Type Theories with Computation Rules for the Univalence Axiom

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```
module main where
```

1 Preliminaries

```
module Prelims where
```

1.1 Functions

We write id_A for the identity function on the type A, and $g \circ f$ for the composition of functions g and f.

```
id : ∀ (A : Set) → A → A
id A x = x

infix 75 _○_
_○_ : ∀ {A B C : Set} → (B → C) → (A → B) → A → C
(g ○ f) x = g (f x)
```

1.2 Equality

We use the inductively defined equality $=$ on every datatype.

```
infix 50 _≡_
data _≡_ {A : Set} (a : A) : A → Set where
  ref : a ≡ a

subst : ∀ {A : Set} (P : A → Set) {a} {b} → a ≡ b → P a → P b
subst P ref Pa = Pa

sym : ∀ {A : Set} {a b : A} → a ≡ b → b ≡ a
sym ref = ref

trans : ∀ {A : Set} {a b c : A} → a ≡ b → b ≡ c → a ≡ c
```

```
trans ref ref = ref

wd : ∀ {A B : Set} (f : A → B) {a a' : A} → a ≡ a' → f a ≡ f a'
wd _ ref = ref

wd2 : ∀ {A B C : Set} (f : A → B → C) {a a' : A} {b b' : B} → a ≡ a' → b ≡ b' → f a
wd2 _ ref ref = ref

module Equational-Reasoning (A : Set) where
  infix 2 ∵_
  ∵_ : ∀ (a : A) → a ≡ a
  ∵ _ = ref

  infix 1 _≡_[_]
  _≡_[_] : ∀ {a b : A} → a ≡ b → ∀ c → b ≡ c → a ≡ c
  δ ≡ c [ δ' ] = trans δ δ'

  infix 1 _≡_[[_]]
  _≡_[[_]] : ∀ {a b : A} → a ≡ b → ∀ c → c ≡ b → a ≡ c
  δ ≡ c [[ δ' ]] = trans δ (sym δ')
```

We also write $f \sim g$ iff the functions f and g are extensionally equal, that is, $f(x) = g(x)$ for all x.

```
infix 50 _∼_
_∼_ : ∀ {A B : Set} → (A → B) → (A → B) → Set
f ∼ g = ∀ x → f x ≡ g x
```

2 Datatypes

We introduce a universe **FinSet** of (names of) finite sets. There is an empty set $\emptyset : \mathbf{FinSet}$, and for every $A : \mathbf{FinSet}$, the type $A+1 : \mathbf{FinSet}$ has one more element:

$$A + 1 = \{\perp\} \uplus \{\uparrow a : a \in A\}$$

```
data FinSet : Set where
  ∅ : FinSet
  Lift : FinSet → FinSet

data El : FinSet → Set where
  ⊥ : ∀ {V} → El (Lift V)
  ↑ : ∀ {V} → El V → El (Lift V)
```

A *replacement* from U to V is simply a function $U \to V$.

```
Rep : FinSet → FinSet → Set
Rep U V = El U → El V
```

Given $f : A \to B$, define $f + 1 : A + 1 \to B + 1$ by

$$(f+1)(\bot) = \bot$$
$$(f+1)(\uparrow x) = \uparrow f(x)$$

```
lift : ∀ {U} {V} → Rep U V → Rep (Lift U) (Lift V)
lift _ ⊥ = ⊥
lift f (↑ x) = ↑ (f x)

liftwd : ∀ {U} {V} {f g : Rep U V} → f ∼ g → lift f ∼ lift g
liftwd f-is-g ⊥ = ref
liftwd f-is-g (↑ x) = wd ↑ (f-is-g x)
```

This makes $(-) + 1$ into a functor **FinSet** $\to$ **FinSet**; that is,

$$\mathrm{id}_V + 1 = \mathrm{id}_{V+1}$$
$$(g \circ f) + 1 = (g + 1) \circ (f + 1)$$

