A Strongly Normalizing Computation Rule for Univalence in Higher-Order Propositional Logic

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Homotopy type theory offers the promise of a formal system for the univalent foundations of mathematics. However, if we simply add the univalence axiom to type theory, then we lose the property of canonicity — that every term computes to a normal form. A computation becomes 'stuck' when it reaches the point that it needs to evaluate a proof term that is an application of the univalence axiom. So we wish to find a way to compute with the univalence axiom.

As a first step, we present here a system of higher-order propositional logic, with a universe Ω of propositions closed under implication and quantification over any simple type over Ω. We add a type $M =_A N$ for any terms M, N of type A, and two ways to prove an equality: reflexivity, and the univalence axiom. We present reduction relations for this system, and prove the reduction confluent and strongly normalizing on the well-typed terms.

We have begun to formalize this proof in AGDA, and intend to complete the formalization by the date of the workshop.

Predicative higher-order propositional logic with equality. We call the following type theory *predicative higher-order propositional logic*. It contains a universe Ω of propositions that contains $\bot$ and is closed under implication $\supset$. The system also includes the higher-order types that can be built from Ω by $\to$. Its grammar and rules of deduction are as follows.

$$
\begin{array}{lrcl}
\text{Proof} & \delta & ::= & p \mid \delta\delta \mid \lambda p : \phi.\delta \\
\text{Term} & M, \phi & ::= & x \mid \bot \mid MM \mid \lambda x : A.M \mid \phi \supset \phi \\
\text{Type} & A & ::= & \Omega \mid A \to A
\end{array}
$$

$$
\frac{}{\langle\rangle \text{ valid}} \qquad
\frac{\Gamma \text{ valid}}{\Gamma, x : A \text{ valid}} \qquad
\frac{\Gamma \vdash \phi : \Omega}{\Gamma, p : \phi \text{ valid}} \qquad
\frac{\Gamma \text{ valid}}{\Gamma \vdash x : A}\ (x : A \in \Gamma) \qquad
\frac{\Gamma \text{ valid}}{\Gamma \vdash p : \phi}\ (p : \phi \in \Gamma)
$$

$$
\frac{\Gamma \text{ valid}}{\Gamma \vdash \bot : \Omega} \qquad
\frac{\Gamma \vdash \phi : \Omega \quad \Gamma \vdash \psi : \Omega}{\Gamma \vdash \phi \supset \psi : \Omega}
$$

$$
\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad
\frac{\Gamma \vdash \delta : \phi \supset \psi \quad \Gamma \vdash \epsilon : \phi}{\Gamma \vdash \delta\epsilon : \psi}
$$

$$
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \to B} \qquad
\frac{\Gamma, p : \phi \vdash \delta : \psi}{\Gamma \vdash \lambda p : \phi.\delta : \phi \supset \psi} \qquad
\frac{\Gamma \vdash \delta : \phi \quad \Gamma \vdash \psi : \Omega}{\Gamma \vdash \delta : \psi}\ (\phi \simeq \psi)
$$

Extensional equality. On top of this system, we add an equality relation that satisfies univalence. We add a new judgement form, $\Gamma \vdash P : M =_A M$, to denote that P is an *equality proof* that M and N are equal terms of type A.

$$
\begin{array}{lrcl}
\text{Equality proof} & P & ::= & e \mid \mathsf{ref}\,(M) \mid \mathsf{univ}_{\phi,\phi}\,(\delta,\delta) \mid \lambda\!\!\!\lambda\!\!\!\lambda e : x =_A x.P \mid P \supset P \mid PP \\
\text{Proof} & \delta & ::= & \cdots \mid P^+ \mid P^-
\end{array}
$$

- For any $M : A$, there is an equality proof $\mathsf{ref}\,(M) : M =_A M$.
- **Univalence.** Given proofs $\delta : \phi \supset \psi$ and $\epsilon : \psi \supset \phi$, there is an equality proof $\mathsf{univ}_{\phi,\psi}\,(\delta, \epsilon) : \phi =_\Omega \psi$.
- Given a proof $P : \phi =_\Omega \psi$, we have proofs $P^+ : \phi \supset \psi$ and $P^- : \psi \supset \phi$.
- Given an equality proof $\Gamma, x : A, y : A, e : x =_A y \vdash P : Mx =_B Ny$, there is an equality proof $\Gamma \vdash \lambda\!\!\lambda\!\!\lambda e : x =_A y.P : M =_{A \to B} N$. (Here, e, x and y are bound within P.)
- **Congruence.** If $P : \phi =_\Omega \phi'$ and $Q : \psi =_\Omega \psi'$ then $P \supset Q : \phi \supset \psi =_\Omega \phi' \supset \psi'$. If $P : M =_{A \to B} M'$ and $Q : N =_A N'$ then $PQ : MN =_B M'N'$.

