A Strongly Normalizing Computation Rule for the Univalence Axiom in Higher-Order Propositional Logic

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Homotopy type theory offers the promise of a formal system for the univalent foundations of mathematics. However, if we simply add the univalence axiom to type theory, then we lose the property of canonicity — that every term computes to a normal form. A computation becomes 'stuck' when it reaches the point that it needs to evaluate a proof term that is an application of the univalence axiom. We wish to find a way to compute with the univalence axiom.

As a first step towards such a system, we present here a system of higher-order propositional logic, with a universe Ω of propositions closed under implication and quantification over any simple type over Ω. We add a type $a =_A b$ for any terms a, b of type A (this type is not a proposition in Ω), and two ways to prove an equality: reflexivity, and the univalence axiom. We present reduction relations for this system, and prove the reduction confluent and strongly normalizing.

1 Syntax and Rules of Deduction

We call the following type theory predicative higher-order propositional logic. Its syntax is given by the grammar

$$\begin{array}{llcl}
\text{Proof} & \delta & ::= & p \mid \delta\delta \mid \lambda p : \phi.\delta \\
\text{Term} & M, \phi & ::= & x \mid \bot \mid MM \mid \lambda x : A.M \mid \phi \supset \phi \\
\text{Type} & A & ::= & \Omega \mid A \to A \\
\text{Term Context} & \Gamma & ::= & \langle\rangle \mid \Gamma, x : A \\
\text{Proof Context} & \Delta & ::= & \langle\rangle \mid \Delta, p : \phi \\
\text{Judgement} & \mathcal{J} & ::= & \Gamma \text{ valid} \mid \Gamma \vdash M : A \mid \Gamma, \Delta \text{ valid} \mid \Gamma, \Delta \vdash \delta : \phi
\end{array}$$

where p is a *proof variable* and x a *term variable*. Its rules of deduction are

$$\frac{}{\langle\rangle \text{ valid}} \qquad \frac{\Gamma \text{ valid}}{\Gamma, x : A \text{ valid}} \qquad \frac{\Gamma \vdash \phi : \Omega}{\Gamma, p : \phi \text{ valid}}$$

$$\frac{\Gamma \text{ valid}}{\Gamma \vdash x : A}\ (x : A \in \Gamma) \qquad \frac{\Gamma \text{ valid}}{\Gamma \vdash p : \phi}\ (p : \phi \in \Gamma)$$

$$\frac{\Gamma \text{ valid}}{\Gamma \vdash \bot : \Omega} \qquad \frac{\Gamma \vdash \phi : \Omega \quad \Gamma \vdash \psi : \Omega}{\Gamma \vdash \phi \to \psi : \Omega}$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad \frac{\Gamma \vdash \delta : \phi \to \psi \quad \Gamma \vdash \epsilon : \phi}{\Gamma \vdash \delta\epsilon : \psi}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \to B} \qquad \frac{\Gamma, p : \phi \vdash \delta : \psi}{\Gamma \vdash \lambda p : \phi.\delta : \phi \to \psi}$$

$$\frac{\Gamma \vdash \delta : \phi \quad \Gamma \vdash \psi : \Omega}{\Gamma \vdash \delta : \psi} \ (\phi \simeq \phi)$$

1.1 Extensional Equality

On top of this system, we add an extensional equality relation. We extend the grammar with

$$\begin{array}{llcl}
\text{Equality Proof} & P & ::= & e \mid \mathsf{ref}\,(M) \mid P \supset P \mid \mathsf{univ}_{\phi,\phi}\,(\delta,\delta) \mid \lambda\!\!\!\lambda\!\!\!\lambda e : x =_A x.P \mid PP \\
\text{Proof} & \delta & ::= & \cdots \mid P^+ \mid P^- \\
\text{Context} & \Gamma & ::= & \cdots \mid \Gamma, e : M =_A M \\
\text{Judgement} & \mathcal{J} & ::= & \cdots \mid \Gamma \vdash P : M =_A M
\end{array}$$

Note that, in the equality proof $\lambda\!\!\!\lambda\!\!\!\lambda e : x =_A y.P$, the term variables x and y and the proof variable e are all bound within P.