```
liftid : ∀ {V} → lift (id (El V)) ∼ id (El (Lift V))
liftid ⊥ = ref
liftid (↑ _) = ref

liftcomp : ∀ {U} {V} {W} {g : Rep V W} {f : Rep U V} → lift (g ∘ f) ∼ lift g ∘ lift f
liftcomp ⊥ = ref
liftcomp (↑ _) = ref

open import Prelims
```

3 Predicative Higher-Order Propositional Logic

Fix sets of *proof variables* and *term variables*.

The syntax of the system is given by the following grammar.

Proof	δ	$::=$	$p \mid \delta\delta \mid \lambda p : \phi.\delta$
Term	M, ϕ	$::=$	$x \mid \bot \mid MM \mid \phi \to \phi \mid \lambda x : A.M$
Type	A	$::=$	$\Omega \mid A \to A$
Context	Γ	$::=$	$\langle\rangle \mid \Gamma, p : \phi \mid \Gamma, x : A$
Judgement	$\mathcal{J}$	$::=$	Γ valid $\mid \Gamma \vdash \delta : \phi \mid \Gamma \vdash M : A$

where p ranges over proof variables and x ranges over term variables. The variable p is bound within δ in the proof $\lambda p : \phi.\delta$, and the variable x is bound within M in the term $\lambda x : A.M$. We identify proofs and terms up to α-conversion.

In the implementation, we write **Term** (V) for the set of all terms with free variables a subset of V, where V : **FinSet**.

```
infix 80 _⇒_
data Type : Set where
```

```
  Ω : Type
  _⇒_ : Type → Type → Type

--Term V is the set of all terms M with FV(M) ⊆ V
data Term : FinSet → Set where
  var : ∀ {V} → El V → Term V
  ⊥ : ∀ {V} → Term V
  app : ∀ {V} → Term V → Term V → Term V
  Λ : ∀ {V} → Type → Term (Lift V) → Term V
  _⇒_ : ∀ {V} → Term V → Term V → Term V

--Proof V P is the set of all proofs with term variables among V and proof variables amo
data Proof (V : FinSet) : FinSet → Set₁ where
  var : ∀ {P} → El P → Proof V P
  app : ∀ {P} → Proof V P → Proof V P → Proof V P
  Λ : ∀ {P} → Term V → Proof V (Lift P) → Proof V P

--Context V P is the set of all contexts whose domain consists of the term variables in 
infix 80 _,_
infix 80 _,,_
data Context : FinSet → FinSet → Set₁ where
  ⟨⟩ : Context ∅ ∅
  _,_ : ∀ {V} {P} → Context V P → Type → Context (Lift V) P
  _,,_ : ∀ {V} {P} → Context V P → Term V → Context V (Lift P)
```

Let U, V : **FinSet**. A *replacement* from U to V is just a function $U \to V$. Given a term M : **Term** (U) and a replacement $\rho : U \to V$, we write $M\{\rho\}$: **Term** (V) for the result of replacing each variable x in M with $\rho(x)$.

```
infix 60 _<_>
_<_> : ∀ {U V} → Term U → Rep U V → Term V
(var x) < ρ > = var (ρ x)
⊥ < ρ > = ⊥
(app M N) < ρ > = app (M < ρ >) (N < ρ >)
(Λ A M) < ρ > = Λ A (M < lift ρ >)
(φ ⇒ ψ) < ρ > = (φ < ρ >) ⇒ (ψ < ρ >)
```

With this as the action on arrows, **Term** () becomes a functor **FinSet** → **Set**.

```
repwd : ∀ {U V : FinSet} {ρ ρ' : El U → El V} → ρ ∼ ρ' → ∀ M → M < ρ > ≡ M < ρ' >
repwd ρ-is-ρ' (var x) = wd var (ρ-is-ρ' x)
repwd ρ-is-ρ' ⊥ = ref
repwd ρ-is-ρ' (app M N)= wd2 app (repwd ρ-is-ρ' M) (repwd ρ-is-ρ' N)
repwd ρ-is-ρ' (Λ A M) = wd (Λ A) (repwd (liftwd ρ-is-ρ') M)
repwd ρ-is-ρ' (φ ⇒ ψ) = wd2 _⇒_ (repwd ρ-is-ρ' φ) (repwd ρ-is-ρ' ψ)
```