The reduction relation. We define the following reduction relation on proofs and equality proofs.

$$(\mathsf{ref}\,(\phi))^+ \rightsquigarrow \lambda x : \phi.x \qquad (\mathsf{ref}\,(\phi))^- \rightsquigarrow \lambda x : \phi.x \qquad \mathsf{univ}_{\phi,\psi}\,(\delta, \epsilon)^+ \rightsquigarrow \delta \qquad \mathsf{univ}_{\phi,\psi}\,(\delta, \epsilon)^- \rightsquigarrow \epsilon$$

$$\begin{aligned}
(\mathsf{ref}\,(\phi) \supset \mathsf{univ}_{\psi,\chi}\,(\delta, \epsilon)) &\rightsquigarrow \mathsf{univ}_{\phi \supset \psi, \phi \supset \chi}\,(\lambda f : \phi \supset \psi.\lambda x : \phi.\delta(fx), \lambda g : \phi \supset \chi.\lambda x : \phi.\epsilon(gx)) \\
(\mathsf{univ}_{\phi,\psi}\,(\delta, \epsilon) \supset \mathsf{ref}\,(\chi)) &\rightsquigarrow \mathsf{univ}_{\phi \supset \chi, \psi \supset \chi}\,(\lambda f : \phi \supset \chi.\lambda x : \psi.f(\epsilon x), \lambda g : \psi \supset \chi.\lambda x : \phi.g(\delta x)) \\
(\mathsf{univ}_{\phi,\psi}\,(\delta, \epsilon) &\supset \mathsf{univ}_{\phi',\psi'}\,(\delta', \epsilon') \\
&\rightsquigarrow \mathsf{univ}_{\phi \supset \phi', \psi \supset \psi'}\,(\lambda f : \phi \supset \phi'.\lambda x : \psi.\delta'(f(\epsilon x)), \lambda g : \psi \supset \psi'.\lambda y : \phi.\epsilon'(g(\delta y)))
\end{aligned}$$

$$\begin{gathered}
(\mathsf{ref}\,(\phi) \supset \mathsf{ref}\,(\psi)) \rightsquigarrow \mathsf{ref}\,(\phi \supset \psi) \qquad \mathsf{ref}\,(M)\,\mathsf{ref}\,(N) \rightsquigarrow \mathsf{ref}\,(MN) \\
(\mathsf{ref}\,(\lambda x : A.M))P \rightsquigarrow \{P/x\}M \qquad (P \text{ a normal form not of the form } \mathsf{ref}\,(_)) \\
(\lambda\!\!\lambda\!\!\lambda e : x =_A y.P)Q \rightsquigarrow [M/x, N/y, Q/e]P \qquad (Q : M =_A N)
\end{gathered}$$

Here, $\{P/x\}M$ is an operation called *path substitution* defined such that, if $P : N =_A N'$, then $\{P/x\}M : [N/x]M = [N'/x]M$.

Main Theorem.

Theorem 1. *In the system described above, all typable terms, proofs and equality proofs are confluent and strongly normalizing. Every closed normal form of type $\phi =_\Omega \psi$ either has the form $\mathsf{ref}\,(\_)$ or $\mathsf{univ}(\_,\_)$. Every closed normal form of type $M =_{A \to B} N$ either has the form $\mathsf{ref}\,(\_)$ or is a $\lambda\!\!\lambda\!\!\lambda$-term.*

Thus, we know that a well-typed computation never gets 'stuck' at an application of the univalence axiom.

Proof. The proof uses the method of Tait-style computability. We define the set of *computable* terms $E_\Gamma(A)$ for each type A, and computable proofs $E_\Gamma(M =_A N)$ for any terms $\Gamma \vdash M, N : A$. We prove that reduction is locally confluent, and that the computability predicates are closed under reduction and well-typed expansion. We can then prove that, if $\Gamma \vdash M : A$, then $M \in E_\Gamma(A)$; and if $\Gamma \vdash P : M =_A N$, then $P \in E_\Gamma(M =_A N)$.

Remark. Tait's proof relies on confluence, which does not hold for this reduction relation in general. In the proof, we prove confluence 'on-the-fly'. That is, whenever we require a term to be confluent, the induction hypothesis provides us with the fact that that term is computable, and hence strongly normalizing and confluent.