∀ {V} {P} → Proof V P → Proof V P → Proof V P
appP δ ε = app -appProof (app₂ (out δ) (app₂ (out ε) out₂))

ΛP : ∀ {V} {P} → Term V → Proof V (Lift P) → Proof V P
ΛP {P = P} φ δ = app -lamProof (app₂ (out (liftExp P φ)) (app₂ (Λ (out δ)) out₂))

--  typeof' : ∀ {V} → Var V -Term → TContext V → Type
--  typeof' x₀ (_ , A) = A
--  typeof' (↑ x) (Γ , _) = typeof' x Γ

propof : ∀ {V} {P} → El P → PContext' V P → Term V
propof Prelims.⊥ (_ , φ) = φ
propof (↑ x) (Γ , _) = propof x Γ

data β : Reduction PHOPLGrammar.PHOPL where
  βI : ∀ {V} A (M : Term (V , -Term)) N → β -appTerm (app₂ (out (ΛTerm A M)) (app₂ (ou
```

The rules of deduction of the system are as follows.

$$\frac{}{\langle\rangle \text{ valid}} \qquad \frac{\Gamma \text{ valid}}{\Gamma, x : A \text{ valid}} \qquad \frac{\Gamma \vdash \phi : \Omega}{\Gamma, p : \phi \text{ valid}}$$

$$\frac{\Gamma \text{ valid}}{\Gamma \vdash x : A} \ (x : A \in \Gamma) \qquad \frac{\Gamma \text{ valid}}{\Gamma \vdash p : \phi} \ (p : \phi \in \Gamma)$$

$$\frac{\Gamma \text{ valid}}{\Gamma \vdash \bot : \Omega} \qquad \frac{\Gamma \vdash \phi : \Omega \quad \Gamma \vdash \psi : \Omega}{\Gamma \vdash \phi \to \psi : \Omega}$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad \frac{\Gamma \vdash \delta : \phi \to \psi \quad \Gamma \vdash \epsilon : \phi}{\Gamma \vdash \delta\epsilon : \psi}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \to B} \qquad \frac{\Gamma, p : \phi \vdash \delta : \psi}{\Gamma \vdash \lambda p : \phi.\delta : \phi \to \psi}$$

$$\frac{\Gamma \vdash \delta : \phi \quad \Gamma \vdash \psi : \Omega}{\Gamma \vdash \delta : \psi} \ (\phi \simeq \phi)$$

```
infix 10 _⊢_:_
data _⊢_:_ : ∀ {V} → TContext V → Term V → Expression V (nonVarKind -Type) → Set₁ w
  var : ∀ {V} {Γ : TContext V} {x} → Γ ⊢ var x : typeof x Γ
  ⊥R : ∀ {V} {Γ : TContext V} → Γ ⊢ ⊥ : Ω ⟨ (λ _ ()) ⟩
  imp : ∀ {V} {Γ : TContext V} {φ} {ψ} → Γ ⊢ φ : Ω ⟨ (λ _ ()) ⟩ → Γ ⊢ ψ : Ω ⟨ (λ _ ())
  app : ∀ {V} {Γ : TContext V} {M} {N} {A} {B} → Γ ⊢ M : app -func (app₂ (out A) (app₂
  Λ : ∀ {V} {Γ : TContext V} {A} {M} {B} → Γ , A ⊢ M : liftE B → Γ ⊢ app -lamTerm (ap

data Pvalid : ∀ {V} {P} → TContext V → PContext' V P → Set₁ where
  ⟨⟩ : ∀ {V} {Γ : TContext V} → Pvalid Γ ⟨⟩
  _,_ : ∀ {V} {P} {Γ : TContext V} {Δ : PContext' V P} {φ : Term V} → Pvalid Γ Δ → Γ

infix 10 _,,_⊢_::_
data _,,_⊢_::_ : ∀ {V} {P} → TContext V → PContext' V P → Proof V P → Term V → Set₁
  var : ∀ {V} {P} {Γ : TContext V} {Δ : PContext' V P} {p} → Pvalid Γ Δ → Γ ,, Δ ⊢ va
  app : ∀ {V} {P} {Γ : TContext V} {Δ : PContext' V P} {δ} {ε} {φ} {ψ} → Γ ,, Δ ⊢ δ ::
  Λ : ∀ {V} {P} {Γ : TContext V} {Δ : PContext' V P} {φ} {δ} {ψ} → Γ ,, Δ , φ ⊢ δ :: ψ
  convR : ∀ {V} {P} {Γ : TContext V} {Δ : PContext' V P} {δ} {φ} {ψ} → Γ ,, Δ ⊢ δ :: φ
```