```
repid : ∀ {V : FinSet} M → M < id (El V) > ≡ M
repid (var x) = ref
repid ⊥ = ref
repid (app M N) = wd2 app (repid M) (repid N)
repid (Λ A M) = wd (Λ A) (trans (repwd liftid M) (repid M))
repid (φ ⇒ ψ) = wd2 _⇒_ (repid φ) (repid ψ)

repcomp : ∀ {U V W : FinSet} (σ : El V → El W) (ρ : El U → El V) M → M < σ ∘ ρ > ≡ M
repcomp ρ σ (var x) = ref
repcomp ρ σ ⊥ = ref
repcomp ρ σ (app M N) = wd2 app (repcomp ρ σ M) (repcomp ρ σ N)
repcomp ρ σ (Λ A M) = wd (Λ A) (trans (repwd liftcomp M) (repcomp (lift ρ) (lift σ) M))
repcomp ρ σ (φ ⇒ ψ) = wd2 _⇒_ (repcomp ρ σ φ) (repcomp ρ σ ψ)
```

A *substitution* σ from U to V, $\sigma : U \Rightarrow V$, is a function $\sigma : U \to \mathbf{Term}\,(V)$.

```
Sub : FinSet → FinSet → Set
Sub U V = El U → Term V
```

We need the following definition before we can define $M[\sigma]$, the result of applying a substitution σ to a term M.

Given a substitution $\sigma : U \Rightarrow V$, define the substitution $\sigma + 1 : U + 1 \Rightarrow V + 1$ as follows.

```
liftSub : ∀ {U} {V} → Sub U V → Sub (Lift U) (Lift V)
liftSub _ ⊥ = var ⊥
liftSub σ (↑ x) = σ x < ↑ >

liftSub-wd : ∀ {U V} {σ σ' : Sub U V} → σ ∼ σ' → liftSub σ ∼ liftSub σ'
liftSub-wd σ-is-σ' ⊥ = ref
liftSub-wd σ-is-σ' (↑ x) = wd (λ x → x < ↑ >) (σ-is-σ' x)
```

Now define $M[\sigma]$ as follows.

```
--Term is a monad with unit var and the following multiplication
infix 60 _[_]
_[_] : ∀ {U V : FinSet} → Term U → Sub U V → Term V
(var x) [ σ ] = σ x
⊥ [ σ ] = ⊥
(app M N) [ σ ] = app (M [ σ ]) (N [ σ ])
(Λ A M) [ σ ] = Λ A (M [ liftSub σ ])
(φ ⇒ ψ) [ σ ] = (φ [ σ ]) ⇒ (ψ [ σ ])

subwd : ∀ {U V : FinSet} {σ σ' : Sub U V} → σ ∼ σ' → ∀ M → M [ σ ] ≡ M [ σ' ]
subwd σ-is-σ' (var x) = σ-is-σ' x
subwd σ-is-σ' ⊥ = ref
subwd σ-is-σ' (app M N) = wd2 app (subwd σ-is-σ' M) (subwd σ-is-σ' N)
```

```
subwd σ-is-σ' (Λ A M) = wd (Λ A) (subwd (liftSub-wd σ-is-σ') M)
subwd σ-is-σ' (φ ⇒ ψ) = wd2 _⇒_ (subwd σ-is-σ' φ) (subwd σ-is-σ' ψ)

--The first monad law

idSub : ∀ V → Sub V V
idSub V = var

liftSub-id : ∀ {V : FinSet} → liftSub (idSub V) ∼ idSub (Lift V)
liftSub-id ⊥ = ref
liftSub-id (↑ x) = ref

liftSub-rep : ∀ {U V W : FinSet} (σ : Sub U V) (ρ : El V → El W) (x : El (Lift U)) → li
liftSub-rep σ ρ ⊥ = ref
liftSub-rep σ ρ (↑ x) = trans (sym (repcomp ↑ ρ (σ x))) (repcomp (lift ρ) ↑ (σ x))