We add the following rules of deduction

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : A}{\Gamma, e : M =_A N \text{ valid}} \qquad \frac{\Gamma \text{ valid}}{\Gamma \vdash e : M =_A N} \ e : M =_A N \in \Gamma$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{ref}\,(M) : M =_A M} \qquad \frac{\Gamma \vdash P : \phi =_\Omega \phi}{\Gamma \vdash P \to Q : \phi \to \psi =_\Omega \phi' \to \psi} \Gamma \vdash Q : \psi =_\Omega \psi$$

$$\frac{\Gamma \vdash \delta : \phi \to \psi \quad \Gamma \vdash \epsilon : \psi \to \phi}{\Gamma \vdash \mathsf{univ}_{\phi,\psi}\,(\delta,\epsilon) : \phi =_\Omega \psi} \qquad \frac{\Gamma \vdash P : \phi =_\Omega \psi}{\Gamma \vdash P^+ : \phi \to \psi} \qquad \frac{\Gamma \vdash P : \psi =_\Omega \psi}{\Gamma \vdash P^- : \psi \to \phi}$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A \to B \quad \Gamma, x : A, y : A, e : x =_A y \vdash Mx =_B Ny}{\Gamma \vdash \lambda\!\!\!\lambda\!\!\!\lambda e : x =_A y.P : M =_{A \to B} N} \qquad \frac{\Gamma \vdash P : M =_{A \to B} M' \quad \Gamma \vdash Q : N =_A N}{\Gamma \vdash PQ : MN =_B M'N'}$$

$$\frac{\Gamma \vdash P : M =_A N \quad \Gamma \vdash M' : A \quad \Gamma \vdash N' : A}{\Gamma \vdash P : M' =_A N'} \ M \simeq_\beta M', N \simeq_\beta N'$$

2 The Reduction Relation

We define the following reduction relation on proofs and equality proofs.

$$\begin{gathered}
(\mathsf{ref}\,(\phi))^+ \rightsquigarrow \lambda x : \phi.x \qquad (\mathsf{ref}\,(\phi))^- \rightsquigarrow \lambda x : \phi.x \qquad \mathsf{univ}_{\phi,\psi}\,(\delta,\epsilon)^+ \rightsquigarrow \delta \qquad \mathsf{univ}_{\phi,\psi}\,(\delta,\epsilon)^- \rightsquigarrow \epsilon \\
(\mathsf{ref}\,(\phi) \to \mathsf{univ}_{\psi,\chi}\,(\delta,\epsilon)) \rightsquigarrow \mathsf{univ}_{\phi \to \psi, \phi \to \chi}\,(\lambda f : \phi \to \psi.\lambda x : \phi.\delta(fx), \lambda g : \phi \to \chi.\lambda x : \phi.\epsilon(gx)) \\
(\mathsf{univ}_{\phi,\psi}\,(\delta,\epsilon) \to \mathsf{ref}\,(\chi)) \rightsquigarrow \mathsf{univ}_{\phi \to \chi, \psi \to \chi}\,(\lambda f : \phi \to \chi.\lambda x : \psi.f(\epsilon x), \lambda g : \psi \to \chi.\lambda x : \phi.g(\delta x)) \\
(\mathsf{univ}_{\phi,\psi}\,(\delta,\epsilon) \to \mathsf{univ}_{\phi',\psi'}\,(\delta',\epsilon') \rightsquigarrow \mathsf{univ}_{\phi \to \phi', \psi \to \psi'}\,(\lambda f : \phi \to \phi'.\lambda x : \psi.\delta'(f(\epsilon x)), \lambda g : \psi \to \psi'.\lambda y : \phi.\epsilon'(g(\delta y))) \\
(\mathsf{ref}\,(\phi) \to \mathsf{ref}\,(\psi)) \rightsquigarrow \mathsf{ref}\,(\phi \to \psi) \qquad \mathsf{ref}\,(M)\,\mathsf{ref}\,(N) \rightsquigarrow \mathsf{ref}\,(MN) \\
(\mathsf{ref}\,(\lambda x : A.M))P \rightsquigarrow \{P/x\}M
\end{gathered}$$

3 Proof of Strong Normalization

4 Future Work