liftSub-lift : ∀ {U V W : FinSet} (σ : Sub V W) (ρ : El U → El V) (x : El (Lift U)) →
  liftSub σ (lift ρ x) ≡ liftSub (λ x → σ (ρ x)) x
liftSub-lift σ ρ ⊥ = ref
liftSub-lift σ ρ (↑ x) = ref

var-lift : ∀ {U V : FinSet} {ρ : El U → El V} → var ∘ lift ρ ∼ liftSub (var ∘ ρ)
var-lift ⊥ = ref
var-lift (↑ x) = ref

subvar : ∀ {V : FinSet} (M : Term V) → M [ idSub V ] ≡ M
subvar (var x) = ref
subvar ⊥ = ref
subvar (app M N) = wd2 app (subvar M) (subvar N)
subvar (Λ A M) = wd (Λ A) (trans (subwd liftSub-id M) (subvar M))
subvar (φ ⇒ ψ) = wd2 _⇒_ (subvar φ) (subvar ψ)

infix 75 _•_
_•_ : ∀ {U V W : FinSet} → Sub V W → Sub U V → Sub U W
(σ • ρ) x = ρ x [ σ ]

rep-sub : ∀ {U} {V} {W} (σ : Sub U V) (ρ : Rep V W) (M : Term U) → M [ σ ] < ρ > ≡ M [
rep-sub σ ρ (var x) = ref
rep-sub σ ρ ⊥ = ref
rep-sub σ ρ (app M N) = wd2 app (rep-sub σ ρ M) (rep-sub σ ρ N)
rep-sub σ ρ (Λ A M) = wd (Λ A) (trans (rep-sub (liftSub σ) (lift ρ) M) (subwd (λ x → sy
rep-sub σ ρ (φ ⇒ ψ) = wd2 _⇒_ (rep-sub σ ρ φ) (rep-sub σ ρ ψ)

sub-rep : ∀ {U} {V} {W} (σ : Sub V W) (ρ : El U → El V) M → M < ρ > [ σ ] ≡ M [ σ ∘ ρ
sub-rep σ ρ (var x) = ref
sub-rep σ ρ ⊥ = ref
```

```
sub-rep σ ρ (app M N) = wd2 app (sub-rep σ ρ M) (sub-rep σ ρ N)
sub-rep σ ρ (Λ A M) = wd (Λ A) (trans (sub-rep (liftSub σ) (lift ρ) M) (subwd (liftSub-l
sub-rep σ ρ (φ ⇒ ψ) = wd2 _⇒_ (sub-rep σ ρ φ) (sub-rep σ ρ ψ)

liftSub-comp : ∀ {U} {V} {W} (σ : Sub V W) (ρ : Sub U V) →
  liftSub (σ • ρ) ∼ liftSub σ • liftSub ρ
liftSub-comp σ ρ ⊥ = ref
liftSub-comp σ ρ (↑ x) = trans (rep-sub σ ↑ (ρ x)) (sym (sub-rep (liftSub σ) ↑ (ρ x)))

-- The second monad law

subcomp : ∀ {U} {V} {W} (σ : Sub V W) (ρ : Sub U V) M → M [ σ • ρ ] ≡ M [ ρ ] [ σ ]
subcomp σ ρ (var x) = ref
subcomp σ ρ ⊥ = ref
subcomp σ ρ (app M N) = wd2 app (subcomp σ ρ M) (subcomp σ ρ N)
subcomp σ ρ (Λ A M) = wd (Λ A) (trans (subwd (liftSub-comp σ ρ) M)  (subcomp (liftSub σ
subcomp σ ρ (φ ⇒ ψ) = wd2 _⇒_ (subcomp σ ρ φ) (subcomp σ ρ ψ)

rep-is-sub : ∀ {U} {V} {ρ : El U → El V} M → M < ρ > ≡ M [ var ∘ ρ ]
rep-is-sub (var x) = ref
rep-is-sub ⊥ = ref
rep-is-sub (app M N) = wd2 app (rep-is-sub M) (rep-is-sub N)
rep-is-sub (Λ A M) = wd (Λ A) (trans (rep-is-sub M) (subwd var-lift M))
rep-is-sub (φ ⇒ ψ) = wd2 _⇒_ (rep-is-sub φ) (rep-is-sub ψ)

typeof : ∀ {V} {P} → El V → Context V P → Type
typeof () ⟨⟩
typeof ⊥ (_ , A) = A
typeof (↑ x) (Γ , _) = typeof x Γ
typeof x (Γ ,, _) = typeof x Γ

propof : ∀ {V} {P} → El P → Context V P → Term V
propof () ⟨⟩
propof p (Γ , _) = propof p Γ < ↑ >
propof p (_ ,, φ) = φ

liftSub-var' : ∀ {U} {V} (ρ : El U → El V) → liftSub (var ∘ ρ) ∼ var ∘ lift ρ
liftSub-var' ρ ⊥ = ref
liftSub-var' ρ (↑ x) = ref

botsub : ∀ {V} → Term V → Sub (Lift V) V
botsub M ⊥ = M
botsub _ (↑ x) = var x

sub-botsub : ∀ {U} {V} (σ : Sub U V) (M : Term U) (x : El (Lift U)) →
  botsub M x [ σ ] ≡ liftSub σ x [ botsub (M [ σ ]) ]
```

```
sub-botsub σ M ⊥ = ref
sub-botsub σ M (↑ x) = let open Equational-Reasoning (Term _) in
  ∵ σ x
  ≡ σ x [ var ]                           [[ subvar (σ x) ]]
  ≡ σ x < ↑ > [ botsub (M [ σ ]) ]      [[ sub-rep (botsub (M [ σ ])) ↑ (σ x) ]]

rep-botsub : ∀ {U} {V} (ρ : El U → El V) (M : Term U) (x : El (Lift U)) →
  botsub M x < ρ > ≡ botsub (M < ρ >) (lift ρ x)
rep-botsub ρ M x = trans (rep-is-sub (botsub M x))
  (trans (sub-botsub (var ∘ ρ) M x) (trans (subwd (λ x₁ → wd (λ y → botsub y x₁) (sym (
  (wd (λ x → x [ botsub (M < ρ >)]) (liftSub-var' ρ x))))
--TODO Inline this?

subbot : ∀ {V} → Term (Lift V) → Term V → Term V
subbot M N = M [ botsub N ]
```

We write $M \simeq N$ iff the terms M and N are β-convertible, and similarly for proofs.

```
data _↠_ : ∀ {V} → Term V → Term V → Set where
  β : ∀ {V} A (M : Term (Lift V)) N → app (Λ A M) N ↠ subbot M N
  ref : ∀ {V} {M : Term V} → M ↠ M
  ↠trans : ∀ {V} {M N P : Term V} → M ↠ N → N ↠ P → M ↠ P
  app : ∀ {V} {M M' N N' : Term V} → M ↠ M' → N ↠ N' → app M N ↠ app M' N'
  Λ : ∀ {V} {M N : Term (Lift V)} {A} → M ↠ N → Λ A M ↠ Λ A N
  imp : ∀ {V} {φ φ' ψ ψ' : Term V} → φ ↠ φ' → ψ ↠ ψ' → φ ⇒ ψ ↠ φ' ⇒ ψ'

repred : ∀ {U} {V} {ρ : El U → El V} {M N : Term U} → M ↠ N → M < ρ > ↠ N < ρ >
repred {U} {V} {ρ} (β A M N) = subst (λ x → app (Λ A (M < lift ρ > )) (N < ρ >) ↠ x) (
repred ref = ref
repred (↠trans M↠N N↠P) = ↠trans (repred M↠N) (repred N↠P)
repred (app M↠N M'↠N') = app (repred M↠N) (repred M'↠N')
repred (Λ M↠N) = Λ (repred M↠N)
repred (imp φ↠φ' ψ↠ψ') = imp (repred φ↠φ') (repred ψ↠ψ')

liftSub-red : ∀ {U} {V} {ρ σ : Sub U V} → (∀ x → ρ x ↠ σ x) → (∀ x → liftSub ρ x ↠
liftSub-red ρ↠σ ⊥ = ref
liftSub-red ρ↠σ (↑ x) = repred (ρ↠σ x)

subred : ∀ {U} {V} {ρ σ : Sub U V} (M : Term U) → (∀ x → ρ x ↠ σ x) → M [ ρ ] ↠ M [
subred (var x) ρ↠σ = ρ↠σ x
subred ⊥ ρ↠σ = ref
subred (app M N) ρ↠σ = app (subred M ρ↠σ) (subred N ρ↠σ)
subred (Λ A M) ρ↠σ = Λ (subred M (liftSub-red ρ↠σ))
subred (φ ⇒ ψ) ρ↠σ = imp (subred φ ρ↠σ) (subred ψ ρ↠σ)
```

```
subsub : ∀ {U} {V} {W} (σ : Sub V W) (ρ : Sub U V) M → M [ ρ ] [ σ ] ≡ M [ σ • ρ ]
subsub σ ρ (var x) = ref
subsub σ ρ ⊥ = ref
subsub σ ρ (app M N) = wd2 app (subsub σ ρ M) (subsub σ ρ N)
subsub σ ρ (Λ A M) = wd (Λ A) (trans (subsub (liftSub σ) (liftSub ρ) M)
  (subwd (λ x → sym (liftSub-comp σ ρ x)) M))
subsub σ ρ (φ ⇒ ψ) = wd2 _⇒_ (subsub σ ρ φ) (subsub σ ρ ψ)

subredr : ∀ {U} {V} {σ : Sub U V} {M N : Term U} → M ↠ N → M [ σ ] ↠ N [ σ ]
subredr {U} {V} {σ} (β A M N) = subst (λ x → app (Λ A (M [ liftSub σ ])) (N [ σ ]) ↠ x
  (sym (trans (subsub (botsub (N [ σ ])) (liftSub σ) M) (subwd (λ x → sym (sub-botsub σ
subredr ref = ref
subredr (↠trans M↠N N↠P) = ↠trans (subredr M↠N) (subredr N↠P)
subredr (app M↠M' N↠N') = app (subredr M↠M') (subredr N↠N')
subredr (Λ M↠N) = Λ (subredr M↠N)
subredr (imp φ↠φ' ψ↠ψ') = imp (subredr φ↠φ') (subredr ψ↠ψ')

data _≃_ : ∀ {V} → Term V → Term V → Set₁ where
  β : ∀ {V} {A} {M : Term (Lift V)} {N} → app (Λ A M) N ≃ subbot M N
  ref : ∀ {V} {M : Term V} → M ≃ M
  ≃sym : ∀ {V} {M N : Term V} → M ≃ N → N ≃ M
  ≃trans : ∀ {V} {M N P : Term V} → M ≃ N → N ≃ P → M ≃ P
  app : ∀ {V} {M M' N N' : Term V} → M ≃ M' → N ≃ N' → app M N ≃ app M' N'
  Λ : ∀ {V} {M N : Term (Lift V)} {A} → M ≃ N → Λ A M ≃ Λ A N
  imp : ∀ {V} {φ φ' ψ ψ' : Term V} → φ ≃ φ' → ψ ≃ ψ' → φ ⇒ ψ ≃ φ' ⇒ ψ'
```

The *strongly normalizable* terms are defined inductively as follows.

```
data SN {V} : Term V → Set₁ where
  SNI : ∀ {M} → (∀ N → M ↠ N → SN N) → SN M
```

Lemma 1. *1. If $MN \in SN$ then $M \in SN$ and $N \in SN$.*

2. If $M[x := N] \in SN$ then $M \in SN$.

3. If $M \in SN$ and $M \rhd N$ then $N \in SN$.

4. If $M[x := N]\vec{P} \in SN$ and $N \in SN$ then $(\lambda x M)N\vec{P} \in SN$.

```
SNappl : ∀ {V} {M N : Term V} → SN (app M N) → SN M
SNappl {V} {M} {N} (SNI MN-is-SN) = SNI (λ P M▷P → SNappl (MN-is-SN (app P N) (app M▷P

SNappr : ∀ {V} {M N : Term V} → SN (app M N) → SN N
SNappr {V} {M} {N} (SNI MN-is-SN) = SNI (λ P N▷P → SNappr (MN-is-SN (app M P) (app ref

SNsub : ∀ {V} {M : Term (Lift V)} {N} → SN (subbot M N) → SN M
SNsub {V} {M} {N} (SNI MN-is-SN) = SNI (λ P M▷P → SNsub (MN-is-SN (P [ botsub N ]) (sub
```

The rules of deduction of the system are as follows.

$$\frac{}{\langle\rangle \text{ valid}} \qquad \frac{\Gamma \text{ valid}}{\Gamma, x : A \text{ valid}} \qquad \frac{\Gamma \vdash \phi : \Omega}{\Gamma, p : \phi \text{ valid}}$$

$$\frac{\Gamma \text{ valid}}{\Gamma \vdash x : A} \ (x : A \in \Gamma) \qquad \frac{\Gamma \text{ valid}}{\Gamma \vdash p : \phi} \ (p : \phi \in \Gamma)$$

$$\frac{\Gamma \text{ valid}}{\Gamma \vdash \bot : \Omega} \qquad \frac{\Gamma \vdash \phi : \Omega \quad \Gamma \vdash \psi : \Omega}{\Gamma \vdash \phi \to \psi : \Omega}$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad \frac{\Gamma \vdash \delta : \phi \to \psi \quad \Gamma \vdash \epsilon : \phi}{\Gamma \vdash \delta\epsilon : \psi}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \to B} \qquad \frac{\Gamma, p : \phi \vdash \delta : \psi}{\Gamma \vdash \lambda p : \phi.\delta : \phi \to \psi}$$

$$\frac{\Gamma \vdash \delta : \phi \quad \Gamma \vdash \psi : \Omega}{\Gamma \vdash \delta : \psi} \ (\phi \simeq \phi)$$

```
mutual
  data valid : ∀ {V} {P} → Context V P → Set₁ where
    ⟨⟩ : valid ⟨⟩
    ctxV : ∀ {V} {P} {Γ : Context V P} {A} → valid Γ → valid (Γ , A)
    ctxP : ∀ {V} {P} {Γ : Context V P} {φ} → Γ ⊢ φ : Ω → valid (Γ ,, φ)

  data _⊢_:_ : ∀ {V} {P} → Context V P → Term V → Type → Set₁ where
    var : ∀ {V} {P} {Γ : Context V P} {x} → valid Γ → Γ ⊢ var x : typeof x Γ
    ⊥ : ∀ {V} {P} {Γ : Context V P} → valid Γ → Γ ⊢ ⊥ : Ω
    imp : ∀ {V} {P} {Γ : Context V P} {φ} {ψ} → Γ ⊢ φ : Ω → Γ ⊢ ψ : Ω → Γ ⊢ φ ⇒ ψ
    app : ∀ {V} {P} {Γ : Context V P} {M} {N} {A} {B} → Γ ⊢ M : A ⇒ B → Γ ⊢ N : A → I
    Λ : ∀ {V} {P} {Γ : Context V P} {A} {M} {B} → Γ , A ⊢ M : B → Γ ⊢ Λ A M : A ⇒ B

data _⊢_::_ : ∀ {V} {P} → Context V P → Proof V P → Term V → Set₁ where
  var : ∀ {V} {P} {Γ : Context V P} {p} → valid Γ → Γ ⊢ var p :: propof p Γ
  app : ∀ {V} {P} {Γ : Context V P} {δ} {ε} {φ} {ψ} → Γ ⊢ δ :: φ ⇒ ψ → Γ ⊢ ε :: φ → Γ
  Λ : ∀ {V} {P} {Γ : Context V P} {φ} {δ} {ψ} → Γ ,, φ ⊢ δ :: ψ → Γ ⊢ Λ φ δ :: φ ⇒ ψ
  conv : ∀ {V} {P} {Γ : Context V P} {δ} {φ} {ψ} → Γ ⊢ δ :: φ → Γ ⊢ ψ : Ω → φ ≃ ψ →
```